Mathematical Finance

Lecture 30: Multivariable Stochastic Calculus, Stochastic Differential Equations

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Hello everyone, they next topic that we will see is multi variable stochastic calculus. So far what we have seen, is you can term it as you know single variable stochastic calculus because we dealt with one Brownian motion case, but in general when you have to consider a more general setup then it may not be a single Brownian motion which will come inside the modeling framework, it could be more than 1, so in that case what you need is this multivariable stochastic calculus. But this is not much difficult once you have understood this single variable, just that it has to be extended to the multivariable case in every result that we have seen.

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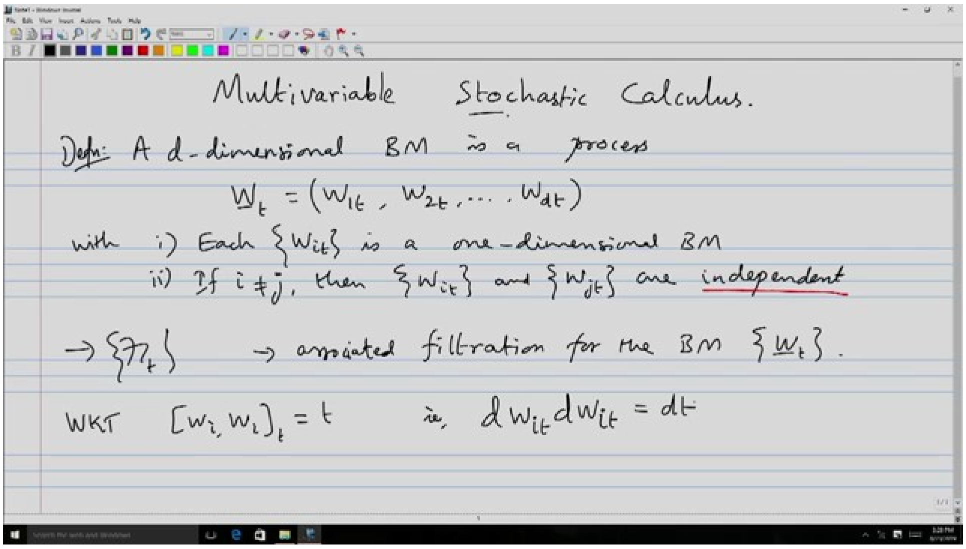

So as you see, then we need to define what do we mean by a multidimensional Brownian motion in specific. We will say a d-dimensional Brownian motion is process say $W_t = (W_{1t}, W_{2t}, W_{3t}, W_{4t}, ..., W_{dt})$, as a vector, as earlier please understand that W_{1t} is $W_1(t)$ with the properties being, one, each of this Wit we want to call it with each of this is one dimensional Brownian motion, which is not very difficult and which is not that major constrain.

But this part what we mean, when in our case when we say d dimension Brownian motion. Then this W_{it} the processes and W_{jt}, the processes W_{it} and W_{jt} you know each one of the Brownian motion and these are independent, this is the crucial condition when we consider the multidimensional Brownian motion that it just the aggregation of dt-one dimensional Brownian motions with additional property that the processes the Brownian motion processes the one dimensional versions are all independent.

Now, again associated with this, so there is a filtration $\mathcal{F}_t$ is basically associated filtration for the Brownian motion, which is W_t. which means it has the same three properties that we defined for the one dimensional process. which are the information accumulates and the addaptivity, when we say addaptivity each one of them we mean and then they independence of the future increments because the future increments this vector d dimensional Brownian motion is independent of the current information that you have.

Now, what if, we have the dependent Brownian motion and which might generally be the case, latter we will see that if you have dependent Brownian motion, it can be expressed in terms of independent Brownian motion as a linear combination and hence considering this would suffice for our purposes. Now, if you look at this d dimensional processes and its properties obviously each one of them is a one dimensional Brownian motion.

So, all the properties that we have are one dimensional Brownian motion would also hold true for each one of them. Additionally, we may also have to look at, because we have concerned of more than one process at the same time their inter linkages will also play a role, but then by assumption that they are independent, so that the interrelationships greatly simplifies, only thing that we need to look at is the quadratic variation property what happens in this particular case when you have such a scenario. We know, (we know that what do we know?) that this one is equal to t, that is what we have. This we know because this each one of them is a Brownian motion, so this is true.

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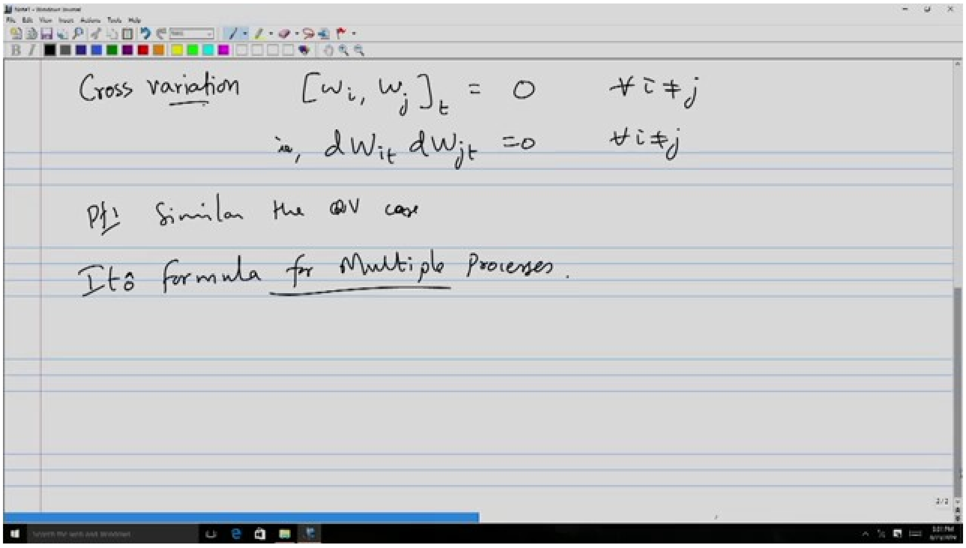

Now, what is the corresponding quantity? What is this quantity? So, if I take $i \neq j$, which means what we called this as cross variation, what is the cross variation of one process with respect to the other process? So, this the important component here, is that you say the cross variation, what is this? which is essentially meaning to say W_i, $W_j(t)$, what happens to this? The claim is this for $i \neq j$. That is you want to see, that this is exactly in the informal way of our writing, this is equal to 0 for $i \neq j$.

Of course, the proof is similar to the quadratic variation case, so it is just that W_i, now you are just replacing W_i with W_j, so it is just the product terms. Then you take its sample cross variation term, take its expectation show that it is 0 and take its variants show that it tends to 0 and hence this term is tending to 0 in the L^2 norm or in the L^2 conversions and hence the cross variations is here.

So, we are not going to look at the proof but it is similar that you can see, what we need is that this result that we might use elsewhere. Now, once we have this now we can also write what is our Ito formula for multiple processes, so far again we have done it for univariate cases which means when only one processes are there.

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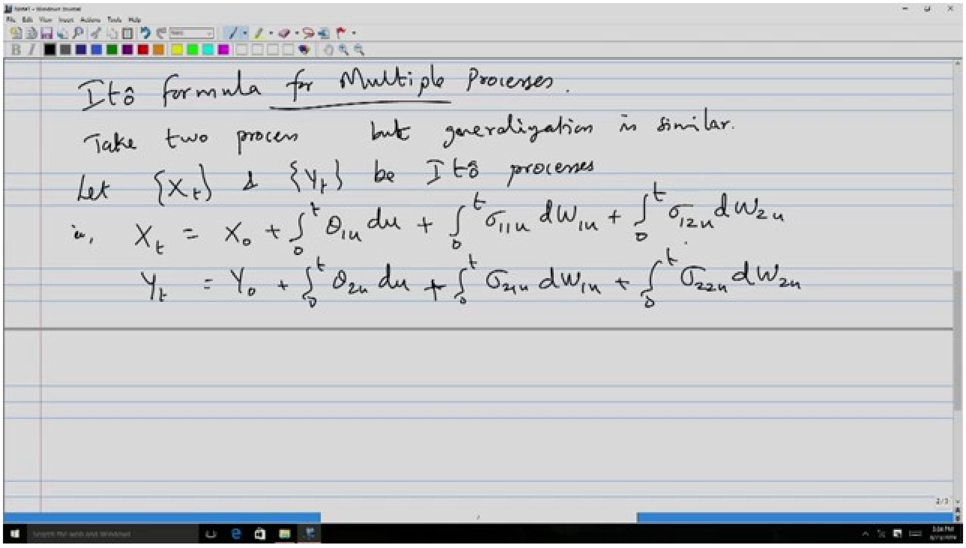

Now, what is the Ito formula for multiple processes? Now, we will take two processes but generalization is similar or analogues, so there is nothing no complex things will come when you go from two to some general n processes or whatever number you want to consider.

So, let us straightaway take, (we will take let) X_t and Y_t be Ito processes, we will directly go for Ito processes. which means what?

$$X_t = X_0 + \int_0^t \Theta_{1u} du + \int_0^t \sigma_{11u} dw_{1u} + \int_0^t \sigma_{12u} dw_{2u}$$

and

$$Y_t = Y_0 + \int_0^t \Theta_{2u} du + \int_0^t \sigma_{21u} dw_{1u} + \int_0^t \sigma_{22u} dw_{2u}$$

So this is what would be there, where this Θ, σ and, σ_{ij} are as per our requirement they are adapted processes. such that all these integrals exist and are well defined, otherwise you cannot write in this form, so that condition that we impose we will assume without loss of generality (we will assume) that those are all true.

Now, but what is this means that is given as you know Ito process in 2 dimension effectively are with respect to two Brownian motion case, is that it is given by a constant plus and what we have here is an ordinary integral plus one Ito integral with respect to W_1 and another Ito integral with respect to W_2 the same is the case with Y_t alternatively.

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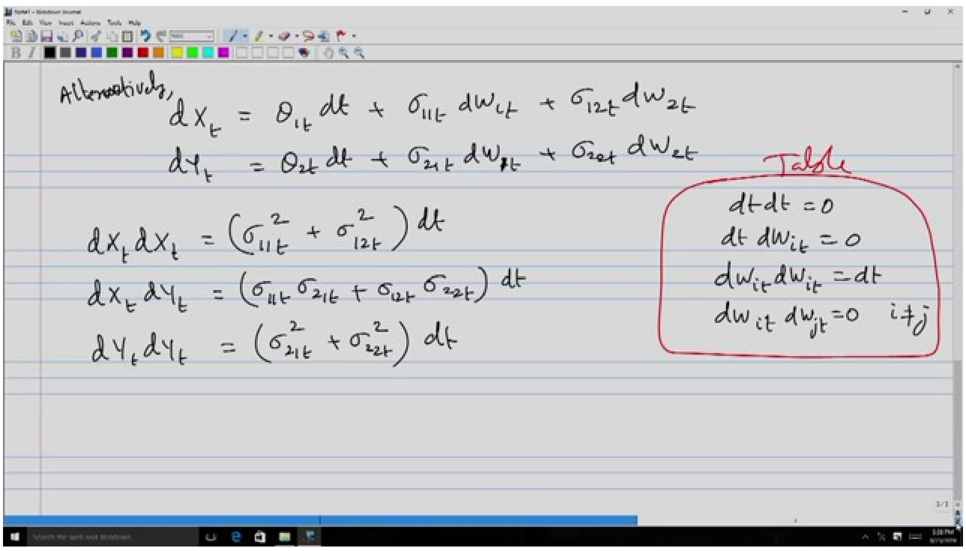

Alternatively, so you can write this in the differential form as we say often that integral form is easier to for pen and paper calculation,

$$dX_t = \Theta_{1t}dt + \sigma_{11t}dw_{1t} + \sigma_{12t}dw_{2t}$$

$$dY_t = \Theta_{2t}dt + \int_0^t \sigma_{21t}dw_{1t} + \int_0^t \sigma_{22t}dw_{2t}$$

So, this is what, as we say pen and paper calculation we always refer to conveniently we will write in the form of differential forms, but it does not have a precise meaning (the precise meaning), it is in terms of the integral form as given in the first expression in terms of integrals. Now, once we have this Ito integrals in the differential form we do not know how to get the quadratic variation and other cross variations and other stuff with respect to this.

We know each of these Ito integrals accumulate quadratic variations and at what rate that also we know, so we simply have to use that and for cross terms of course, when we have a result just now we showed what happened to the cross variation and that part we will be using. So, that means suppose if I look at now the quadratic variation of

$$dX_t dX_t = (\sigma_{11t}^2 + \sigma_{12t}^2)dt$$

$$dX_t dY_t = (\sigma_{11t}\sigma_{21t} + \sigma_{12t}\sigma_{22t})dt$$

$$dY_t dY_t = (\sigma_{21t}^2 + \sigma_{22t}^2)dt$$

The formulas that we have used are written in the table. So, this is the table that we have used to arrive at this, which we can write in the form of the differential form. Of course, we can write in the terms of this integral form to have a meaningful expression, we keep saying that of course this is what it is. (Of course) again if you want to prove this, in a similar way like earlier one can do this.

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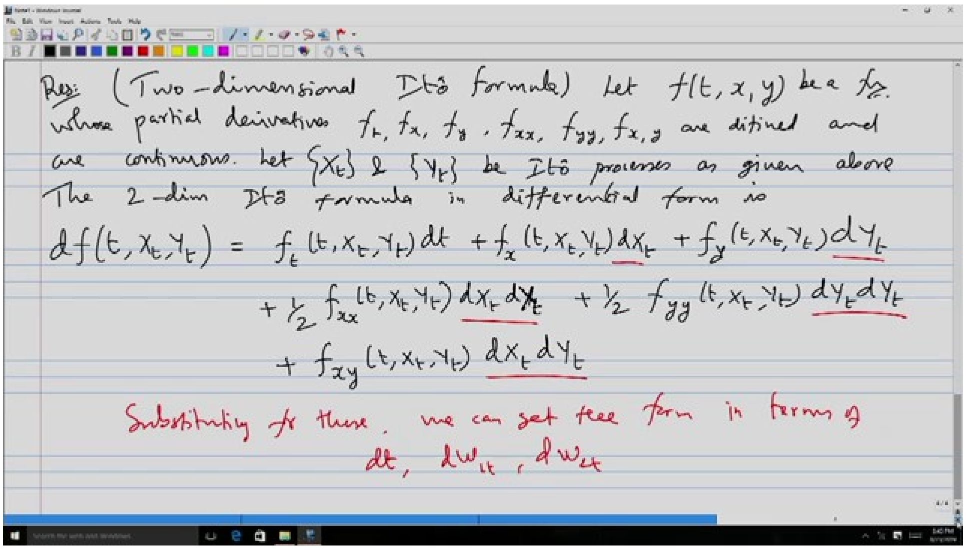

Now, that a part, we now go into once we have these terms for these products and everything it is done. we can now give the result for, in this particular case the result is two dimensional Ito formula, so what you have? You have function $f(t,x,y)$, now it is a function of three variables be a function whose partial derivatives $f_t, f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ are defined and are continuous. Let X_t and Y_t be Ito processes as defined or as given above, it is not defined per say or as given above.

Now, the two dimensional Ito formula in differential form is essentially the following, it is should be clear by now like how we arrive at this, after having looked at the previous Ito formula with respect to the other cases, so this is now given by the above expression (see pic).

So, you can see that we have, this actually this will be $(1/2)f_{xy}$ terms and $(1/2)f_yx$ terms and because of our assumed conditions, that this is essentially for functions whose second partial derivative exist and or continuous. $f_{xy} = f_{yx}$ and hence this half and half get added and you get one times f_{xy} times this with this product term. Of course, so the differential if you want dx dy and all these things you know if you want to expand in terms of the differential form, you write it explicitly and then you can write down the Ito's formula in terms of dt and dW_{1t} and dW_{2t}, because you already know what are these quantities in terms of that. So, the differential form in that way you just have to write it down.

So, if you substitute for all so one can get substituting for this we can get the form in terms of dt, dW_{1t}, dW_{2t}, so that is all the things you can do, which is not you know not difficult but it just say it will be a lengthier formula when in terms of all the processes that you have here. So, essentially when we say Ito process as given above if you want to expand then you will use that form otherwise for general Ito process, this is what will be your Ito's formula and the proof and all other stuff is precisely exactly same as the earlier case that we have consider.

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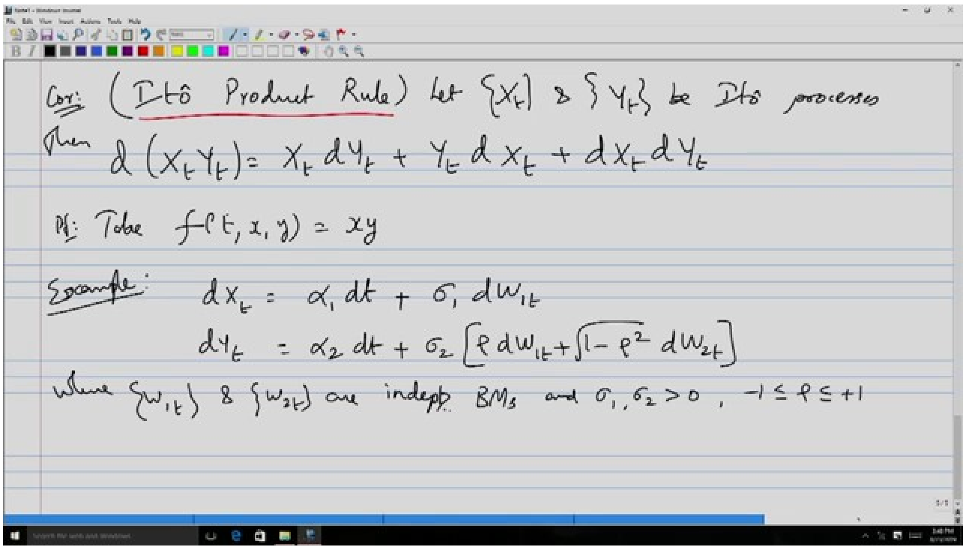

Now, as a corollary or by using this what we get is the following. As a corollary, what you would obtain is called as Ito's product rule, what is this? If let you are the usual X_t and Y_t be Ito process then $d(X_tY_t) = X_tdY_t + Y_tdX_t + dX_tdY_t$. This is (you know) the proof is basically you take the function of $f(t,x,y) = xy$ that is all we did. Like this now any function of x,y,t included, any function that you have you know you will be able to apply this Ito's formula to get the corresponding expansion at least in the differential form very easily. so that you will use this product formula, this is what is known as the Ito's product rule when you are dealing with which will be differential of the product of two Ito processes that is what given here.

Now, the final point what we will see, is that we have defined into the case of the Brownian motion, the d dimensional Brownian motion as consisting of the independent Brownian motion that could be used to model in fact when we wrote the Ito processes formula, which we wrote it as terms of W_1 and W_2, where W_1 and W_2 are independent. Now suppose, what if, we have dependent processes and how they could be related to independent process in what way.

So, let us look at this example, what is that, that you have? Suppose, if you have two processes say given by dX_t and dY_t (see pic). ,where W_{1t} and W_{2t} are independent Brownian motions, so the independent, so, this is what is the form that we wrote the Ito process form and my Σ_1, $\sigma_2 > 0$ and $-1 \leq \rho \leq +1$, is what you have and α_1 and α_2 are any constants any real numbers that you have it here.

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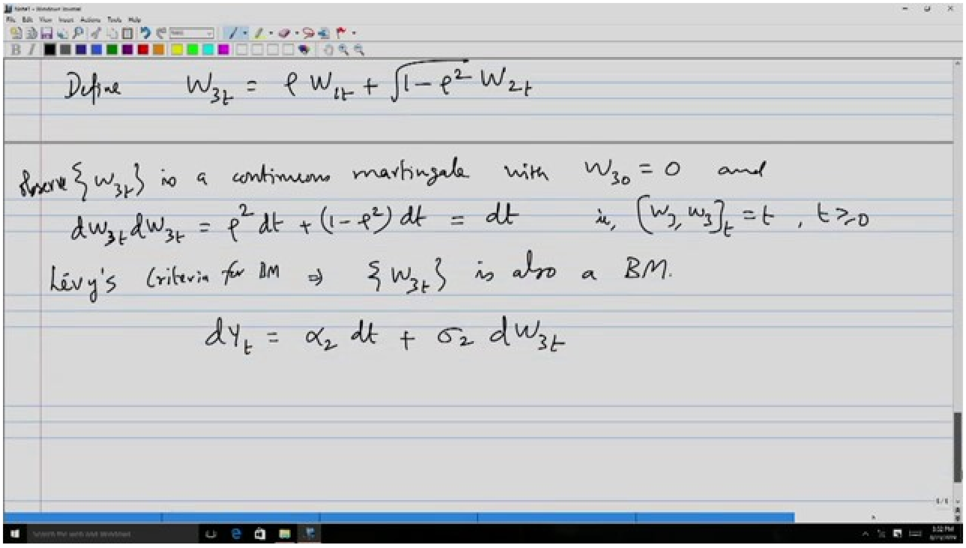

Now, to analyze the process why, what we do is, how to get the relationship (what we do) is that we define, define another process say W_{3t}. Then W_{1t} W_{2t} are martingales, so you can observe that W_{3t} is continuous martingale , because you can easily see this is a martingale with W_{30} as 0 it has continuous paths because W_1 and W_2 has, because they are martingale, W_3 also is a martingale and if you look at so we have defined an new process W_{3t} and we are looking at its properties observe.

So, this is we are basically observing and your dW_{3t} which is equal to your rho square dt from the first and from the second, your W_1 and W_2 are independent the cross products terms will go away. And the quadratic variation of W_{2t} will give you $(1-\rho^2)dt$, which gives as dt, that is W3, W3 the quadratic variation of t for all t greater than or equal to 0.

Though, all these properties we have already mentioned result, when we did Brownian motion as something called as Livy's criteria. So the Livy's criteria for or characterization for Brownian motion implies that my W_{3t} is also a Brownian motion, so what one can write the Y process, the second Ito process because we can write the you know we can write the Ito process Y_2 in this form now $\sigma_2 dW_{3t}$, so if you look at these two forms the whatever we have written within the square bracket is what we defined to be W3t and we showed that that is also a Brownian motion so then you have this.

Now, if I look at my x and y, both are Ito processes but now they are expressed in terms of W_1 and W_3 where this W_1 and W_3 are correlated whereas W_1 and W_2 are uncorrelated. this is you can trace back to the earlier case that we have done wherein you know we did the independent normal random variables and how we got the dependent normal variables, dependent normal variables how we expressed as in terms of independent normal variables, you can trace this idea, this result back to that.

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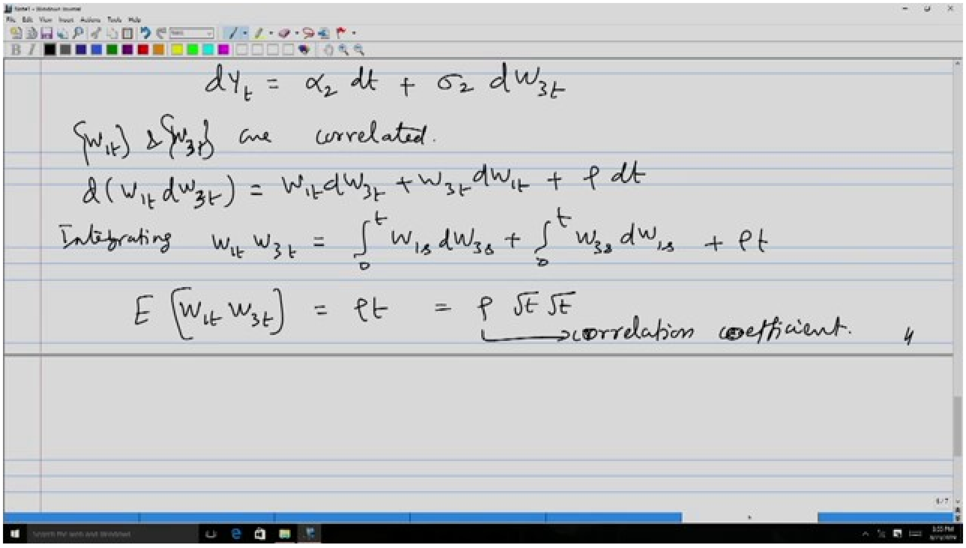

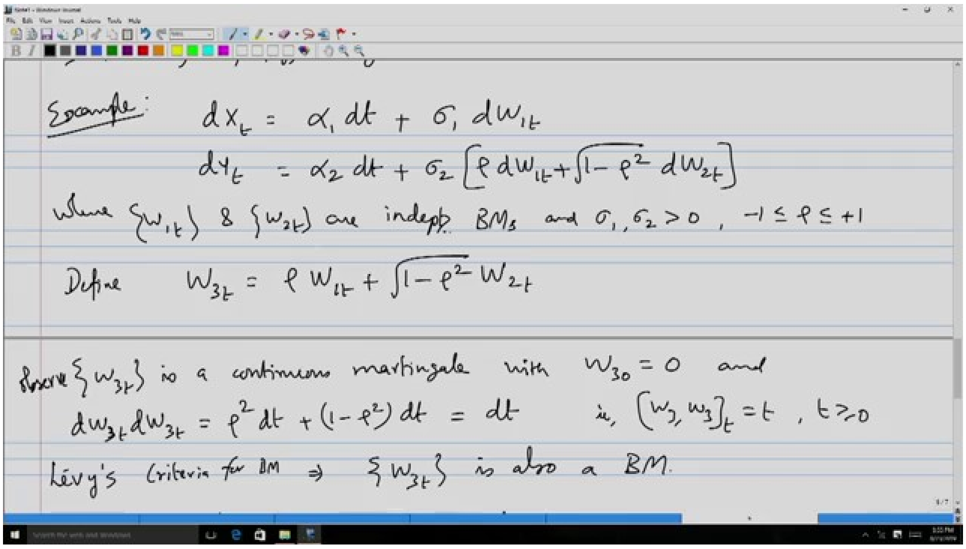

Now this we say, now we can say that this W_{1t} and W_{3t} are correlated, now we will see like what is correlation? so your dW_{1t}, dW_{2t} by Ito's product rule if I imply then what you would end up with $W_{1t}dW_{3t} + W_1$ and W_3 is what when we are looking at, W3t dW1t plus dW1 and dW3, which will give you ρdt. Now if I integrate, what you would end up with W1t W3t as 0 to t W1s dW3s plus integral 0 to t W3s dW1s plus rho t.

Now, if I take expectation if I take $E[W_{1t}W_{3t}] = \rho_t$ and hence this rho is the correlation coefficient. So, what do you observe, is that you know if you have correlated Brownian motions W_1 and W_3 with correlation coefficient W (then of course they are standard Brownian motions), then you can have another Brownian motion you can defined in terms of this quantity W_2 such that W_2 and W_1 would be uncorrelated, so this tells us that if you want to if you have the correlated Brownian motion you can express or you can consider in terms of independent Brownian motion as well.

So, assuming that you know we are d dimensional Brownian motion is independent, is sufficient because if we have a correlated processes we can always express in terms of uncorrelated processes in terms of the independent. So this what we have the multi variable stochastic calculus part that we have. So this is about the Ito integral for the multivariable processes and how one can consider the correlated and uncorrelated Ito processes how they will be related through uncorrelated and correlated Brownian motions case.

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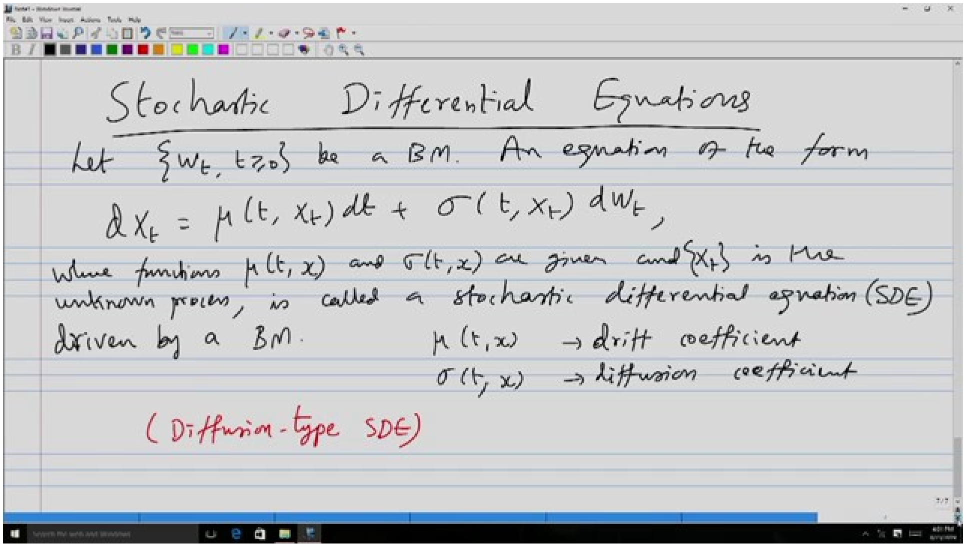

The next topic, that we will consider is what, is generally called as stochastic differential equations. You guys have seen the stochastic differential cases as we went through, so differential equations as you all know are used to describe the evolutions of systems, now to deterministic, when we say deterministic differential equation, if you add a random noise then what you get is stochastic differential equation, if I have to say in simple form so we consider what is the form of this and then we will see like what we have we can do and what we can do with that.

So, essentially what we say, is if you can add a random noise. so here in the random noise in our material is in terms of the Brownian motion of course there is a process called white noise process. which is what actually we are adding but you know you will not go into the very detail but you will see what is the SDE and what is the mean by a solution and some properties and some examples and how to solve this, that is what briefly we will see.

As usual you take a Brownian motion, be a Brownian motion now an equation of the form, which we call $dX_t = \mu(t, X_t)dt + \sigma(t, x_t)dW_t$. So an equation of the form, where this is what it is. This is called a stochastic differential equation. Which we will abbreviate as SDE driven by a Brownian motion. and this mu is what we call it as sigma, this is we will call as the drift coefficient.

And this is the diffusion coefficient and such an equation also has a special name, the above is called the diffusion type SDE, because one can have much more general SDE but for all our purposes simple scenarios that we are going to model this is sufficient the diffusion type SDE. so this is what the diffusion type SDE that we consider.

So, this is what we called it as SDE. where this is some functions μ andσ are the unknown. so , on the right hand side you have d_t and dW_t term, if the sigma term is 0 ,you can observe if the diffusion coefficient is 0 then or diffusion rather than,(so you can call this as coefficient does not matter,) sometime diffusion term what it is called.

This diffusion term see this sigma is 0 then what you get is an ordinary differential equation. but now that we are perturbing with a Brownian component by introducing there randomness there and hence it becomes a stochastic differential equation this is only one form, of course there are many forms.

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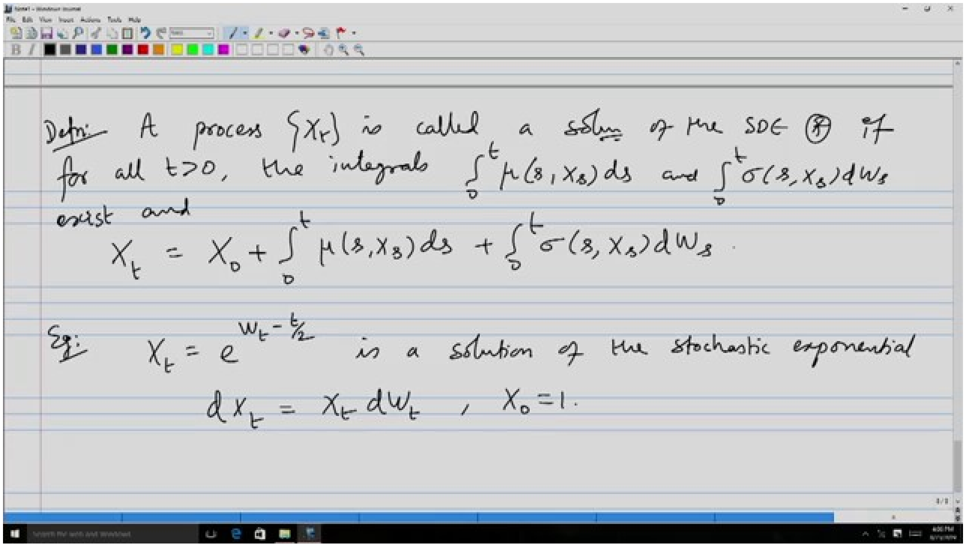

Now, given this what do we mean by? So solution to this we can define a process X_t is called solution, here actually we mean a strong solution but without that distinction we will say solution. we mean strong solution in some sense, because there are two notions of solutions to such an SDE but we are worried about only the strong solution and hence we simply call this as a solution.

Solution of the SDE, say let us give a equation to this or the star, equations to be star, if for all $t \geq 0$ the integrals, say $\int_0^t \mu(s,X_s)ds$ and $\int_0^t \sigma(s,X_s)dW_s$ are exist, and your

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s$$

So, we say any process X_t is called a solution to this SDE in the interval 0 to t, 0 is our starting point, then t is our ending, in the interval 0 to t we are here looking (for it) for all t these integrals are defined exist and different and first one is an ordinary Lebesgue integral and the second one is the Ito integral. they exist and your X_t can be represented in terms of this s in this manner. So this is what we call it as a solution to such a SDE.

Now, let us look at some simple example, so of course you would see this solution would be some functional of some initial value the time t and the Brownian motion path up to time t starting from 0. Because dW is X_s and X_t is connected with that, so it means it will be some function of t as well as the Brownian from time 0 till time s, so you can all solution you can rewrite as if involving in that form that is much easier.

Now, we have seen certain things for example this one

$$X_t = e^{W_t - t/2}$$

a solution of the stochastic exponential, now we will see what is this stochastic exponential but you can assume that, that is also an SDE stochastic exponential. what is that? This is actually $dX_t = X_t dW_t$ with $X_0 = 1$, so if I look at this differential and this is the solution.

Now, one way to verify, is actually verify that X_t satisfies this that is what. Now this, if you look at this, this is not in the simple case, so this is the function, X_t is a function of t and W_t alone but in general this will be a function of t and W from 0 to t. that is what, we said in the other case, which you may able to see in some other cases of expression.

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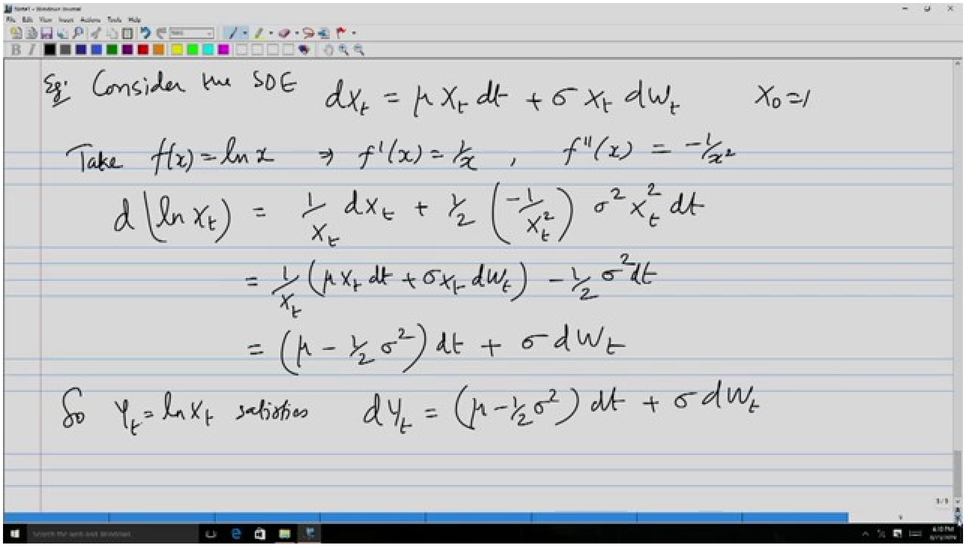

Now, there are certain SDE's which may be of more interest to us. so let us look at some other example, so this is really, so this is actually an SDE and similarly, this is consider the SDE, say

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

with $X_0 = 1$. So this is the SDE. Now you can pretty much solve this, how do we do go about is that you know you take the function $f(x) = logx$, then my $f'(x) = 1/x$ and $f'' = -(1/x^2)$.

Now, this quantity then the differential of log of this function f, which is differential of log Xt would be given by the expression in the pic. you can easily see that.

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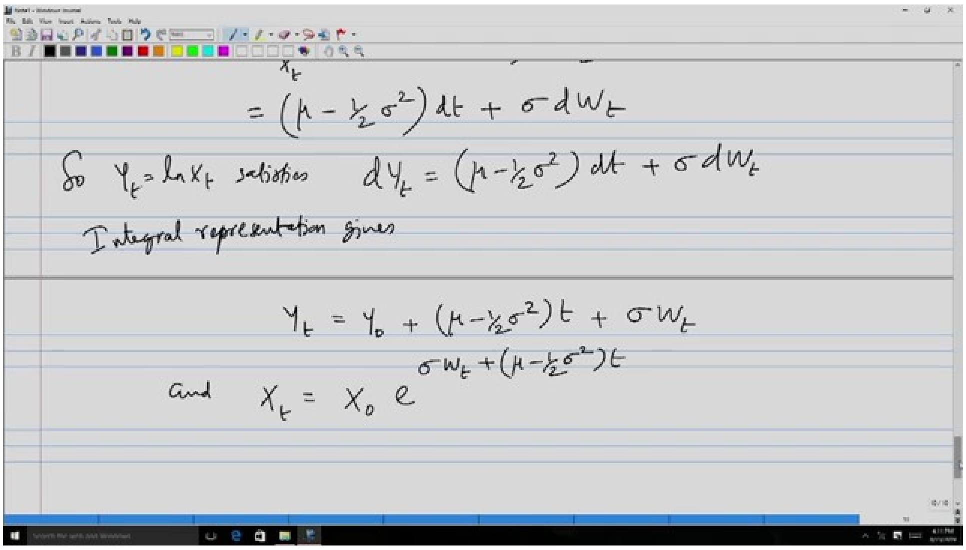

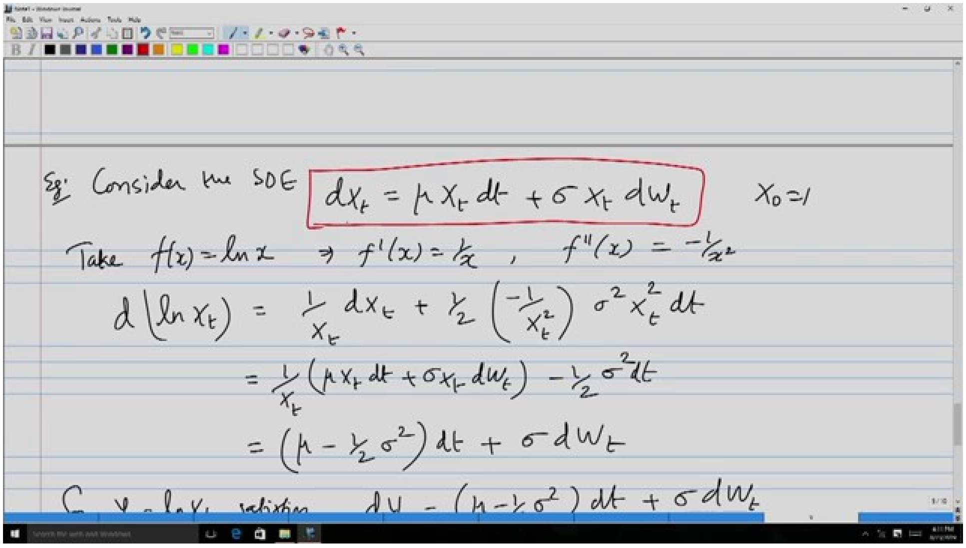

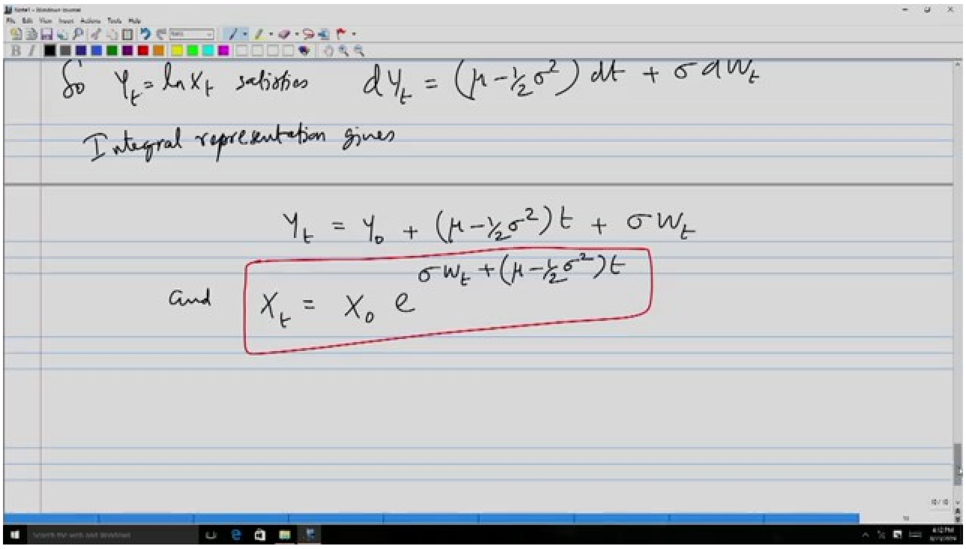

So, the integral representation, what it gives? It gives

$$Y_t = Y_0 + (\mu - (1/2)\sigma^2)t + \sigma W_t$$

and hence my $X_t = e^{\sigma W_t (\mu-(1/2)\sigma^2)t}$.

So, if my X_t is given by this, then its differential form is given by this, so the differential form and this is the process explicit form, in terms of the Brownian motion, so this is what it is. so this X_t is what is this process and this X_t satisfies that SDE or if you are given that SDE now you solve that SDE to get to this process X_t. so this is what is the prime example and now you can easily identify this X_t process we have already encountered when we talked about the Brownian motion and you recall that W_t is Brownian motion and normally distributed and this e to the power normal distributed one would be log normal distributed. And this log normal distribution arises in the limit for the asset prices, when you take the limit of the asset price process in the binomial model, we have seen that.

So this is exactly you know this example is exactly fitting into that framework, so your X_t is given by this and this X_t satisfies this SDE. So now, if you have to describe the evolution of the process either you can say that the processes is given at time t is given by the explicit expression X_t or you can write the differential to describe the evolution of the quantity X_t.

So, this is in either case. That is what you will have as an example. And this is the most interesting model or the first that we are going to consider a little later, when we make the asset price process model that and this is what we are going to use. of course, there could be even other examples, say for example as I check interest-rate model example that we have considered earlier in the lecture in the previous lecture, so that could also be written and its solution could be given and so on you can have.

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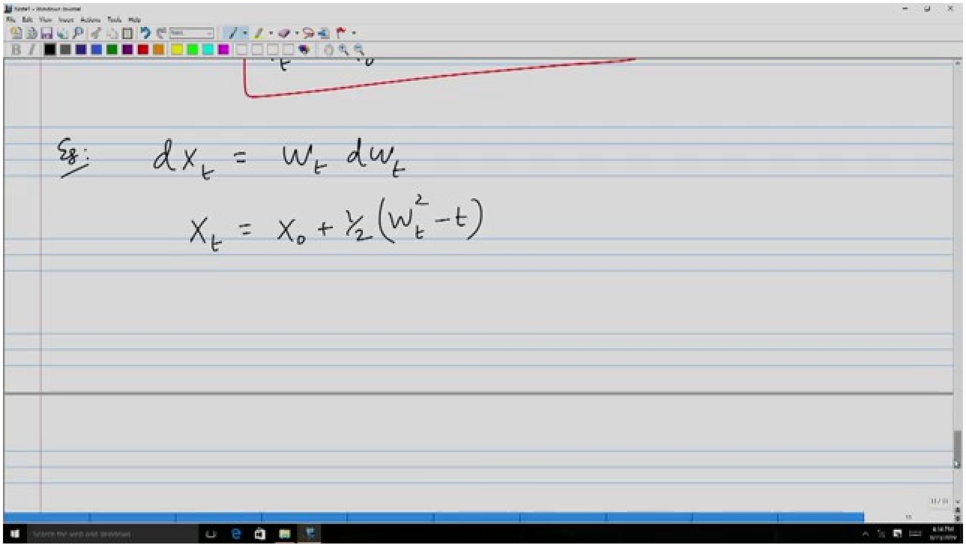

Or you could have another example simpler example, so if you have $dX_t = W_t dW_t$, now what is it solution? Its solution is $X_t = X_0 + (1/2)(W_t^2 - t)$. you can verify by using integration by parts and Ito integral formula or whatever want to keep. So that is what you know we have. Only like, as far as solutions are concerned with respect to the SDE's, so, this is what the another example, which we can verify. so only some classes of SDE's admits very explicit closed form solutions and what we have seen is one and there could be other examples that we can write it down, but of course we will not in to the details.

In general, one can compute the solutions, if we know at least you know the existence and uniqueness of the solutions just like in the differential equation case even in SDE case also we can have existence and uniqueness theorem. Which under conditions on the coefficient of this diffusion type SDE guarantees that there is a unique solution and in that case at least you are guaranteed to have a solution which may or may not be unique and again uniqueness condition is satisfies, then it will be unique solution that you can obtain in this case.

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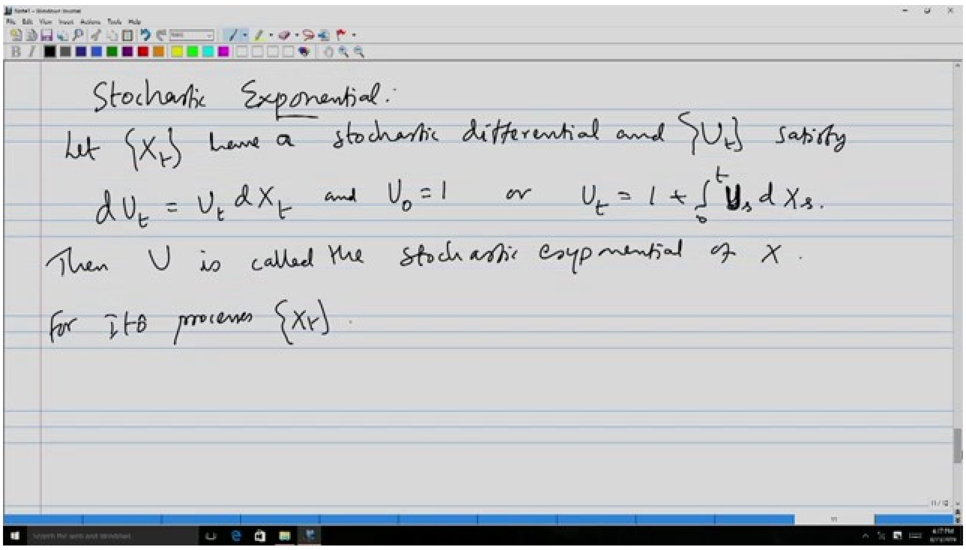

Now, in the simple case we have a specific name for the particular type of SDE, which is what we call as the stochastic exponential. just previously like you know we have mentioned this name in the first example that we have considered. so if let the process X_t have the stochastic differential and another process U_t satisfy this expression $U_t dX_t$ and my $U_0 = 1$ or equivalently you can write it $U_t = 1 + \int_0^t U_s dX_s$ then U is called the stochastic exponential of X.

If obviously, if X is a finite variation process, then we know exactly that the solution is simply e^{X_t} you will get. But since here X_t could in general be a (Ito integral) so for Ito processes , which is that X_t.

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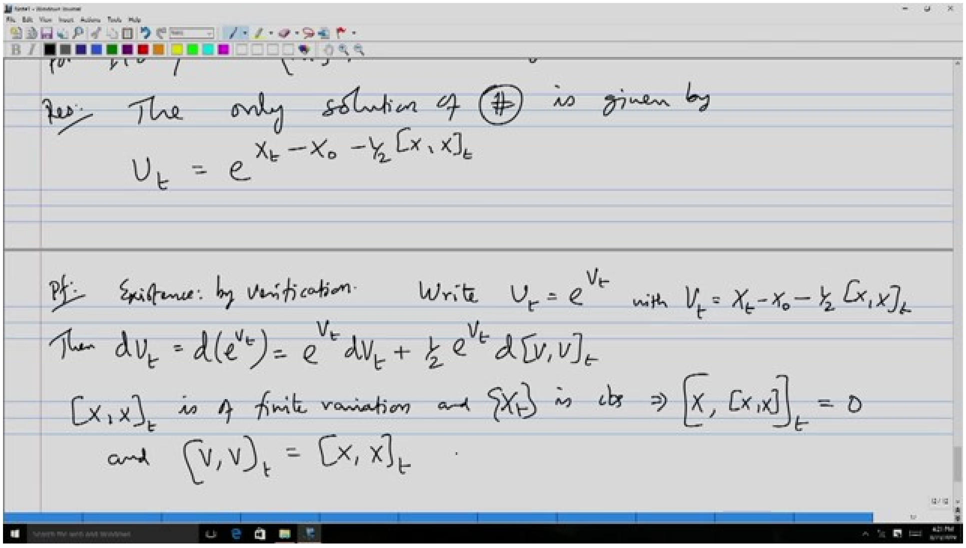

So, the solution is actually given by the following result, so, how to get solution, so the only solution of this one, suppose if, I call this as a hash, is given by Ut given above, and you know simply that the for finite variation processes this quadratic variation term that we have given in the exponent is 0, so and hence, the remaining part only would be the solution. but in for a general Ito processes, for Ito processes the following holes and hence the previous example that we have seen like how we are ensuring that that is the only solution or the solutions exist in this manner. that is what you know we are trying to see.

So, the proof actually goes is not really difficult. So, it is just that existence is basically consisting of verification because we have given that this is the solution. So, if you verify that this is the solution, so existence is by verification, so write $U_t = e^{V_t}$ with $V_t = X_t - X_0 - (1/2)[X,X]_t$. Then if you write $dU_t = d(e^{V_t})$ which is again given as follows.

Now, one can observe that this is equal to this is an Ito process. so this is a Ito process, this is a finite variation process, because you can go back and see that this does not have any dW term in the for an Ito process the quadratic variation term, so it is just that the integral over 0 to t of the square of the integrand in the Ito integral component of the Ito process times dt is what you get.

So, this is the finite variation and X_t is continuous implies, the quadratic variation of X with this one I am taking quadratic variation of X with the quadratic variation of X with X to that process, this will be equal to 0 and this quantity, which is equal to this then this will be given simply given by because you are taking $V_t V$ with V so you are looking at X with t, and obviously in that case this seems a finite variation process.

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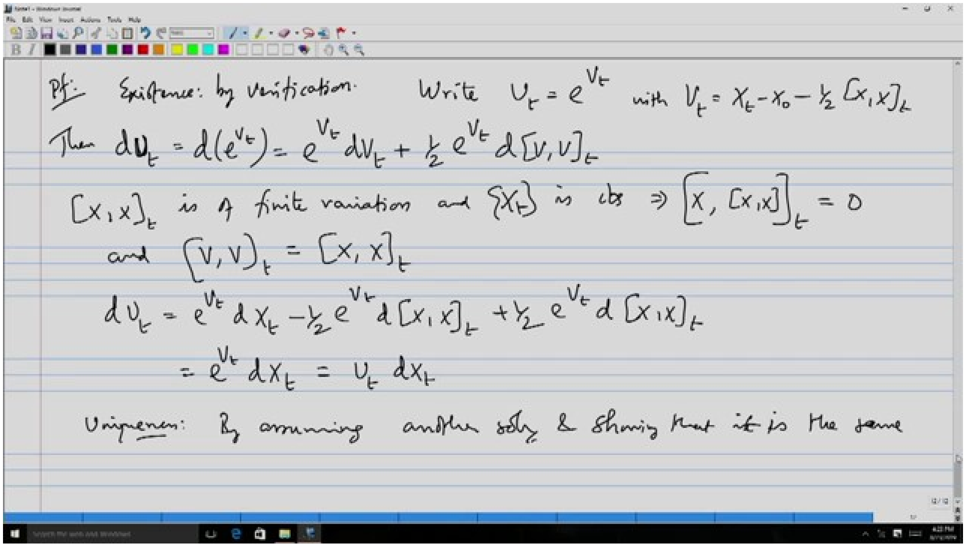

Now, if we use this expression here what you would see, is that this d sorry this is U), so this is $dU_t = e^{V_t}dX_t = U_t dX_t$ and that is what your SDE.

So, this basically $U_t dX_t$, is what then you are saying, so and hence this the given solution satisfies this and the uniqueness proved in the usual way by assuming another solution and showing that, it is the same. So, that one can show of course you know this is what, is in the case you have it here, so that means the processes we call as stochastic exponential, is e to the power the usual quantity.

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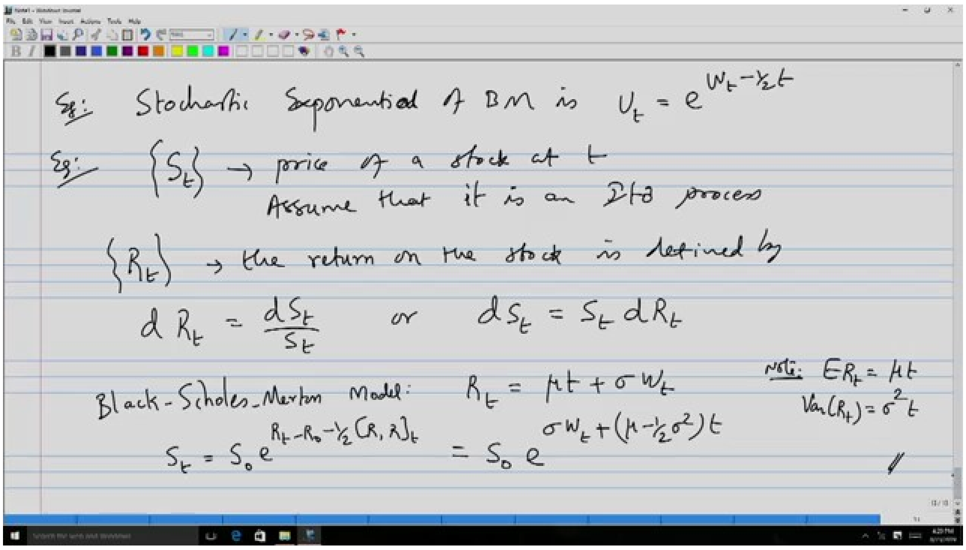

So, of course if you take a Brownian motion, say example if you look at it for stochastic exponential of Brownian motion, that is $U_t = e^{W_t-(1/2t)}$, which is what you have seen and it satisfies the SDE.

Now, what is the application in finance for this process, for this stochastic exponential? If you let, if you let S_t denote the price of stock at time t and assume that it is Ito process, that is it has a stochastic differential with the appropriate form, then the process of the return on the stock, then R_t, which is the return on the stock is actually defined process. Defined by a process is return process and this is the stock price process, this is defined by dR_t is equal to dS_t by S_t or dS_t equal to $S_t dR_t$. This is what will turn up to be and returns are easier to model from the first principles.

So, that is what you know for example in the Black Scholes model that we are going to consider next. so the returns are going to be independent and non-overlapping intervals and we will have finite variances, is what we are going to assume. So this assumption for example black we will write first time, I mean in this, so we are going to consider this next.

So, in the Black Scholes model what we are assuming is $R_t = \mu t + \sigma W_t$, this is nothing but the return process, then the stock price process then if I assume that this is the return process with drift as you know $\mu\sigma$ as the vitality, mean rate of return. If I take $E(R_t) = \mu t$, variance of R_t is basically $V(R_t) = \sigma^2 t$. So, this μ is what will be called mean rate of return, a rate of return per unit of time and sigma square is the vitality term so sigma is what we call as the vitality of this stock price process, so return is basically given by this, which is basically you will get that in non-over lapping intervals because of this property of this W the returns are independent. and will have finite variance and it is given by this, then this S_t is nothing but if I apply their previous result, we can get the result given above.

So this is what it is, so essentially what the stochastic exponential is the stochastic exponential of the return process is nothing but the stock price process that will be the model in the Black Scholes model case. So, that is what we mean. where this mu and sigma will have meaning as the mean rate of return and the vitality of the stock price process that we have it here.

So, this is what we use and this what is the connection that you have of course there are many more quantities with respect to SDE and other things but since we make consider only BSM model, so we restrict our attention and the existence and inverse theorem we are not even stating.

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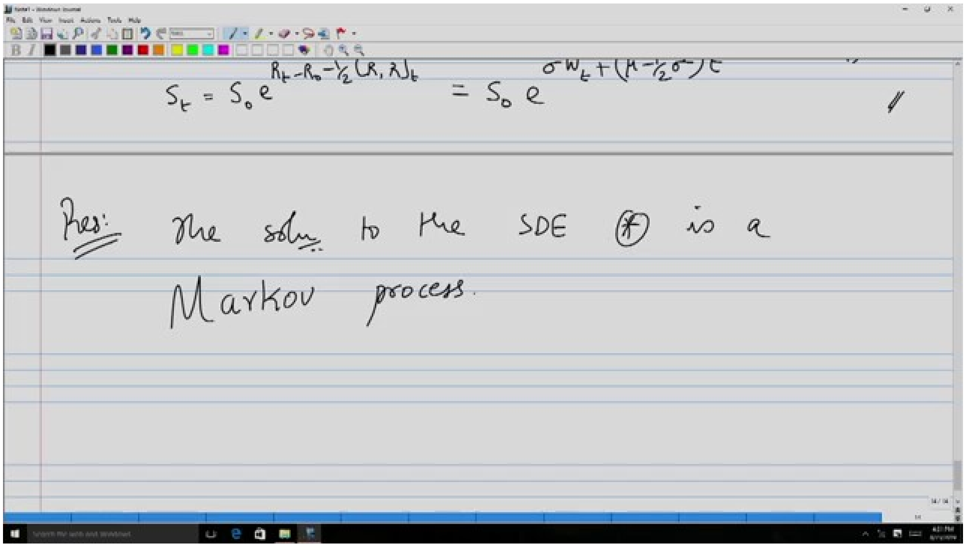

But one important property that you know we will use, is without proof we will take, is essentially the following. Which is if you have so the first SDE that we wrote. so this the solution to the SDE I suppose this is the original SDE that we wrote, which we have given as star, SDE star is a Markov process, so this one particular result will be very useful, latter we will and this result we will take without proof and we assume that this is true.

So, this is existence and uniqueness of course we can state and we have shown at least for the simple stochastic exponential type SDE. the unique solution is given by that expression and that is what with our model we are considering. But the solution one can also show that this is actually the Markov process. Again you can see here from the expression here itself $S_t W_t$ is the Markov process, so at any given point of time you only need t so for future evolution of course because the increments of Wt are independent, so you will Markov property follows immediately from here.

For this particular example, it is easy to see but in general also to the original diffusion type SDE, which you have written in the first earlier, when we started the SDE discussion that solution is Markov process. This is true for this diffusion type SDE. So this is what you know result what we will take and with this we end the discussion on our the stochastic calculus required for to handle the continuous time asset pricing models. Later then we will use all these materials and some more from the stochastic calculus itself as and when is required in the asset pricing processes, which we will consider latter, thank you.