Mathematical Finance
Lecture 31: Black-Scholes-Merton (BSM) Model, BSM Equation, BSM Formula

Professor N. Selvaraju[1] and Professor Siddhartha Pratim Chakrabarty[1]

[1]*Department of Mathematics, Indian Institute of Technology Guwahati, India*

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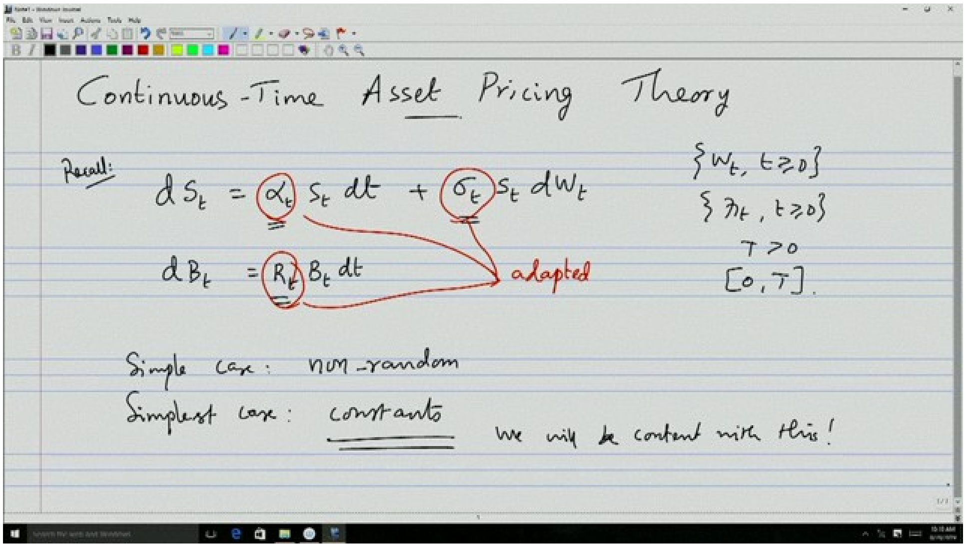

Hello everyone, the next topic that we will see is the continuous time finance or continuous time asset pricing theory or risk neutral pricing in continuous time. Before we go to this neutral pricing, you know, first we will try to apply you know the similar process as we have done in discrete time. Recall, that in discrete time we first considered the binomial asset pricing model and we considered derivative, which has an underlying as there is 3 assets in the binomial asset pricing model.

And through replication and with the help of no arbitrage pricing theory, we could find out what is the initial prize and prices at all intermediate times along with the delta hedging rule, which will replicate the short position that you had taken in the under derivative. So, you just recall that portion. So, in a one period model, what we have done is that, there were two quantities Δ_0, X_0 which needed to determine. Which we determined by equating the evolutions and setting it up a 2 by 2 system of linear equations and by solving which you know, (we get the) we have got the solution, which is basically what is the initial price. and what is the replicating strategy for replicating the short position in the underlying asset.

Similarly, in a two period model. If you recall, there were 6 unknowns, 3 of them are the underlying positions and 3 of them were the prices during that time 0 and time 1 in a 2 period model. So, this is what we gave via the binomial pricing algorithm and we found out this quantities and hence because this is the price that is needed along with the hedging strategy to replicate the short position in the derivative.

So, the short hedge, we simply call that as the short hedge, replication of short hedge or the value or the wealth that is required to replicate the short hedge position, what we called as the no arbitrage price. because that is what will give you a fair price. Because that does not give any one party, whether this long or short order any undue advantage, is not being given by such a price. So, that is what we call as a fair price. The fair, the idea in our context is the no arbitrage price. So, based on the no arbitrage principle, so that is why, that is call, we call that is arbitrage pricing theory approach to find the prize of the derivative.

Later, we did put this after defining the 2 quantities $\tilde{p}$, $\tilde{q}$ which we called as the risk neutral pricing measure. We try to put this in that framework and give the risk-neutral pricing formula. Here also in continuous time also, we are going to do almost you know, similar thing. First what we will take up is the first part of it, which is basically the APT approach or by replication or by equating the evolution. How do we find the price or what is the price?

So, that is a part that we will take. So, that is what the first part of this ideas that we will carry over because then you can connect with your discrete time model, how you done that. So, what we have now, is that the continuous time, all the assumptions that was there in the discrete time model holds true here also, which means if there is a frictionless market where we assume that the bid-ask spread is 0, any amount of quantity , it can be bought it can be sold and the interest rate for buying and selling, which is borrowing and lending is the same. So, all such assumptions that we had it in binomial model remains true.

So, what is the different assumptions that we are making here is about the evolution of the risk free asset and the risky asset. Because there the time was discrete, here the time is continuous. So, you need to describe how the prices at various times are given. It could be given simply as complete expression or it could be given in an equivalent way to, if you describe how things evolve from one time to the other. Then that is an equivalent way of description. And in finance as you might have understood or you might have seen that you know, we prefer the later one to describe the evolution, to give the asset price process.

So, in that regard, in that regard recall this processes that we have mentioned. So, we said that dX_t or now I may call this as dS_t, where this, this and also you could have a $dB_t = R_t B_t dt$, which is , this three quantities. So, this three quantities, α, σ, R_t if they are adapted. And what we have the underlying one, underlying one is, there is a Brownian motion and there is associate filtration or there is a filtration for the Brownian motion. And there is a timeline so in which you are considering in this model.

And we said that this model, of course, we know what is the expression of S_t in this particular case and what is the extension of B_t in this particular case. So, this dS_t expression is what we said that can be used to describe almost all processes with the properties that they have continuous paths, has no jumps and driven by single Brownian motion. Whenever α_t, σ_t or adapted processes.

Similarly, the second quantity, which we wrote here is in general. It could be taken as adapted processes. So, this is you could think of as the, you know, the evolution of the risk free asset. Which is less random than the randomness in S_t. Because the randomness in S_t is also contribution from these dw term, which is Brownian term, which can oscillate very to a large extent in a even small interval of time. Because this is normal distributor random variable in oscillations. So, because of this property. So, this is the general setup. So, this is what along with assumptions, if you assume that there is a risky asset, which follows the dynamics given by the first equation and there is a risk free asset, which follows the dynamics given by the second equation. Then this becomes a more general model for, you know, which can be applied in almost for to denote or to define the evolution of almost all processes which, with the required characteristics for S and V.

But you know for simplicity because this will involve α_t, σ_t and adapted processes. A simpler case could be when these three quantities are nonrandom. (The simple cases), these are nonrandom quantities this, what we talked about this alpha, sigma and R. Then it could be, we could make simply this as time-dependent. Now, the simplest case is they are all constants.

So, the simplest case of this dynamics is what they are all the constants. And we will be content with this, in this particular course. So, you can make it slightly more general nonrandom and theory is not going to be so complex. But you know the expression, writing, and everything. So, this is constants that we are making it.

So, that means, when we have the constants here. (Then what actually) if you recall the solution of this, the first equation and the second equation. The first equation, then it gave simple geometric

Brownian motion and in the second equation, it is simply the, simple ODE in the particular case and that gave solution as E to the power RT along with the 1.

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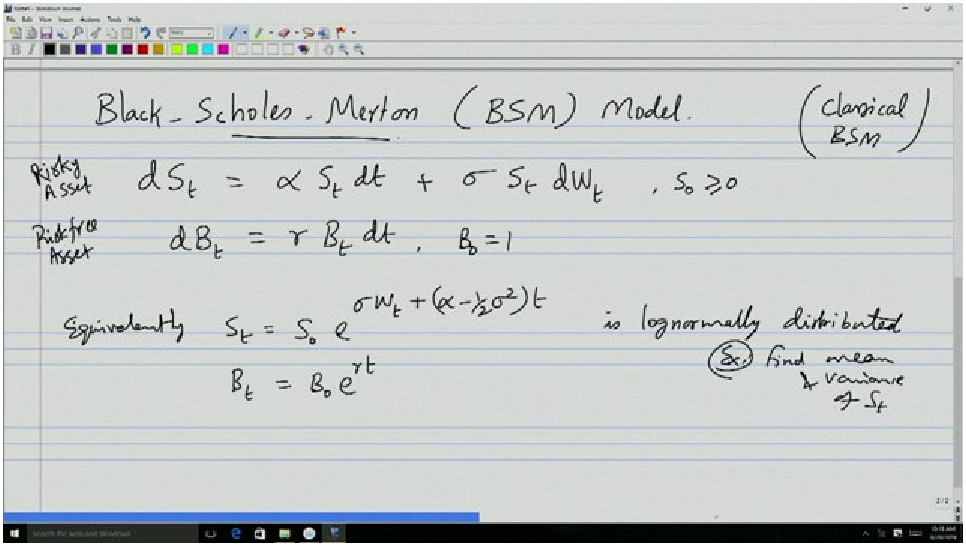

So, let us describe that as the model that we are considering and that model is what will be called as or BSM. As I is oppose to the binomial model of the discrete time case, what we have here is the Black Scholes Merton model, which was traditionally is to be called as simply as Black Scholes model. that is how you will find in many books and references and places. But if you go back and you will see that actually the Noble prize was awarded for the works of all people, all three. So, of course you might like to call them as simply as Black Scholes Merton model.

So, this is the Black Scholes Merton model, which is the continuous time thing that we have. So, now what we have the description is, you have a risky asset, whose description is given by

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

and the risk-free asset is given by

$$dB_t = rB_t dt, B_0 = 1$$

Because if as you know in these two particular cases equivalently, this can be represented as$S_t = S_0 e^{\sigma W_t + (\alpha - (1/2)\sigma^2)t}$ as $B_t = B_0 e^{rt}$.

And r you are writing in terms of the, the previous two equations, which are effectively SDEs, of course, the second one is actually an ODE but you can still call that as an SDE. So, this is given by the SDEs.

So, to make a meaningful discussion we simply assume that S0 is greater than 0, it is not necessary that you know this is the case here. So, it could also be $S_0 \geq 0$. And similarly for $B_0 = 1$ for convenience we take, it could be any starting point and S_1 solution is simply this, in this particular case. So, this is what the description.

All other assumptions with regard to the market under which, you know these two assets exits, remains the same. So, what we consider? So, this is in some sense called the classical BSM, I mean in sometimes, if we mean that the constant coefficients. The, in the SDE for risky assets the drift coefficient is constant and this diffusion coefficient is constant and the SDE for the risk-free asset the coefficient is also constant. And what is the risky assets that we have in mind when we are writing down this, this is typically a stock. Of course, we could use this model to model any other price process, which will follow this pattern,$S_t$is given by this.

And SDE is what? Distribution now, it is a log normally distributed, essentially what. So this is log normally distributed random variable, this$S_t$is log normally distributed. Because there is a W_t here, (which is the), which follows a normal distribution and this is E to the power some constant times

W plus some constant. So, that is essentially a log normal distribution. (S_0, S_T), by the way, this S_0 is typically assume to be non-random that is we also make it clear. And$S_t$is log normally distributed random variable, you can find its mean, variance yourself.

So, exercise is find mean and variance of ST, you can try and see like what it its mean behavior, what is its variance, how it is distributed as. So, this is the model. So, this is the classical BSM setup, where there. and risk-free asset by the way. We mean it could be a money market account, it could be a bond, it could be anything something which has this as the dynamics. Like this, there is a confusion here that the sigma$S_t$is DWT. So, there is a Brownian motion.

So, you have a probability space on which we have defined a Brownian motion and there is a filtration for this Brownian motion with required properties as we defined earlier and we have the assumptions about the market and the assumptions about the behavior of or the evolution of risky and risk-free asset. All this thing is what defines or describes the BSM model. So, now that we have described the BSM model.

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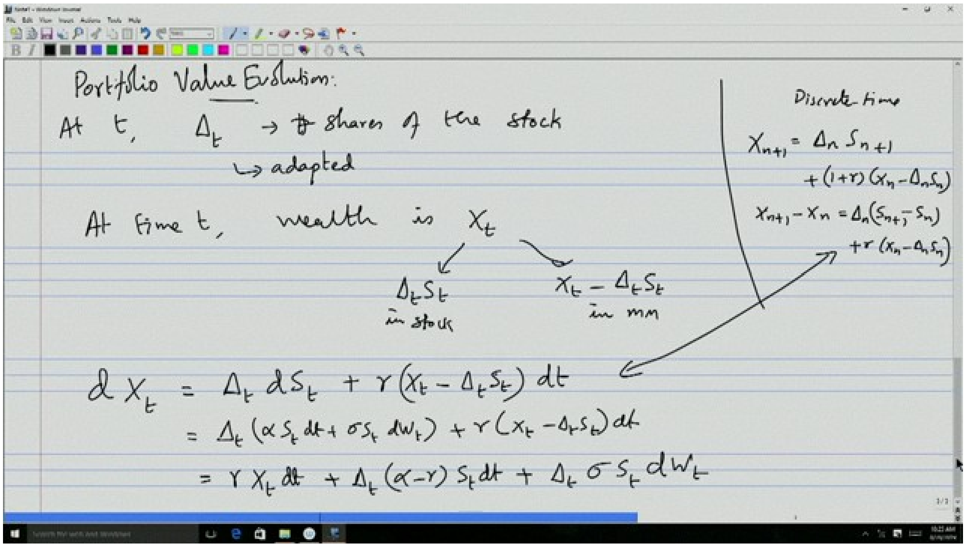

Let us look at the case, where the first part of this exercise that we will try to replicate, what we will see. Now, let us look at as the first part. How the portfolio values evolve, in such a dynamics case. So, what we have seen portfolio or the wealth. So, what we have? We have these two dynamics, this dynamics given and we have what we take is at t, Δ_t to be our position in the risky asset, which is stock. So, this is basically the number of shares of the stock, which in this particular case is the risky asset.

So, this position can be random, but what we require is in that case is this adopted, this be adopted to the filtration. So, this position can be random, but when you reach time t, this value of this should be known, so that from time t to t+, you are going to take this position. So, this position should be known at the point of time.

Recall, in discrete time binomial model we have that. So, you say that you have at time t, you know, so the wealth or the portfolio value. Suppose if, we call this as X_t. Then you take Δ_t as your position. So, which means out of these X_t how much you are going to invest in the risky asset. It is $\Delta_t S_t$in stock and the remaining amount, which is $X_t - \Delta_t S_t$ in say stock we said no, so in money market or bond or risk-free. This is this much you are going to invest in the risk-free asset. So, this is what. So this is your decision, what this Δ_t portfolio position process, Δ_t gives, is that at time T, if your wealth is X_t. Then, out of this X_t you are going to invest $\Delta_t S_t$in stock and the remaining amount is in X_t. So, you are not keeping anything with you. So, (you have to) you are investing everything there into this.

Now, so how this wealth evolves? We can write down the wealth, which is changing in a small interval after t, which is what is, you know typically writes in informal way as the dX_t. Now, the contribution of this would be from two sources, one is the in stock, our position in stock, the other is our position in risk-free asset. In stock, we had a position of $\Delta_t S_t$, which means Δ_t number of shares ST.

Now, this S_t has moved from S_t to $S_t +$ something. So, which means that moment is essentially. So the evolution of this component of our wealth is basically the $\Delta_t dS_t$ plus the other component. Since, it grows at the rate rdt. So if you have dB_t then it is $rdB_t dt$. .

Now, you can draw a parallel to your discrete time thing. So, let us see, in discrete time what you had, you recall.

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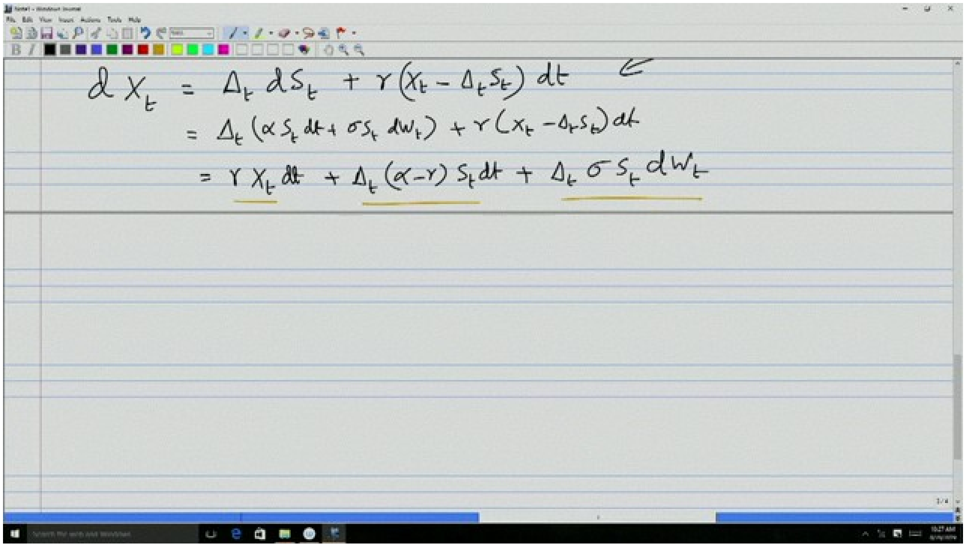

Now, this is you know the terms appearing on the hand side here, you know, you can try to understand what each one of them mean. So, there is a first-term, there is a second term, there is a third term. So, what is this actually it depict. as you can see the first term is basically the, the average underlying rate of return for the portfolio, which is given by $rX_t dt$.

And the second term is basically the risk premium, for going to the asset S_t, this $\alpha - r$. R is the rate of return of the risk-free asset and alpha is the rate of return of the risky asset. So, for picking the risky assets, this is the risk premium that you are having it here. which is obviously proportional to the position that you are taking and the third term is basically a pure volatility term proportional to your again, your position and the size of the stock investment which is delta dS_t, is what it is.

So, this is a pure proportional term, you see , dW is essentially W is normal. So dw, so it could be the difference in the normal. So, this is a pure volatility term. Whereas, the first two terms, which has dt terms. I mean, you know (now by now you know you understand) the meaning of this dt terms. So, this more certain than dw term because this appears in the short interval, you can predict that is how the ODs are formed and if you perturb with the randomness is what you get an S_t, so this is of that nature.

So this three terms, you can understand as if you know this the evolution of a portfolio value is given by three components, one is the, the average that you expect, the other is the risk premium, the other is the volatility term . Now, we often shall consider the discounted stock price process and discounted portfolio process. So for which what does that mean? So, we often consider rather than X_t and S_t, we consider the discounted value of it, so, that you know you are bringing everything at single time point unit you are looking at it.

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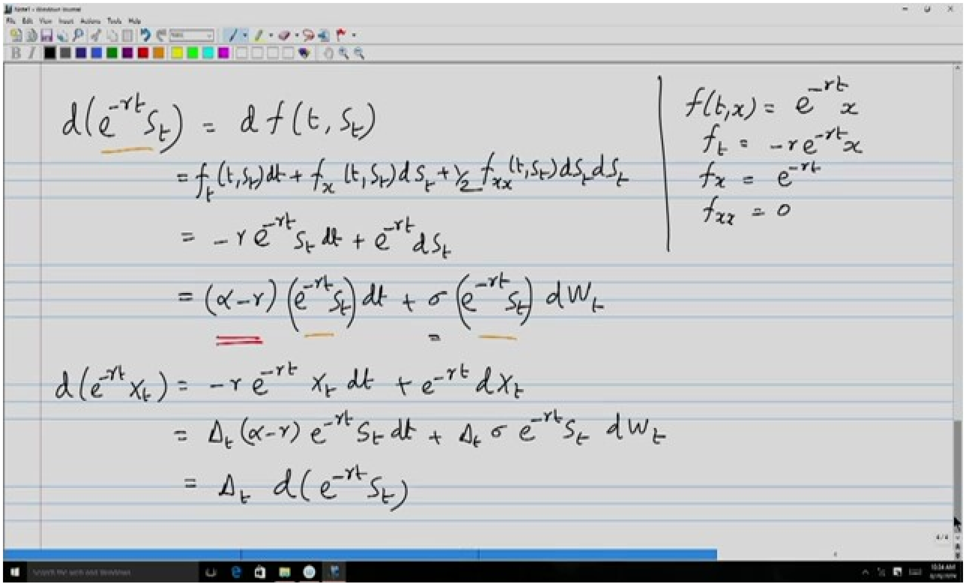

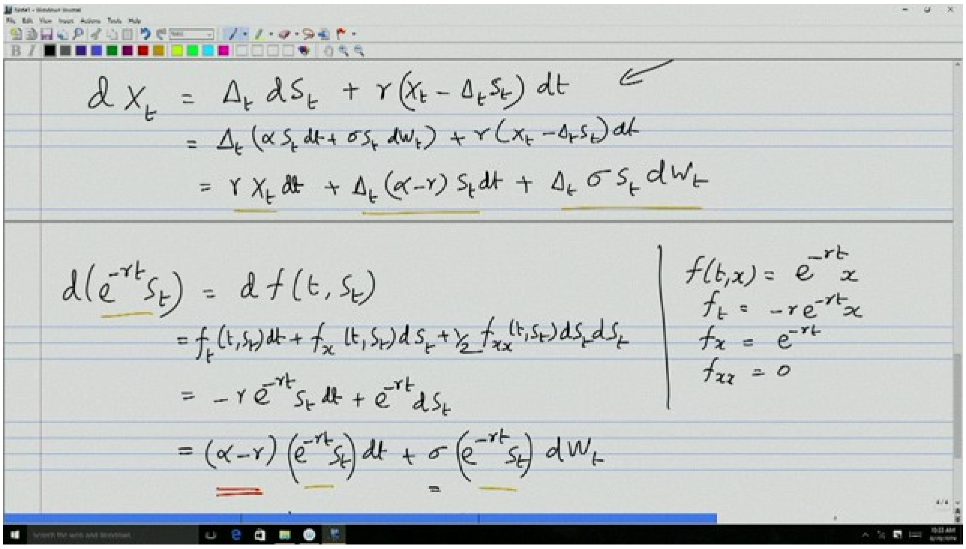

Now, so this is essentially, what we mean is, this quantity, is what we often look for $(e^{-rt}S_t)$. If I consider this to be a $f(t,St)$. So, what is the function here now? We have $f(t,x) = e^{-rt}$. We can see the partial derivatives now. Now, if I apply ITO's formula here what we will get is that.

But the change that happens is in the mean drift term. So what actually, when you have a process and if you look at the discounted process what happens? The mean rate of return is reduced by the discount factor, which is R here. So, alpha is getting reduced by alpha minus R. So, these are just observation that you make a note, (which is) this is what, is going to be useful also, when we go ahead with this.

And similarly we can compute $d(e^{-rt}X_t)$. So, this line if you look at it and if you compare with the previous line here, so what you have dX_t is given by this line here. This is the expression.

Now, this dX_t, when you discount the first-term, which is what we call the mean date of return term vanishes. And then the rest of the term remains. that is what this two terms, the second and third terms is what will give you this. And this could further be written down as $e^{-rt}S_t$. So, in a self-financing wealth process or the portfolio evolution process, see the discounted value or the, you know the evolution of the discounted portfolio value process is simply due to the evolution of discounted as a price process. There is no other component which contributes to this. That is what you may need to understand from the last line that we have written from here.

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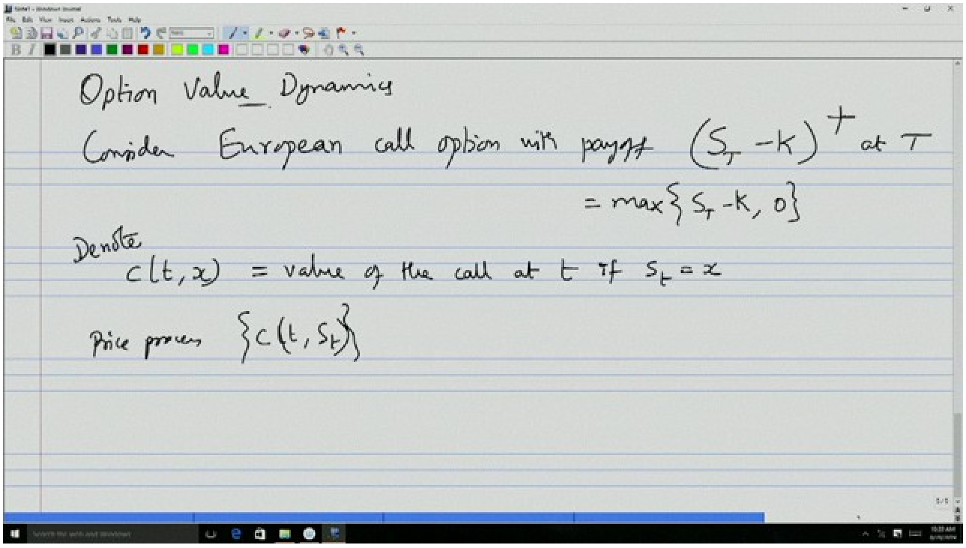

So (this is a), we are now done the evolution of the portfolio values. How the portfolio evolves from time, one-time to the other we have seen it. Now, the next thing that we will see is the option value evolution or evolution of option value, option value dynamics, evolution means. So the, dynamics we can write use the word, so that dynamics of the option value.

Now, what we consider? We consider a European call option with payoff $(S_T - K)^+$, which you know, what is this? So, this is payoff, this is at time T, which means the maturity or the exercise time for this European call option is T and we are considering the interval 0 to T as the timeline.

Now, this is the dynamics. So, the strike price K is some non-negative constant. So, what Black Scholes and Morton argued is that, (at any,) the price of the call or the value of this call at any time T prior to its maturity should depend on that time and the underlying asset price process at that point of time. Of course, apart from the other constants that are involved in this, which is α, σ, R, K. Apart from this, but out of all those things. Since this, all this things K, α, σ, R these are all we are assuming constant for the interval 0 to T.

So, the only two variables are the time and the underlying asset price, which in this particular case is the risky asset, the risky assets price at that point of time. So, following that reasoning, suppose if you denote $c(t,x)$, as the that is the argument that they had denote and later on, you will see why that argument which they gave is actually true. Once, now that you know little bit about stochastic calculus you can, we will see later that why that particular thing is true.

So, but for the time being assume that this is the value of the call at T if the, if$S_t = x$. The value of the European call at time T if the value of the underlying asset is value is X. if that is the case, this is what. So, there is nothing random about this function, this functional form is known. The randomness comes through the X. X is if you replaced by the quantity S_t, corresponding quantity at any given point of time. Then what you get is the price at that point of time, for that particular value of S_t.

Now, since this$S_t$is a random process which could take any value in its states base. So, this total value $c(t, S_t)$becomes random. Otherwise. If you look at the price process. So, the randomness comes through this S_t. So, when you replace this X by this, what you get is the price process.

Because$S_t$is unknown (in looking). When we are looking into the future, so this process is unknown, but we know the functional form of this $c(t,x)$. So, there is nothing random about the function $c(t,x)$, but this process $c(t, S_t)$is random process. So, that is what the price process in this particular case. So, basically what we have? We if you know the formula for it, then given any point of time, then we will be able to get the price at time T. That is what would be the thing that we might look for into the future.

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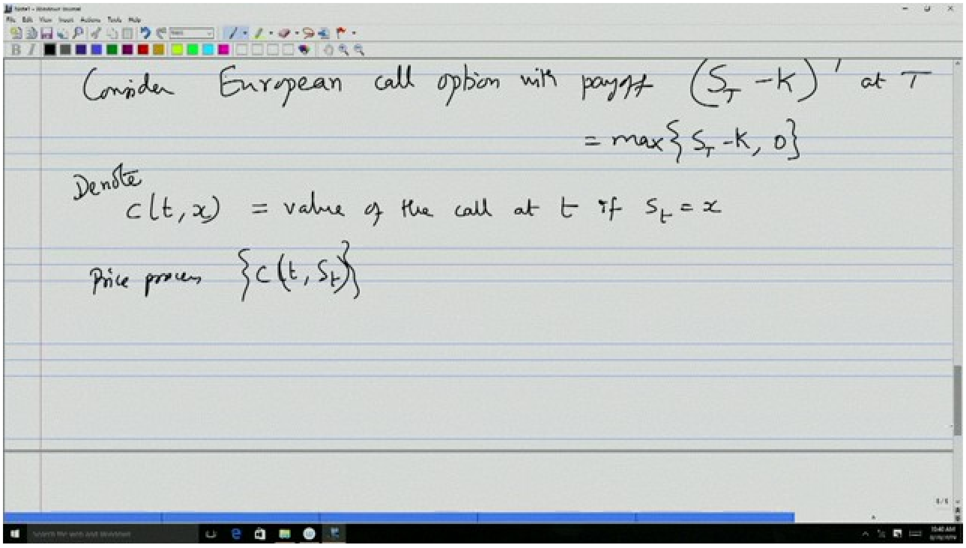

Now, so our aim is at least to get a formula for this function C assuming that this is true. Now I mean, you know by now, like why this $c(t,x)$ when X is replaced by $S_t$is itself $f(t,x)$, we can give the reasoning ourselves, since you know stochastic calculus. We have seen that this price processS_t is a Markov process because it is the solution of diffusion type of SDE, which we have said that the solutions are a Markov process and hence this S_t is a Markov process, which even otherwise by looking at the expression for $S_t$in terms of W_t, you can know that is W_t is a Markov process and hence this is a Markov process.

So, for a Markov process, for future evolution all that you need is the time or the value of the process at that point of time. So, knowing onlyS_t given the information of the filtration in terms of $\mathcal{F}_t$ which is the σ filler time T in the filtration, given $\mathcal{F}_t$ the only relevant information that you require for the evolution of the , $c(t,s_t)$i s simply the value of S_t. which is $S_t$and hence this is a Markov process and hence this argument was, I mean that is what is the reasoning why, you know they have done this.

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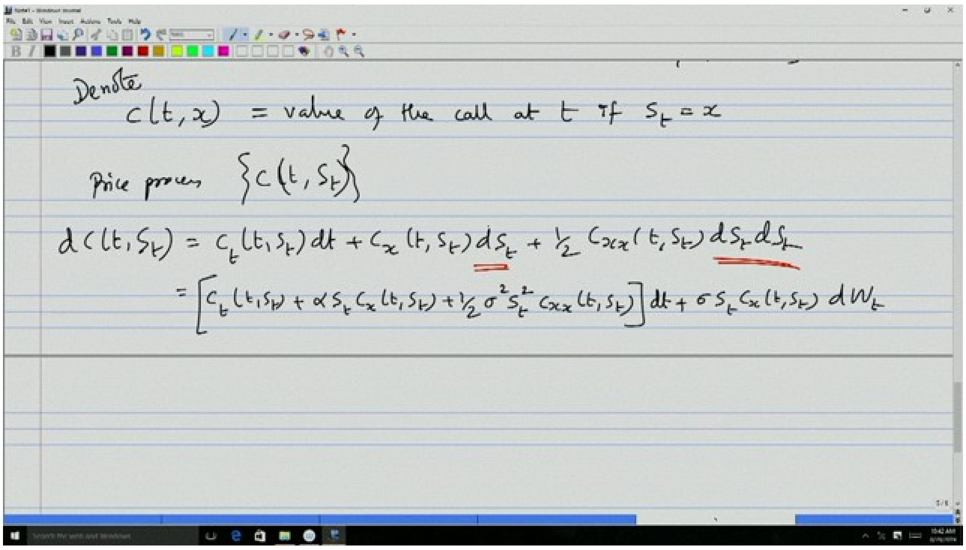

So, now let us look at the differential of this. because we need to equate the how these two portfolio and the price process, we need to equate by equating that only we form the set of equation to solve for this. Precisely same thing we are going to, so we are going to look at the differential. Now, what is $dc(t,S_t)$? The detail calculation are shown above.

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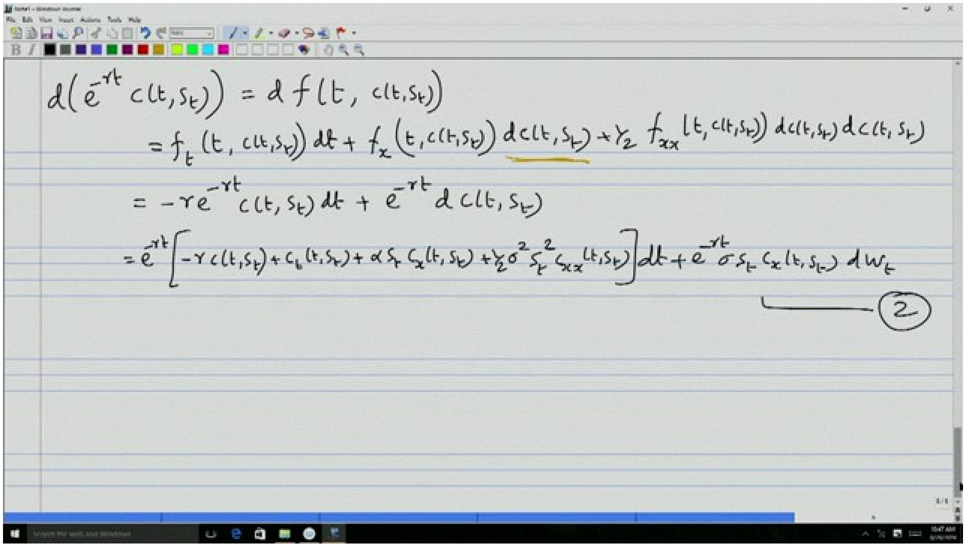

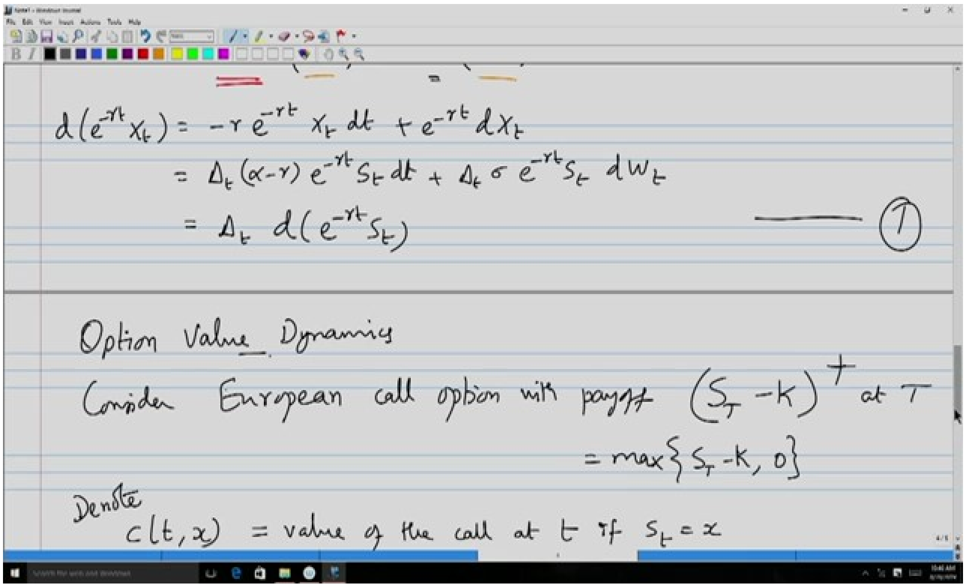

Now, we next compute the discounted value of this $c(t,S_t)$again, you know, we could think this as a function of the same $f(t,x) = e^{-rt}X$, where $X = c(t,s_t)$.

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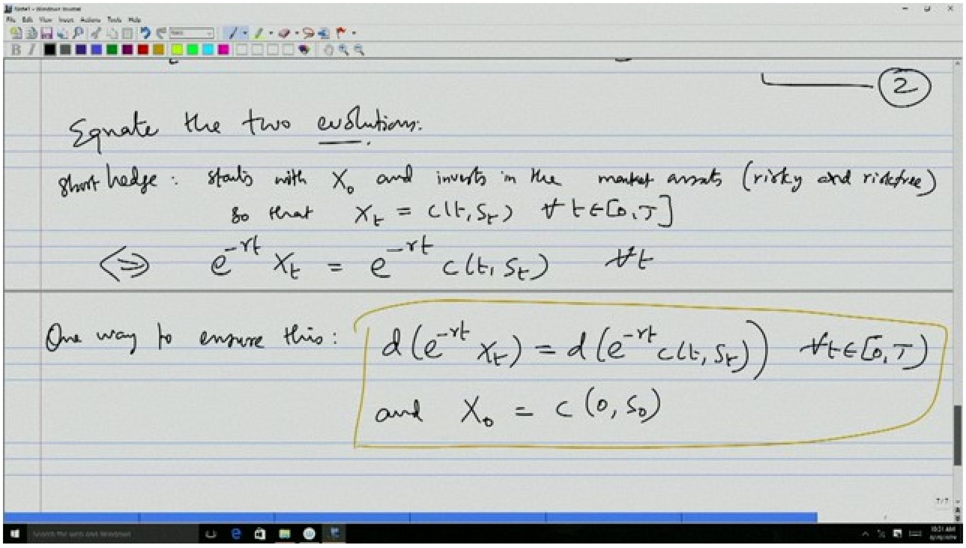

Now, what we have to do, (we just have to,) how do we find the position and the prices, recall the discrete time model, which we just have to equate the two evolutions. So, what we have is that a short option hedging portfolio starts with some initial capital X_0 and invest in the stock market and stock and the money market account as risky and risk-free account, risk-free assets. So, that the portfolio value $X_t = c(t, S_t)$ agrees at each time with the corresponding wealth process. The portfolio value agrees with the prize of the option at that point of time.

So, basically what we need is that a short hedge. What is this? This is starts with some X_0 and invests in the underlying assets, which is basically risky. and underlying I would not use here because this has a different meaning in the assets, in the market assets, risky and risk-free. So, that $X_t = c(t, S_t)$ for all $t \in [0, T]$, so this is what we want. If we do this, you start at each time 0.

Ultimately you want $X_T = c(T, S_T)$. But how we want to do that: if you agree that at every time point this, of course, that is what the no arbitrage we are doing it, otherwise there will be arbitrage in the middle, which one can take advantage. So, this happens. So this is basically if and only if you can say $e^{-rt}X_t = e^{-rt}c(t, S_t)$. This happens only if and only if this.

Now, one way to ensure this, what is this, is to make sure that your $d(e^{-rt}X_t) = d(e^{-rt}c(t, S_t))$ and $X_0 = c(0, S_0)$ So, by using this fact, you would see that you will end up with the previous line that we have describing here. And we get the desired in equality.

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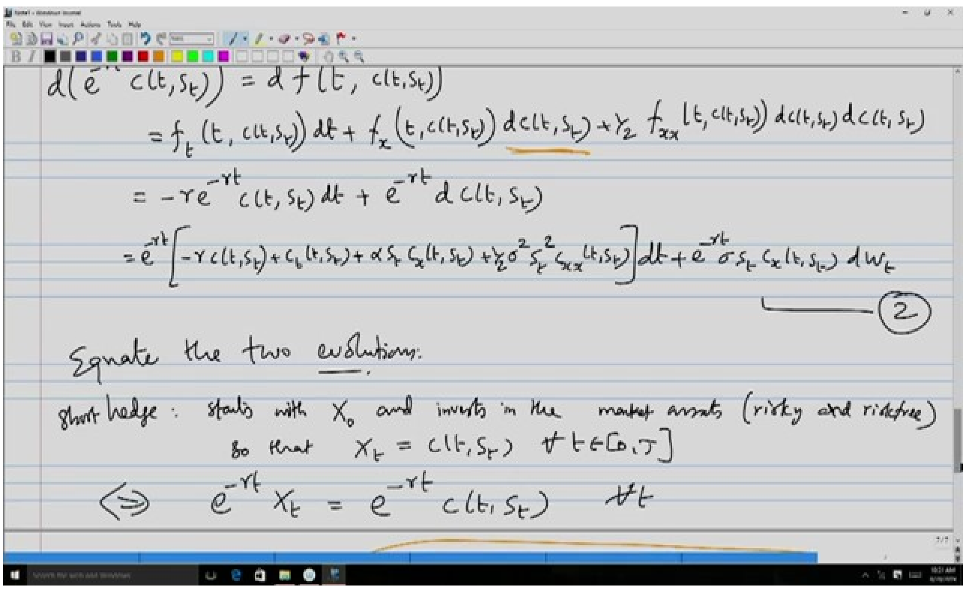

Comparing ① & ②

$$\Delta_t(\alpha - r)S_t\,dt + \Delta_t \sigma S_t\,dW_t$$

$$= \left[-r c(t,S_t) + c_t(t,S_t) + \alpha S_t c_x(t,S_t) + \tfrac{1}{2}\sigma^2 S_t^2 c_{xx}(t,S_t)\right] dt + \sigma S_t c_x(t,S_t)\,dW_t \quad \text{— ③}$$

First look at dW_t terms:

$$\Delta_t = c_x(t,S_t) \quad \forall t \in [0,T)$$

Short hedge: starts with X_0 and invests in the market assets (risky and riskfree) so that $X_t = c(t,S_t)$ $\forall t \in [0,T]$

$$\Longleftrightarrow \quad e^{-rt}X_t = e^{-rt}c(t,S_t) \quad \forall t$$

One way to ensure this:

$$d(e^{-rt}X_t) = d(e^{-rt}c(t,S_t)) \quad \forall t \in [0,T)$$

and $X_0 = c(0,S_0)$

Comparing ① & ②

$$\Delta_t(\alpha - r)S_t\,dt + \Delta_t \sigma S_t\,dW_t$$

$$= \left[-r c(t,S_t) + c_t(t,S_t) + \alpha S_t c_x(t,S_t) + \tfrac{1}{2}\sigma^2 S_t^2 c_{xx}(t,S_t)\right] dt + \sigma S_t c_x(t,S_t)\,dW_t \quad \text{— ③}$$

First look at dW_t terms:

$$\Delta_t = c_x(t,S_t) \quad \forall t \in [0,T)$$

delta-hedging rule

Now, going back, now if we compare the expression that we have written here, 1 and 2, 1 is for the discounted value process. So, comparing 1 and 2 that we have written in the pic.

If you equate this 1 and 2 what you get is this. So, 3 is what should hold. Now, what is required for 3 to hold but comparing 1 and 2 if you know as per this process one way to ensure that, you know you achieve what you said to achieve is to do this, starting with an X0 which is given by $C(0,S_0)$ and the making the evolutions equal, which means that this two must be equal. Now, what is require in order to make this hold? First, you know you look at dW_t terms, dW_t terms gives you what you have Δ_T. if I compare the left side dw terms, you look at term wise dw terms, dw terms and you look at dt terms. So, you look at this two terms and then see, if this two terms, term wise if they are equal. Then you are done, this what we are trying to do. So, this Δ_t, if I compare $\Delta_t \sigma S_t$ is here, so $\Delta_t = c(x_t, S_t)$, for all $t \in [0,T)$.

So, if this is true, if Δ_t is equal to this, then this is true that the dw terms would be equal throughout. So, this is called as the delta hedging rule, what this gives? Now, you can see that the position that you take, what you see from here is at the position that you take at time T is given by the first derivative of the function $c(t,S_t)$ with respect to S_t.

So, at each time prior to expiration, now what is the position that you need to hold is given by this Δ_t. And this quantity is what called the C_x, is what called the delta of the option and if you are looking for more of the Greeks what they call in general, this is one of the Greeks and this is a delta of the option. What is delta of the option? Means its position that delta is what then you need to use to construct a delta hedging strategy. What is the delta hedging strategy is precisely having this as your position.

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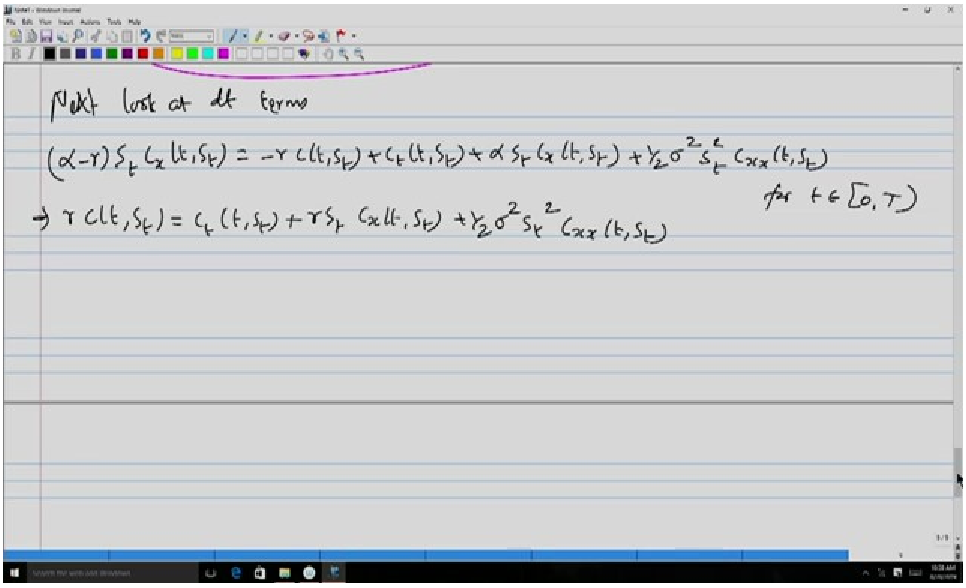

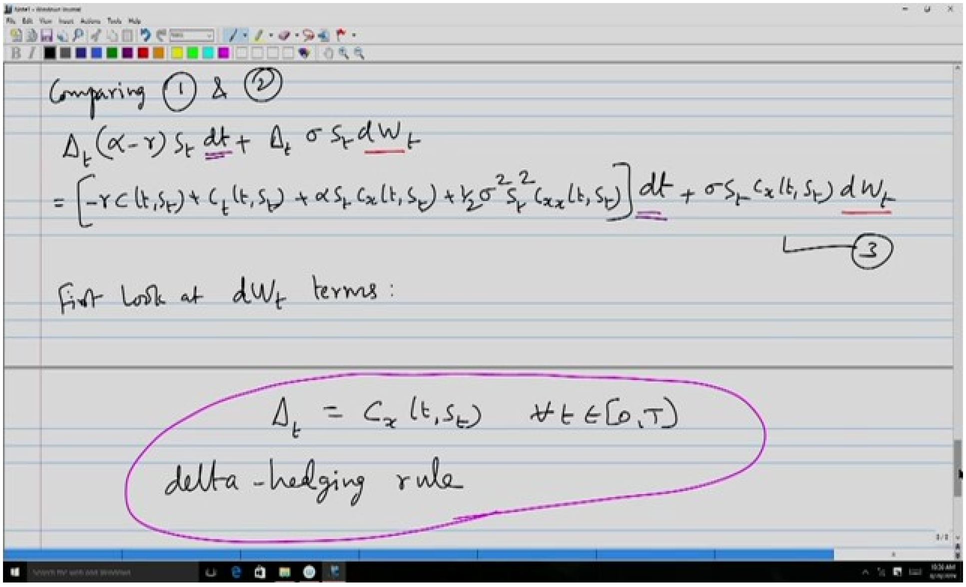

Now, next look at dt terms. Then what you would get is now Δ_t is given by C_x, Δ_t is given by C_x term. So, the left side is essentially $(\alpha - r)S_tC_x$ and the right side complete terms would remain the same. So, what you have is given above. This is for T in 0 to this.

This, you can now cancel this alpha term on both sides. So, you left with only the remaining terms, which actually gives us R; to for all T, for T in this things. So, (what we need) this needs to be satisfied, we want a function that satisfies, that is satisfying this in this interval.

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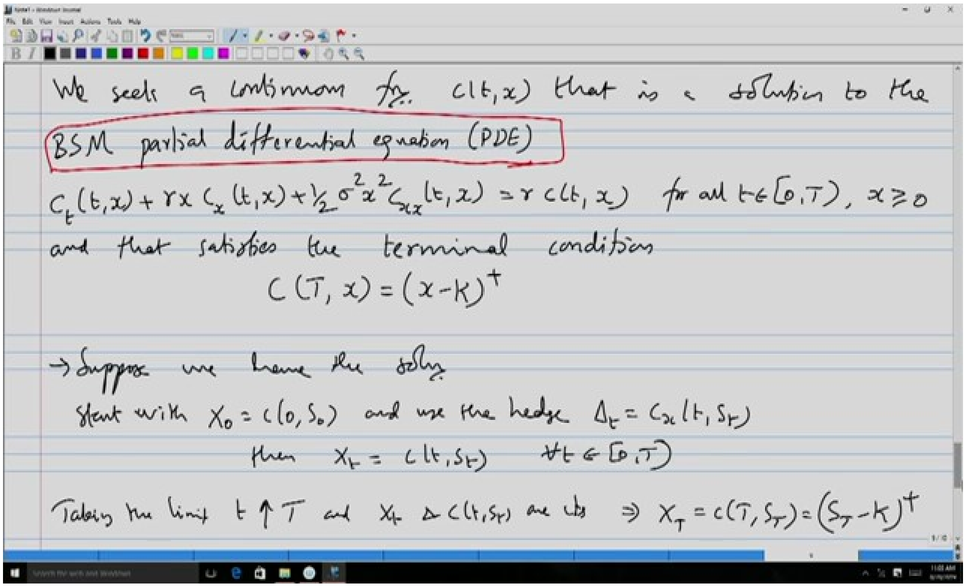

And also satisfying the condition that at time T, what is required. So, in a sense, in conclusion what we need, is that we seek a continuous function say $C(t,x)$ that is a solution to the (what we call) BSM partial differential equation (PDE), what is that PDE? This is given above.

And that satisfies (what we call as) the terminal condition $C(t,x) = (x-k)^+$. All we need basically this is the BSM and PDE that we have. So, this is the solution to this BSM, PDE which is what given by this and this, so if we have. Suppose we have found a solution to this BSM, PDE. Then and what we have done.

Suppose if we have the solution, then we can start with, starts with say an X_0, which is given by $C(0,S_0)$ and adopt what and use the hedge $\Delta_t = C_x(t,S_t)$. Then what we are going to have, so then my

$X_t = c(t,S_t)$. Now, taking the limit as T increases to T and the fact that both X_t and C are continuous, we can conclude and X_t and $c(t,S_t)$ are continuous implies that my X_t would be equal to $c(t,S_t)$, which is$(S_t - k)^+$ the positive part.

So, this means the short position has been successfully hedged by this process. So, what we need, if the delta hedging strategy, which is given by this. But again for the delta hedging strategy C_x, you need the function C. So if you can solve this PDE, by some means, you get the functions form $c(t,x)$. Then you have everything. that the stock price process or the derivative price process at time T is given by $c(t,S_t)$ and the hedge is basically the first order derivative with respect to X is what would give you this.

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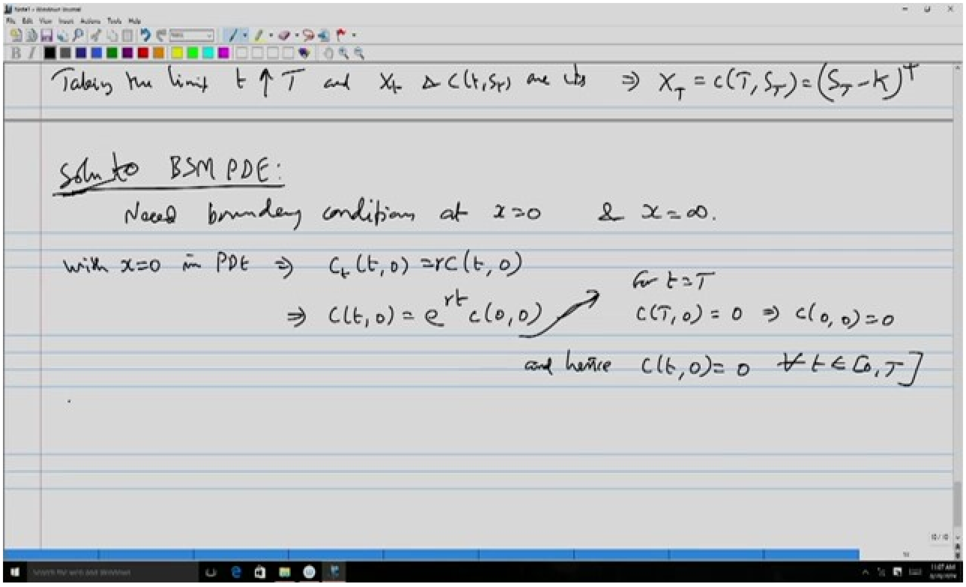

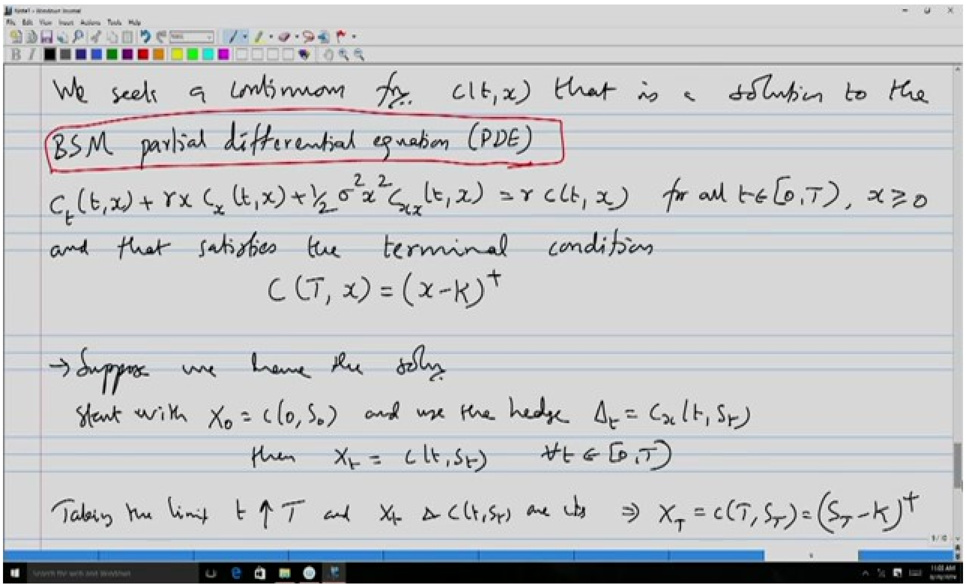

Now, what is the solution? We will see this and then we will end. So, this BSM and PDE solution to BSM, PDE. So, here I am going to solve the solution the PDE right now here ,because later on, we will give a different way of solving this, but you know, you can solve this BSM, PDE and then get the exact same expression that what we are getting here. So that is, you know, you can take.

Now, what we want is the BSM, PDE to hold for all X greater than or equal to 0 and for all T starting from 0, including 0 but up to T, but excluding capital T. So, that is what it is the case, but we know that if X is 0, then this is going to remain 0 throughout the stock price process. If X is greater than 0. So, the stock price process is going to remain positive throughout.

So, those information is what then we might use to look at that. But before starting of that, so let us specify whether the condition that is specified here with respect to the PDE and the terminal condition is sufficient. If you look at this is a backward parabolic PDE and which for such an equation, you also need the, in addition to the terminal condition, a boundary conditions at, need boundary conditions at X equal to 0 and X is equal to infinity. So we will see, you know some such condition that we have to develop in order to solve this PDE, so let us briefly give this.

Now, if you substitute X is equal to 0 in the PDE, with X is equal to 0 in PDE you will get as $c_t(t,0) = rc(t,0)$ is what is the expression that you would get. So, this is an ordinary differential equation which with solution as $c(t,0) = e^{rt}c(0,0)$. Substituting this $t = T$ because this is again should satisfy that $t = T$. So which means that for $t = T$, what you would get is $c(T,0) = 0$. So, that implies $c(0,0 = 0)$. And hence from here, you know we are moving to here. (and hence) what you would get, and hence your $C_(t,0) = 0$ for all T in 0 to T. So, this is one boundary condition, which is X at 0.

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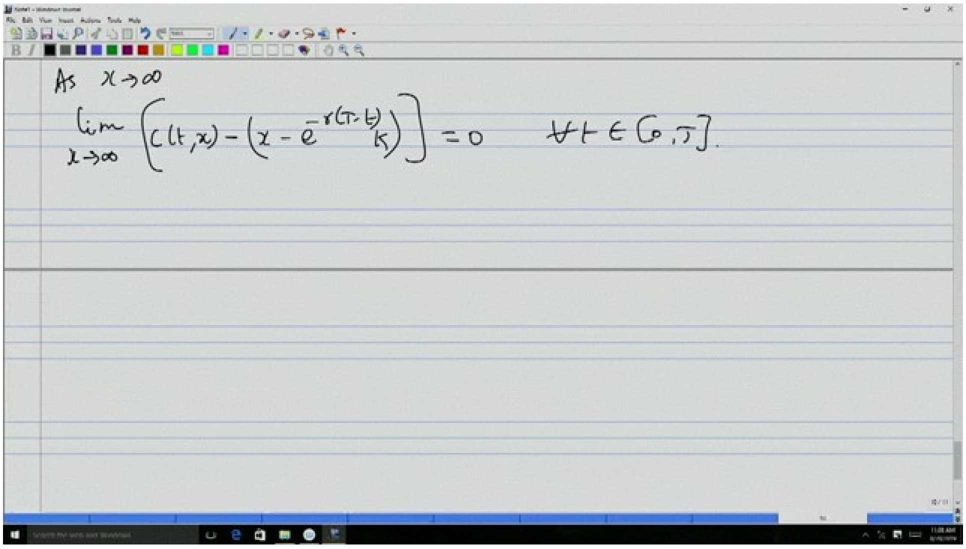

Now, as X tends to infinity, the other boundary condition that we might look for is essentially, if you look at X very large value, so this in that the $c(t,x)$ also grows without bound. In such a case we give the boundary condition by giving rate of growth at infinity by specifying the rate of growth. This is basically we are saying by giving this condition, this is equal to 0 for all t in 0 to T.

So, in particular what we are saying is $c(t,x)$ is growing at the rate as X. because for large values of X because the European call is going to be very deep in the money. and the deep in the money is exactly same as X, because $(x-k)^+$. No matter whatever be the K value, for X very large, $(x-k)^+$ is almost equal to X that is precisely what we are saying by this condition.

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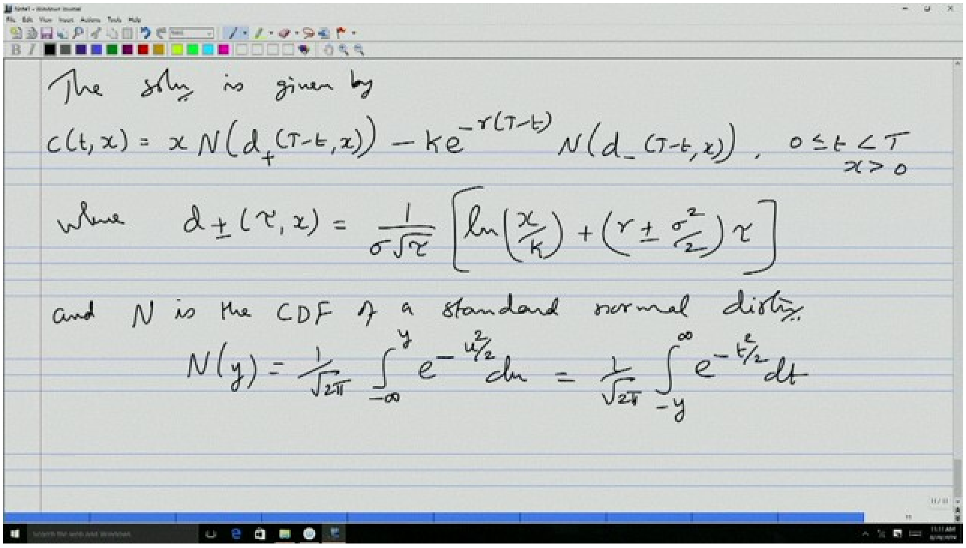

So, with this two boundary conditions, then the solution is given by, so we have a BSM, PDE with one terminal condition and two boundary conditions. The solution is given by, we simply give the solution later, you know, we will try to prove or derive the solution here.

Where my d_{+-} is given above, and N is the CDF of a standard normal distribution, CDF function, which are normal distribution.

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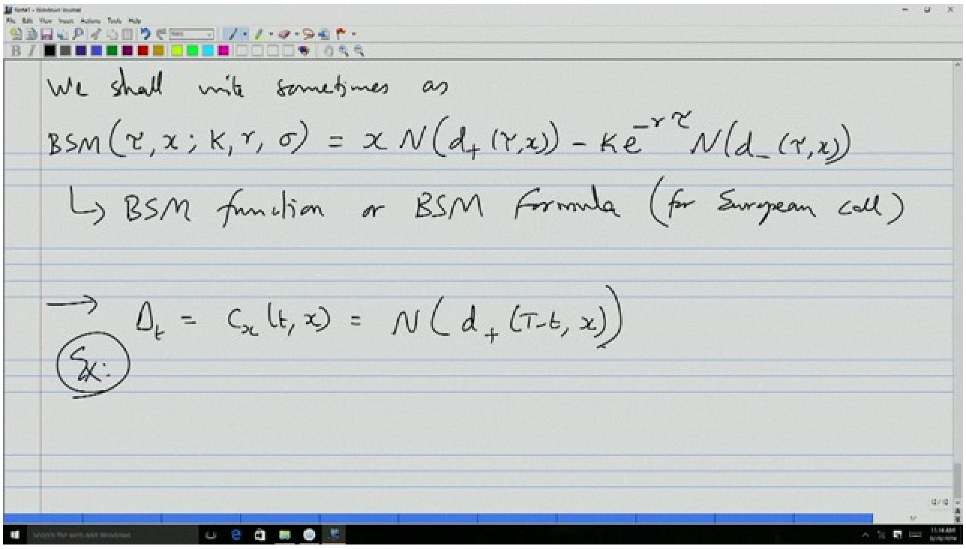

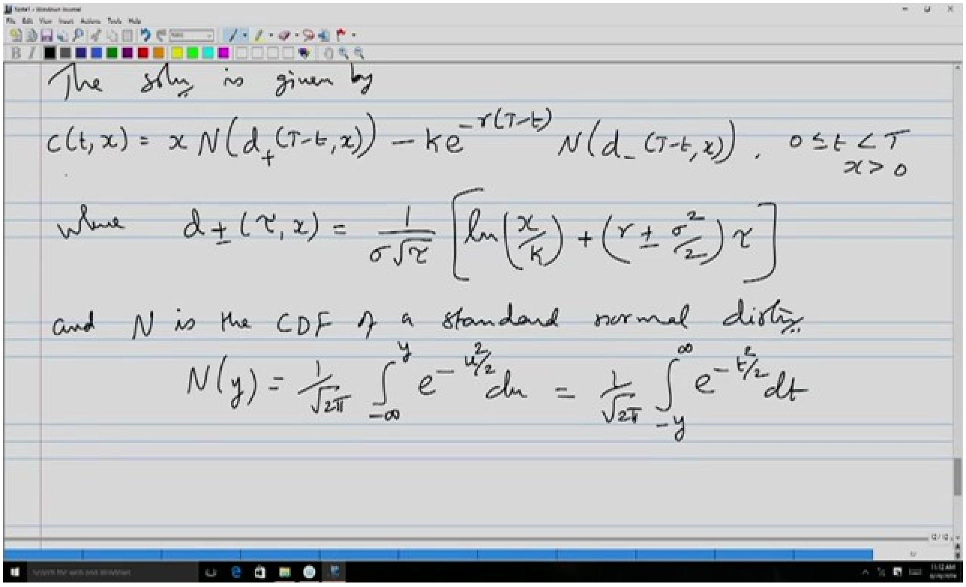

Now, what we see here is that. So, we shall write sometimes, because this were going to need later as $BSM(\tau, x; k, r, \sigma)$. And call this as the BSM function or BSM formula for the European call option price. Is what we call this for, you know notational convention. So, now let us look at this, of course this is a solution that we have written down here, which is given in terms of some quantities d_+, d_-, in some books you would findd_1, d_2 they will use does not matter, it is only the notation. That two quantities and $N(d_+)$ of and $N(d_{)}$ is what the normal CDS replaced in this manner. And this, what is the expression that they give. So, this is what then you would get.

Again, if you look at here the solution is given for not at T, small t equal to capital T and $X = 0$. And in both the cases, of course, they are defined by continuity of the function. So, that it can be verified that, you know the solution, which is satisfied by continuity, if we define what you get is actually, is the solution that you would get here.

Now this one, the delta hedging strategy here, which we said that $C_x(t, x)$, in this particular case we will turn out to be $N(d_+)$ this. (We can) you know this is an exercise you can try that the delta hedging strategy is actually given by this quantity. This is the explicit form of the delta hedging strategy that we have got now for this formula. So, this is what you know, we have discussed the APT approach or the replication approach to pricing a European call option in the Black Scholes Merton model. We will later put it up in the risk neutral framework and continue further in the next lecture. Thank you.