Mathematical Finance

Lecture 35 Multidimensional BSM Model, Fundamental Theorems of Asset Pricing

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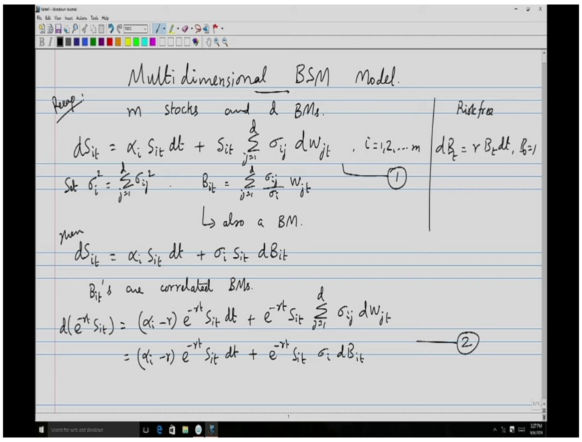

Hello everyone! In the previous lecture we just started this multidimensional Black-Scholes-Merton model with m stocks and d Brownian motions. Remember the stocks here whatever we are assuming so far in the classical BSM framework means they do not pay out any dividend. So this is dividend free stocks is what we are considering okay. And then in this multidimensional case especially we have m stocks and each of them driven by d Brownian motions with the corresponding coefficient, division coefficient given by σ_{ij} is the constant.

And this is the description in terms of the d dimensional Brownian motion which is given by w_j where there are d of them. Now, once we set this define B_{it} which we also showed that this is also a Brownian motion and in terms of B it we can also write dst as given here which is basically the expression that we have seen here, right. So this is the expression that we have right where σ_i is nothing but the mortality of i-th stock, and this B_{it} are typically correlators and we have seen the correlations we are expressing in terms of this $\sigma_i, \sigma_k, \sigma_{ij}, \sigma + kj$ and so on which we have seen.

And this is the equation 1 is the description of risky assets, there are m of those, m stocks, which pays no dividend, and there is free assets the description is as given on the right side here which is usual. If you look at the discounted asset-price process of the risky assets then that is given by the expression 2. We can interchangeably refer to either the first expression on the right hand side or the second expression on the right hand side as so referring the equation 2 right. This is what we have done in the previous class or when we started this discussion on this. Just quickly we will go through this part before we go further.

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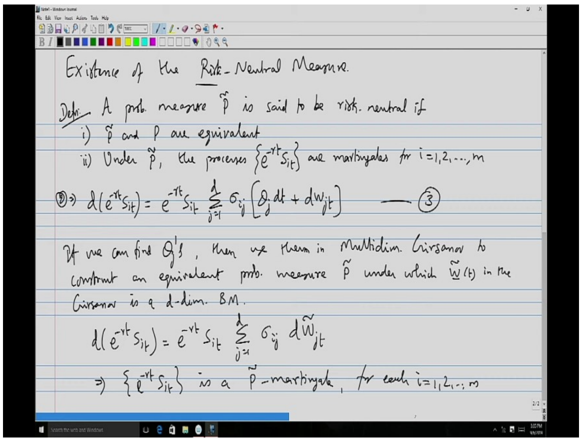

Now, the next thing that we will look at is basically existence of the Risk-Neutral measure. This is what we look at next okay. Now, we will precisely define, we have already defined it but in this multidimensional set up, we will precisely define what is a risk-neutral measure which is true even in the one-dimensional case. So, a probability measure say $\tilde{P}$ is said to be risk-neutral if it satisfies two conditions as we already know, one $\tilde{P}$ and P are equivalent which means if for any A the probability of A is equal to zero then $\tilde{P}$ of a particular set A also must be equal to zero, right. So they agree on what we call null sets of probability measure.

Now, under $\tilde{P}$, the processes which is basically $e^{-rt}S_{it}$, they are all martingales or i is equal to 1, 2, m. As we said already the risk free asset is already a martingale. We do not need to say that we can take that as the discounted factor. So that will be a constant one throughout. So, all the risky assets we need to be martingale if this is ensured then such a probability measure is what we call it as the risk-neutral measure, okay.

Now, in order to make the discounted stock price process or martingales, we would like to rewrite 2 in the following form. So, now 2 implies that your $d(e^{-rt}S_{it})$. So this is suppose if I call this as 3, equation 3 right. Now if we can find theta j now recall from one dimensional process what is this meant in our case there that this is, the market price of risk quantity.

So this is here the market price of risk quantities because there are 3 of those, okay. If we can find Θ_j such that this 3 holds true with one such Θ_j for each W_j for each source of uncertainty, then we can use the multidimensional, okay. So, if we can find theta j's right, one each for each W_j corresponding to that that makes three trues means two must equal three. If we can find theta j we can write in this form then we can use these Θ_j then use them in Multidimensional Girsanov.

If we can use them as the theta's required in the multidimensional Girsanov theorem to construct an equivalent probability measure, use them to construct an equivalent probability measure. Suppose, if we call that as $\tilde{P}$ under which the $\tilde{W}_t$ in the Girsanov is a d-dimensional brownian motion. So, this permits as to write the above that is equation 3. So, because we said this is a vector each one of them, there are d of those in the Girsanor so that is what we have.

And what this implies, and this implies because this has only dW term there is no dt term, this implies that $\{e^{-rt}S_{it}\}$ is $\tilde{P}$ martingale, this is true for each i, this is true for each i. So, the what we have now is that if we look at the discounted price process, if we can find theta j then we can use those theta j's in the multidimensional Girsanor to obtain or to construct an equivalent probability measure $\tilde{P}$ under which this process which is what essentially in 3 whatever is there in the square bracket is what we call as dW tilda jt.

So that will be our martingale, sorry, so that will be a brownian motion and hence we could write this in this form which makes this as the $\tilde{P}$ martingale as required. Because Girsanor by default gives with z with appropriate properties of z that it gives an equivalent probability measure. We only need to ensure the second condition which is ensured if there is a solution to this set of equations that we have here, right. So that is what we are having it.

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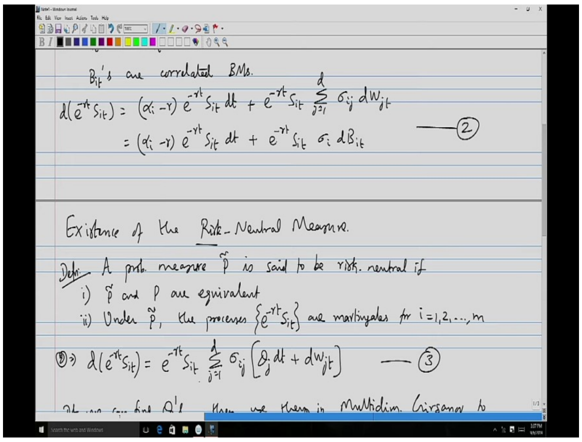

So, now the problem of finding the if we can if have we have theta j then all of this equals to true. So that means to find $\tilde{P}$, now what we need to know, what we need to do, is that we need to find theta j such that 2 and 3 are equal, 2 can be written in terms of as equal to 3. Now, if you look at 2 and 3 okay, so look at the first line. For example, of 2 and 3, you would see that the dW terms are the same, okay. To look at this expression here and 3 here, you look at the dW terms. These are exactly the same right that means then for this two equal then dt terms are also has to be the same. So, that gives, that means the coefficient here and the coefficient here for the dt terms must equal.

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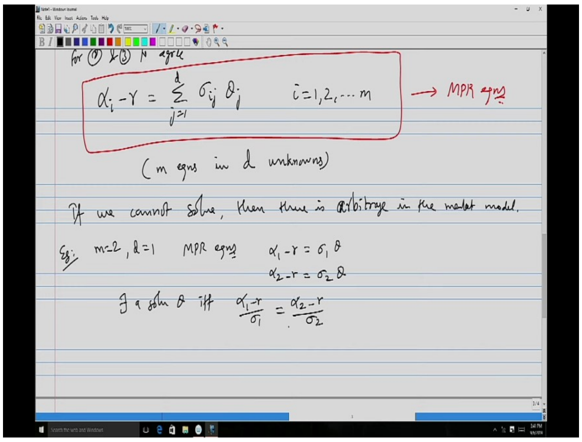

So, that means so for 2 and 3 to agree what we need is that this must be true,

$$\alpha_i - r = \sum_{j=1}^{d}\sigma_{ij}\theta_j$$

for i is equal to 1, 2.., m, okay. So remember this is the main equation which we call MPR Market Price of Risk equations. And here how many we have we? We have, this is a set of m equations in d unknowns, m equations in d unknowns.

If we can solve this for theta, if we can solve this for theta then we can find use that theta to find a $\tilde{P}$ which is the risk- neutral probability and that is what our aim. Now if we can solve for this that is good but if e cannot solve okay then what happens then there is arbitrage in the market model, okay. To see this let us quickly look at this example so that you can understand how the arbitrage opportunities exist if we do not have. Remember this is an m equation in d unknowns, as long as the number of equations is less than the number of unknowns, you have more chance of getting solutions here right.

But the moment it is the other way around then you have less chance of getting the solutions right in general, right. So that is what we will have here. So here if you look at an example of that nature so if then there is arbitrage will be there in the model and that model is bad and we should not use the model for any other purpose, especially we are looking at for pricing aspect so it should not be used for it.

So, now, let us take the case of m is equal to 2, d is equal to 1 which is the other way around. In that case what will happen so the MPR equations right. So there exists a solution theta if and only if this ratio

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$

If this does not hold, if this does not hold, then there is no solution to this because this is one unknown in two equations. So which is equivalent solution only if this true if not there is no solution.

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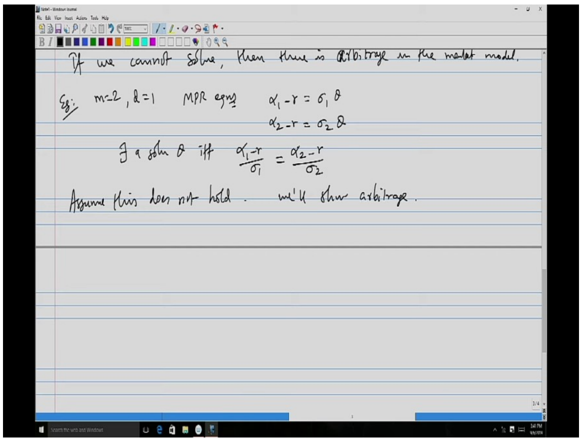

So, now if this does not hold, if this does not hold, right, this is equality, then what happens then one can arbitrage one stock against the other to make a risk free profit. Let us see how we can do that okay.

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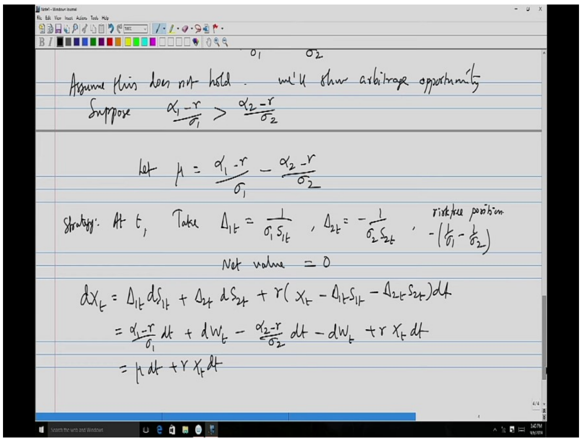

Assume right, assume that this does not hold, now we will show arbitrage opportunity, we will show arbitrage opportunity in such a scenario. For which first let us suppose, for simplicity you assume that the first one, the first ratio is greater the second ratio. And this difference let us call that as for simplicity mu alpha 2 minus r let us call this difference as simply mu.

Now, what one can do the strategy is you know at take Δ_{1t} and Δ_{2t}. So this, we are taking a long position in a stock which has the higher ratio $\frac{\alpha_1 - r}{\sigma_1}$ which is the MPR higher market price ratio. And we are taking short position in the other stock. Now to construct at this time so these two position, right, these two position would require the initial capital as $(1/\sigma_1 - 1/\sigma_2)$.

So the risk free position is essentially the $(1/\sigma_1 - 1/\sigma_2)$ case, but this is what is required for the stock positions here so the risk free position would be negative of this. So the negative of this would be the risk free position which means if for this $(1/\sigma_1 - 1/\sigma_2)$ is positive then without this minus, the quantity which is inside the bracket, if this is positive, then we will borrow that much from risk free and if this is negative that much we will invest in the risk free position.

So, this is the position, so the net value of at time t is 0. So, which means at time t the net wealth after making this strategy is 0. Now, look at what is the evolution of such a portfolio which will be dX_t.

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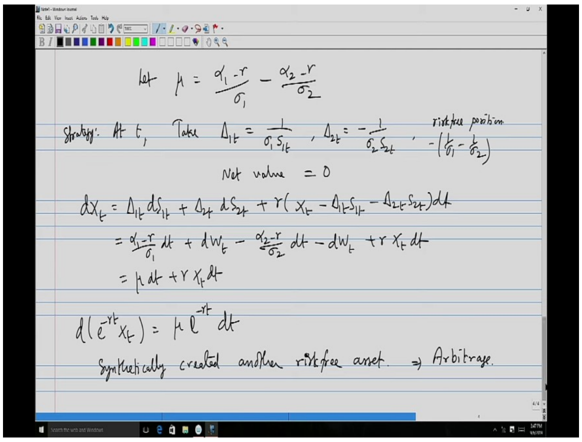

So, if I look at the discounted value of this, this will be mu e to the power minus rt dt, okay. So either from the last line or from the previous line, you can (clear) see that we have synthetically created another risk free asset, right, another risk free asset because this rX_t is what then we should be having it such situations right but we have created another one which grows little bit more than r.

If mu is positive of course in this case mu is positive that is how we have taken that is what our position? So, this μ is positive, this X_t right gives a return which is more than the risk free rate of return. So and hence, which implies there is arbitrage opportunity in this such a situation. Now, if you created another risk free asset, now the arbitrage opportunities are endless. So one can go ahead to create more and more situations in this particular scenario that you can look at.

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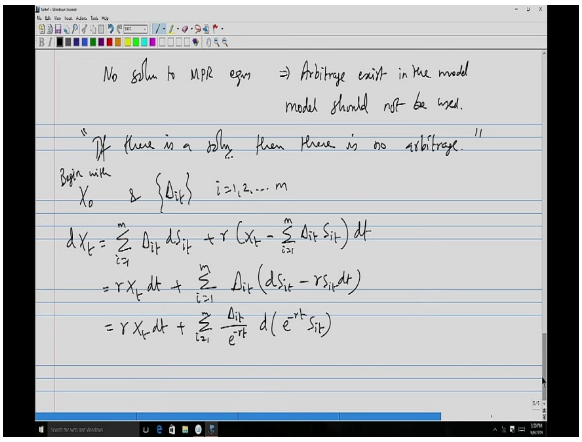

So, which means that now what we mean that if there is no solution to MPR equations right that implies arbitrage exist in the market in the model. And the model should not be used because arbitrage is there. If there is a solution, if there is a solution, then there is no arbitrage. So, this is what you we need to now show. So, to show this essentially what what we do is the following, you look at the situation where we begin with some initial wealth X0 okay begin with some initial wealth X_0 and Δ_{it} processes for 1 is equal to 1, 2, m.

You have m portfolio position process is given by delta it one for each stock, then the portfolio evolution will be given by this $\sum_{i=1}^{m} \delta_{it} dS_{it}$ by the position in these m stocks plus rX_t minus summation is equal to 1 to m delta it S it times dt right. So the second term is the risk free assets contribution and the first term is the m stock contribution which you can see you can write in that form.

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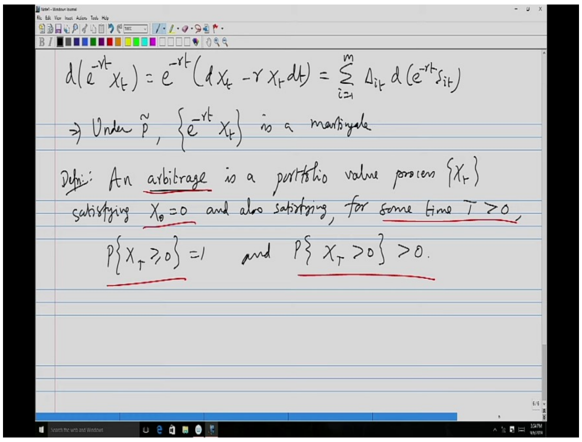

So, now, if look at the discounted value, the discounted value of this then would be $e^{-rt}(dX_t - rX_tX_tdt)$. Now, we already know the expression for this in terms of the W_j so if you write that we would see what this is. So, what we have now is under, what does this means under $\tilde{P}$ $\{e^{-rt}X_t\}$ is a martingale, because, his you go back to the expression that we have written before this example for this e to the power minus.

You see look at this expression here, d to the power this expression here. So this is what gives us you know the expression that this is true. So basically what? If you have so if you have a $\tilde{P}$ or if you have a risk neutral measure $\tilde{P}$, X_t would be the portfolio value process corresponding to a portfolio which gives position for all the m stocks points at each time point. And hence, the risk free assets position too because what we are forming is self-financing portfolio portfolios.

Then under that $\tilde{P}$a this is the martingale okay. Now, so this is what then as the result what we got now. So, the discounted value process portfolio value process is a martingale is what we have got. Now, we will define the arbitrage also in a precise manner before we state our next result, which is which was one of the important results. An arbitrage is a portfolio value process, now that we have given the portfolio value process we are defining it now otherwise the ideas are clear before for us. Portfolio value process X_t satisfying $X_0 = 0$ and also satisfying for some time capital T which is strictly greater than 0, the following two conditions.

And what is that $P(X_T \geq 0) = 1$ and $P(X_T > 0) > 0$. So this is what is the arbitrage conditions or (this) for some T greater than 0, right. So an arbitrage is a portfolio value process satisfying this condition what does that mean? You start at 0 wealth there is some time in future at which you have probability of losing its 0.

That is what the first condition says in terms of probability which is probability of non negative wealth is 1 and second part says that you have positive probability of earning a positive amount. So which means the probability that X_t is strictly greater than 0 is strictly greater than 0, which means you come with zero wealth, you have no way of losing money and you have positive probability of earning some positive money. Such a situation is what we call it as an arbitrage opportunity here.

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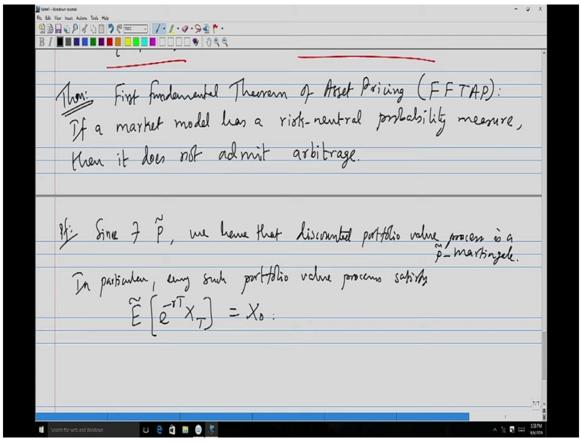

So, now, once we define this arbitrage opportunity now we are into our one of the main result which is called First Fundamental Theorem of Asset Pricing or, FFTAP, we call its short. What is that? That is the following, what does the First Fundamental Theorem of Asset pricing say. If a market model has a risk-neutral probability measure, then it does not admit arbitrage. Remember this is one way implication which is what will be the general result.

In special cases of course this may mean both way which means existence of a risk-neutral measure implies and implied by the no arbitrage condition but in this particular case which is more general, it is one way implication that existence of risk neutral measure implies no arbitrage okay. So the proof is simple we can go through, we may not go through the other proofs but at least this is important result just like you know fundamental theorem of any other fields so this is also called a Fundamental theorem of asset pricing or sometimes fundamental theorem of finance, it is also sometimes used. So, now what we have, since there exists a $\tilde{P}$ which is a risk neutral measure then we have that discounted measure by property is that it makes the discounted portfolio processes so the discounted okay so the discounted portfolio value process is a martingale under is a $\tilde{P}$ martingale that is what we have just shown. In particular, every such portfolio value processes satisfy what $\tilde{E}[e^{-rT}X_T] = X_0$. How did we get this? Because this is a martingale, martingales have constant expectations and hence this is true.

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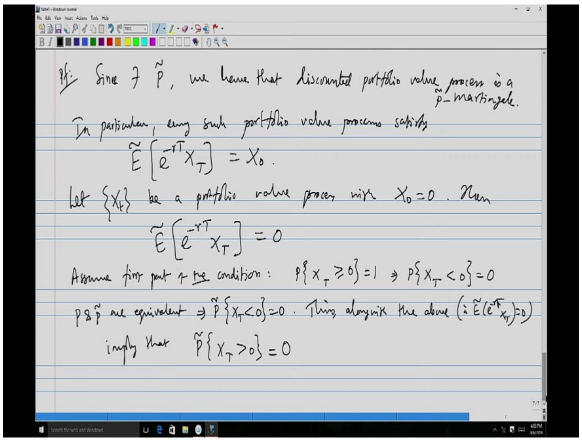

Let X_t be a portfolio value process with $X_0 = 0$. Then, what we have, and then we have E tilda this expression becomes $\tilde{E}[e^{-rT}X_T] = 0$ okay. Now, so this is what we have. Since there is a risk neutral probability measure, you know let us take a portfolio value process is equal to 0 then for that portfolio value process at that time capital T, this expectation under this $\tilde{P}$ is 0.

Now, what is the arbitrage opportunity, so look at this $X_0 = 0$ we have taken care. Now, this both this condition also must satisfy okay. Suppose, if we show that the first condition satisfied implies that the second condition will not be satisfied then we are done that there can be no arbitrage in this case.

Assume the first condition now. Now suppose the first part of the condition assume the first part of the condition, means the condition for the arbitrage which is basically $\tilde{P}$ right.

So, which we have written here for any probability but still we are talking about arbitrage in that sense so that is what you have okay, so we will show with P. Because anyway this is all so this is true, you will know never mind you will know what we are talking about. This is greater than or equal to 0 is 1 this implies probability of X_t less than 0 is 0, probability of X_t less than 0 is 0 is what we have.

Now, P and $\tilde{P}$ are equivalent implies $\tilde{P}(X_T < 0) = 0$ because they should agree on the sets of probability 0. Now this along with the above which is basically the $\tilde{E}[e^{-rT}X_T] = 0$ imply that my $\tilde{P}$ of X at Capital T greater than 0 must also be equal to 0. Otherwise if this is not the case what will happen? You can quickly see.

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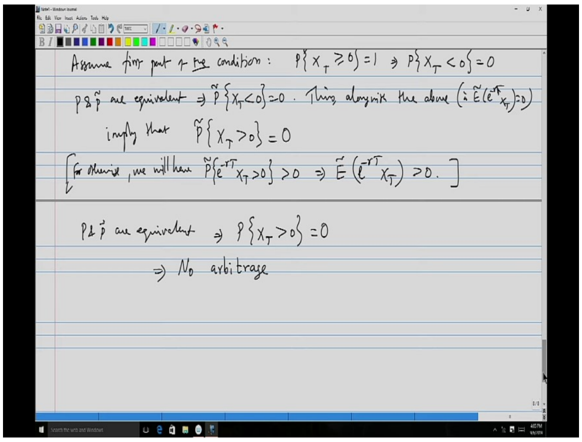

If this is not the case if this is not the case we would have otherwise the discounted value of this $\tilde{P}$ of E to the power minus capital T, okay. So let us write that or otherwise, what will happen we will have $\tilde{P}(e^{-rT}X_T > 0) > 0$ is greater than 0 right because we assumed that $\tilde{P}(X_T < 0) = 0$. So that would imply this is what is true, which would imply further right that this is $\tilde{E}(e^{rT}X_T) > 0$.

So if this not the case then what that would imply is essentially that this is this expectation is strictly greater than 0. Now, again P and $\tilde{P}$ are equivalent implies P of X_T greater than 0 is also equal to 0 which means the second part of the condition which means no arbitrage. Hence, X_T is not at arbitrage possibility right. So, in fact, there cannot exist any arbitrage because in every portfolio that starts at 0 would follow almost the same path same argument so there cannot be arbitrage in the market. So this is what is the first part of the our ideas that what we wanted to do which is essentially that the first fundamental theorem which says that if the market model has a probability measure then there is no arbitrage in the model. And this plays a crucial role because we want to build a model where there is no arbitrage. If there is arbitrage lurking in the model which may be explicit as we have seen in the previous example or which may not be explicit.

But there is arbitrage in the model then any answer or anything that you derive out of this model will not have any meaning at all because it will give rise to so many problems in terms of the answers or prices or anything it gives because it will not make sense. So one should never use prices derived from a model which has arbitrage opportunity. So your foremost primary concern should be that to build a no arbitrage model so that there is fair price being derived from by using the model for whatever purpose that we are using it here. So that is why this plays a major role.

And if you see if you go further, in especially, in the interest rate model you know you take this as the base so you specify the model itself under a no arbitrage condition or under this risk neutral probability measure. So that is what is the advantage, so this first fundamental theorem of asset pricing gives a condition okay that when the model will be arbitrage free means we should have a risk-neutral measure which boils down to finding a solution to the mark price of risk equations.

If you can find a solution to the market price risk equation models right then there is no arbitrage in the model, I mean you will conclude that way. So this solution exists as you see is that number of equations is less than or equal to d is what a more general situation is that you have a more possibility or a more chance that a solution to such a system exists right. There will be infinite number of solutions does not matter at least there is a solution exists.

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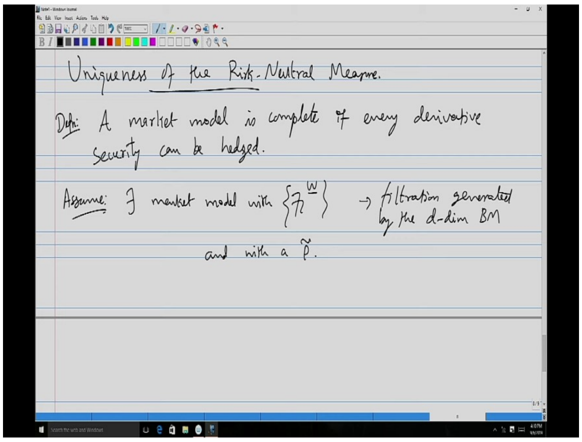

Now, let us look at the other aspects of this theory which is basically the connected with the uniqueness of the Risk-Neutral measure. We talked about existence, now we talk about uniqueness, before we go we define a definition which is we have already said but we still define it. A market model is complete it is already what we mean by this if every derivative security can be hedged. By stating this itself, you could now feel that where e are connecting this uniqueness or anything. It is with respect to the hedging component that we are looking at here.

So, what we have now for this again just like in the one dimensional case, we assume that there is a risk-neutral measure and we also assume that the filtration is generated by the d dimensional brownian motion because that is what you know we did in the case of one dimensional model yes well. So, what we have. So our assumption is, so you can bring down to the previous case itself okay. Let us suppose there is a market model with filtration generated by the d dimensional Brownian motion.

So this is F bar W means the filtration generated by the d-dimensional brownian motion and with a $\tilde{P}$ with a risk neutral measure, we assume that there is neutral measure. That means what we have solved for the market price risk equation to get a solution , which we have used in the Girsanor theorem to come to get an equal probability measure $\tilde{P}$ that is what is the risk neutral measure because that makes the underlying asset pricing as martingales and makes the market also as arbitrage free so that is what you know.

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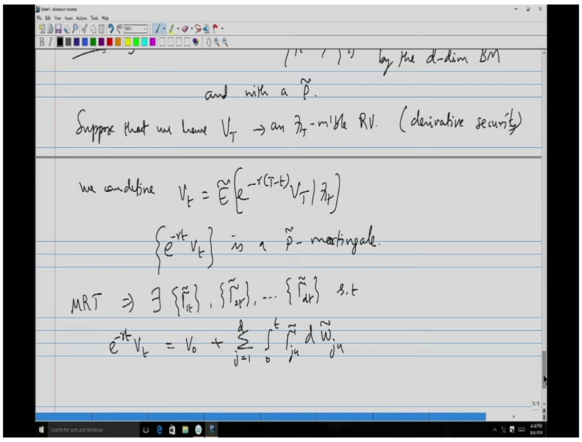

Now, we consider, suppose further that we have, suppose that we also have V_T and $\mathcal{F}_T$ measureable. So, here this is what this our $\mathcal{F}_T$ that we are picking it up right so then FT measureable where this is the d-dimensional brownian motion that we have which is what in our sense is derivative security that is what we have here. So, which is the pay off of some derivative securities what then we call it as V_T, now it could be $(S_T - K)$ or it could be, you know, now there are multiple assets so it could be a multidimensional asset derivative whatever it is, it is the an $\mathcal{F}_T$ measurable by a d-dimensional Brownian motion.

Now, we can define, now what we wanted to do we can, we want to make sure that we can hedge a short position in this derivative security whose pay of off is what is this V capital T. So, we can define as earlier our $V_t = \tilde{E}[e^{-r(T-t)}V_T|\mathcal{F}_t]$ and observe for that what we have here my $\{e^{-rt}V_t\}$ is a $\tilde{P}$ martingale. So, by this expression we have already seen in the one dimensional case is exactly the same that we are going to repeat in the program.

And just as we have done earlier so this is $\tilde{P}$ martingale that is what we see. Now my MRT implies there exists processes say $\{\tilde{\Gamma}_{it}\}$ right such that your $e^{-rt}V_t$ is equal to that.

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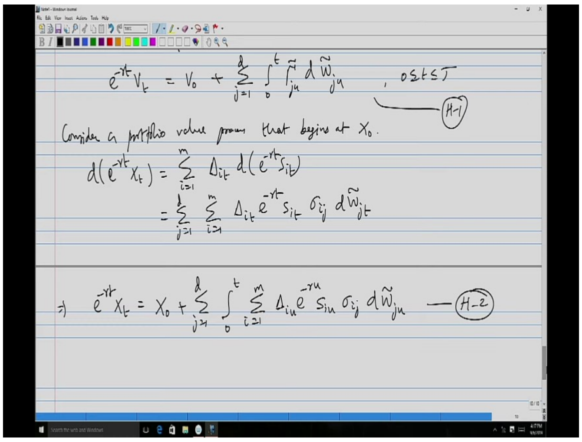

Suppose if, if you look at this 0 less than or equal to t less than or equal to capital T. So, this is the bi MRT what we have is that since this process is a $\tilde{P}$ martingale so that means there exists some d adopted processes, there exists adopted processes such that you can present this $\tilde{P}$ martingale in terms of a constant plus some d number of ito integrals, one with each with respect to each of these W_j Brownian motion.

Now, this part is the first part that we have done Now similarly, consider a portfolio value process that begins at X_0. Now, according to the earlier expressions that we have written, the discounted value of this we have written down. Recall, in this itself, in the earlier part we have written. This means now we write in the integral form. Then finally, we get the expression for $e^{-rt}X_t$.

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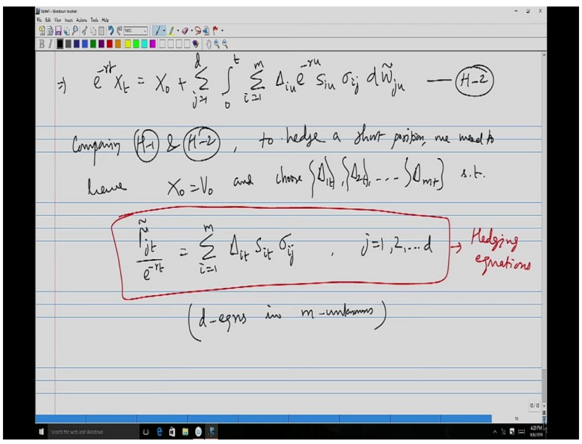

Now, let me call this as an equation say H-1 this equation H-2, now if I compare H1 and H2 so in order to hedge a short position right what we need is that so comparing H-1 and H-2, so to hedge a short position, we need to have $X_0 = V_0$ and choose Δ_{1t}, Δ_{2t}, and so on. All these things are processes, you can write this as processes such that what we have divided by e^{-rt} or I could write this in the upper part.

So, by comparing these two to hedge a short position, we have $X_0 = V_0$ and $\Delta_{1t}, \Delta_{2t}, .., \Delta_{mt}$ is m processes are to be selected such that this equation is true and this is a what we call it as the Hedging equations. So in the one dimensional case there was only one equation, so we simply wrote Δ_{it} to be equal to something which is exactly the similar form if I assume there is only i here. So this is what is the multidimensional version of this.

Now, if I look at here these are d equations in m-unknowns, so this is what is Hedging equations. The market price of risk equations were m equations in d-unknowns and hedging equations or d-equations in m-unknowns. Now, here more chance for solution if m is greater than or equal to d, if the number of unknowns is more than the number of equations then there is more chance for solutions whereas it works just opposite in the other case market price of risk equations where we have the opposite clearly.

So that is what the contrasting thing you would see is. Now, if you want an arbitrage free model you will tilt towards the other side. If you want a complete model you will tilt towards this side but then the question is which is more important, arbitrageness or completeness. So, in reality you see that the arbitrageness is more important in the sense that you know it should be fair, and in reality most of the markets that you find are that exist are all incomplete markets. I mean which is not complete in the sense that what we generally term it as incomplete markets. So, that is what is the case that you would find, so which one is giving more importance in those case.

So, now this one this hedging equations a solution to this hedging equation ensures that the market is, which means that you can find delta it and hence the derivatives security can be hedged. So that means as far as for the moment we forget about the arbitrageness, so we want to look at the hedging equation and the completeness so that a short position in every derivative can be hedged. Now when that is possible when there is a solution to this set of equations. So that is what we call the hedging equations.

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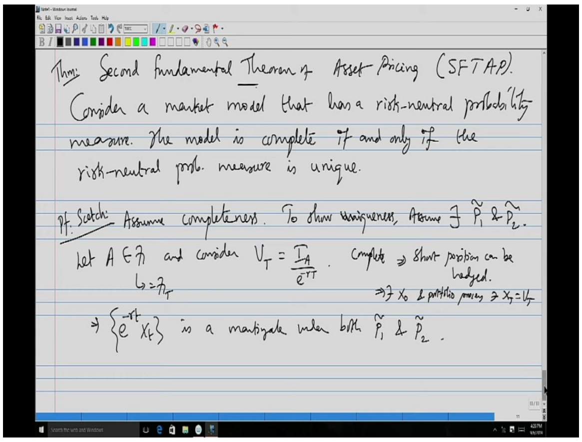

And the result connected to that is what we call as the Second Fundamental Theorem of Asset Pricing. Now, what does it says is the following. Consider a market model a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability is unique right. So, when we can do that when the model is complete is when this risk-neutral measure is unique.

Now, proof we will not do completely we will just sketch the proof, first part is easy. So assume completeness which means the model is complete which means every derivative security can be hedged. Now we want to show the uniqueness of the risk-neutral measure. How do we show? As usual that there are two risk-neutral measures and show that they are one and the same. So to show uniqueness, assume there exists $\tilde{P}_1, \tilde{P}_2$, right.

Now, let us me pick and consider a derivative security V_T which is essentially indicator function A times e^{-rt} divided by the discount, okay. Now, model is complete which means that the short position in this particular derivative can be hedged the short position in this by the way this $\mathcal{F}$ is actually equal to $\mathcal{F}_T$ so that this is fine. Short position in this can be hedged that means what there is a portfolio value process with some initial wealth condition that satisfies $X_T = V_T$.

So this means that what then if I look at my, so completeness this implies that short position can be hedged which means that there is an X_0 under delta process such that you have $X_T = V_T$. And this so that also implies since both of them are risk neutral measure. So these are this process is $e^{-rt}X_t$, so that is what there exists an X_0 okay. And the portfolio process such that $X_T = V_T$. Now, if I look for that X discounted value process is a martingale under both $\tilde{P}_1, \tilde{P}_2$ because both are risk-neutral measures. So it must be true under both this is true.

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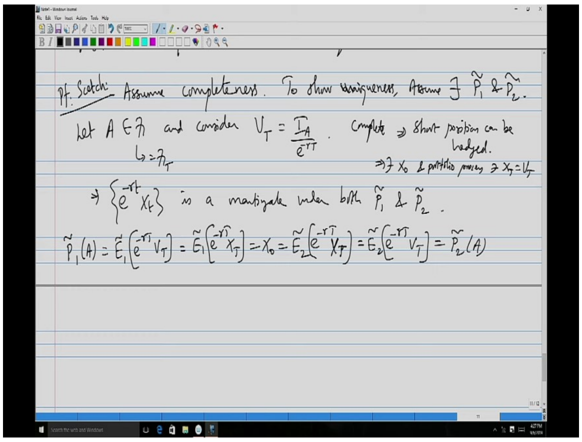

Now, I look at what is this

$$\tilde{P}_1(A) = \tilde{E}[e^{-rT}V_T] = X_0 = \tilde{E}_1[e^{-rT}V_T] = \tilde{P}_2(A)$$

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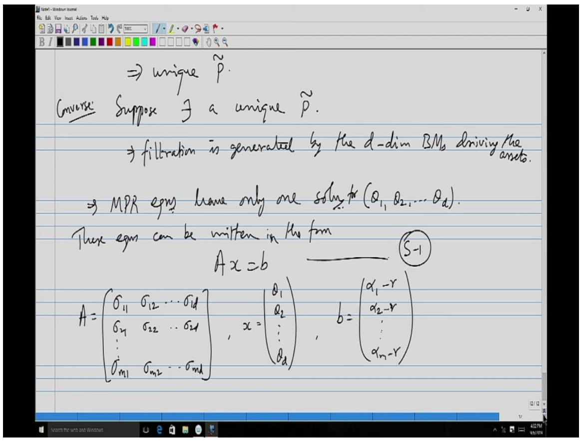

Converse is basically suppose there exists a unique $\tilde{P}$ okay there is a unique $\tilde{P}$. So this means first of all that the filtration for the model is generated by the Brownian motion alone by the d-dimensional Brownian motion driving as such. If not if that is not true then what happens one could assign arbitrary values to the other processes by keeping the same d and get different this neutral probability measure so that is not possible.

This means the first part is that this is basically so this implies so that the filtration must be generated by he d-dimensional Brownian motions which are driving the assets okay. It cannot be otherwise, otherwise then we will have multiple risk-neutral measures. Now, because the d-dimensional Brownian motions are the only source of uncertainty the only way multiple risk-neutral measures can come, so this is the only way that the risk-neutral measures come is via so this means the multiple solutions to the market price of equations okay.

Market price of, so if this is the case then the only way multiple $\tilde{P}$ can come is that is only through the market price of risk equations. Now, if the market price of risk equations has multiple solutions then corresponding to each theta you will have one $\tilde{P}$ and hence, you will multiple solutions. Now, the unique $\tilde{P}$ means the MPR equations have only one solution which means there is a unique solution to these MPR equations and hence you get the only one solution which is the $\theta_1, \theta_2, ..., \theta_d$. Only one solution for this theta d and hence there is only one $\tilde{P}$.

So uniqueness implies that the MPR equations give a unique solution. Now, what is this equation, these equations you can write in the matrix form. These equations can be written in the form say $Ax = b$, where A, bx are given as in figure. So, this is what you have as the vector form, so what we are saying is that this is say suppose if I call this as S-1.

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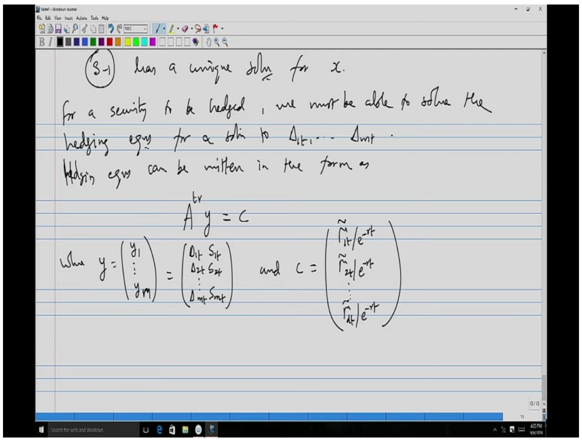

So, what we have, what we have said is that S-1 the equation S-1 has a unique solution for x, it is a thing we are looking at so this is for a fixed time, for a fixed T and omega this is what you will get here means allowing a single path. Now, in order to be assured that the derivative security to be hedged completely. Now for a security to be hedged we must be able to solve the hedging equations for a solution to delta 1t and so on and delta mt for a fixed time t, no matter what the other quantity appears.

Now, this equation so this hedging equation can be written in the matrix form as. How we can write? $A'y = c$, A is the same as the earlier, $y = (y_1, ..., y_m)$. And C is basically this quantity.

So, if as derivative security should be hedged in order to be assured that the market is complete, then we must be able to find Δ_t and this hedging equation then can be written in this form. Now if no matter what is the whatever value of your y whatever sorry whatever value of C. If this system can be sure to have a solution okay then we are done no matter what the C is then we can find a portfolio processor delta t such that every derivative security can be hedged.

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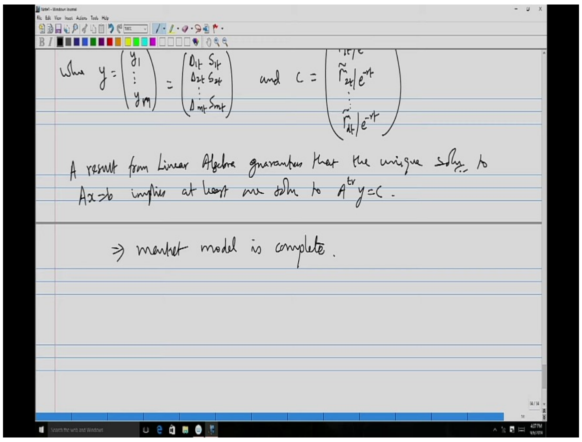

A result from Linear algebra guarantees that the unique solution to Ax is equal to b implies at least one solution to A transpose y equal to C for whatever. And hence, the uniqueness to $Ax = b$ implies the existence of the solution y for any value of C that is given here. So, this implies that the market model is complete. For someone who wants to look at this particular result from linear algebra or you can look in the appendix of the reference text, which gives the proof of that part so that guarantees this. And hence, this is what the market model is complete.

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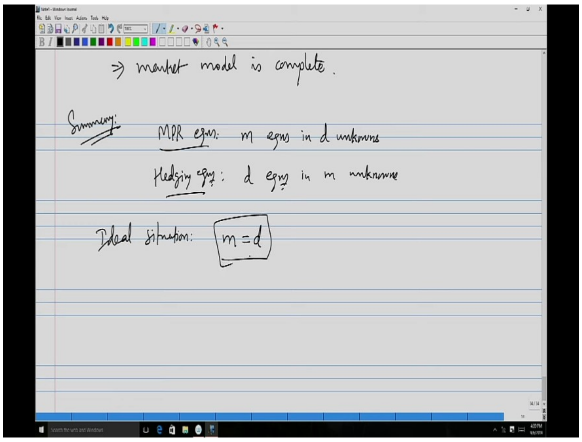

Now, what is the summary here? You have the MPR equations, which have d, first we will write the equations. M equations in d-unknowns, to count for this theta 1 to theta d, then hedging equations is essentially, now you have d equations in m-unknowns. Now, out of which you give importance generally to MPR equations because you want the model to be arbitrage free, so you want the solution to this m equations in the d-unknowns.

And so that gives rise to at least solution if you can have. So, the ideal situation is m equal to d, in this case what you will have, you will have a unique solution, you want to ensure at least one solution, so you want to have at least one solution. And also if by ensuring the one solution you are ensuring the market's completeness. So, this is the ideal situation which one looks for, so which we will have, which means the number of assets and the number of driving brownian motion that you take in the model must be equal. If not you give importance to this MPR equation to have this arbitrage free model, which means the number of unknowns is more than the number of equations, which means d is more than m. So, if you have more unknowns and less number of equations you have more chance that you can get a one solution, it could be infinitely many solutions that is not a problem. It will have a one solution, the ideal situation is this.

Otherwise, if you have multiple solutions to MPR equations means the model is incomplete, the market model is incomplete. In that case what will happen is that you cannot completely hedge a derivative, every derivative security. There will be some derivative security which you will be able to hedge and other derivative securities which you may not be able to hedge completely and hence the market model is incomplete.

So, these two fundamental theorems gives the conditions for arbitrage freedom of the market when you can ensure or when you can show that the model is arbitrage free and when the market is going to be complete. So, this is what these two fundamental theorems now gives us. Thank you!