Mathematical Finance

Lecture 33: Girsanov Theorem, Risk Neutral Pricing of Derivatives, BSM Formula

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Hello everyone, we will continue our discussion on this change of measure rate idea. Just recall from the previous lecture the basic ideas that we started, when we are introducing this risk-neutral pricing in continuous time that we have a probability space on which we have a Brownian motion and we want to define a new probability measure. Which is equivalent to this, which is $\tilde{P}(A)$, which we define as given here in this expression. And with that Z we can define a process is a Z_t, which is the R-N derivative process or Z is what RN derivative is.

Now, for any y which is $\mathcal{F}_T$ measurable, for any fix t between 0 and T. We have these two Lemmas, which is what is basically just the continuous time version of what we have seen in the discrete time model. So, that is what we have seen. Just recall all these points so whatever we have written down here these two Lemmas and the ideas of how to be defined $\tilde{P}(A)$ and Z_t, Z and what are the properties of this Z.

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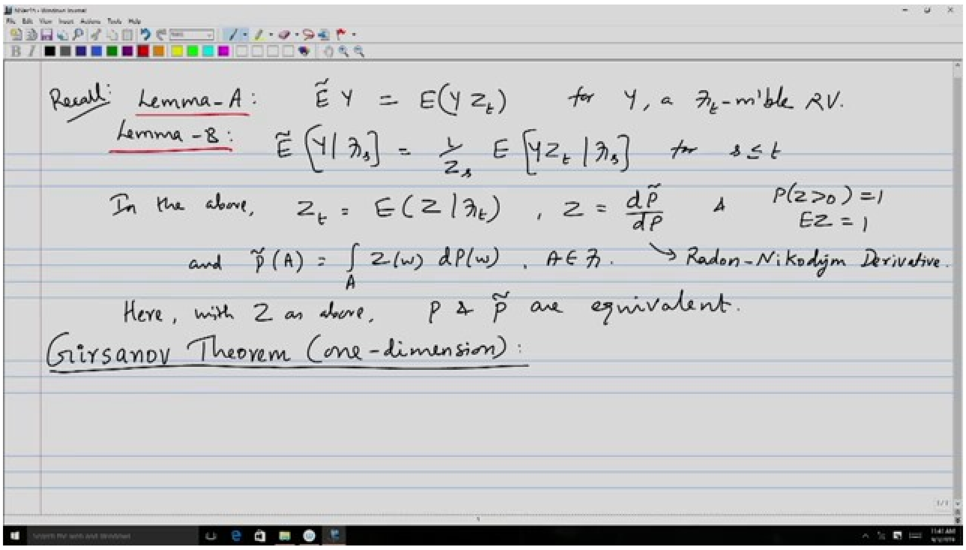

With these properties of Z of course these two are equivalent. P and $\tilde{P}$ are equivalent probability measure, which we know by what does that mean. it means that they agree on what is possible and what is impossible means agree on sets of probability 0 and conversely sets on probability 1 as well. Now, keep in this mind with these two lemma in mind now let us state the Girsanov Theorem, which is going to play a major role while we continue forward our the continuous time finance model.

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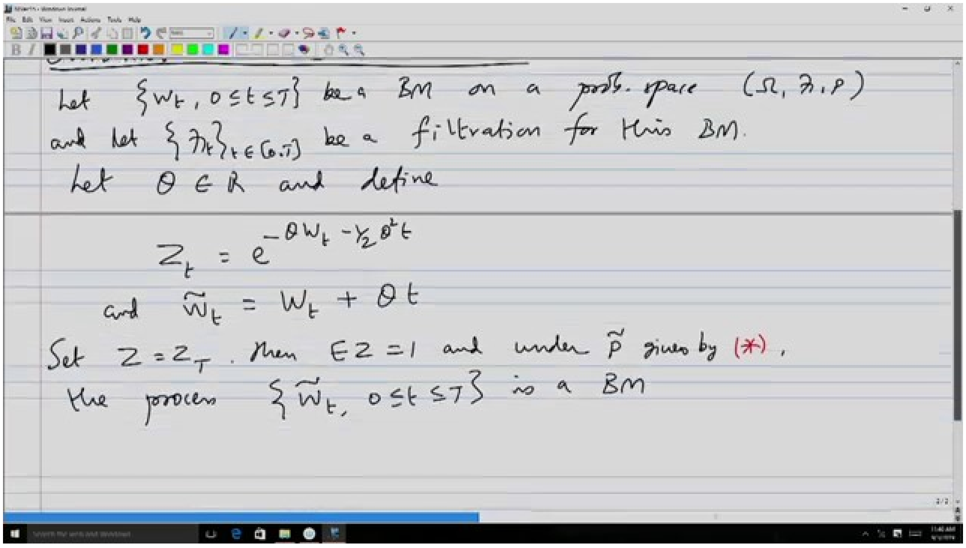

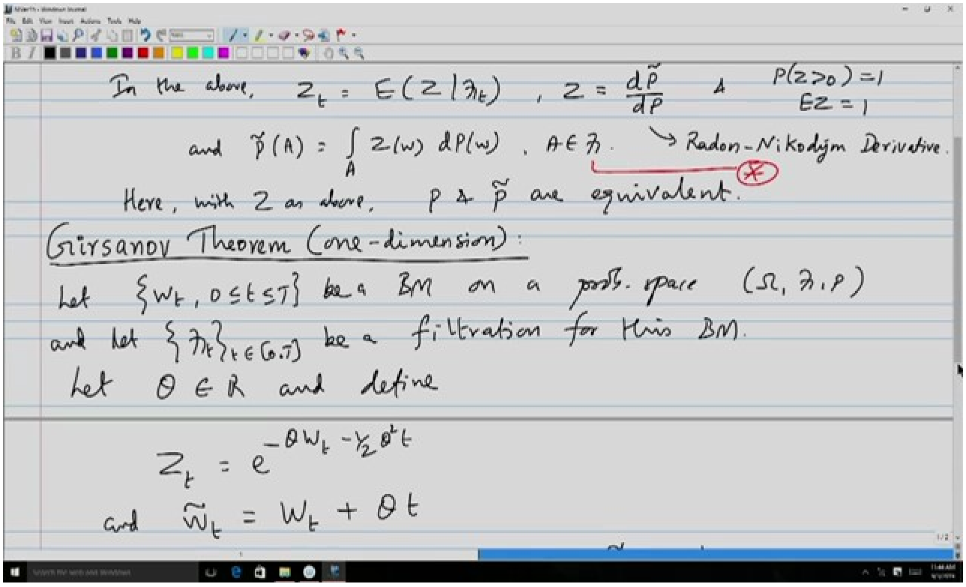

Now, what do we have here? So, you have a Brownian motion now we are BM, Brownian motion on a probability space say $(\Omega, \mathcal{F}, P)$ and this is and let $\mathcal{F}_T$, T is also in this be a filtration for this Brownian motion again. It has the properties that we have defined when we say filtration for the Brownian motion. Now, what we do is that we will pick a real number let theta belongs to R and define the process $Z_t = e^{-\theta W_t - 1/2\theta^2 t}$ and another process $\tilde{W} = W_t + \theta t$.

We may also assume some conditions for under when we actually try to prove this but that we will not state as part of the result. So, here what we have picked, we have picked a real number theta this the same process the way of defining could be generalized even if theta is a time dependent quantity but nonrandom or even to the case where theta is an adopted process

But for our purpose, since theta constant would do the purpose for our discussion, we will restrict the statement of the theorem or the theorem itself to the constant theta case. Now, what we do set $Z = Z_T$ then you can see that $\mathsf{E}(Z) = 1$. and also probability of Z greater than 0 is also equal to 1. and under $\tilde{P}$ given by say suppose if, I want to give a name to this let me give this as this red star given by this, the process which is $\tilde{W}_t$ is a Brownian motion

This is the main result of this. So, what you do? You have a Brownian motion defined on a probability space endowed with a filtration $\mathcal{F}_t$. You pick a theta, you define this Z_t and $\tilde{W}_t$ then you pick this Z to use a new probability measure $\tilde{P}$ and under that probability measure $\tilde{P}$ this W tilde process which is what, it is basically $W_t + \theta t$, recall W_t if I simply look at the distribution it is normal$(0, t)$.

Now, what we are doing? we are shifting this W_t by so this is basically a Brownian motion with drift process. Now, this drift one you know by adjusting the probabilities you are bringing back to the $\tilde{W}_t$ will be again normal$(0,t)$ under a different probability measure $\tilde{P}$. Now, how to get that $\tilde{P}$, is what the process described. So, how to get a Z and so on.

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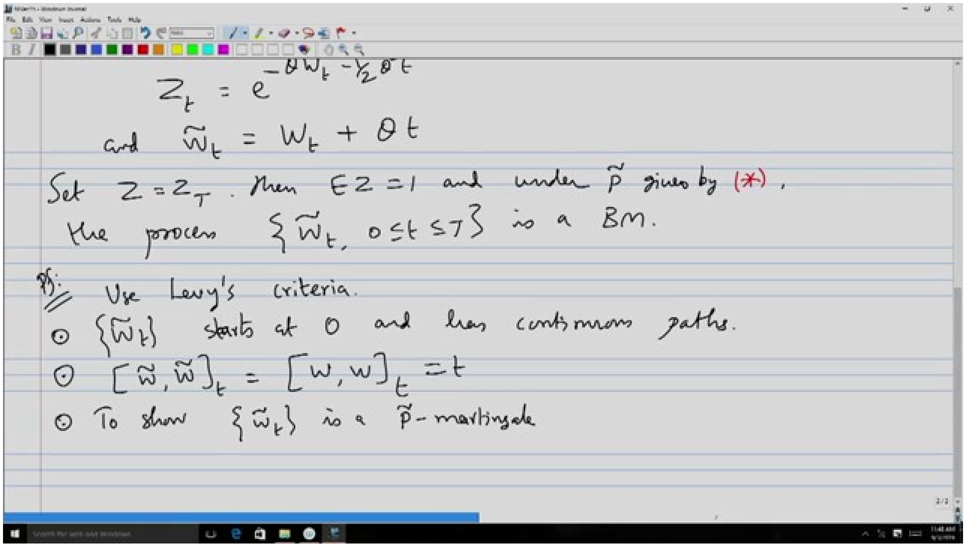

Let us quickly go through the proof though it is. In our discussion for Finance this may not be learnt but this and this is simple and this you must understand that how this is coming. Now, how do we show that some process is a Brownian motion? We will use the Levy's criteria. Levy's criteria, which means a continuous time martingale starting at 0 having continuous paths, having quadratic variation one per unit of time is the Brownian motion. This precisely that what we will see.

So, the process one can easily see this process, this is starts at 0, because at time $t = 0$, $W_0 = 0$, this is also 0 so this is starts at 0. and has continuous paths as well, that is easier to see and next this is one part next part is also pretty simple that if I look at the quadratic variation oh sorry of $\tilde{W}$ would be same as the quadratic variation of W, which is equal to t.

And you can easily write down in terms of stochastic differential of this $\tilde{W}$and you can compute and see that this is actually the case. So, this part is also clear. Next what we share is to show, next what we are going to look at is that to show what is remaining (to show that this is a) since this is a martingale under $\tilde{P}$, so we want to show that this is a $\tilde{P}$ martingale if we show this then we are done.

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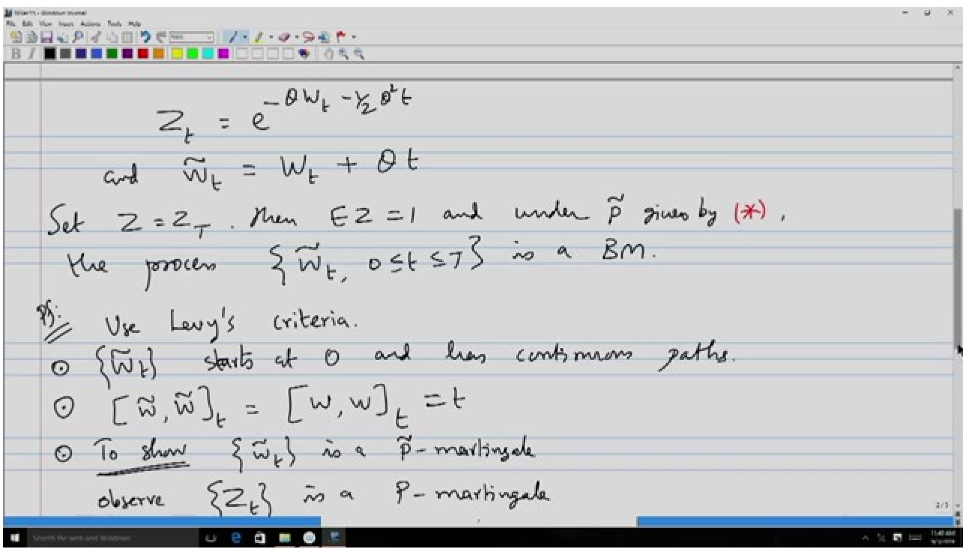

Because by Levy's criteria then such a process so this is continuous path, quadratic equation now next to show that this is what required. So, this is what now need to show that this is actually the case. Now, first thing that you can observe is that how do we show is what the following. Observe that Z_t the way we have defined is a P martingale, you go back and see this how the expression for this Z we have written this.

So, Sigma now you can assume this as simply as minus theta since also real number and so this expression is actually what we called earlier the exponential martingale that is what occurring here. This is a martingale under the measure in under which, W is the Brownian motion, which is P, so this is a Pmartingale you can easily see.

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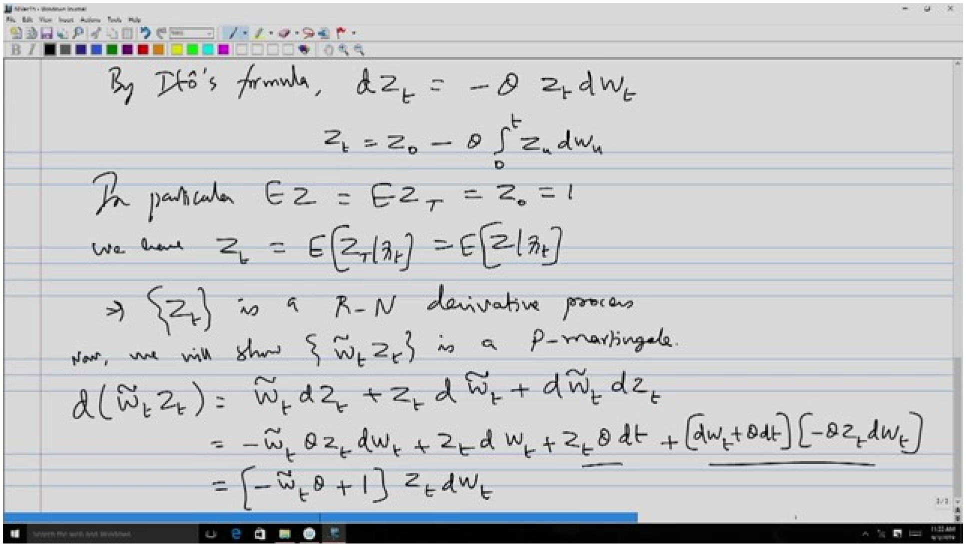

Otherwise another way to see that is you know if you use Ito's formula. What you would get is that, if you compute this stochastic differential this will be equal to this quantity. So, this also tells us , $dZ = -\theta Z_t dWt$. This also tells us that you know this is nothing but if you the Ito integral component, there is no dt term in the expression first stochastic exponential and hence this is a pure Ito integral, the right-hand side is also pure Ito integral and Ito integrals are martingale and hence Z is a martingale. You can easily see this basically that is what.

You can also see by this as well by looking at the form of dZ_t, if it does not have Z_t terms. So you can write this Z_t as above, so this is a martingale constant times minus or plus whatever is the case. this is a addition of a constant, so this is a martingale. So, by this also you can see that this is what is true. In particular, what we have? In particular, our expectation of EZ, which is expectation of EZ at capital T since this is a martingale, martingales have constant expectation, so this is across all-time points this is true which is equal to Z_0 which is equal to 1.

So, we then have what we have because Z_t is a martingale under P and EZ equal to this we have Z_t which is $E[Z_T|\mathcal{F}_t]$ which is expectation of EZ this. So, this EZ is nothing but this quantity Z_t. So, what do we have then Z_t we have defined there is actually what is R.N. derivative process. So, that implies or the shows that Z_t is a R.N, Raydon-Nikodym derivative process and hence the Lemma, two Lemmas that we have seen, Lemma A and Lemma B applies to this particular scenario, where to this Z_t if we pick any random variable then we can do that computation.

So, usually we just have to use that computation to see that what we are aiming to get. So, next what we will show, we will show this is, we have observed this. Now, we will show that this process is a P martingale. So, how do we see? So, now if I simplify this, so what you would get, is $[-\tilde{W}_t\theta + 1]Z_t dW_t$. So, what will happen to the other term? This third term and this fourth term that we have written on the right-hand side will get cancel, so this term and this term will get cancel and dt dWt is 0 and dw dw will give you this.

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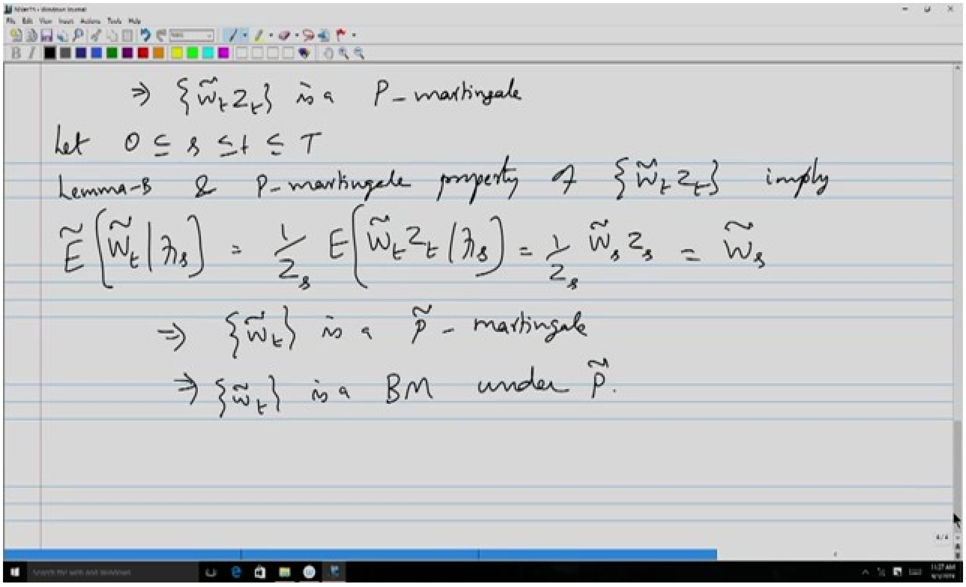

This implies your $\{\tilde{W}_t Z_t\}$ is a P martingale is the next step of this proof. Now, you pick two time points s less than or equal to t or less than or equal to capital T. Now, Lemma 2, what we have written? A B we have written, so the Lemma B and the P martingale property of this process $\{\tilde{W}_t Z_t\}$ imply the following. what we wanted to show? We wanted to see that this is what it is. This is exactly using the Lemma B. Now, we know that this property W tilde Z is actually P-martingale. So this means this is equal to W tilde s Zs, which is equal to W tilde s, which implies W tilde t is a $\tilde{P}$ martingale.

And hence $\tilde{W}_t$ is a Brownian motion under $\tilde{P}$, we have defined here, hence the proof. So, this is what is the Girsanov Theorem, which you know will be using a lot whenever we want to make or construct the probability measures and so on. So, with this tool in hand now we can actually go to the continuous type model, which we called as BSM model.

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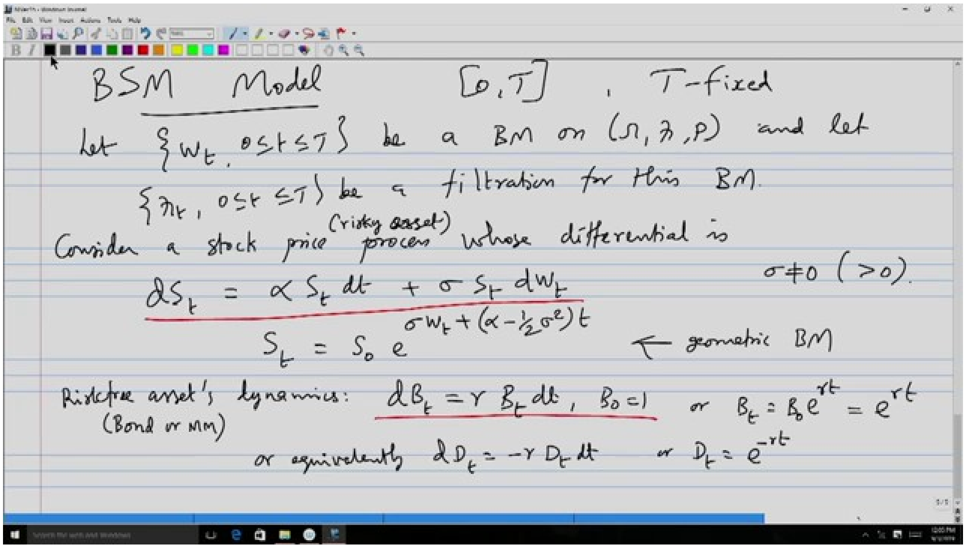

Which we have seen in some way that you know, if you take a limit of the binomial model what we end up with this a geometric Brownian motion model. and that is what will have in this BSM model. All the assumptions that we had with respect to the market in the discrete time model, in the binomial model case are also holds true here. (Those are) except the evolution of risk free and risky assets, because that is what the time is, continues time and discrete time model differs all other assumptions is true.

Which means there is no bid-ask spread, there is you know lending rate and borrowing rate is the same and any asset is available to be bought or sold at any amount of those assets are possible. So, all those assumption which we generally put under the name frictionless market is true, even in the case of this. So, what is difference is only the evolution of the risky and risk free assets which we give now. So, what we have? So, we have the underlying Wt, so it is basically 0 to T, is the interval in which we are looking at it.

So, this be a Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. and let $\mathcal{F}_t$ be a filtration for this Brownian motion. The simplest case is that take this filtration to be the one generated by the Brownian motion itself, because that is what we will be going to assume and we go forward but it could be in general. If it is in general then this needs to satisfy the condition for being a filtration for this Brownian motion.

Now, so consider stock price process whose differential is given by or the dynamics is given by $dS_t = \alpha S_t dt + \sigma S_t dW_t$, we generally assume σ to be strictly greater than 0 quantity but we need to assume that it is non zero. For all analysis purpose we need to assume that this is non zero but in general it is greater than 0. The requirement is this but actually this is greater than 0 in practice. But for the analysis purpose this non zero thing will suffice. So, here what is alpha, what is Sigma, we have already seen. this is the mean rate of return and how this is interpreted the mean rate of return, we have seen and Sigma is the volatility.

In general again, α and σ could be a time-depend quantities or it could even generally be adopted process but again for our purpose since we have this GBM in mind, so what we have is this process are equivalent. what is this means? This means that my St is given by this process that is all it is. e to the power Sigma Wt plus alpha minus half Sigma square times t, this is what we are saying.

So, when we say either differential form or in this explicit form, what I mean is one and the same. So, this is the second line, so this is what basically the geometric Brownian motion, which we call and we obtained this as a limit of you know a properly scaled binomial asset pricing model so that is what it is scaled. So, instead of giving an expression for St, we are writing its dynamics of St and that is typical way of doing here.

So, this is about the stock price process, then the risk-free asset. This is risky stock price in general essentially the risky asset is what then we are looking at it. Then there is a risk-free asset whose dynamics are given by you can write it as dB_t as $rB_t dt$ with typically $B_0 = 1$ or equivalently B_t as in general $B_0 e^{rt}$, which with $B_0 = 1$.

This is you can say assume this is a bond, say bond or money market just imagine that this are the case that we are having in mind. So this is the dynamics is given by this or equivalently Bt is given by this factor by which what call. Sometimes or equivalently we can also write this in terms of the discount factor. which is minus r Dt dt, which is essentially Dt is e to the power minus r t. so instead of writing this, this is one over Bt essentially it is one over Bt if you apply the Ito's formula to that function one over Bt you will get this expression does not matter.

So, this is second one is sometimes convenient to look at the discount factor rather than the growth factor only in that sense. So, what we have, two dynamics one for the risky asset and one for the risk-free asset. So, in binomial model how the price evolves from one time to other time is given by the binomial model. In this case by these two quantities, that is the difference.

Now, once we have this, you can see that in the risky assets, stock price dynamics has a dW term whereas the risk-free asset as a bonds dynamics has no dW term. and you see that bonds dynamics $B_t = e^{rt}$, which is what you know we associate to the risk free asset, because there is nothing random there. and S_t has a W_t which has a quadratic variation property, which mean it is random with a positive quadratic variation that is what it is, so that is where it is more random. So, at any given point of time in any dynamics, say this is predictable in some sense but this is in a way more random are purely random in that particular, in this particular example.

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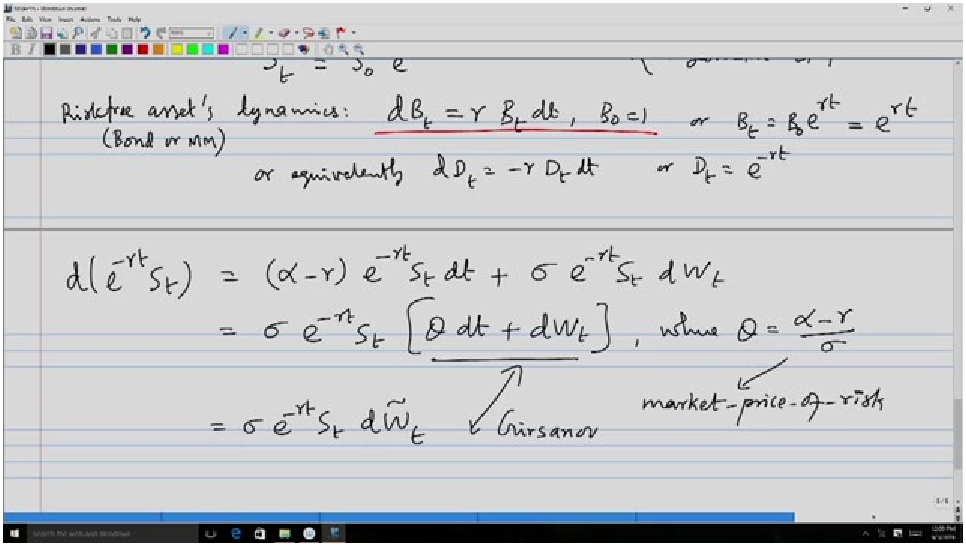

Now, once we have this now we can look at the discounted stock price process which is $d(e^{-rt}S_t)$, which we have already obtained in the earlier BSM equation derivation or you can simply apply Ito's formula again to arrive at this expression. which we are reproducing, again what we see is essentially the mean rate of return when you discount stock price process by the rate all by the mean rate or the mean rate of the asset s is reduced by the rate r, so it becomes $\alpha - r$. So, this is what we have observed this is what we know we could see here as well.

Now, this can be written as $\sigma e^{-rt}S_t[\theta dt + dW_t]$, where what is our $\theta = \frac{\alpha - r}{\sigma}$. which we call it as the market price of risk. So, what is this mean? People who are aware this will also called as Sharpe ratio. if you wanted to say but this is more populated your with this market price of risk. so alpha is the mean rate of return of the risky asset alpha minus r is the risk excess return over the risk free asset and Sigma is the volatility, which is we call risky a measure of risk.

So, per unit of risk what is the excess return that you get in risky asset, is what is called the market price of risk and this is what this theta is. Now, if we write for this particular theta, now we can see that we can create a Brownian motion say $\tilde{W}$ and with that we can write this process as $d\tilde{W}_t$. So, how do we get from here to here we are using Girsanov. So, recall Girsanov what we have to do that we supply pick a theta, define an Z_t and use that Z_t to reach EZ and from EZ you till define a $\tilde{P}$ and under $\tilde{P}$ this process this θ_t, $\theta t + W_t$ process would be Brownian motion and that is what we called as $d\tilde{W}$.

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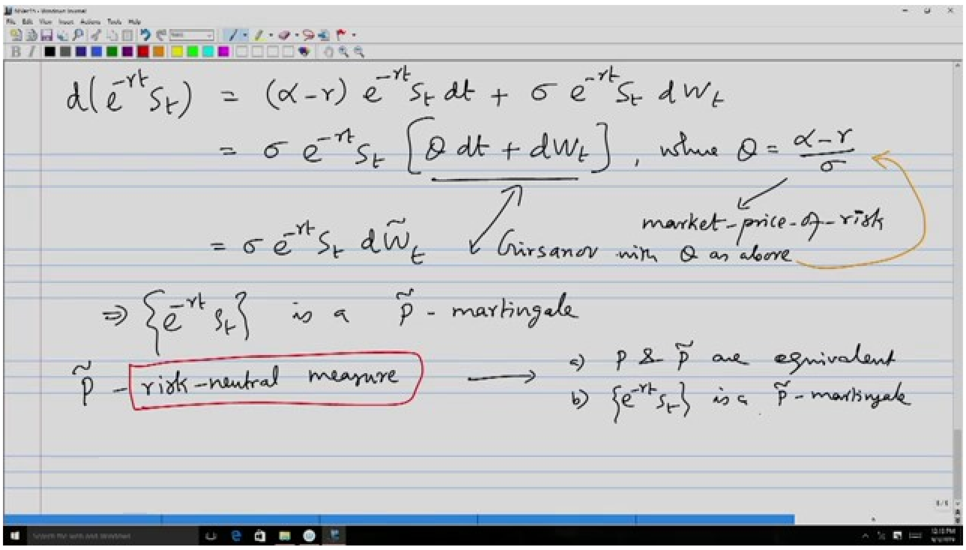

So, this is a you know Brownian motion, which by Girsanov by applying to Girsanov we see that with this theta you know as above as above means in this particular case this is the theta that we have here, keep in mind. With this particular theta you supply in Girsanov you get a $\tilde{P}$ Z and W tilde and this is that W tilde that you get here. So, this is what then you see, so what does that mean? This means that this process, the discounted stock price process is a $\tilde{P}$ martingale, this is a $\tilde{P}$ martingale that is what you know this expression means.

Now, (you would see that you can derive this is you can) you define this $\tilde{P}$ is what we call it as risk neutral measure, risk neutral probability measure. So, what is this? So, what are properties that it should have? It should have two properties one is a P and $\tilde{P}$ are equivalent and the b it makes this one, $\{e^{-rt}S_t\}$ is a $\tilde{P}$ martingale.

If it makes this discounted asset price process as a martingale and if it is equivalent to P any probability measure which is equivalent to P and which makes the discounted asset price processes as martingale. In this, actually there are two assets risky and risk-free but risk-free asset if you look at it, it is e^{-rt}, if you discount by that it become just one, so it becomes a constant value. So, you look at only the risky asset, which is $\{e^{-rt}S_t\}$ if this is a $\tilde{P}$ martingale then back measure is what you call it as the risk neutral probability measure or risk neutral measure.

Recall, now you can make a connection to this discrete time model what we have done. In discreet time model we define $\tilde{P}$ by using the parameters of the binomial model, which are actually U D and R. So, we define $\tilde{P}$ and Q tilde which make up the risk neutral probability measure by defining using these quantities by defining them using this, those parameters.

Here also, we are actually using the parameters of the model to define a new probability measure but it is in a much more complex way in the underneath like you know description it is coming because here we cannot define for each one of those paths or anything. Each one of those elements in the in an uncountable probability space. So, we define for each set A using this R.N. derivative. there it is simply ratio here it is actually an R N derivative which is a continuous process. And hence things are involved that is only thing.

But we are doing the same thing, you see theta, theta we are taking $\alpha r \sigma$ this are all the parameters of the model. $\alpha r \sigma$ all of them are coming from the model to make up this theta. θ you supply in Girsanov to get $\tilde{P}$ and W tilde, so that $\tilde{P}$ is what your risk neutral probability measure, so that is what. So, what are these properties, it should satisfy these two properties. Later we will define more precisely. But this is what you have here.

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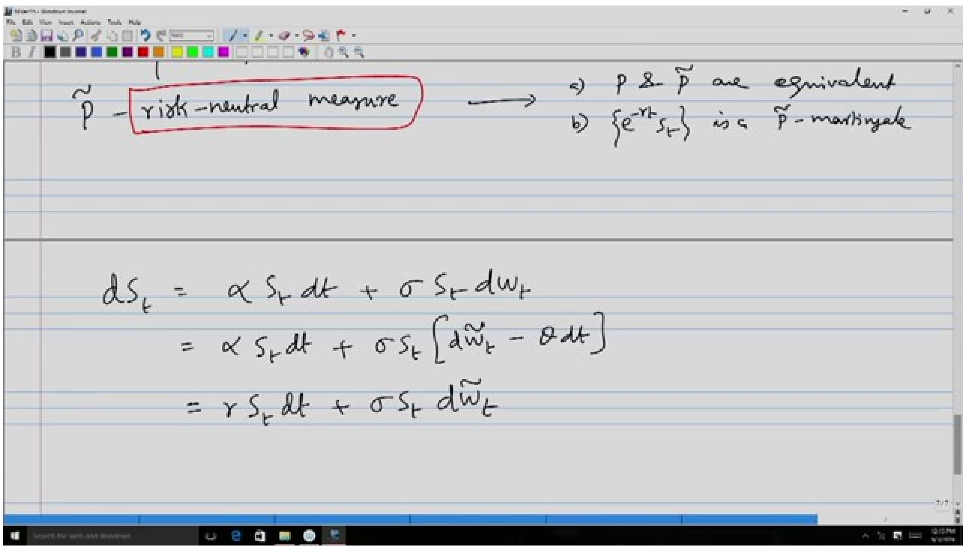

Now, (if you look at the) so the discounted price process is a $\tilde{P}$ martingale. So, what is happening to the undiscounted price processes? So, this undiscounted price process under $\tilde{P}$ what is its dynamics, we said dS_t. Now, dW_t you substitute in terms of $d\tilde{W}$, so what you would get, right.

So, which is $\alpha S_t dt + \sigma S_t dw_t$ is finally $rS_t dt + \sigma S_t d\tilde{W}_t$.

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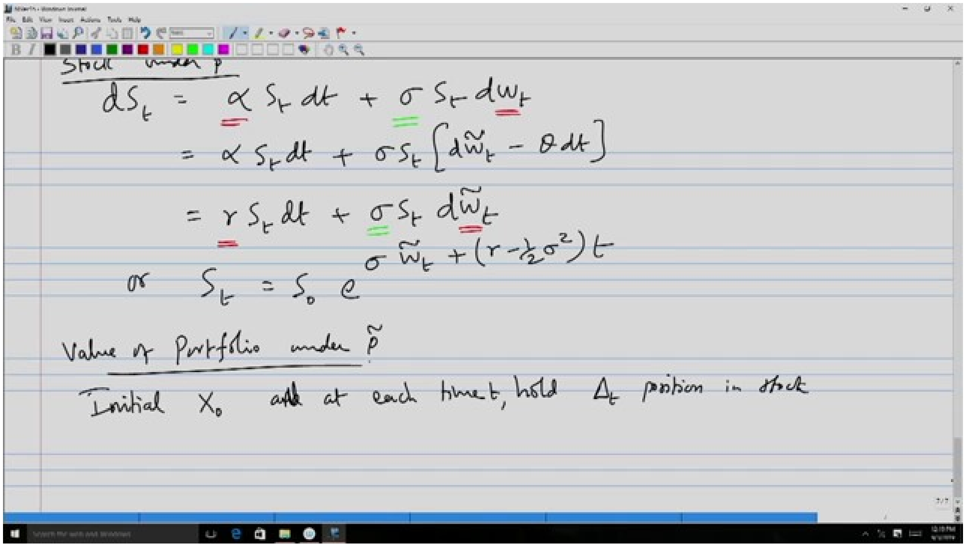

Or

$$S_t = S_0 e^{\sigma \tilde{W}_t (r-1/2\sigma^2)t}$$

is the solution to this. So, what is happening here? dS_t is essentially this quantity. you look at this what happens under W under W tilde. And you also look at this this σ. So, look at this under W, S_t has mean rate of return α under volatility σ under $\tilde{W}$ or under $\tilde{P}$. This stock price process has a mean rate of return or volatility the same Sigma.

So, now you can see like when you discount this by r, this first term goes away leaving out with only the dW term, which makes that as a martingale, so that is what you see here. So, this is what happens to the undiscounted stock price process under $\tilde{P}$. So, what is its mean rate of return, it is r instead of α. So, it changes the mean rate of return to r, so how does it being done?

So, this is in some way similar to what happen in the binomial tree, what is changed? (Binomial tree like it did not change the probability sorry it did not change the) when it change from P to $\tilde{P}$ in the binomial tree what we had it just changed the probabilities of that but it did not change the up and down movement which is in a way similar here that volatility term is what is basically the which stock price paths are possible or not, not possible. so it keeps the same the only thing is it changes the mean rate of return.

Wherever this alpha is greater than r which is typically the case normal times that is what we expect risky asset to give more rate of return than the risk free asset. What it does? It changes the probabilities by assigning higher probabilities to those paths which have you know lower return, so that you know the mean rate of return becomes or downs to or rather than alpha. That is what is happening here. So, now having seen this now we will look at the value of portfolio or portfolio value process under $\tilde{P}$. So this part is, this is S stock under $\tilde{P}$ what happens, now this is what we observe.

Now, we look at the portfolio process that we have. Recall what is the portfolio process that you have? So, you start with an initial X_0 and at each time hold Δ_t position in stock. Investing and borrowing in risk-free as required, so this is what the portfolio position Δ_t at each time t, at each time t hold this much position in stock. So, the rest is in risk-free, so if you look at a self-financing portfolio process of such nature then what is the dynamics that we have seen?

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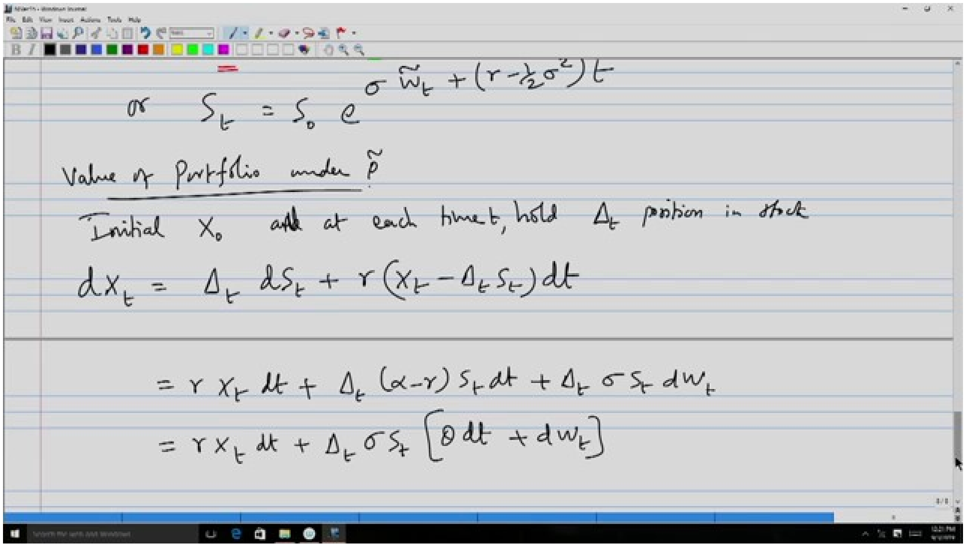

We have again already define or already derived this. So, maybe I can start the first-line itself so which is basically $\Delta_t dS_t + r(X_t - \Delta_t S_t)dt$. This is what we have written earlier. We have also shown that this is equal to by you know one or two steps if you write, you will get the final result given above.

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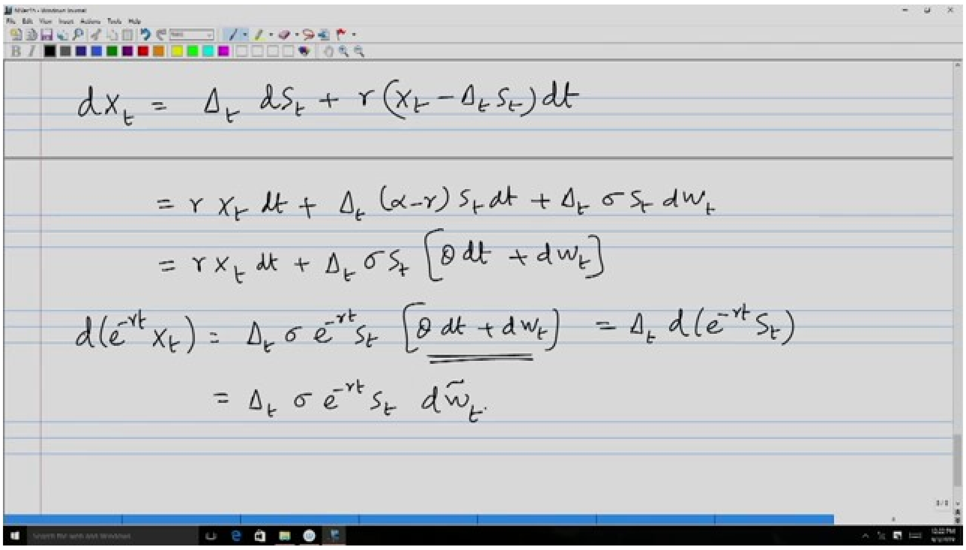

Now, if I look at now Ito's or the discounted value of this, which is also again we have obtained earlier. which will be equal to, so this first term will go because that is what is the effect of that the other terms will be there so that means that my $\Delta_t \sigma e^{-rt} S_t[\theta dt + dW_t]$, which can also be seen to be $\Delta_t d(e^{-rt} S_t)$, which means the evolution changes in the discounted value of the portfolio entirely or entirely due to the fluctuations in the discounted value of the underlying stock multiplied by the position. This we can use the earlier quantity and we can write this as.

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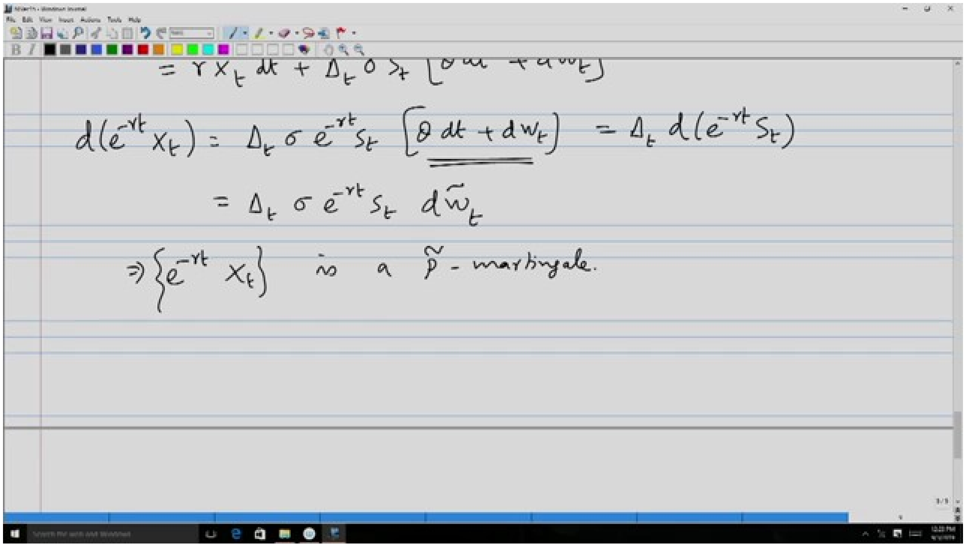

So, what we have now this quantity $\{e^{-rt}X_t\}$ is a $\tilde{P}$ martingale. That is what we have. So, what you have in front of U for investment that there is a money market account with rate of return r and there is a stock with the mean rate of return r under $\tilde{P}$. Regardless of now how you invest, so the mean rate of return for the portfolio is going to be r under $\tilde{P}$ which means with the proportion of the investment that you decide between risky and risk-free does not matter because the portfolio is going to give you a rate of return of r.

Recall exactly same thing we have seen in the discrete time model binomial model as well. So, that is precisely, so that is what we said understanding there is much easier. Here you know we just have to put the things in the proper frame work of continuous time model other than that everything is the same here.

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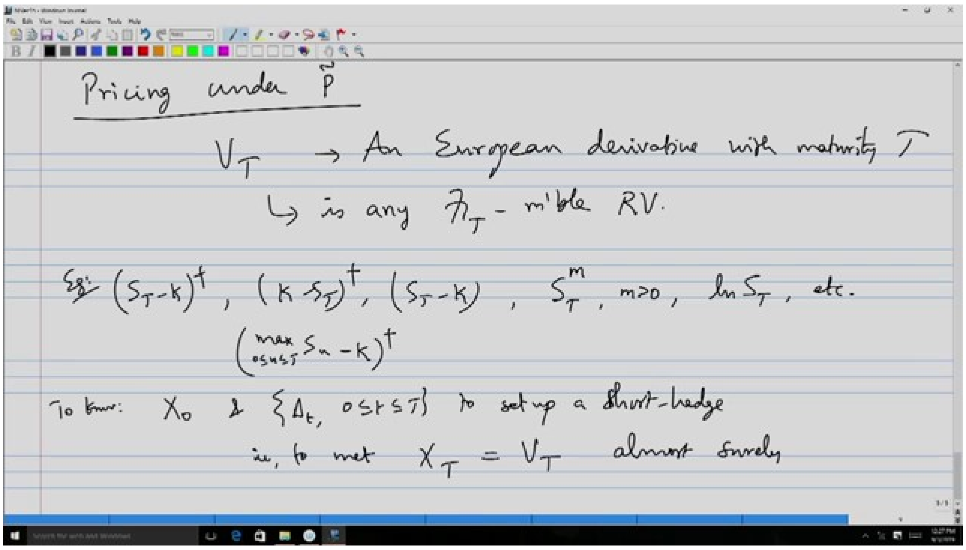

Now, we will compute the pricing aspect pricing under $\tilde{P}$ or under the risk neutral or measure or risk neutral pricing formula. So, what we have? Recall when we started this discussion we took the European call option as a derivative and we obtain the Black-Scholes-Merton PDE along with the terminal conditions solving, which we got the BSM formula.

Now, let us look at a generic European type derivative, which is V_T, which is what an European derivative with maturity capital T and in our case what does this mean? This means that this is simply, technically what this means? This is simply is any $\mathcal{F}_t$ measurable random variable, that is what is V_t for us. We will not worry what is the form, if it is $(S_T - K)^+$, it is European call option.

Again you could have path dependent, you could have anything, anything is what you know we say is an European derivative with maturity T. Just like in discrete time all these functions could be framed into this. So, we allow this could also be the path dependent something like you know Max of Su for u lies between this, all these things. so this are all could be the path dependent or path independent everything.

Now, what we wanted to know? We want to know an X_0 and a portfolio process Δ_t, so we wish to know the what is the initial capital require and what is the portfolio process that this needs to adapt to set up a short hedge. Means short position that he is taking the underlying derivative is need to setup this hedge, which is what we call short hedge.

To setup the short hedge, that is to meet what I should have X_T equal to V_T almost surely. that is what you know we need. If it is surely it is done, fine, but in continuous time we always require this to be almost surely case. So, with probability one if I can achieve V_T by replicating this by using this Delta hedging strategy Δ_t and starting with some initial wealth X_0 that is what you call, you need to know.

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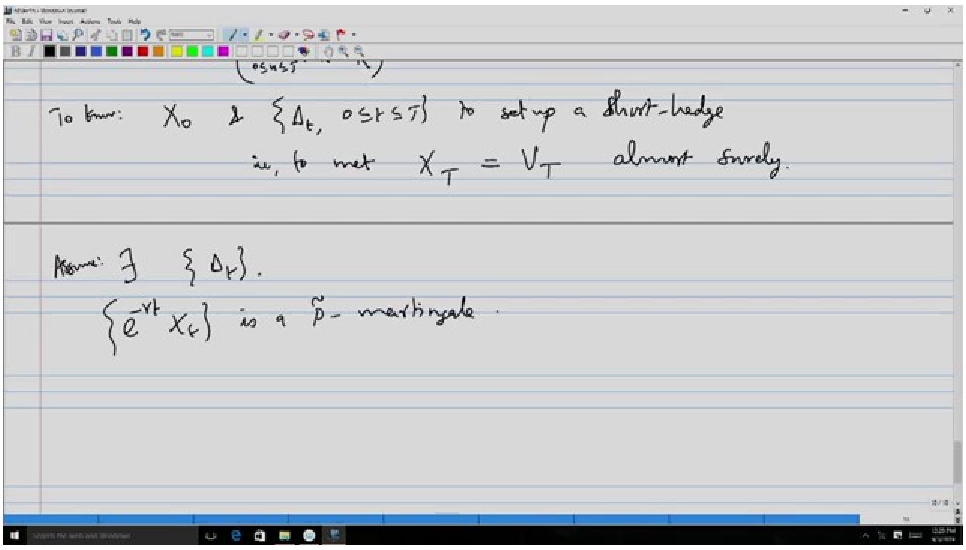

So, this he wants to do this. So, in a moment little later we will see that this Δ_t whether there exist such a Δ_t. we need to be assured of that, that will be answered little later for the moment you assume that there is some Δ_t that is available. Then X_0 and is what you need to know to set up the hedge. So, this can be done if you see, now once you could have, so, (there exist) assume there exist such Delta t process. Now, if that is the case if that can be chosen, now what we see this fact that we just now saw that $e^{-rt}X_t$ is a $\tilde{P}$ martingale.

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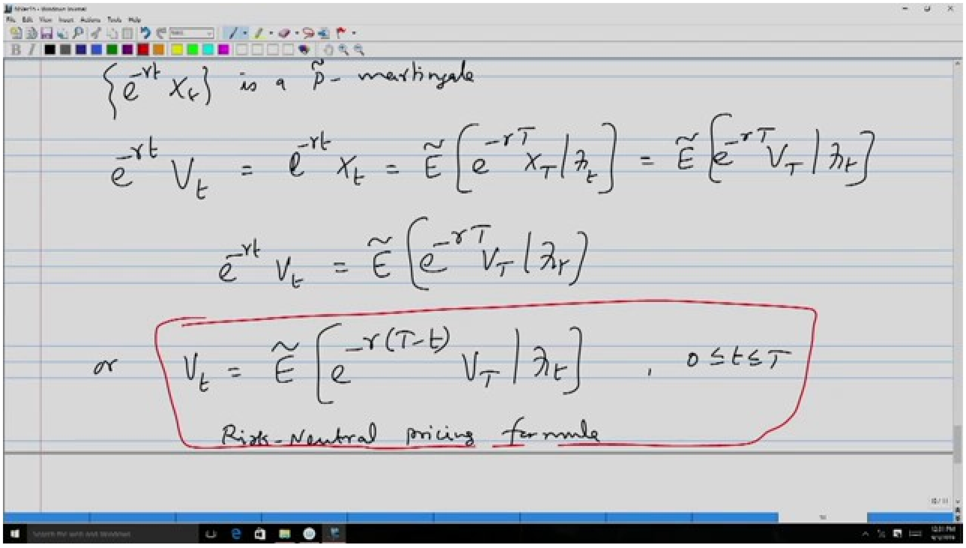

So, what is this means? So, this means the following that since martingale have constant expectation, so what this would mean is essentially $e^{-rt}X_t$ is equal to $\tilde{E}[e^{-rT}X_T|\mathcal{F}_T]$. So, this is what the martingale property gives. Now, since we want this X_t to replicate V_T, this is also equal to $\tilde{E}[e^{-rT}V_T|\mathcal{F}_T]$. Now, since X_t is the value that is required at time t to hedge your position from small t until capital T and hence this is the value of the derivative at time t and hence you would call this as V_t. so this is what is called as the risk neutral pricing formula. We have seen something similar in discrete time. So here also you have exactly the same, which is what the important one is.

So, this is what, is important one, this is the risk neutral pricing formula. So, we have, how did we arrive at just quickly go through once more. We observed that the discounted wealth process is $\tilde{P}$ martingale and the martingale property means these a middle equation, middle part of this equation and since we want X_t to replicate V_T, so this X_t is actually is equal to V_T.

Now, if I look at the first and the last, second and the last term here. you will see that X_t is the wealth that is required at time t to replicate V_T and hence X_t must be the price, no arbitrage price of V_T at time t and we call the price to be V_T and hence we get this expression and hence we get this risk neutral pricing formula that we have here. So, this is what we have risk neutral pricing formula that we have here, this is what is important.

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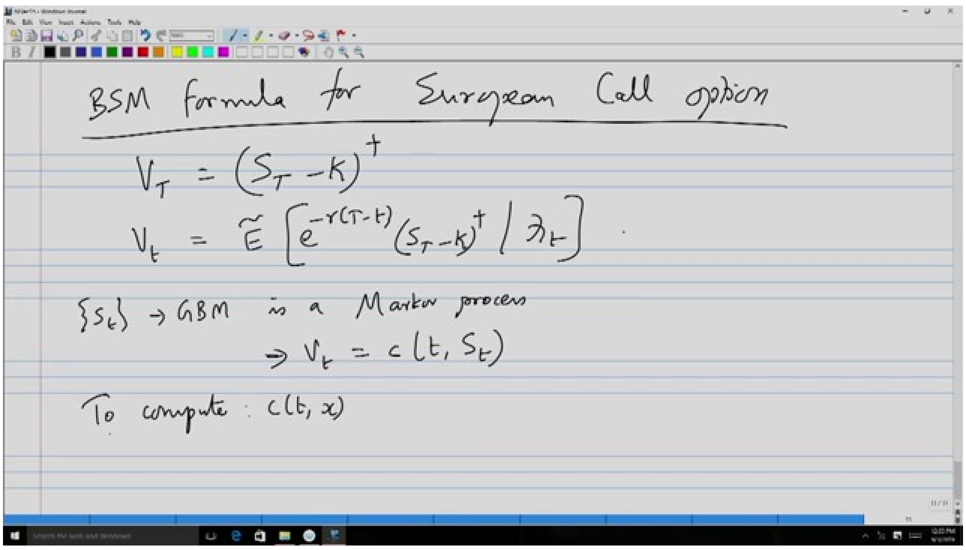

Now, given any derivative what you need to do? You just need to evaluate this expression to get what is the price at time t. Now, let us see quickly how this BSM formula for European call option, which we have already said that we gave the BSM and BDE along with the terminal and boundary condition and we said this is the solution and we did not derive the solution. If you are comfortable with solving PDE you can solve the PDE along with the boundary and terminal condition and see that that solution actually occurs. What we are going to give now is an alternative using risk neutral pricing formula, which is what how do we obtain BSM formula for the European call option, how do we arrive it. So, what we have here, call option here so my $V_T = (S_T - K)^+$, Now what we see is that S_t the process is GBM. So, S_t which is geometric Brownian motion is a Markov process.

We can also notice that W_t is a Markov process, geometric Brownian motion is also a Markov process. (So, the right-hand side is then nothing but) so this implies V_t will be some function of t and S_t at time t because what is this, what is the definition of Markov process? If S_t is given to be a Markov process then some function of S_t for any every function of S_t there is another function G such that if that satisfies. So, this is given to be Markov process because GBM we have already seen W_t is Markov, this GBM is Markov, so this is Markov.

So, because geometric Brownian motion is a Markov process, so this expression, the right-hand side expression depends on a function of t and S_t which we call $c(t,S_t)$. So, we have to now compute, the program is now to compute $c(t,x)$ is what you know we need to do. Now, how do we do? We will use the; how do we compute this, because here now, if you look at here this V_t expression you see that S_T that the S at maturity time t and F at small-time t. if I look at it this S_t is neither measurable with respect to $\mathcal{F}_T$ nor independent of $\mathcal{F}_T$.

So, it is a general conditional expectation. Now, in that case how you will you evaluate. we will try to evaluate that you break this S_t into pieces of two or more random variables some of them are measurable with respect to $\mathcal{F}_T$ and some of them are independent of $\mathcal{F}_T$ and then use independent Lemma to evaluate the conditional expectation that is the idea. So, use how to compute is now use independence lemma.

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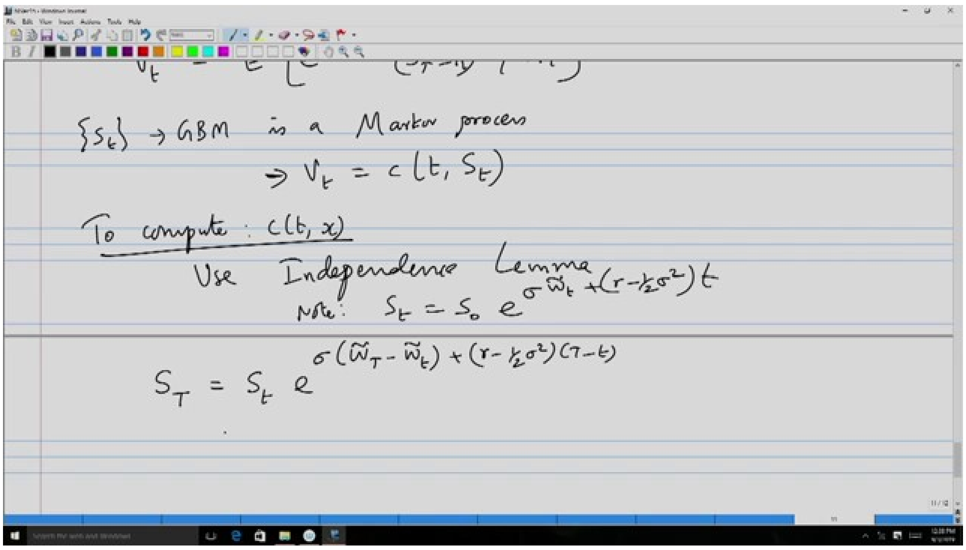

So, if you independence lemma if you want to use, now you look at what is my S_T. We know S_T is given by the above formula.

You take S at t, S at capital T, so what you are taking S at capital T divided by S at t, so you will get this expression. This is otherwise you know you can directly write down also this expression if you want to obtain from the previous line you can obtain this. So, you can segregate up to t and then the remaining portion.

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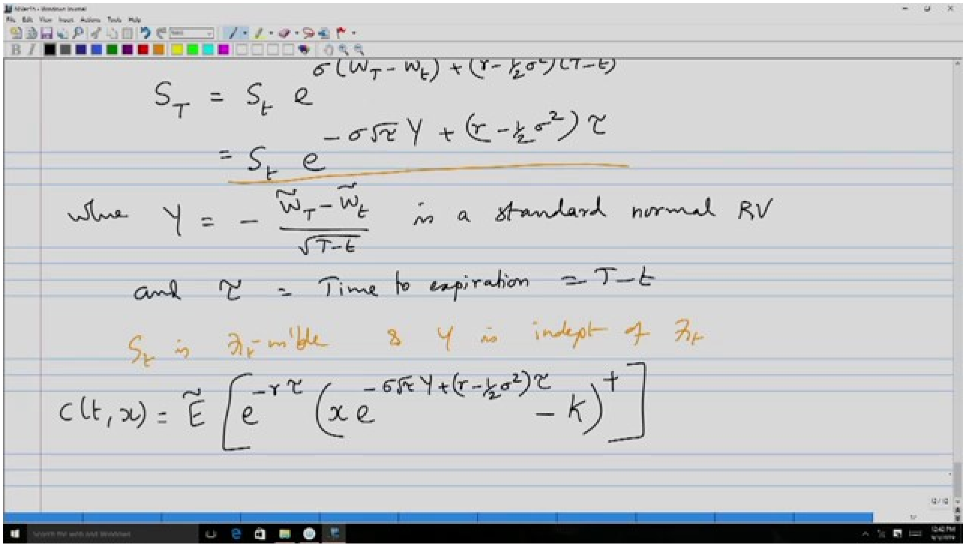

This you can see can be written as can be written as S_T as that, where $Y = \frac{\tilde{W}_T - \tilde{W}_t}{\sqrt{T-t}}$ is standard normal. Or the difference has the mean 0 variants $T - t$. So, if you divide this you will get the standard normal, if EZ is standard normal minus Z is also standard normal and hence Y is standard normal. And we also use this notation tau for simplicity, tau is the time to expiration or the remaining time or time to expiry, which is basically T minus t, we denote it by tau because everywhere now tau, tau, tau we will be use rather than T minus t, this is what will be using it.

So, (now once we like St in this form), now if you look at inside the conditional expectation risk neutral pricing formula, you see that you can look at this particular line, you can look at say this particular line expression for this. Now, Y is a standard normal random variable which is independent of $\mathcal{F}_T$, so what you observe is, now S_t is $\mathcal{F}_T$ measurable and Y is independent of $\mathcal{F}_T$. So, you have expressed this capital S_T, which is neither measurable with respect to $\mathcal{F}_T$ nor independent of $\mathcal{F}_T$ as a product of in terms of two random variables one is measurable with respect to $\mathcal{F}_T$ the other is independent of $\mathcal{F}_T$.

So, that is what you have done precisely. now with this, so this is Y is independent and hence this whole complete thing is e to the power something something, everything is independent of $\mathcal{F}_T$. So, we can now compute $C(t,x)$ by again using the independence formula.

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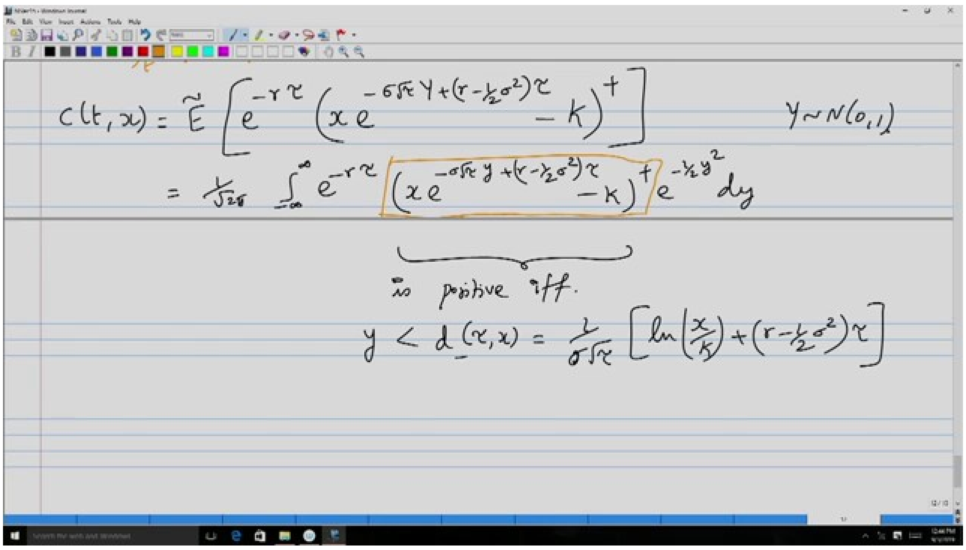

Now, we just need to evaluate this function where Y is normal 0, 1. So, we need to evaluate this conditional expectation to obtain the function $c(t,x)$. so this is then nothing but the expression given above. So this is what we have here.

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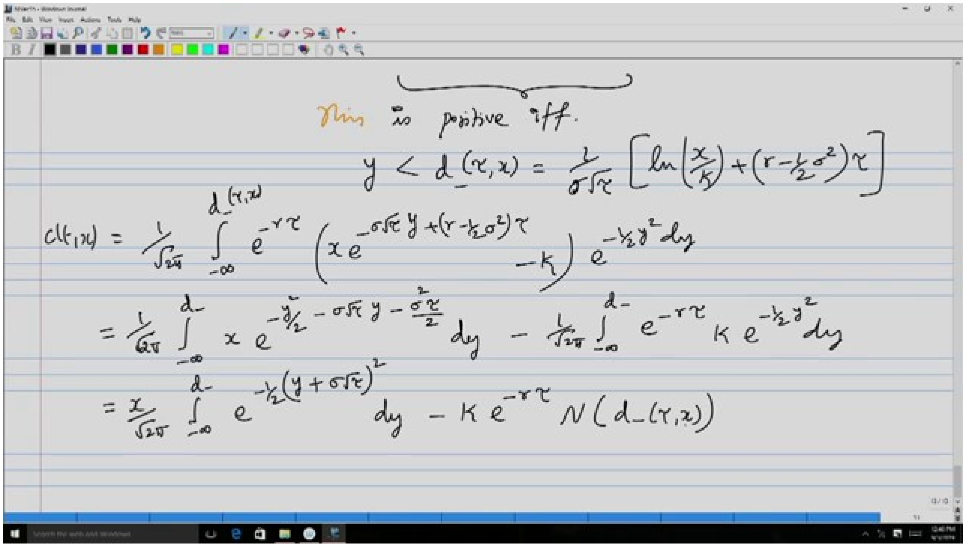

So, if we use this, if this is positive if and only if this is true. so that means my $c(t,x)$ now is one over under root 2 pi minus infinity to d minus of tau and x up to that limit I will take, because the other one is 0 anyway. So, I can write this inside one e to the power minus Sigma under root tau y plus half Sigma square times tau minus k without the positive part that is what you know you would do dy.

Now, this can be further split with this minus you can split it into two part, one is under root of two pi minus infinity to so d minus for the simplicity I am just using simply x minus x. Now, this e to the power you can see that this is y square by two minus Sigma under root tau y minus Sigma square tau by two times dy minus; so I will write in this form itself for the time being for this step minus infinity to d minus e to the power minus r tau, k e to the power minus half y square dy.

Remember, in the first term e to the power minus r tau just outside and inside e to the power r plus tau gets cancelled the remaining terms is what collected here. and this could be written as a perfect square. So, you can see that this is x by under root two pi of minus infinity of d minus e to the power minus half y plus Sigma under root two pi this square dy minus k e to the power minus r tau of N of d minus of tau x. So slowly we are getting into that.

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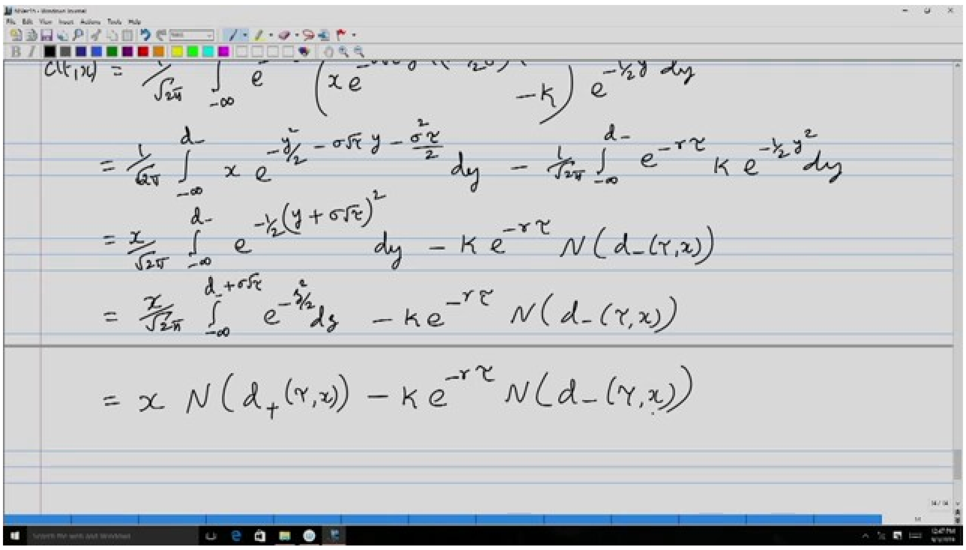

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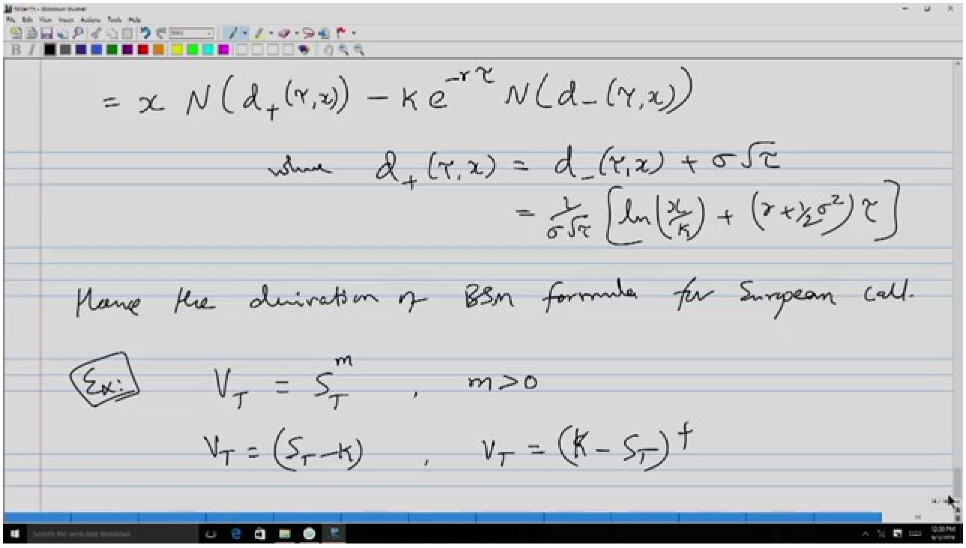

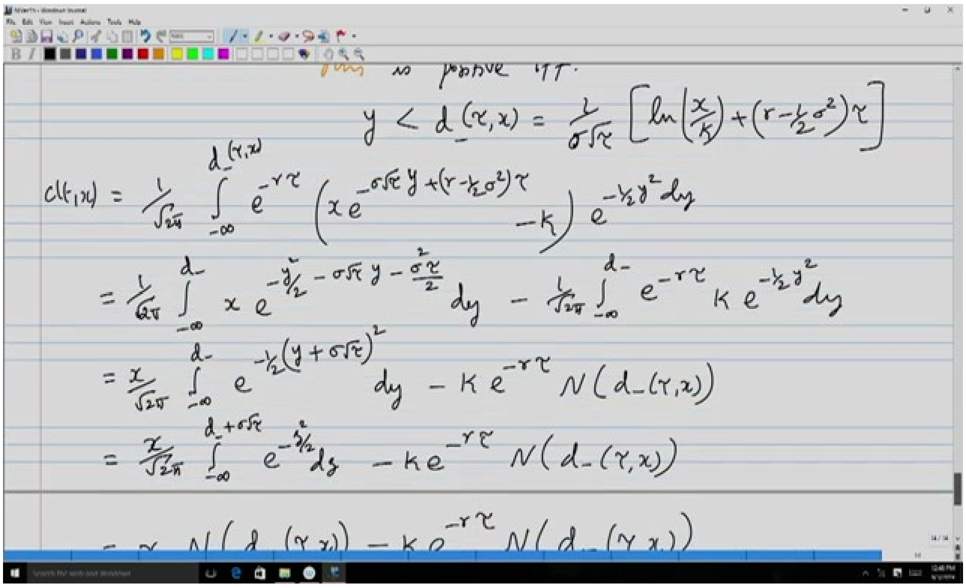

Now, see this is the technique that you have seen. How we derived the European? Now, you have the forward contract price you have the foot price. Exactly same process if you repeat instead of $V_t = S_t - k$ you put appropriate quantity and tried to follow the same process you will be able to derive the price for the corresponding European derivative. So, you can exercise for example this quantity. Suppose if, this is m is some, you can take any real number or m greater than 0, so this is exercise. You can try V_T equal to $S_t - k$ for there will be second you already have the expression S_T to the power m for some m real number or in general m positive in particular you can obtain, what is the expression, everything you know you just have to derive in a similar way you get the corresponding BSM formula for the corresponding option.

So, what we have derived is BSM formula for the European call option which we have given as a solution earlier we did not provide how to obtain the solution so now we have obtained the solution that we have. So, we have given risk neutral pricing formula and we have used it to obtain the solution as how to obtain the exact expressions.

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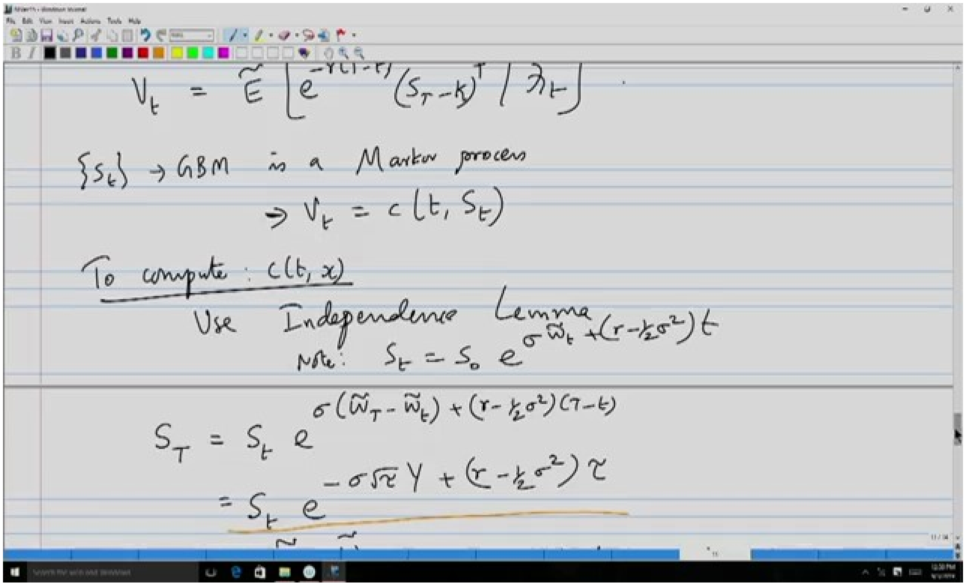

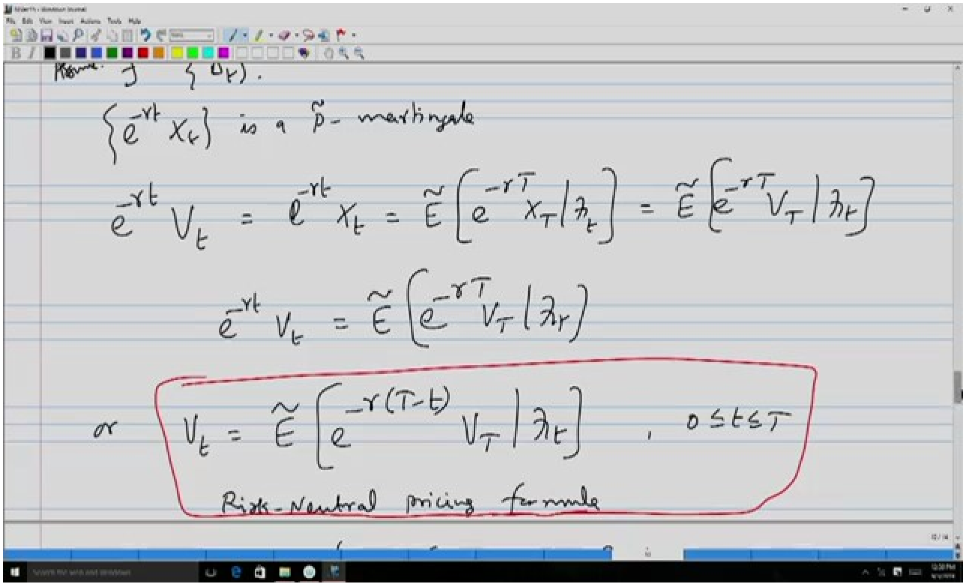

So, remember this is a based on the assumption that you can construct a short hedge and that depends that there is a portfolio process Δ_t. Now, the existence of Δ_t is what we need to prove to complete the circle, otherwise the risk neutral pricing formula that you are giving has no meaning unless it is accompanied by the short hedge expression are at least the proof that there exist such a process. So, that is what we will do next, thank you.