Mathematical Finance
Lecture 34: MRT and Hedging, Multidimensional Girsanov and MRT

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Hello everyone, what we have seen so far is the classical BSM model, classical BSM model by which we mean that with one stock and with constant parameters. Wherein the price of that one stock, the price process essentially is given by the geometric Brownian motion. And the growth factor is simply given to be the continuous time compounding factor, which is basically.

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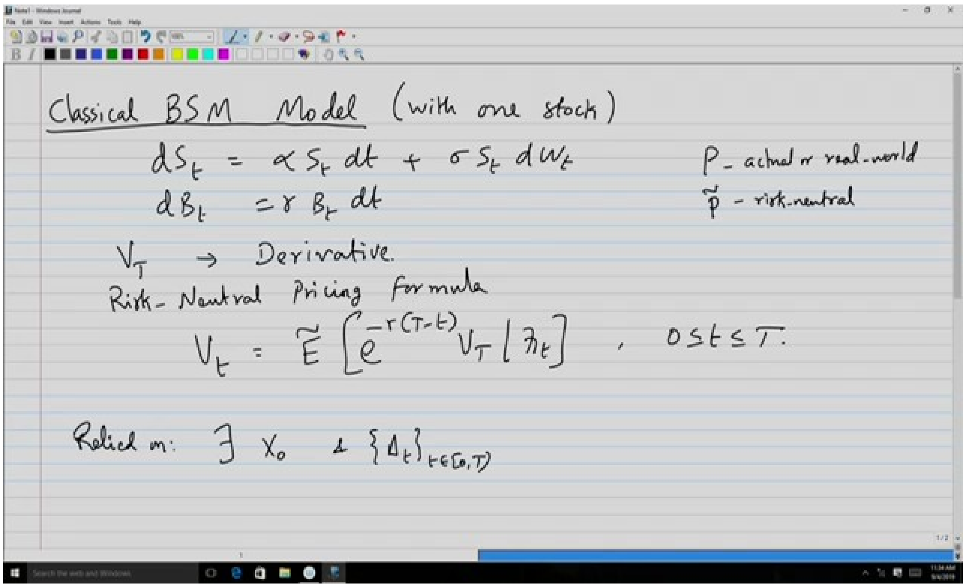

And we described or we described this model with these two equations which is under the actual or real-world probability measure P. And we considered V_T to be a derivative, which is simply an $\mathcal{F}_T$ measurable running variable. It could be a path independent or it is path dependent. Then we derived the risk neutral pricing formula, which is given by this expression here.

Which is

$$V_t = \tilde{E}[e^{-r(T-t)}V_T|\mathcal{F}_t]$$

which, so this is the price at a time t, this is no arbitrage price at time small t of a derivative, which is going to mature at capital T. Of course, this is European derivative what we have, then its price at time t is simply given by this particular conditional expectation right. So, see now where the conditional expectation we are placing.

So, the price of (before time) at any time before maturity of a derivative is given in terms of the conditional expectation. So, that is why we said the conditional expectation place a critical in major role in this risk neutral pricing theory. Of course, if you are using replication in other way around of course which is what called APT approach to the pricing. Then of course you will not encounter this risk neutral pricing measure right. So, where we also have two probability measure P, which is actual or real-world probability measure. $\tilde{P}$, which is basically the risk neutral probability measure. and what is the properties of this risk neutral measure? We have seen it should be equivalent to P and it should make the discount as a price processes in the model F martingale. That is what the properties of neutral measure precisely. that is what you have seen it in the discrete time model also.

Recall $1+r = u\tilde{P} + d\tilde{Q}$, is what we had in the discrete time. So, similar things is what is happening here? That it reduces the mean data of return to R under $\tilde{P}$ okay. Now we have derived this and we have considered at least one case of V_t, which is the European call option and we have derived the BSM formula. Now given any V_t you just have to exactly follow the same process to derive the risk neutral pricing formula for any other derivative, which is in our case simply is an $\mathcal{F}_T$ measurable random variable.

It could be path independent like an European call input and forward contract or any other such thing or it could be even path dependent, which is what will happen exactly, if you consider the more you go out of this plain vanilla options. plain vanilla options we mean this simply European call input cases. If you go out of that than what you get is? What we call generally an exotic option? And for any exotic options to get the price what you need to use is simply this material pricing formula.

You just evolve it by plugging in this V_t and you get under this model setup the price. which means assuming the geometric period in motion evolution of for the underlying risky asset you will get the V_t. Now, the risky assets could be even changed to some other things of course, that is what you know you go as you go along but we will not get into that.

So what we have seen is? This is the risk neutral pricing formula okay. We have given you can as an exercise you can try for some other simple V_t functions. which are functions of S_t right like European put and forward contract, which you know already the formula but you can try to derive them using risk neutral pricing formula okay.

Now, coming back to this derivation of this formula, this derivation of this formula relied on a key fact that you could be able to find an initial wealth X_0 and a portfolio process Δ_t, which tells your number of shares that you hold at each time t such that exact capital T is the wealth at capital T is equal to is equal to the payoff of the corresponding derivative. (So, that means that this is) So, there is a portfolio process, is what the existence of this portfolio process what you have?

So, this means that relied on, this formula relied on the fact that there exist an X_0 and a portfolio process Δ_t you can you do not need at capital T but you can even if you have it does not matter. Then such that you know your exact capital T is equal to B at capital T. And hence you are done. So, this corresponding wealth at each time is what you call it as the price of the derivative?

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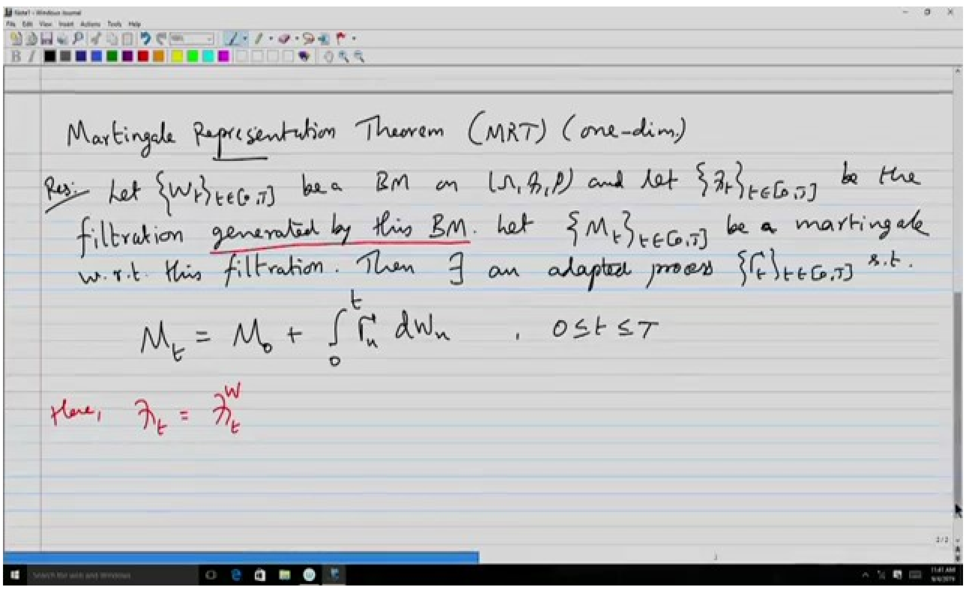

Now, the question is like how we ensure, so because as we see their risk neutral pricing formula to be a combine by the corresponding hedge if not then the formula might be questioned, right. So, what is the hedge? Whether there exists a hedge? What is the question that we are trying to answer? So, in this case, where there is you know one stock driven by one Brownian motion.

So, let us look at the case first and later we know we will generalize this to the multiple or multi-dimensional BSM model. Now, this relies on what is called based on a result, so this the question that we have aroused can be answered with the help of a result, which is called martingale representation theorem, we might simply called MRT. And this is in one-dimension, is what we are looking at it, means with Brownian motion case that we are looking at it. So what is the result? Of course, you are not going to prove this result because the proof is involved. So we will just take the result as given and we will try to use this in our case to justify whether there about the existence of this Δ_T. So what we have? Here let you have a Brownian motion be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t$ the filtration generated by this Brownian motion.

Let, you have a process M_t be a martingale with respect to this filtration, okay right. Then there exists an adopted process, which is in more generality. Some gamma t such that you have $M_t = M_0 + \int_0^t \Gamma_u dW_u$, so this is the result. So, now there are two cruising points that you need to look at that the filtration is generated by this Brownian motion.

So, here what we have? Your $\mathcal{F}_t$ is actually $\mathcal{F}_t^W$ in our earlier notation that we have introduced that this F_t is actually $\mathcal{F}_t^W$, which is the filtration generated by the Brownian motion only. So, that is what you know you need to understand that the critical point that you know we have here whereas Girsanov theorem, it could be any general filtration, which means the filtration in Girsanov theorem could be larger than the one generated by the Brownian motion.

There could be other random variables, which put also the parts of the generating class for say filtration. But in this particular case, it has to be precise it has to be the same as which means that what? Since we knew F the filtration as information contents, so what this is saying is? That the information that you have by absorbing Brownian motion starting from 0 up to time t is exactly the same as the one, which is there in the filtration.

So that is what it is? Now so what this says? Is that now in this setup if you have a martingale with respect to this filtration then this martingale can be represented, that is how the name come. As a constant plus and ito integral with respect to some adopted process gamma, so that is what? So this is an ito integral.

This martingale can be represented as an ito integral, this is simple setup, this is what you have as martingale representation theorem, okay. There relevance of this result to our problem, which is hedging is that since $\mathcal{F}_t$ has only one W, which generates this $\mathcal{F}_t$. So there is only one source of randomness, the only source of uncertainty in the model is the underlying Brownian motion and hence there is only one set of source of uncertainty that is to be removed by hedging right, so that is what it beings.

And since M_t in this particular case if you look at, so these are some implies that you know because this is given in terms of ito integral and since ito integrals are continuous. So, this martingale cannot have jumps. And if you need to have as jumps, of course, you need to introduce more random sources than W that you have, right so, that is what it would be.

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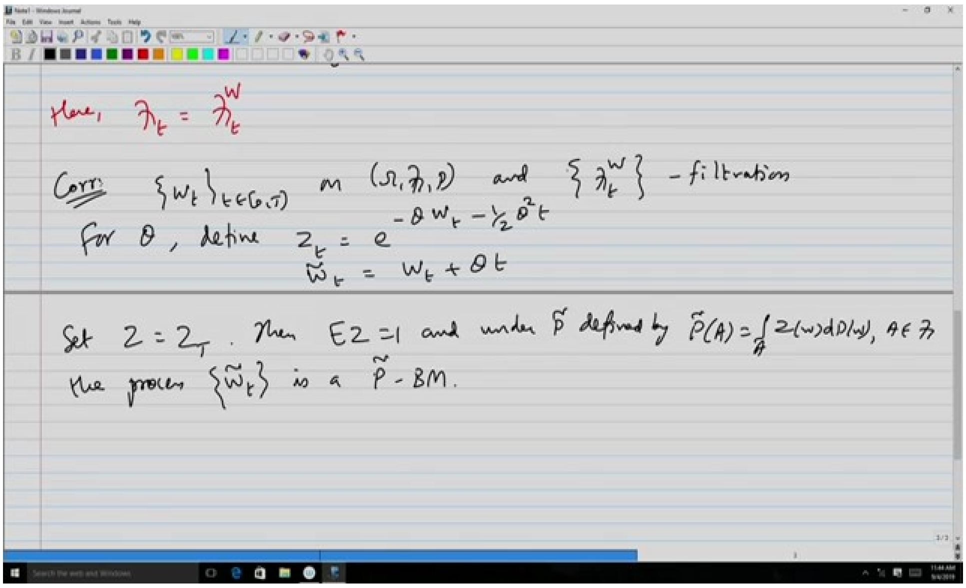

Now, the fact, so what is the restrictive assumption here, is this filtration. Now this and the other gives a now combinely can be written as a corollary. If you write a corollary, which combining these two, right, even in the case, so what you have? So you have a W_t in 0 to T on $(\Omega, \mathcal{F}, P)$ and you also have this $\mathcal{F}_t^W$ this is the filtration. So, this is the W t is there and then F t W is a filtration.

Then, if you pick then for theta we define my Z_t and $\tilde{W}_t$ as above. Then set is at a set equal to T then expectation of set is equal to one and under $\tilde{P}$ defined by this or usual definition. The process $\{\tilde{W}_t\}$ is a $\tilde{P}$ of Brownian motion. This is what the first part; this is exactly the Girsanov except that the filtration now we have shrunk a bit to make it equal to $\mathcal{F}_t^W$.

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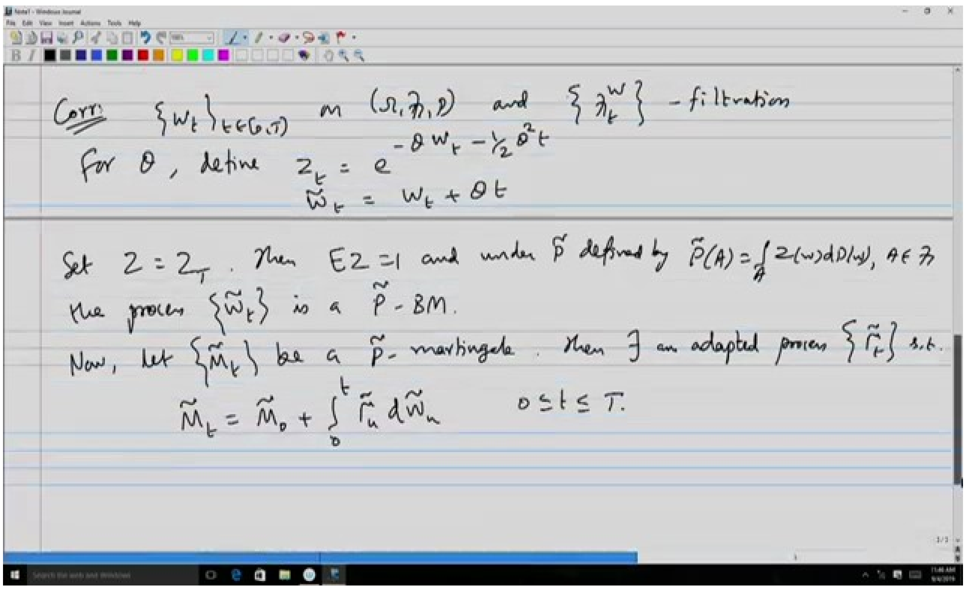

Now, in this particular case now let $\tilde{M}_t$ be a $\tilde{P}$ martingale, then there is an adopted process, say $\tilde{\Gamma}_t$ such that all this time periods are all $t \in [O,T]$ that you remember. $\tilde{M}_t = \tilde{M}_0 + \int_0^t \tilde{\Gamma}_u d\tilde{W}_u$. At the outside it might look like we are just combined Girsanov and MRT by reducing the filtration to the one generated by W. But if you look at it little bit more closely this is little bit more than that because the first part is find there is no problem.

But the second part when you pick it up you know you are looking you are taking is not in terms of P but in terms of $\tilde{P}$ and the W is a Brownian motion under P and $\mathcal{F}_t^W$ is the filtrationed by that W. But even then you know this result holds even under $\tilde{W}$. so that is what is the little bit detail that you can cut it, but of course that is if you wanted to look at little bit more closely what exactly is happening that is what is happening in this case okay.

So, this is the corollary part that we have which is true, so which mean this martingale representation theorem right. When we come to this $\tilde{P}$ we are still taking not $\mathcal{F}_t^W$ but $\mathcal{F}_t^W$. There is the filtration and even under the another filtration this adopted process means this adopted to that filtration and still this is true , that is what it is? Of course, the proof you can see if and one how and when and it is true okay.

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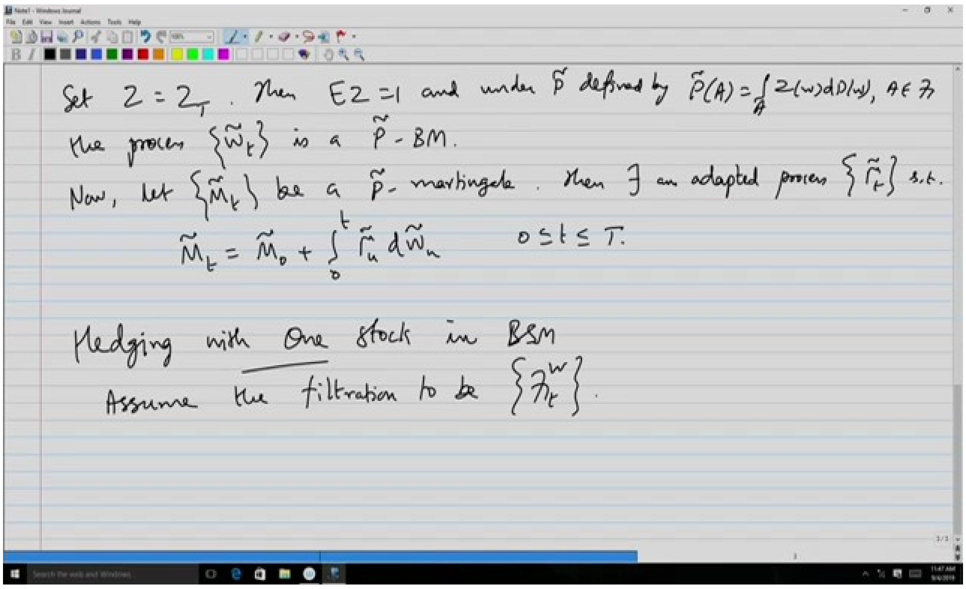

Now, let us come back to the hedging problems, so this is the result. Now we have combined and you written this corollary martingale theorem is what? Is going to help us you know to establish that there is a portfolio process Δ_T and hence the risk neutral formula is valid and it holds true.

So, let us come back to this and then now we come back to hedging with one stock in BSM of course the classical version right. Now, assume the filtration to be $\mathcal{F}_t^W$ only. In the earlier we did not put any restriction. But now if you want to show the existence of Δ_T, then you need to put this restriction, so this is what it is now?

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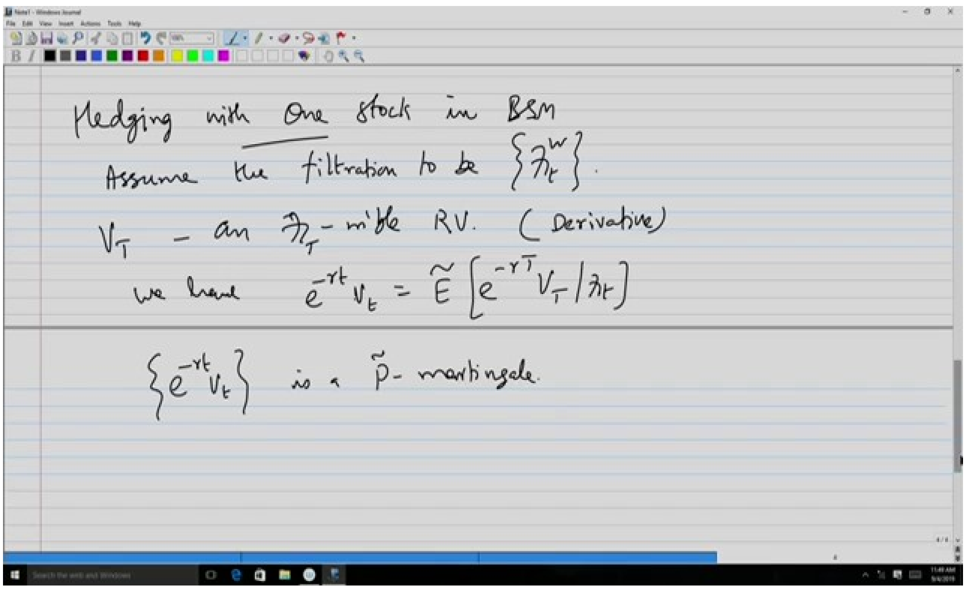

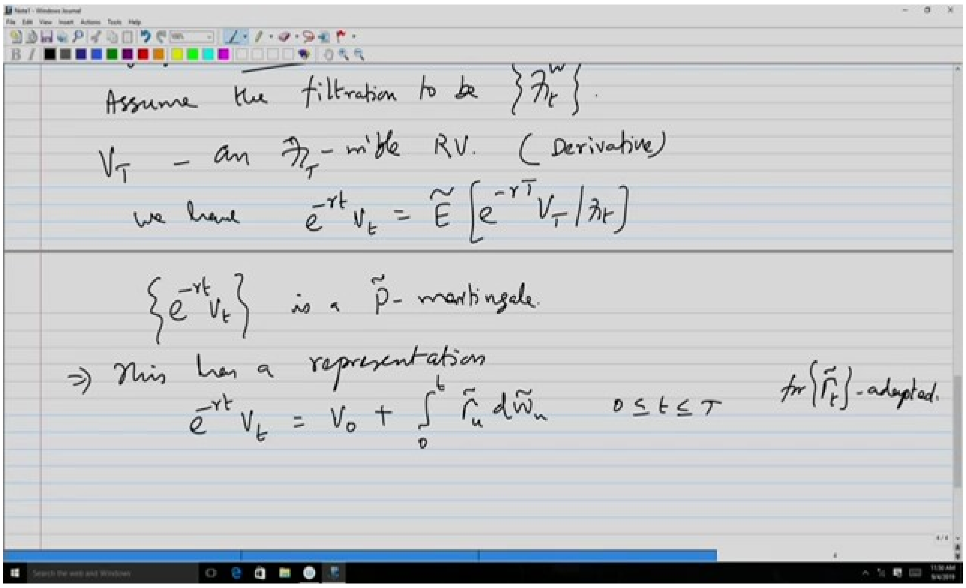

Now, we take V_T and $\mathcal{F}_T$ measurable random variable. so which is what the derivative in our case. And V_T from the risk neutral pricing formula, so we have

$$e^{-rt}V_t = \tilde{E}[e^{-rT}V_T|F_t]$$

,this is what we have? So this quantity is a $\tilde{P}$ martingale. Because that is what exactly this statement also would mean or you can even show that you know you can take this left side condition on $\mathcal{F}_s$ for $s \le t$ under $\tilde{P}$ and so that this is what you mean?

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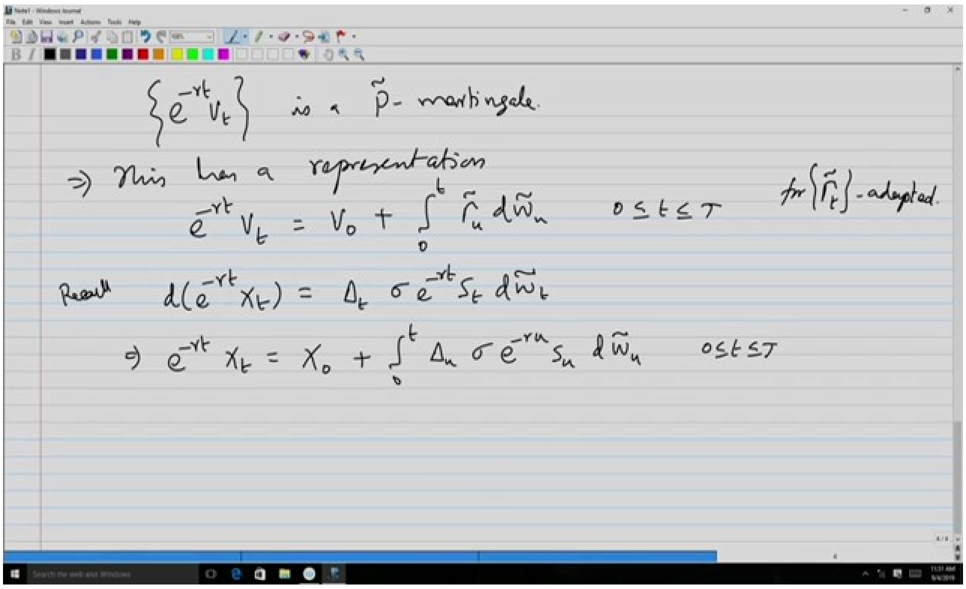

So, this is what we have, suppose if I call this as a let us give the name little later okay. So, this is what is the first fact that you. Now, on the other hand if you know we have also seen for any portfolio process what is the differential of the discounted value process? So what you recall? Recall that your $d(e^{-rt}X_t)$. After simplifying we got that expression. So, this expression we have obtained earlier, the discounted wealth process of any portfolio for any Delta. Now, what we need to have? In order to have $X_t = V_t$ for all t, we should choose, what we have to do? X_0 equal to V_0 and choose Δ_t to satisfy, what we have? $\Delta_t \sigma e^{-rt} S_t = \tilde{\Gamma}_t$.

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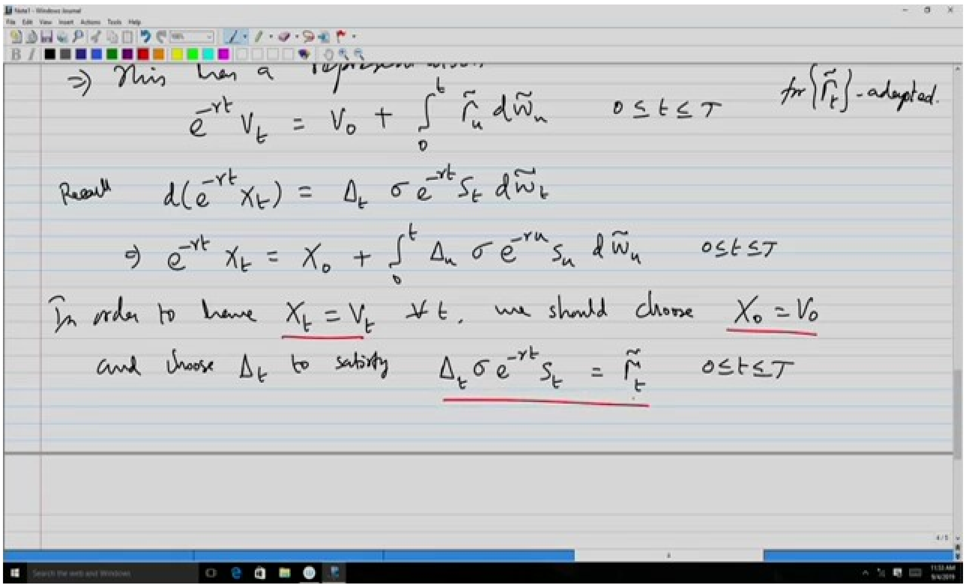

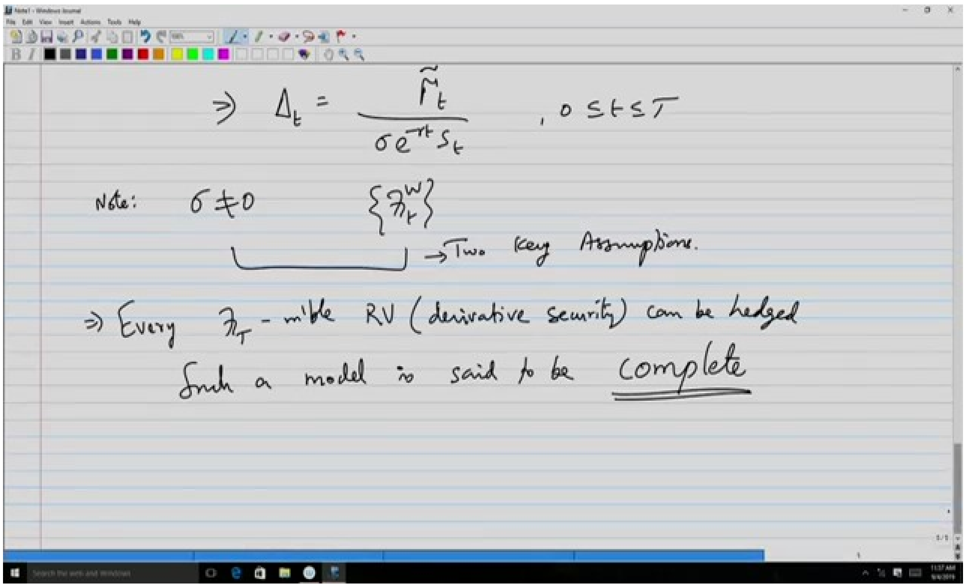

Now, how did we do here, so you look at the representation of $e^{-rt}V_t$, which is this V_0 plus this ito integral with the integrand.

Now, if we need to have which means that we want to ensure that there is a Δ_t, which ensures which gives or which say gives us an initial wealth as well as a Δ_t such that what we want is? X_t equal to V_t. which means we want, this can be achievedthrough making it X_0 equal to V_0 and look at the integrand here and the integrand here, if these two are equal then this is what we have done?

Now, so what is this means? This means my $\Delta_t = \frac{\tilde{\Gamma}_t}{\sigma e^{-rt} S_t}$. If I pick my Δ_t to be equal to this with this choice of Δ_t because I know there exists the $\tilde{\Gamma}_t$ and σ is given by the Sigma or S all this condition are given in the model. So if I take as the position, then we will have X_t equal to V_t at all-time t and hence X at capital T would be also V at capital T, which means we can completely hedge, so this is the thing.

So, the two key assumptions that makes this happen is that Sigma is a nonzero. Note, Sigma is nonzero is one of the assumption, if Sigma is zero it stocks itself like Sigma is zero means that is gone. So then there only risk free asset that you have you really do not have the risk free assets. So, but typically we said it Sigma, is strictly greater than zero. because it gives the water at a term. And that means that this quantity is well-defined Delta t right.

The other assumption is $\mathcal{F}_t^W$, the filtration. So, this is what the two key assumptions? For existence of this hedge because there is only one randomness F_t generated by the underlying Brownian motion. which means that the there is no randomness in the derivative security apart from the Brownian motion randomness, which can be hedged by trading in the underlying stock itself.

So, under these two assumptions, every which means that every F_T measurable random variable right, which is basically every derivative security can be hedged in this model. Such a model is said to be what we call Complete. So, this is the meaning you know of derivative security here many European type of course. So, this is every derivative security can be hedged, which means that there exits an initial wealth and there is a portfolio process, which will hedge the position the short position derivative security. And if the model allow such a thing to happen in that model is said to be a complete model and obviously we would want that we have a model which is complete. So that, we can hedge and we can arrive at the price. Because the price is justified, only when you show that you can hedge the short position in the underlying derivative security right. So this basically the justification but what is lacking here is that you look at this Delta t this Δ_t gives only an expression, which involves $\tilde{\Gamma}_t$.

But a $\tilde{\Gamma}_t$ we do not have an expression for gamma tilde. What MRT says is? That there exist a gamma tilde t that means since we cannot expressively compute from there what is gamma tilde t? So, the existence of such portfolio, which will hedge this short position in that like derivative can be proved using martingale representation theorem. But you really cannot you know directly get what is that position? For positions of course we have to rely on other methods of course one of them is we have seen earlier. At for at least European call option you have seen the Delta hedging at a G's. What is given by that Delta Hedging rule okay? So only MRT guarantees the existence but for computation method of finding either gamma of tilde t or Delta t.

You know you have to rely on some other techniques that we have it here. Now, so this is what, is our discussion on the single Brownian motion case or a classical BSM model with constant parameters and one Brownian motion, which is driving the randomness. And there is one stock that is what we had it. We in this classical we have obtained the risk neutral pricing formula and we have justified it by establishing a hedge, that is what the theory?

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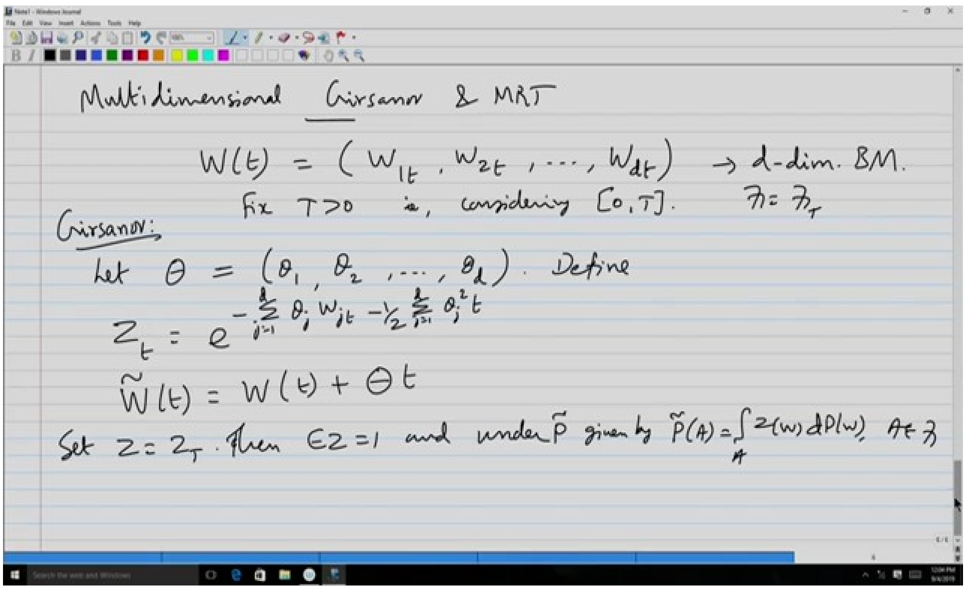

Now, we will want to move to the multidimensional case but for which first we need the Girsanov, multi-dimensional Girsanov and MRT before we specify or describe what is the multidimensional BSM model that we have we are going to look at it. Now, in this multidimensional so what we are having now is? Let us for the time being write this as within bracket, which essentially means $(W_{1t}, W_{2t}, \ldots, W_{dt})$ which is a d dimensional Brownian motion.

And we know what exactly we mean by a d dimensional Brownian motion. Each one of them is a

Brownian motion and in addition to that they are independent right. So, that is what the case that we have? So, d dimensional Brownian motion typically remain each another Brownian motion and they are independent of each other. that is what we have now here. So, we as usual will have you know this filtration and other stuff.

Now, what is this particular case, Girsanov, how will look like, in this case Girsanov will be that we are fixing. So we are also fixing at T, which is greater than zero, so that is we are considering, you know, the interval 0 to T as our time period. Now, what we have is that if this multi-dimensional Girsanov, so what we have? We can write this as θ_t, which is essentially $(\theta_{1t}, \theta_{2t}, \ldots, \theta_{dt})$, is the d dimensional vector or the d dimensional quantity. So, now actually we do not have the, so time dependency. So remember in one dimension we just picked one theta. Now, we are picking a d values of theta d theta's some in a vector theta. Now, define Z_t as above.

There are $\tilde{W}$ and W they are all d-dimensional, so actually the theta is all d dimensional, so actually there are d equations here essentially what we are looking at it okay. Now, you set $Z = Z_T$, then you have the usual one and that $\tilde{P}(A) = \int_A Z(\omega) dP(\omega)$ for $A \in \mathcal{F}$. Under this given by this process $\tilde{W}_t$ is a d dimensional Brownian motion.

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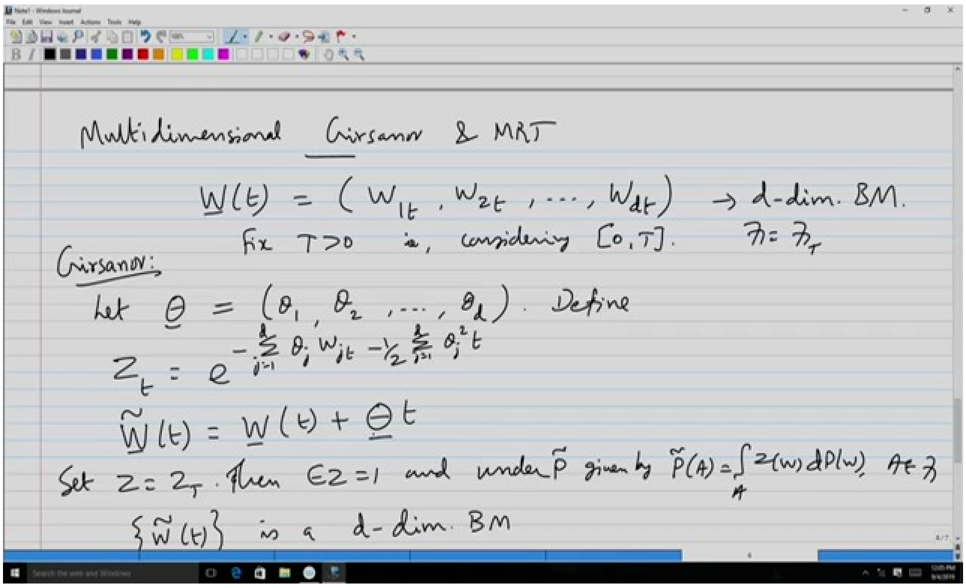

We see that we can also always write this under score to denote to the vector, so there may be they can do that. This is a vector, this is a vector, this is a vector. so that there is no confusion that is in unidimensional or multidimensional. So this is a vector is a d dimensional Brownian motion, is what then you would see, so this is what is the multidimensional Brownian motion.

So, here it is simple in the case of theta but if this is if you want to generalize to a more generic one of course there are little bit involved thing. But here it is type forward that what you are seeing? In the similar way, as the other case that you have it here okay, so this last equation is, this is actually the $\tilde{W}(t) = W(t) + \Theta t$ is actually a short here notation for the d equations. So this is what you have it as a multidimensional?

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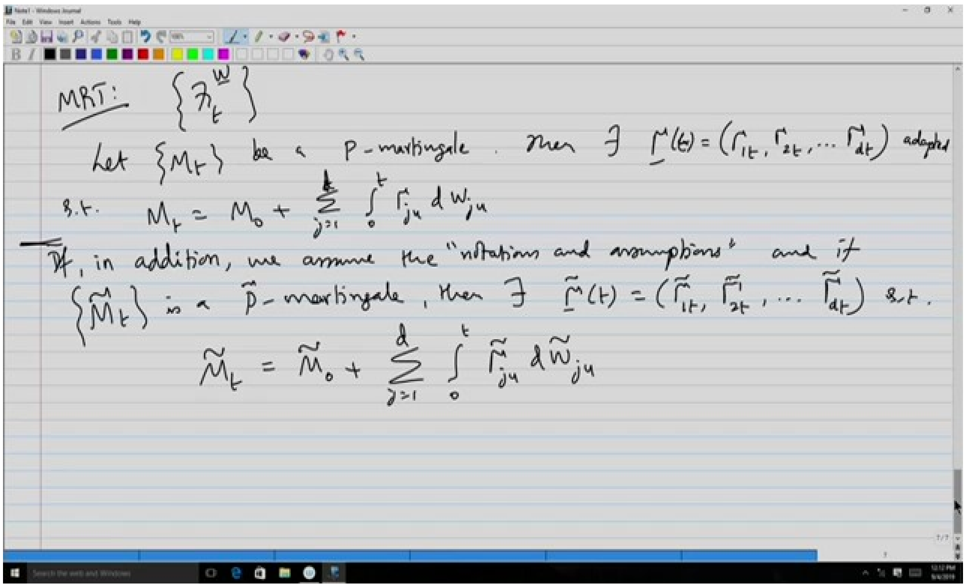

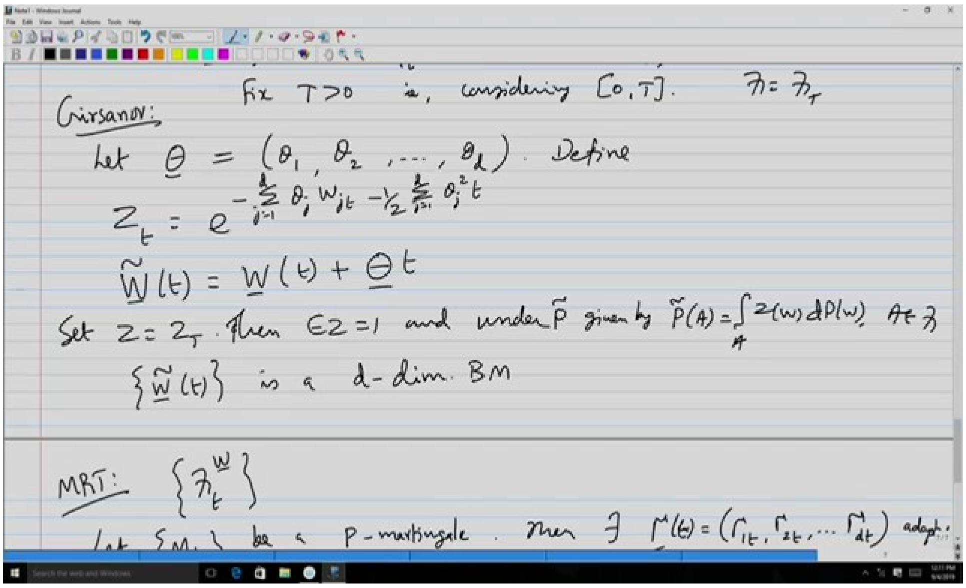

And MRT in this particular case, is that of course, now in this case we are taking $\mathcal{F}_t$ divided by I mean generated by this vector process W_t right d dimensional a filtration generated by the d dimensional Brownian motion W bar W. Now, let in this particular scenario let M_t be a P martingale, right. Then there exists a Γ_u, which is actually Γ_t.

Let us take $\tilde{\Gamma}_t$ d dimensional adopted process such that your $\tilde{M}_t$ is given by that. So, this is what will be the case that you know you are writing it as so what is the representation? Any P martingale can be represent as a constant plus d number of ito integrals.

If in addition, we are trying to write the corollary like earlier. If in addition, if we assume the so this is actually the corollary part, MRT is done, so this is actually corollary condition, we assume the notations and assumptions as earlier. And if your $\tilde{M}_t$ is a $\tilde{P}$ martingale then there exists a adopted process which is 1t, 2t and so on dt right such that you will have $\tilde{M}_t$ is that.

So this is what we meant use, when we use the so this is the multidimensional version and this is the Girsanov multidimensional version what we may have to do? That is what is the case right. So, which means that for getting this $\tilde{P}$ which is equivalent to P in a multidimensional case what you need is you need to pick some d constants, θ_1, θ_2, θ_d define this Z_t and set Z equal to Z_T then you can see that expectation of Z equal to one and probability of the greater than zero is one.

And hence, this Z is actually the RN derivative and the Z_T is RN derivative process and under that Z you can define a P tilde which is as per this, and this will be an equal probability measure to that, and this under W tilde S given here would be d dimensional Brownian motion which means each one of them in that component is a Brownian motion and they are still independent.

Similarly, MRT a multidimensional version is what then you have? So, the representation is that who you have an constant plus some of constant plus d ito integrals is what you are going to have when you look at the multidimensional case that you have.

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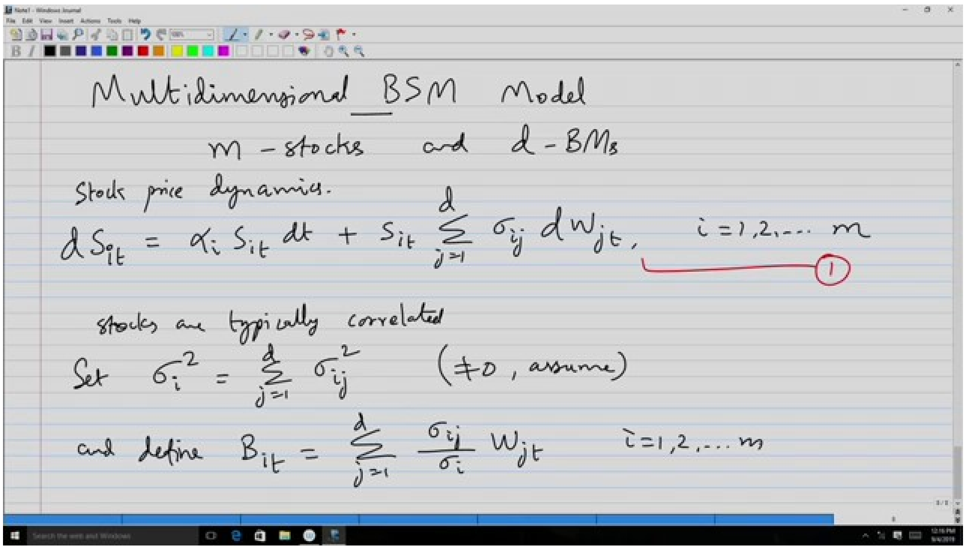

Now, let us move to what we have this multidimensional market model so which is what we are going to see now? Let us describe what is the multi-dimensional BSM. You can also called this is a classic because this is constant parameter one. But still we will not call in that way, so multidimensional BSM model okay is what we will refer to general because so what we have here?

What is the generality that we have over one dimensional BSM model? Now we have more number of stocks and more number of randomness that we are bringing inside the mode. So what we have now is that there are we are assuming there are m stocks and d Brownian motions is what is the randomness count in our case okay.

So, what we the dynamics? The stock price dynamics, how they are given. This is See the expression of dS_{it}. So, let us call this as the description or the description of the evolution of the stock price processes. So here we see this σ_{ij} is typically assume to be a nonzero quantity is such that this metrics of σ_{ij} is or invertible.

So that is essentially you know you can justify that by assuming that you know you are I mean trying to have inside the model. The quantities which are not really can be expressed in terms of other ones. If one stock price process can be expressed as a linear combination of the other obviously then you will not keep that because may not be useful right.

So you will have a truly independent or little bit of that nature. So naturally then you will get this required quantities but you will not you know worry too much about that. So these stocks are typically correlated. So they are not, so it is not linear perfectly correlated but it will have some correlation coefficient that is the situation that we have it in mind. So these stocks are, the stocks are typically correlated. Now, how do we see?

What you will do is that we set or we define

$$\sigma_i^2 = \sum_{j=1}^{d} \sigma_{ij}^2$$

,and we will assume this is not equal to 0, we assume because we are going to divide and so on. And define a quantity which we call B_{it}.

Now, you see that the stocks are typically correlated which you want to see. So we as define a σ^2 which is nothing but the sum of σ_{ij}^2 for j some over j and we assume that to be not 0. And, define B_{it} there are m such processes, B_{it} m such processes using this quantity and σ_i we which we just defined okay.

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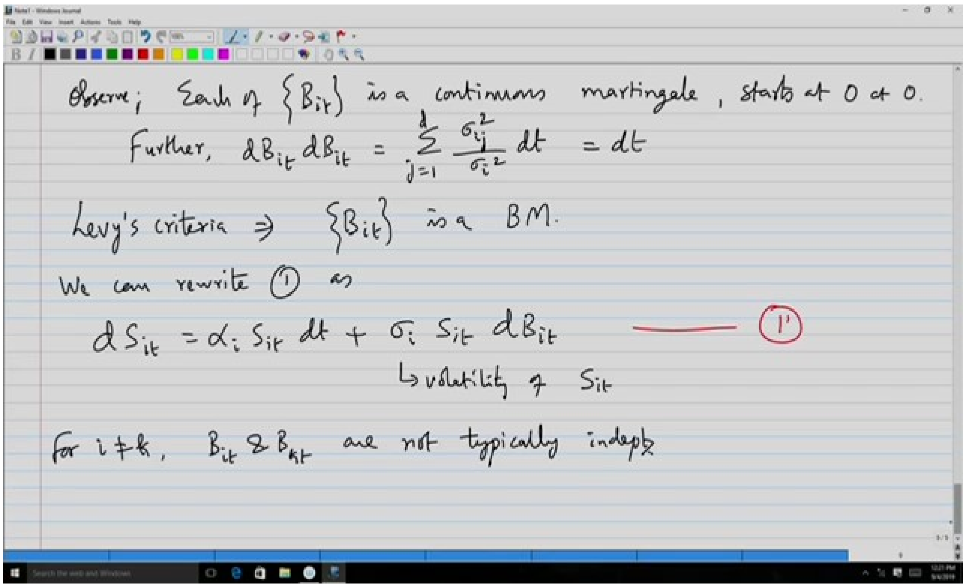

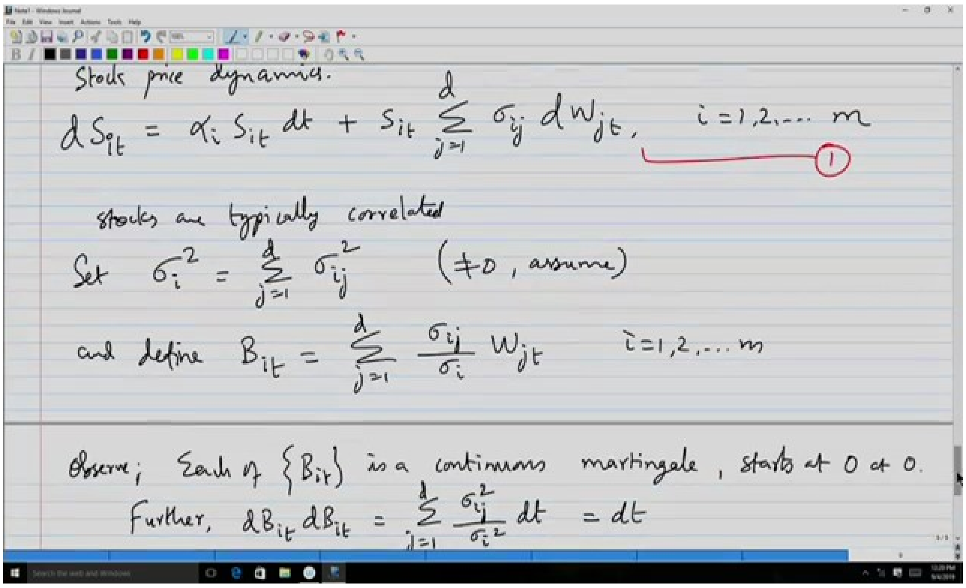

You can define from this using d W's we are defining M Bs, now what are this Bs? You know, let us look at little bit more closely, observe that each of this B it is continuous. Easily you can see because they are sum of W's which are continuous and also how they are martingale which also you can see. Further, you can see that dB it what you would get because you will have only the corresponding terms will come because W i's are W j's are independent.

So, will use the standard formulas to arrive at this σ_{ij} square by $\sigma_i^2 dt$. So only W_j will remain, so this is what you will get? And this is nothing but dt which means the quadratic variation of B_{it} itself or the quadratic variation of the B_i process is one per unit of time right, which is a continuous martingale starts at zero at time zero. And quadratic variation is this and all these things put together, now what this is imply?

My Levy's criteria or Levy's theorem implies that each of this is a Brownian motion right so this is what we are using? So the Levy's criteria tells that each of this, so we can rewrite one as the original equation as dS_{it}, of course this I can call this as one dash, where this this is now you can easily see volatility of S it this process, volatility of this process.

So we have written their so this is an expression which we have written in terms of W j's which are independent Brownian motion. So you have this coefficients and this along expression. Now, by defining B it which is also Brownian motion, B it is by the way is also Brownian motion if you want to write, then basically you can for ith stock, what is the volatility if you want.

Because from the previous one, you can compute this σ_i which as we have defined there and this is the volatility of the ith stock as you can see from here. Now, for i not equal to k, for i not equal to not equal to k, so this B_{it} and B_{kt} are not typically independent. If they are independent of course there is a much simpler situation that we are thinking about. But generally they may not be typically independent.

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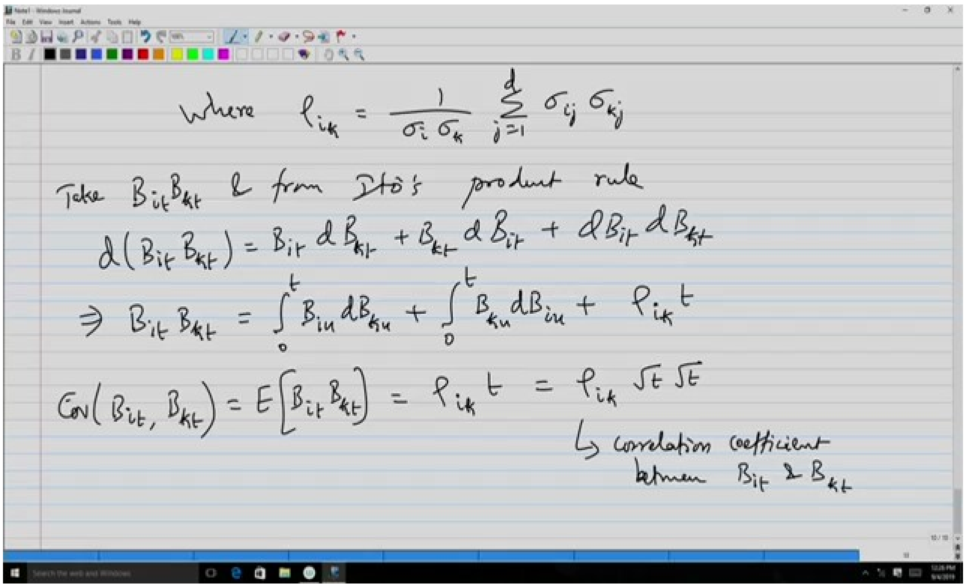

So, how do we see? Note or how do you define or what you get as their dependency. Look at first B_{it} and B_{kt} which is from here is essentially you can see that from the definition of B_i this, so this is j is equal to 1 to d σ_{ij} and σ_{kj} by $\sigma_i Sigma_k$, dt which let us call this as ρ_{ik}.

Now, take ito's, take this quantity B_{it} into B_{kt} and from ito's product rule what you get is $d(B_{it}B_{kt})$ is equal to that. Now, you have seen, you know why we computed that quantity to be, now this if you integrate, if you integrate this what you would get? Now, we can write the covariance term.

So, then what is this means? So this is means that this is the correlation coefficient between the processes B_{it} and B_{kt}, so this is the correlation coefficient. So, you can see that the covariance is given in terms of this ρ_{ik} times because each Bit's are Brownian motion. So, in an integral of 0 to t they have the variance t, so standard derivation square root of t so this is that is what we written here so this means that this coefficient what we have written is actually the correlation coefficient of the processes B_i and B_k.

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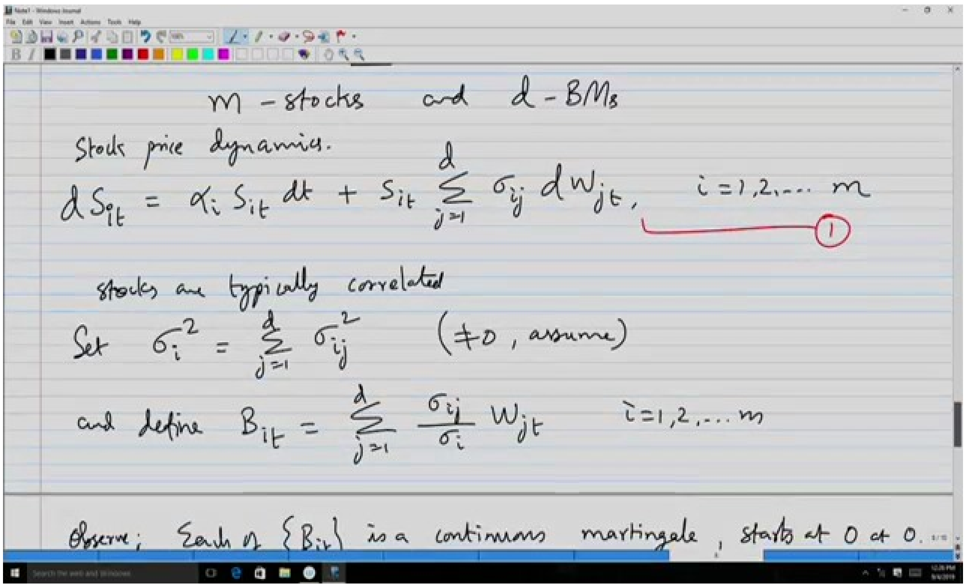

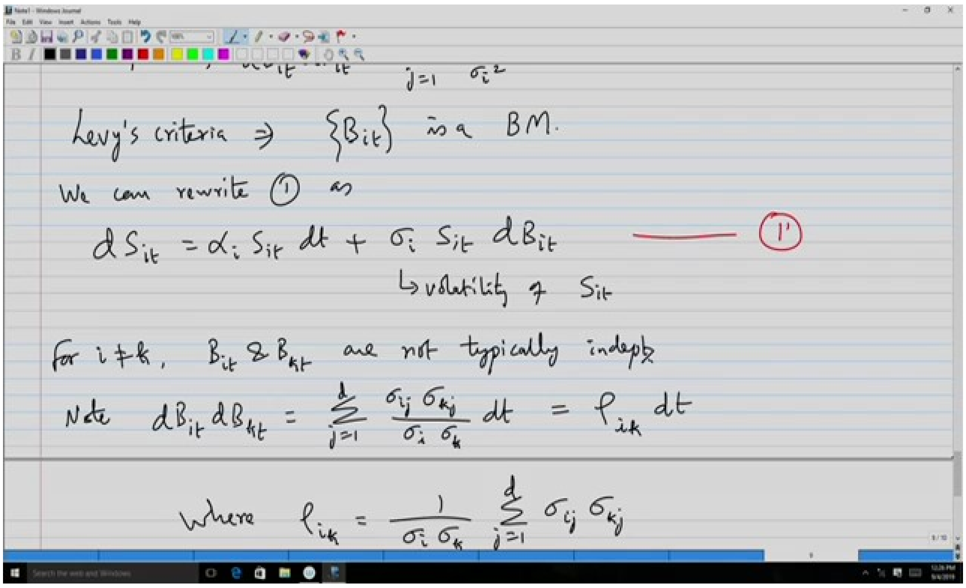

So, you can represent you can represent the multidimensional market okay so just quick recap. The first step is essentially you can de asset, you can represent in this form where W_{jt} are independent Brownian motions or you can write as it is in one dash with Sigma i as its volatility parameter and through this as d is essentially the dynamics is given by through this as d.

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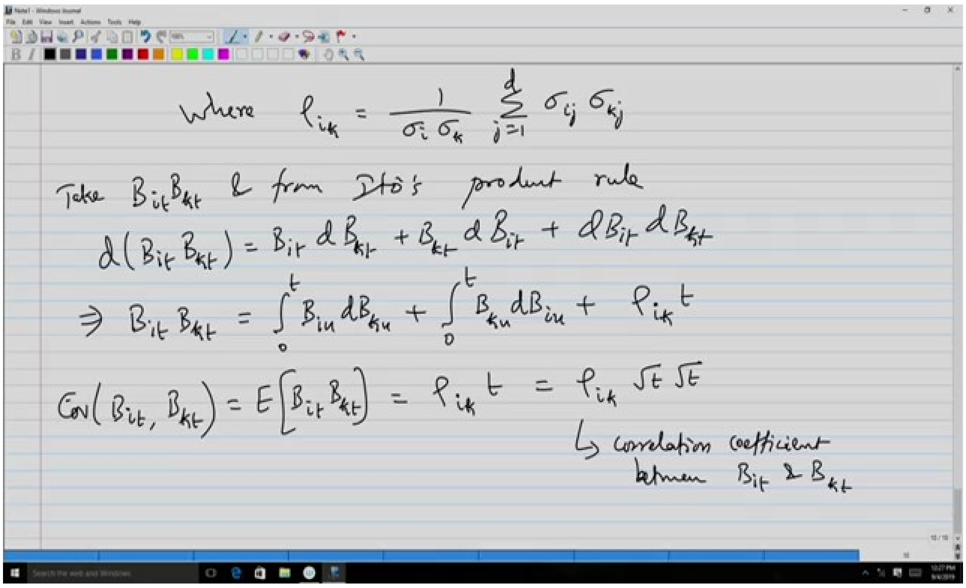

Then B_i's are correlated with the correlation coefficient ρ_{ik} as given in this expression which again you can write in terms of σ_i and σ_{ij}, σ_{kj} and so on so this is what you will have? So what is the description of your risky assets case?

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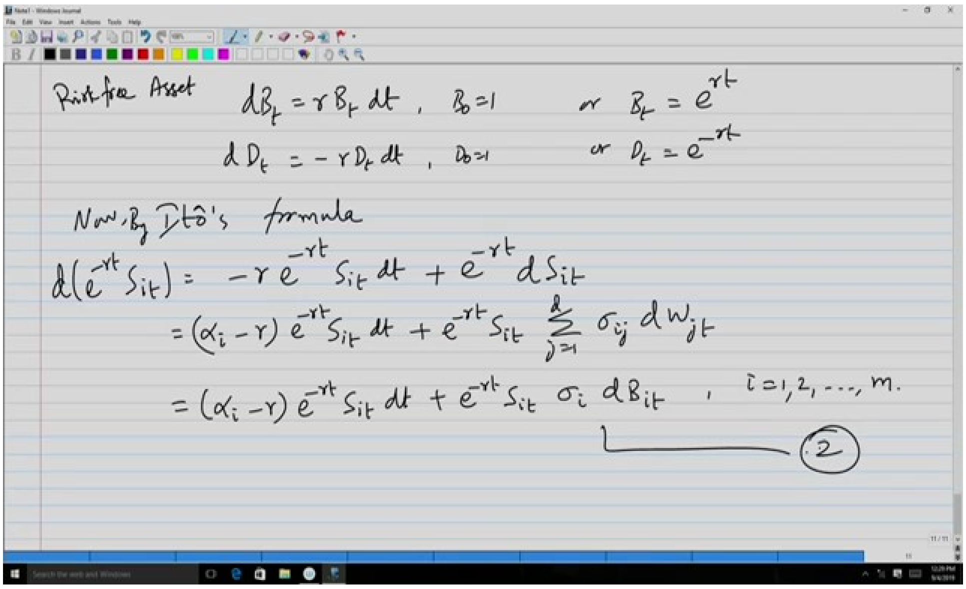

Now, we also need to give as usual the risk free assets, as usual the risk free asset which is which could be either a bond or which is same as the single dimensional model there is no doubt, B_0 equal to one or $B_t = e^{-rt}$, and $D_t = e^{-rt}$.

Now, we look at the, we look at the discounted stock price process, what is expression?

$$d(e^{-rt}S_it) = -re^{-rt}S_{it}dt + e^{-rt}dS_{it}.$$

This is what you get by ito's formula? By making this function if you will look if you write then that is what this is what you will get, which you have seen already.

Or you can write it in terms of the

$$(\alpha_i - r)e^{-rt}S_{it}dt + e^{-rt}S_{it}\sigma_i dB_{it}$$

Let us called this is equation two. So we had equation one, now this is equation two that we have which is the discounted cost price process either in terms of W or in terms of B that we have written earlier.

Now we have this description, now we can next what you will look at is basically the first part of the two part story which is basically the existence of this neutral measure.

If what does that imply and then then the second part of that is that whether the how many such risk neutral measures are existing and what as that imply is what then we look at in generality. Of course, you know, we will take that discussion in the next lecture okay. So, you recall remember these things, of course, we will recall and then we will proceed the next lecture that you. Thank you.