Mathematical Finance

Lecture 28: Ito Integral and its Properties

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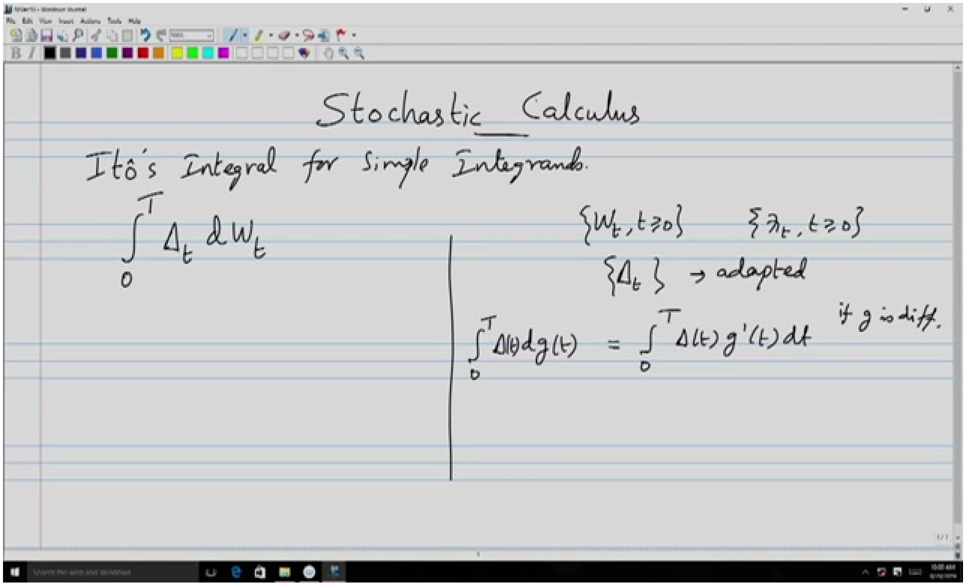

Welcome back, to the next topic that we are going to discuss and this is lies at the heart of the continuous time finance theory, which is what is called as stochastic calculus. So, basically what we would define, we would define what is called as an Ito integral and analyse their properties that leads to the famous formula called as Ito formula that lies at the heart of all the calculations that you do in stochastic calculus and then you know we will talk about, little bit about other things about its applications, and the calculus is basically based on the Ito's formula which is, know you can make analogy with the ordinary calculus formulas.

So, anyway let us start, so basically what we want to do first is basically, what we call as Ito's integral and first what we will consider is basically Ito's integral for simple integrands. Later we will generalise to a more general integrands under certain assumptions. So basically what we want to see is we want to make sense of quantity of something like this as I said Δ_t and this similarly for the W_t.

So, the basic ingredients here are, you know you have Brownian motion process, along with a filtration for the Brownian motion and we say filtration motion for Brownian then you know what it means exactly as per the definition. and you have a process Δ_t which is adapted, which is adapted to this filtration. Again since this is Brownian motion, Δ_t is adopted to $\mathcal{F}_t$ and $\mathcal{F}_t$ is a filtration for the Brownian motion, so we mean that by time t you know the value of Δ_t should be determinable, I mean when you go forward you would see this would be similar are this all represents among other things, on of the quantities that you know, which will denote our positions in the underlying stocks, which are all will be adapted process as we said earlier.

And since this is the Brownian motion for $\mathcal{F}_t$ is filtration for the Brownian motion. Even this Δ_t should give no clue about what is going to be your Wt. That is what you know we will keep that in mind, you may recall that you know this integral is something like in the ordinary calculus that you might counter and you may see that as a digress. Suppose if you look at some integral of some function of T and $dg(t)$, say I will write it here in this form so this also I will write it in this form.

And if g is differentiable, this is equal to

$$\int_0^T \Delta_t dg(t) = \int_0^T \Delta_t g'(t)dt$$

, and the right hand side is nothing but your ordinary lebesgue integral or Riemann integral when both are exist then Riemann at the same, so this is what would mean. but here look at this function W_t, is non-differentiable function, that is why the problem comes that this cannot be defined in this manner as you does in the ordinary calculus theory and hence we need to have a different way to you know give a meaning to this quantity, which is what will keep occurring in stochastic calculus. where you know the integral, where this function g in the ordinary calculus that you look for, so this will be taken by this W but the paths of this Brownian motion nowhere differentiable .

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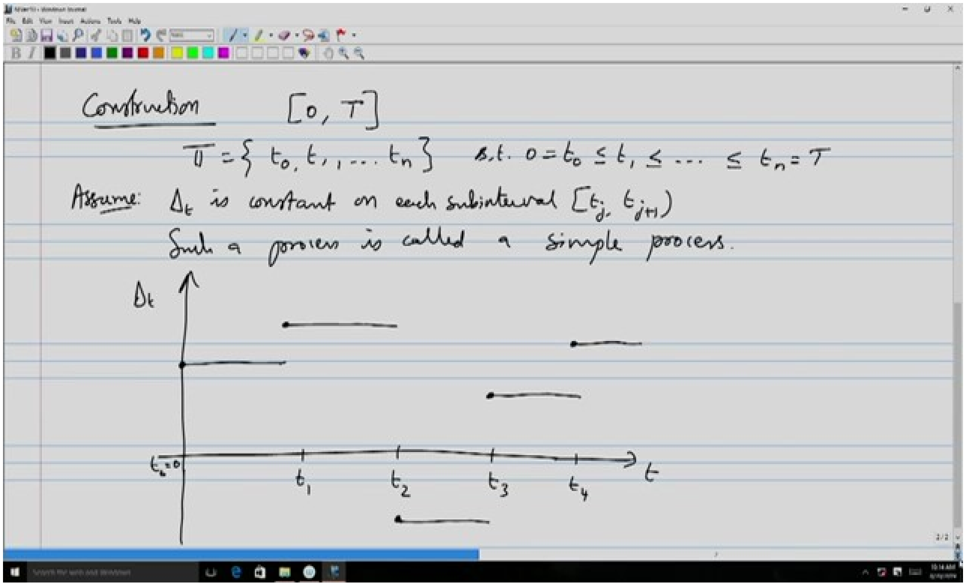

So, that is where the things will come. Now let us look at the construction, how we can construct. such an integral construction of the integral. so what we have this construction is what was done by Ito, who devised a way to give meaning to such an integral in the following way around the non-differentiability of the Brownian paths .

So, we first define as we said for the simple integrands, then we consider and extended to non simple integrands, as a limit integral of simple integrands, so enhance the understanding that we require and in understanding the basic integral, in terms of the simple integrands is crucial and hence I would suggest you know, you spend some time in understanding, so that meaning is clear what exactly we mean. So let us take you know as usual, you know we have an $[0,T]$ and you take your partition of this interval say $\Pi = \{t_0,t_1,...,t_n\}$, such that $0 = t_0 \leq t_1 \leq t_2 ... \leq t_n = T$ and so on or all the other points in the partition and we also assume, Δ_t is constant with respect to t on each sub interval, which is tj to tj plus 1. You know such a process is what we call it as simple process.

So, if I look at how this will look like, so if I take this, now what this is, so this is along with, x axis you have the time and along the y axis you have your Δ_t , which might be your this. So what this says is that you know at time 0 which is t_0 here, this is 0 and this is say t_1, this t_2 and this is t_3, suppose if this is the case and 1 more thing, so this is t_4, so what this says is that Δ_t is constant in each sub interval so between 0, which is what is my t_0 and t_1, this is constant.

So, you take the position starting at t and hold it until upto to t_1 but not t_1, just before t_1 want until that. So which is what this position would be, then a t_1, you know may change your position to here and then until t_2 you may hold it and the t_2, you know, it could be you know any position. So even it can take negative value so at t_2 you have this and until upto t_3 and the t_4, you hold you know, take this values and then until up to t_4 and then from t_4 this is starting.

So it is, if you look at simple without looking at the bringing the stochastisity in mind, this is the step function that you talk about, so which means, which has this specific property, you know this Δ_t is constant in each sub interval $\mathcal{F}_t$ close right open interval of this sub intervals that you are considering. So which means that it will be up this form, just prior to t_1, this will take this value which is on the lower 1 and at t_1 it will take higher value, which is what is meant to this.

So, this is what will look like simple path. So, this path again you know, where the randomness comes in because this path would depend on the omega that you pick, so you, this would depend on for each omega you have 1 such path and hence you know you have this as a simple process otherwise it is called simple function but this is simple process because for each of this omega, observation you can also make that this first part will remain the same as long as you know t_1 remain the same along any path because it is Δ_0 ,so that is what you would will observe. So, this is what a simple process means. Now, let us look at the interplay, we think of the interplay between the Δ_t and wt in the following way.

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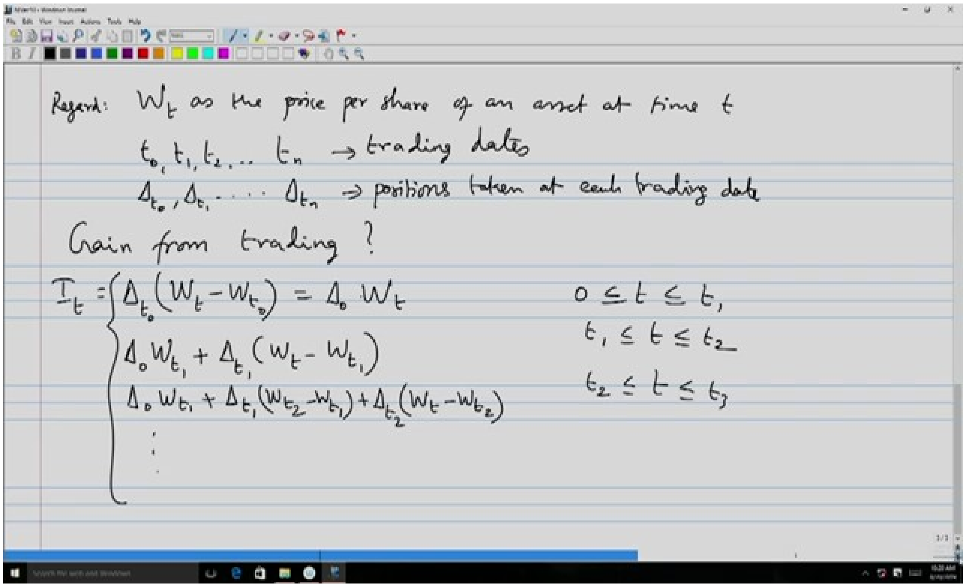

Now, what we do is that, just regard, you know it is regard this W_t as the price per share of some asset at time t, this is to give meaning to this otherwise you know it is just a mathematical construction but you just regard so that you know we can associate a meaning to such a quantity that we are talking about here. And you know like this will not be true in reality because W_t by definition is a normal random variable, which can take in any value between minus infinity to plus infinity and associating these with the, the price of a particular share is not going to be a meaningful quantity. Because it cannot take negative value W_t whereas in reality, the price is non negative quantity that is what you would see. So this is just for our understanding purpose, so regard this and Δ_t in a similar way and this 1, this time points like t_0, t_1, t_2 and so on, these are all you regard this as a trading dates which means these are the dates or the time points at which you know you do make trade and think of Δ_{t_0}, Δ_{t_1} and so on Δ_{t_n}. As the positions, we mean the number of shares that you hold positions taken at each trading date at each trading date. Then, if you regard the gain from trading, what is this quantity if you regard? suppose if, I call this is as It, this will be simply because my position at the beginning at Δ_{t_0}, t_0 equal to 0, it is Δ_0, then $W_t - W_0$, which is nothing but $\Delta_0 W_t$, this is true for all $t \le t_1$, because I start holding the position of Δ_0 at time 0 and I buy the asset at the price W_0, which is 0 here and by time t this W_0 would have grown to this W_t and my position is this.

So, this difference is what the difference or the appreciation that you gain by holding each of this share and you are holding Δ_0 number of shares, so the total gain is $\Delta_0 W_t$. this is for t lies between this. Now suppose if I take my t lies between say $t_1$and t_2 then what would be this quantity? This quantity would be $\Delta_0 W_{t_1}$ until that the gain from 0 to t_1 is this plus at t_1 I change my position to Δ_t 1 and the difference in the price of the underlying asset between $t_1$and t for t lies between t_1 and t_2 would be given by this.

So, this is what the gain, so this is the total gain, delta0 1W t_1 a is the gain that I have from 0 to t_1, this is from t_1 the second term is from t_1 to t, similarly if my t_2 less than or equal to t 3, then this is $\Delta_0 W_{t_1} + \Delta_{t_1}(W_{t_2} - W_{t_1})$ then again I change my position at t_2 to Δ_{t_2} $W_t - W_{t_2}$.

So, this way you can see that you can extend for each of the intervals if my t lies between this and this, then this is what is It, which is basically the gain from the trading that, this quantity and if t lies between say t_2 and t 3 and you know you can extend in this way the, you can define for all t between 0 to any value that you are thinking.

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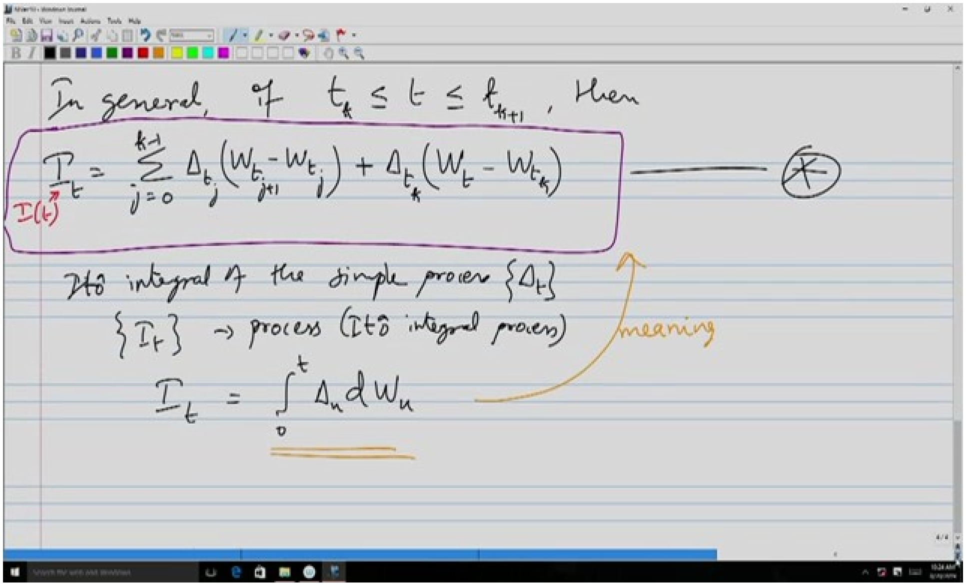

So, in general let us write a general expression, if my $t_k \le t \le t_{k+1}$ then my It which is you know you can also remember that this could also be written as $I(t)$. So this is same as this so but we as we said for simplicity we are writing in this form (see the above pic).

Will be interested to keep in mind as I_t. So this is what we call the Ito integral of the simple process Δ_t.

So, this is what we write it as $I_t = \int_0^t \Delta_u dW_u$ is what and this is what we call the Ito integral of the simple, so what is the you know, meaning of this. So the meaning when we write, this quantity the meaning for delta u simple process is essentially given by this summation, that is what it is, this is for the simple integral, that is what we define.

Now, we wanted to make sense of $\int_0^T \Delta_u dW_u$ but we not just for 1 particular capital T we had defined, we had defined for all t which lies between 0 and t what is the meaning of is and that is what given by this expression and if you think of this upper limit of this integral as variable then you would get this is as a process which is what we call it as this Ito integral process.

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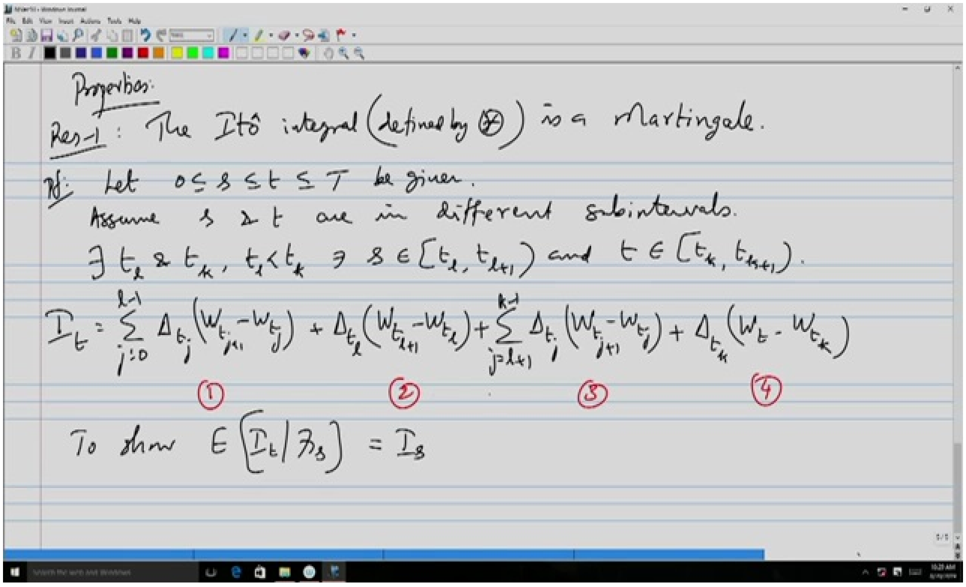

So, when we look at this is as a process then we can look at its properties now, some of this simple properties you know, we can you look at from, the simple integrand case, now you already know the W being Brownian motion is a Martingale. Now you are trading in a in martingale asset, Brownian motion which is a martingale so you are trading in a martingale asset.

Now, since martingales has no tendency so either rise or fall, if you trade in a martingale asset, what do you expect the gain, the gain also should be in such a way that it neither rise nor fall. And that is what the first fact, that we will give that the Ito integral, the definition is here given above that the Ito integral, is essentially defined by star, is a martingale. So, you keep this summation in mind, this is what the meaning of this Ito integral for any t that we have. So this is what we keep repeating and hence say this is what important you need to remember. Now let us look at, how do you prove this, let us do the simple proves later you know we will going to take the research for granted for unique general Ito integral but at least the meaning should be clear from the simple cases.

Now, we will assume that s and t, so, they are in different sub intervals otherwise what will happen the following prove that we make will become simpler, that is the only thing. So which means that what, (so you know there is this) so which means that there exist t_l and t_k such that $t_l \leq t_k$ and so these 2 are such that my in $s \in [t_l, t_l + 1]$ and $t \in [t_k, t_{k+1}]$.

Now my I_t, I can write it in the given form. So this is what the term. There is first, second, third, fourth term here. So, let us call this as term 1, 2, 3, and four. Now, look at one by one what happens in this case, so what we have to show,

$$E[I_t|\mathcal{F}_s] = I_s$$

, this is what we need to show. Now let us look at these four terms that we have here and look at one by one. We have $\mathcal{F}_s$, and my s belongs to the interval t_l to t_{l+1} and the all the terms which are there inside the first term, the inside quantities which is there in the first term, there all you would see measurable with respect to fs, because all of them are time before, time t_l and t_l is less than or equal to s, so all of them are $\mathcal{F}_s$ measurable.

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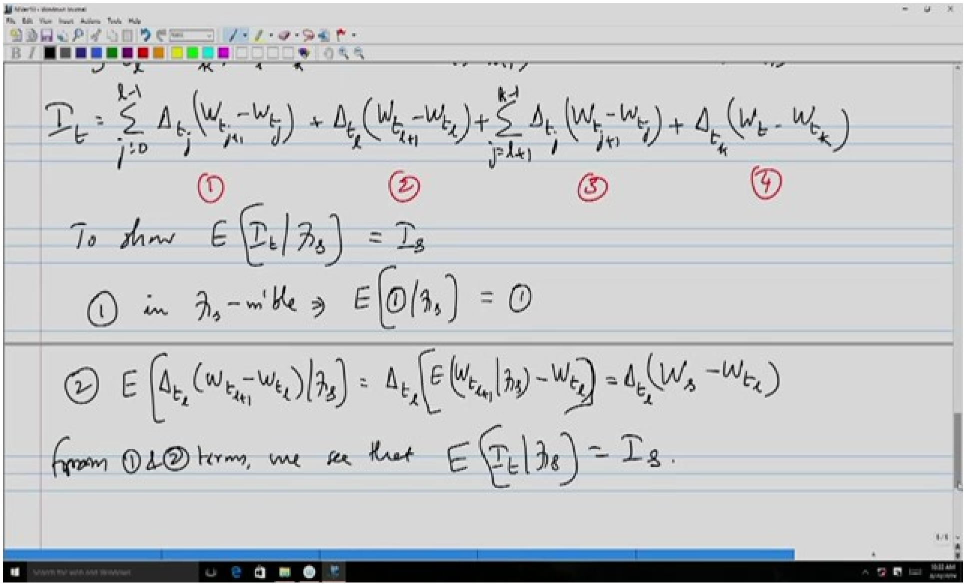

So basically, what we have, so all the qualities we want, this is fs measurable, so, which means when I take this conditional expectation that is what you are observing. Now for 2, if I take the conditional expectation, so this is basically we have to take these 2 quantities, so we will take the conditional expectation so fs is measurable, so basically now if I take the conditional expectation of the terms 1 given fs would turn out to be the complete terms 1, so that is what you know we have here.

The 1 that we have that is, whatever you do like with regard to the conditional expectation of that given fs because this 1 is fs measurable, so whatever you have it will come to, as for the second term is concerned, you can see that. Now you can see out of this terms your W_{t_l} is $\mathcal{F}_s$ measurable, so you can take that out. And you can take the expectation inside and you see W_{t_l} is again $\mathcal{F}_s$ measurable whereas this quantity is not $\mathcal{F}_s$ measurable so it will remain as it is, $E(W_{t_{l+1}}|\mathcal{F}_s) - W_{t_l}$so this is what we have. Finally we get $\Delta_t(W_s - W_{t_s})$

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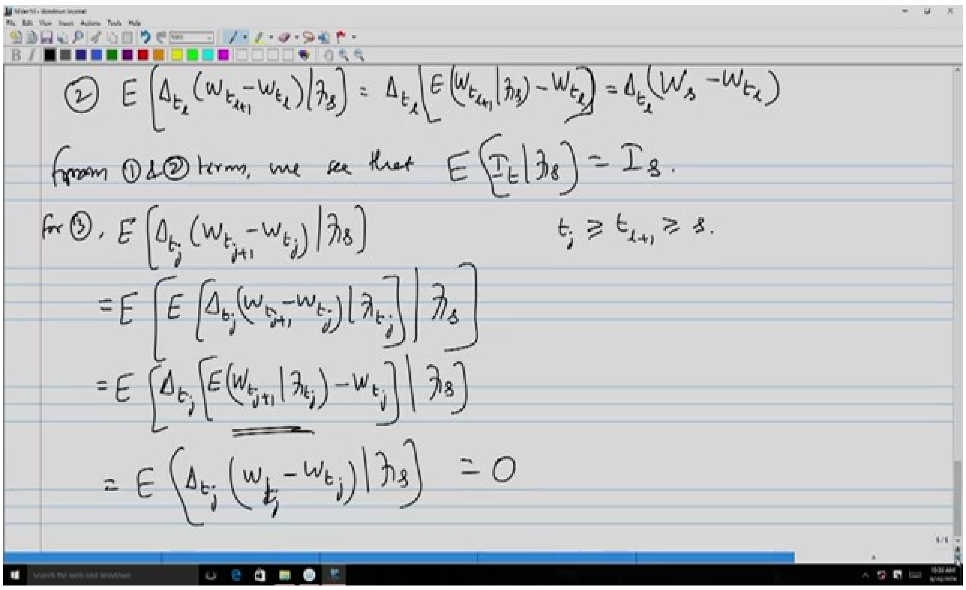

Now, you can easily observe that from this 1 and 2 terms, we see that immediately this $E[I_t|\mathcal{F}_s] = I_s$. Now, you can immediately see this is true, so will then have this as the result which is what we wanted but this would be true only if the other two terms that we have here, if they are 0 when you take $E[I_t|\mathcal{F}_s]$ the third term and fourth term must be 0 then only this is true. So let us see how third term and fourth term are 0. Now for 3, for the third term, what you can do, you can just pick up one of the terms and take the conditional expectation and since when you take conditional expectation of the sum the sum of the conditional expectation by linearity property, so the sum would follow.

Now, for the third term consider the 1 term, which is $\Delta_{t_j}(W_{t_{j+1}} - W_{tj})$, given $\mathcal{F}_s$, this is one of the term, remember here we have, my $t_j \geq t_{j+1} \geq s$, so that you need to remember.

So this is basically what we are using we are using the tower property of the conditional expectation or the iterate conditioning property of the conditional expectation, where $\mathcal{F}_t$j would be greater than or larger sigma field then fs, so this is what we are having. Now if I, if I take the inner quantity alone, Δ_tj is $\mathcal{F}_t$j measurable I can take that out and I can apply the previous idea that how exactly we used it, so this will be Δ_{t_j} and this quantity would be finally $E[\Delta_{t_j}(W_{t_j} - W_{t_j})|\mathcal{F}_s] = 0$.

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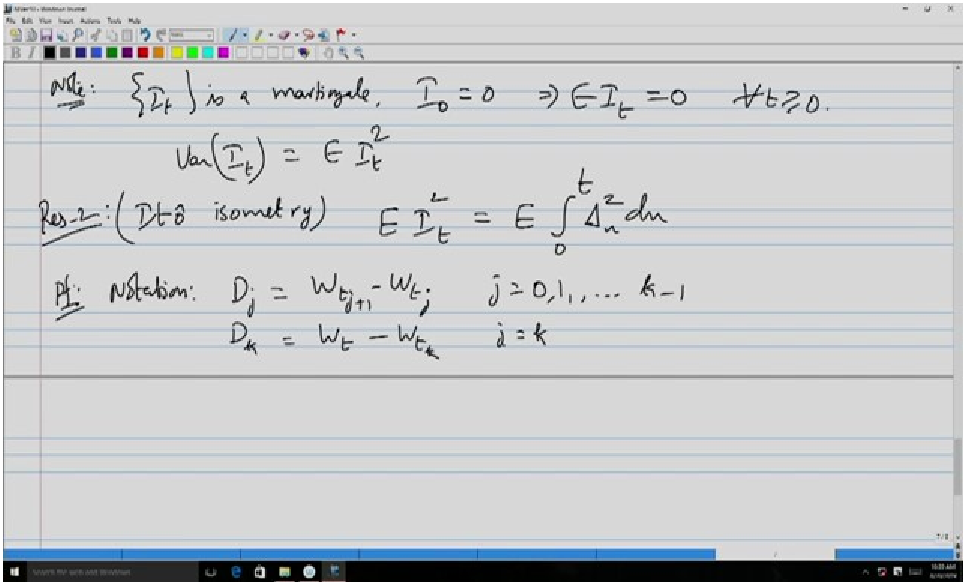

So, this implies 3 for 3 the expectation of the terms 3 given $\mathcal{F}_s$ would be 0 and similarly for 4 exactly the same argument you know you would see 4 given $\mathcal{F}_s$ would also be equal to 0 and hence the proof that I_t is a martingale.

This is the first result, because now we know that this is a martingale and$I_0 = 0$ implies $E(I_t) = 0\ \forall t \geq 0$. Then variants of this process It, you would then nothing but this quantity. so, what is this quantity equal to, is what then we are going to see next.

Which is called as ITO isometric property, so what we have here, exp

$$E[I_t^2] = E\int_0^t \delta_u^2 du$$

So, this is what the second result, which is giving exactly the variants of this Ito integral process. Since mean is 0 and this is variance. This is what then we are looking at it that we have. Now again, this idea of the proof will be along similar line as the earlier 1 to simplify you know proof notations to the simplify the notations we will use

$$D_j = W_{t_{j+1}} - W_{t_j}$$

$$D_k = W_t - W_{t_k}$$

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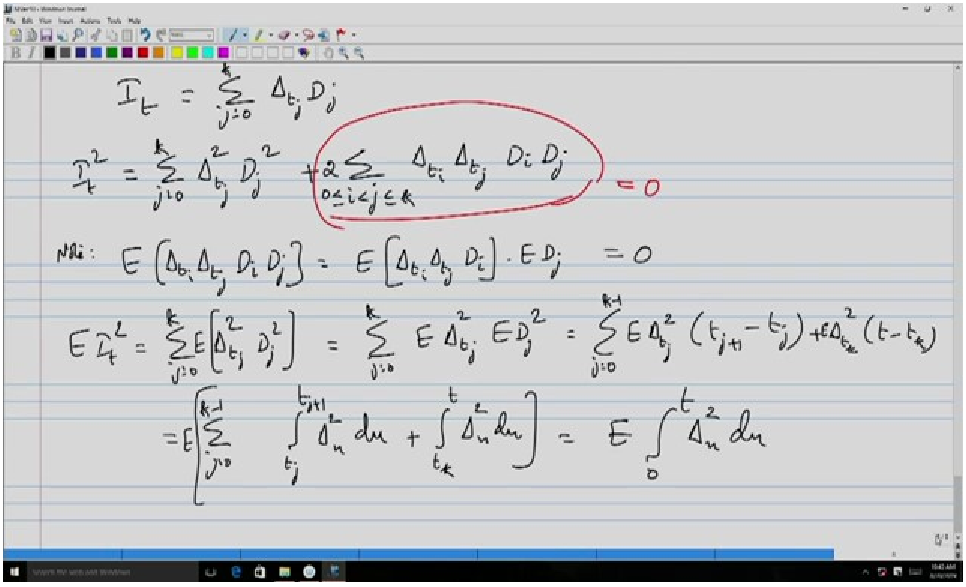

So that, what we can write

$$I_t = \sum_{j=0}^{t} \Delta_{t_j} D_j$$

We can write now what is this I_t^2. Now, what we want to compute the expectation of this, so expectation of this is what we wanted and the expectation of this one, so these terms are going to be 0. You can easily see, note, that expectation of Δ_ti, $\Delta_{t_j} D_i D_j$ if I look at it, so out of this Δ_{t_i}, $\Delta_{t_j} D_i$ are $\mathcal{F}_{t_j}$ measurable and while the Brownian motion increment D_j is independent of $\mathcal{F}_{t_j}$. So that gives us that this is equal to essentially $\Delta_{t_i} \Delta_{t_j} D_i$ times $E(D_j)$ is going to be 0. So this is 0, so (all the you know) this complete (this whole) term will be equal to 0.

Now we get

$$EI_t^2 = E \int_0^t \delta_u^2 du$$

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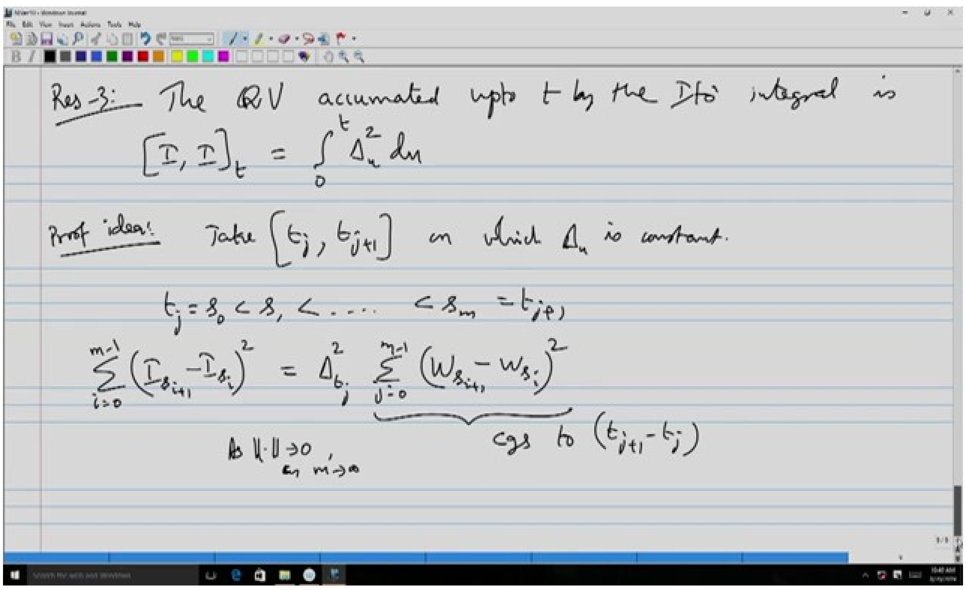

So, we now turn to the third result, which is important also is that the quadratic variation of accumulated up to t by the Ito integral but the Ito integral is given by this I of t again this is a process we will just write it in this manner $\int_0^t \delta_u^2 du$.

Now finally you see that see in case of simple Brownian motion you saw the quadratic variation is also t and variance also t but here you see the difference that you know the variance is expectation of this quantity variance co-integration is this because it is vary from path to path, so this quantity also will vary from path to path. like if you take a larger position Δ_u along one particular path this quantity will be larger, if you take smaller values of Δ_u along another sample path then this quantity going to be smaller whereas our variance is an expected measure, which is going to be a constant.

So, here really you see that this is what happening. So, this again we can compute the quadratic variation accumulated. So this proof-idea, I mean we just give an idea about the proof. So basically take you know t_j an interval sub interval $[t_j, t_{j+1}]$ on which my Δ_u is constant , in that it is a constant. Now you choose a partition point a $t_j = s_0 < s_1 < ..., s_m = t_{j+1}$. Now consider this some, i s i plus 1 minus i s i and then it is square, which will turn out to be the constant, which is written above (see the pic).

Now as the usual limiting process which means as the norm of this partition of this sub interval tending to 0 that means as m infinity and the norm of this partition tending to 0 this quantity, so specially as for example if you look at this quantity, this converges to as norm of this partition tending to 0. That is m times infinity and the norm of partition to 0, this quadratic converges to the quadratic variation of this W in this interval which is again $(t_{j+1} - t_j)$. This is what you would see.

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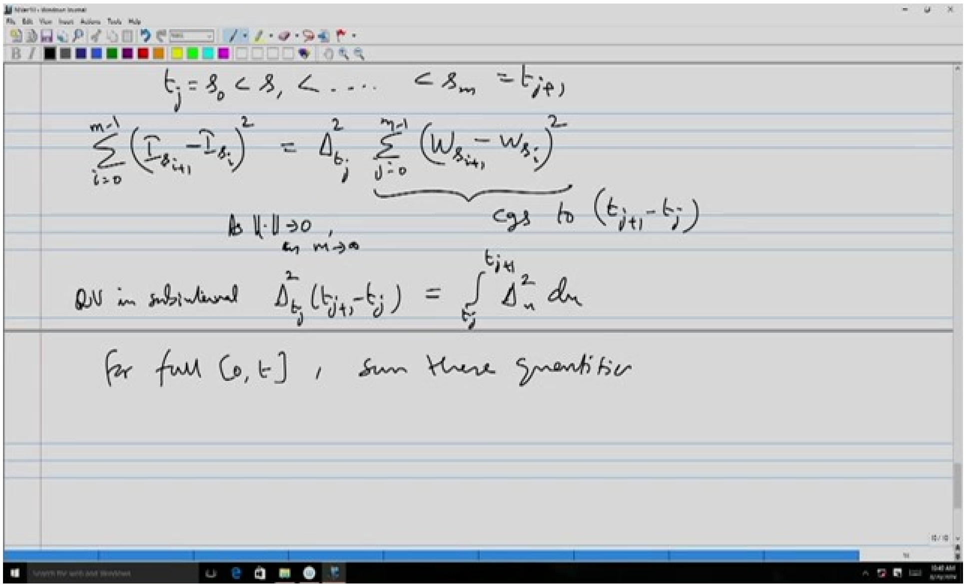

So, what we see is that the quadratic variation in the sub interval, is essentially quadratic variation in the sub interval is $\Delta_t^2(t_{j+1} - t_j)$, which you can write it as $\int_{t_j}^{t_{j+1}} \Delta_u^2 du$. Now for full $[0, T]$, you get it sum these quantities.

So, that is the idea how the proof that you know you will get, so for once you find this for the sub interval then you will get, so where the overall quadratic equation which proves the result. So finally what you have seen is that the three basic properties of the Ito interval which is martingale property, Ito isometric property and the quadratic ration property of what you have seen.

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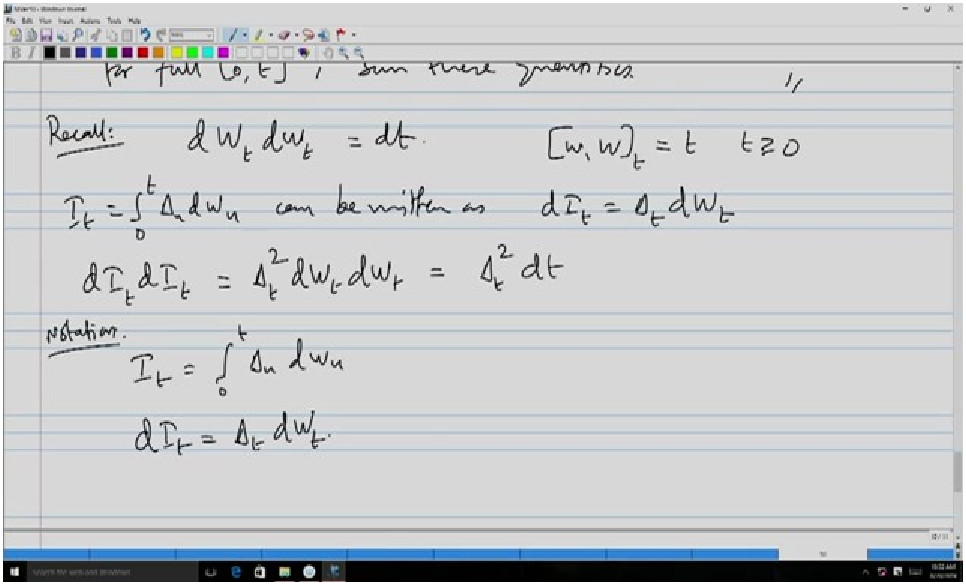

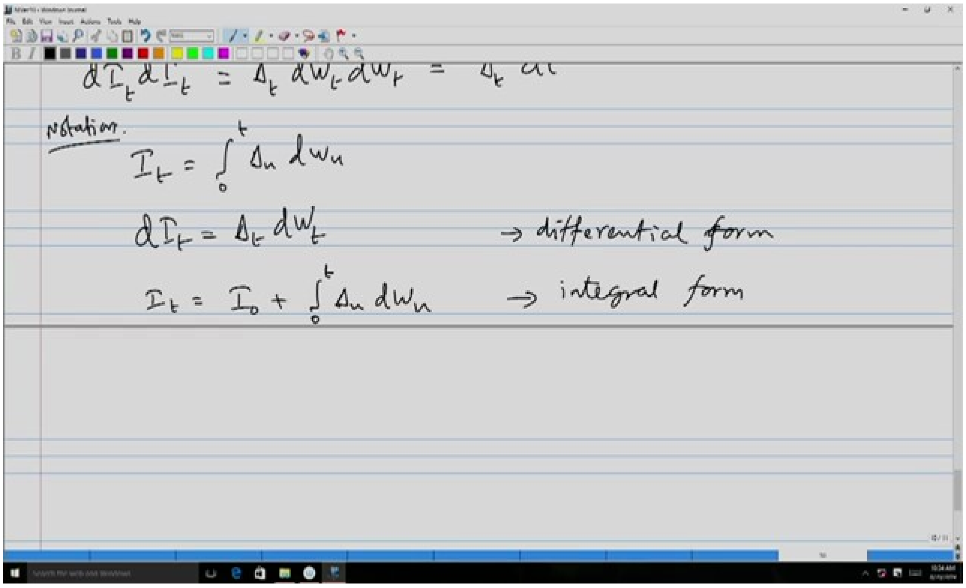

So now, recall that earlier the bread and butter expressions for the quadratic variation, informal way of writing is this. So this is what we write informally to mean this. Remember this is informal way, this is our square bracket formal way of writing this. Now, the Ito integral, which is this $\Delta_u dW_u$ can be written as informal way as

$$dI_t = \Delta_t dW_t$$

Now, what we write for the quadratic variation of I we can take this quantity which is essentially

$$dI_t dI_t = \Delta_t^2 dW_t dW_t = \Delta_t^2 dt$$

This is what the meaning that you know Ito integral accumulates quadratic variation at the rate Δ_t square per unit of time. So, the data of accumulation is what it is, this what is the previous result you would have you know in the pen and paper calculation to easily understand and see what is this quantity, you can also look at in this manner. now for the notation which you know we will be using this pen and paper calculation, this kind of differential formula then the integral form, so we can see that this notation as we wrote earlier this is $I_t = \int_0^t \Delta_u dW_u$ and $dI_t = \Delta_t dW_t$.

They mean almost the same thing but the first has a precise meaning, the second one does not have a precise meaning it is just that, but second one is more intuitive what is happening to I, which means the change in I can be given in terms of Δ_t times the change in W, so it has that informal meaning of what happens whereas it says definition is given by as in the first one, that is what and second one has an imprecise meaning as we just described.

It also has a precise meaning, which one obtains by integrating this differential over you know both side over and integral and put an integral put in a constant, so it is basically $I_t = I_0 + \int_0^t \Delta_u dW_u$,so this is basically is the differential form and this will be called as the integral form. The difference between the first one and the third one is that you know you have a constant I_0 generally constant here and then here it is I_0 is assumed to be 0.

So, you can write the differential form in this means with the assumption that $I_0 = 0$ then it becomes the second one but then the second one for any generic one has the meaning, I mean the integral form is given here, so these are differential form and integral form whenever we have a process we describe in terms of the differential form we can also we should also understand that the corresponding or the precise meaning is in terms of the integral form that we have in the third quantity here in to this case. So this, what is all about the data integral for simple integrand,

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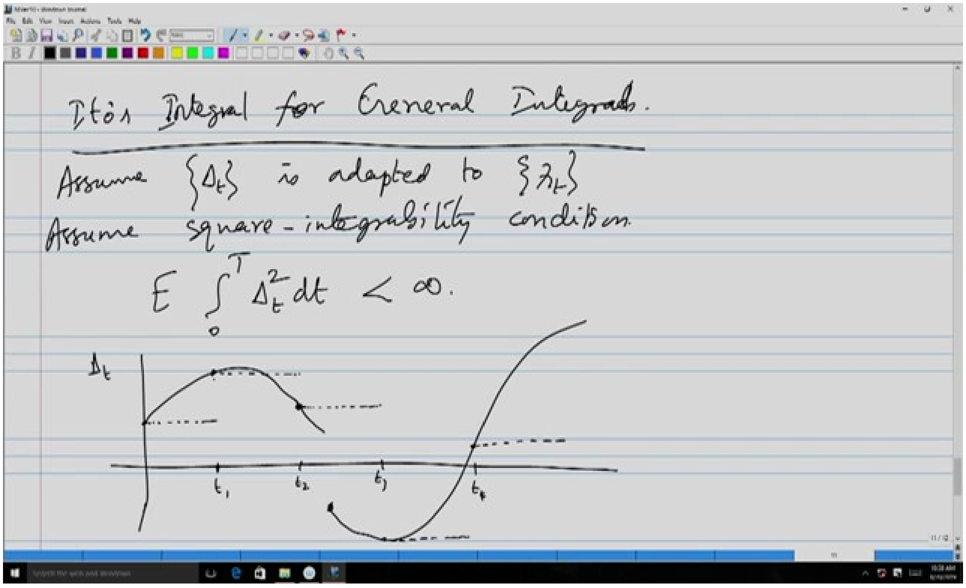

Now, let us define the Ito integral for general integrands. So, what we have, is now a general integrand but we need to put some condition, so what is the condition that we assume that we will follow through out, is essentially you know we have a general integrand is adapted. That is clear, adapted to this $\mathcal{F}_t$ just like the simple integrand case and we also assume what we call the square integrability condition. This is

$$E[\int_0^T \Delta^2 dt] < \infty.$$

This is what is square integrability condition, in fact one can define it into integral with respect to you know a condition, which is weaker than this, that is without this expectation but in that case the expectation is not guaranteed to be a martingale and since in our things we consider only such cases in our case, all Ito integrals will be martingales so that the required integrability condition be imposed.

So, this is what the square integrability condition of for this process that we have here. Now how do we (you know) get for a general integrand the Ito integral. So, what we do is that we approximate this given Δ_t by a simple process and then we make this approximation finer and finer at the maximal step size of the partition approaches 0 the approximated integral will be better approximation in some sense.

So essentially, what we are looking at if I depict graphically in a simple manner, so what I have is this, the given process could be something like this. so it is coming up to that point and then form here then again it goes like this. Now this is the general Δ_t, now how does this simple process will look like, suppose the simple process might you know up to this might come.

So this is my first partition point and at this point it will take this particular value and it will continue up to certain point and at this point again, suppose if this is my second point t_2, at this point it is starting from here. It will take up to say this value and this is my t_3 and this is my t_1 and at this point then this will be equal to this and at this point say this is my t_4, it is this value it will be equal to this. This dotted lines says what is this simple integrand, this straight line is what the general integrand. We will try to take such approximation and then we take this approximation make bit finer and finer.

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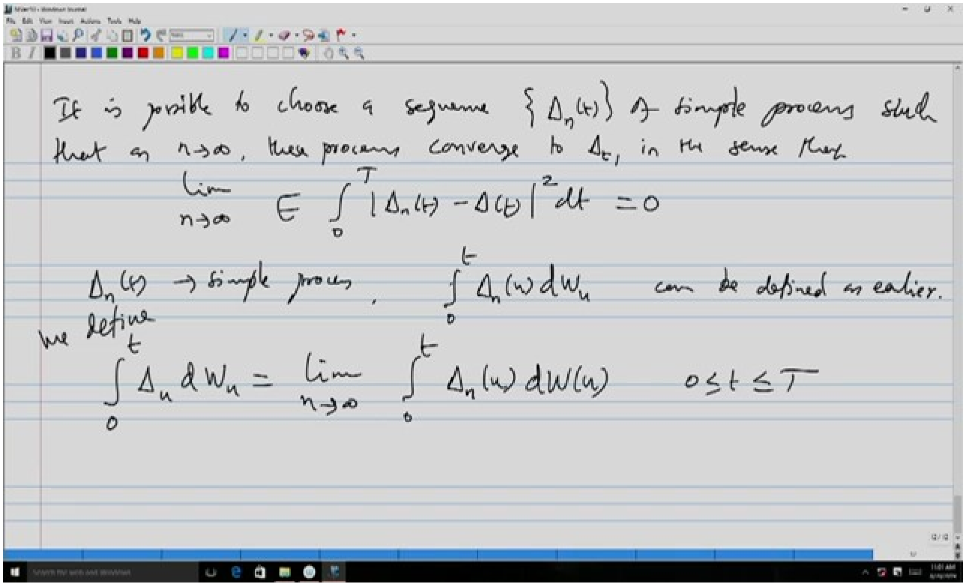

So it is basically, so it is possible in general to choose a sequence $\{\Delta_n(t)\}$ of simple processes, such that as $n \to \infty$ the, these processes (the simple processes) converge to Δ_t, in the sense that limit enter in to

$$\lim_{n\to\infty} E[\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt] = 0$$

So, this is what we mean when we say things converge.

Now, since each of this, this are all simple process, for each one of this we can define $\int_0^t \delta_n(w) dw_u$, so this can be defined, so this can be defined as earlier. Now for each simple process this can be defined, then we define the Ito integral of the general process to be the limit we define the Ito integral to be $\int_0^t \delta_n(w) dw_u$ for each of this simple process δ_n you can compute the quantity and the limit of that enters the infinity is what then define to be the Ito integral of the general quantity that we have here.

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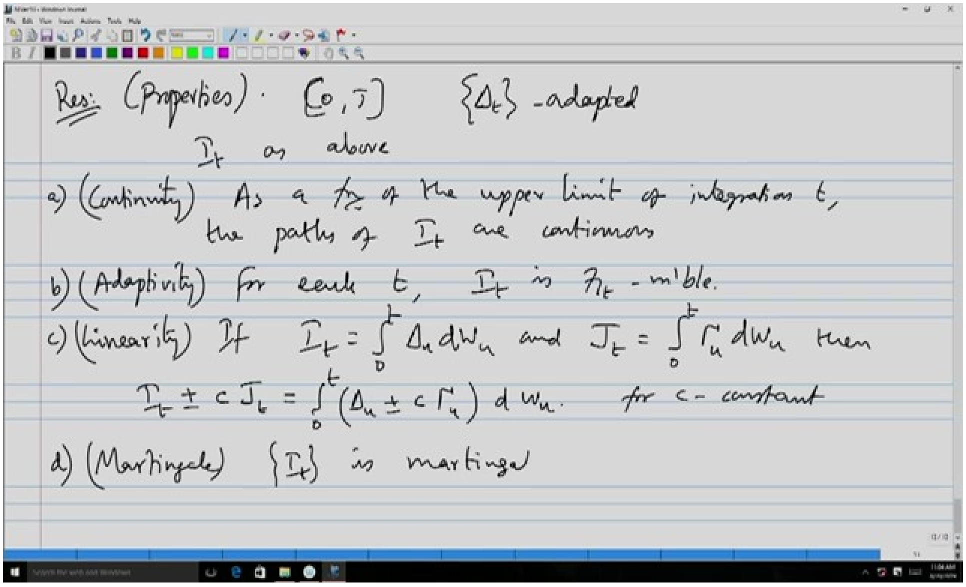

Now, let us look at its properties, the Ito integrals general properties, so we have you know a positive constant I mean an interval, so 0 to t is what you are looking at it and you have an adapted process.

And you define, you define It as above as a limit of Ito integrals of simple integrands then the properties are

a) the first one is continuity property which says the following, as a function of the upper limit of integration t, the paths of I_t or continuous, essentially meaning that this I_t has continuous paths,
b) adaptivity which means that for each t It is $\mathcal{F}_t$ measurable,
c) Third property is linearity.
d) The martingale property.

Then, this is, in fact you can see with respect to the simple integrand these are also true, then the properties that we prove for simple integrands and which is inherited by this limiting process, this is a martingale.

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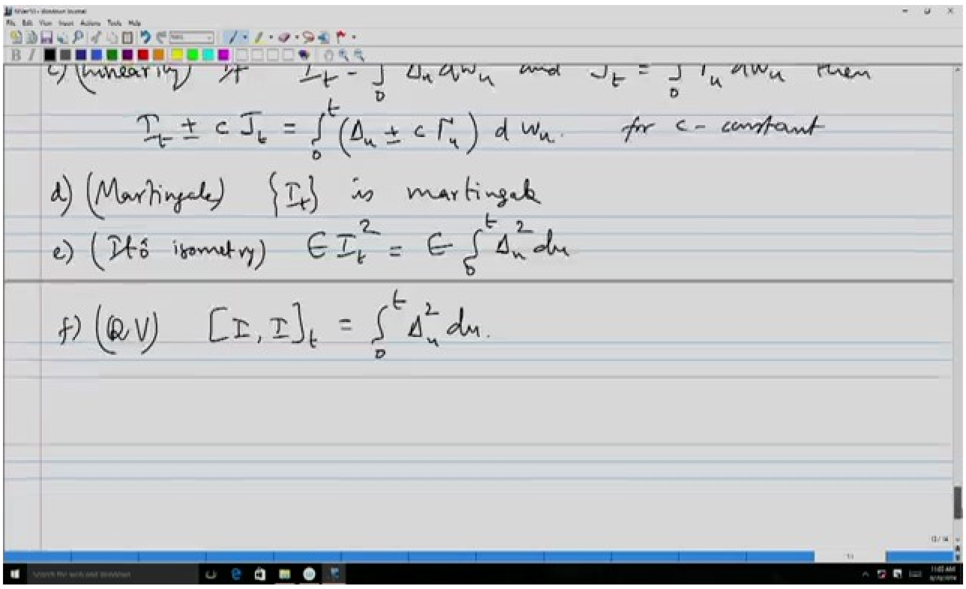

e) Ito isometry, the Ito isometric property
f) quadratic variation, which is,

$$[I,I]_t = \int_0^t \Delta_u^2 du$$

So all these are properties of this Ito integral which we will use elsewhere too and this properties of course now martingale I say isometric quadratic variation property we are going to prove at but this is what would be the case.

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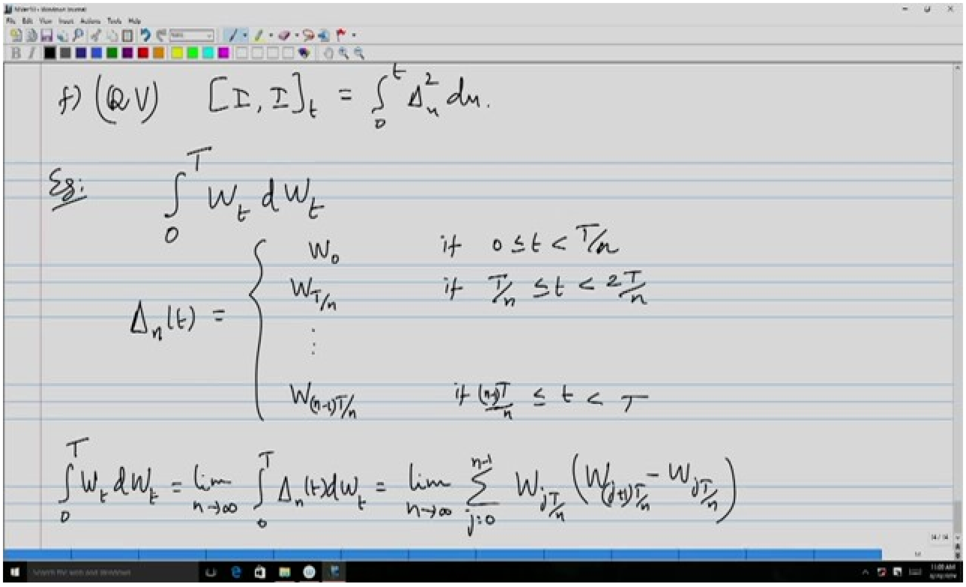

Let us take a simple example to understand to close the discussion this so what we can do is look at is $\int_0^t W_t dW_t$ is what then we wanted to do. So what is this quantity is and from where this is coming, now this you can define or this one, so what is we have to do, how do we have to compute if we have to apply the basic principles that we have to define this quantity by in terms of limit of integrals of some simple integrands which approximates W_t.

So what is the approximation, the approximation here (see the pic).

So, if I take this to be the case, this to be the approximating the simple integrands then we can write this as $\int_0^t W_t dW_t$. The next steps shown in the pic.

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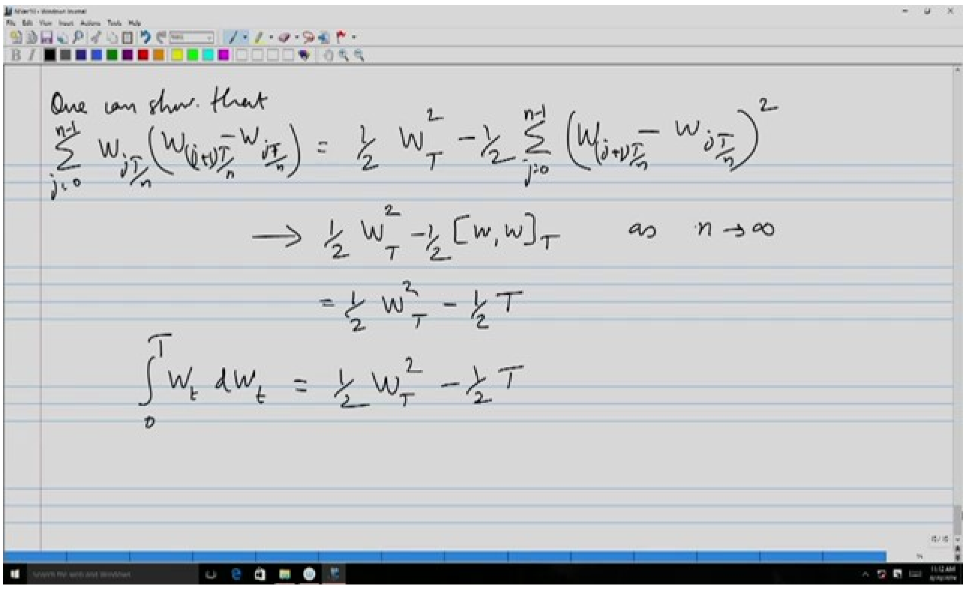

Now, one can you know by simple algebraic manipulations one can show that the quantity $\int_0^t W_t dW_t$.

So, what you can do is that you can consider the quantity in the right hand side this summation and you try to express this in terms of some algebraic manipulation, if you do you can show that this quantity is equal to this. Now if you let the limit tends to because this quantity now you have said that this is equal to this, to the right hand side.

Now, this right hand side quantity, you can see that this tends to $(1/2)W_T^2 - (1/2)[W,W]_T$ as n tends to infinity. This is equal to again $(1/2)W_T^2 - (1/2)T$. So $\int_0^T W_t dW_t = (1/2)W_T^2 - (1/2)T$.

Now, contrast this with your ordinary calculus where this W suppose for some function g which is differentiable function with $g(0) = 0$ you will see that this will be only the first term on the right hand side is what then you would get. And the second term is from where this is coming. it is because of the positive quadratic variation that the Brownian motion process have and as you see there is $(1/2)T$ is essentially is coming from the quadratic variation of this W.

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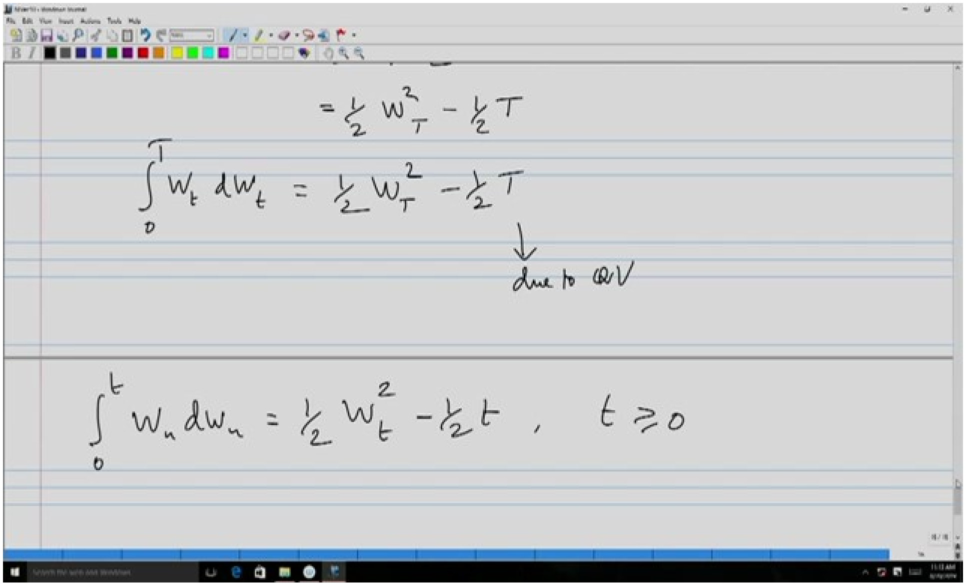

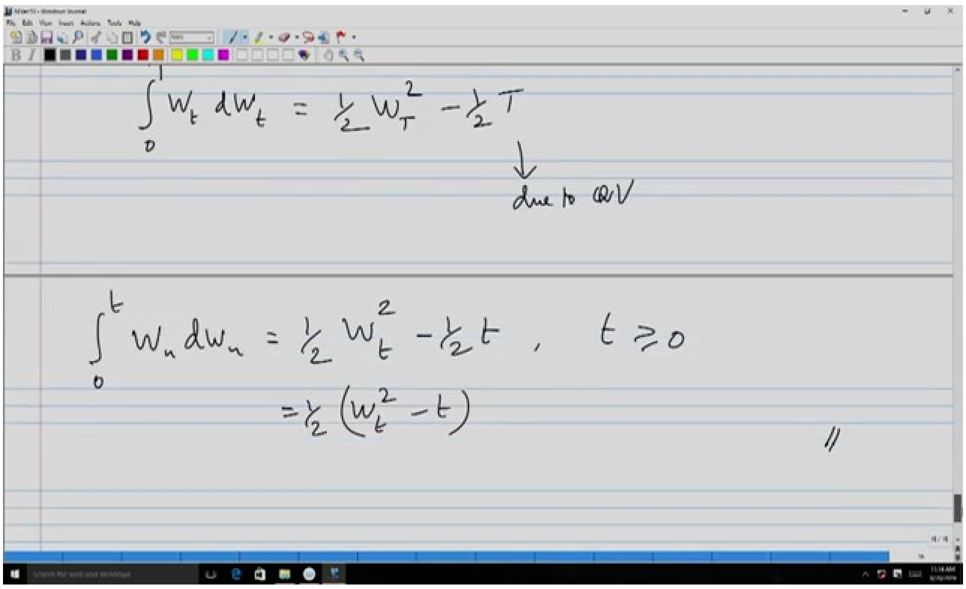

So that is what gives rise to this term, And this is you know true for the case when you have the integrals defined in the Ito s way, so this is basically due to quadratic variation, if this not the case, the ordinary calculus where W_T, I mean generally if you take to be differentiable function then you will see $\int_0^T g(t)dg(t) = (1/2)g^2(t)$ that is what you would get. But here this minus half term T is essentially this coming from the quadratic variation term, alright.

So since this the upper limit can be arbitrary so we can define or we can write this $\int_0^t W dW_u$ to be half of $(1/2)(W_t^2 - t)$

So to make this case then you are actually $E[W_t^2] = t$ and this is what the case, so these is an example that you can see which comes and this is what the difference between the ordinary calculus and stochastic calculus that the quadratic variation of the Brownian motion plays a crucial role in determining what is this quantities are equal whereas this might vanish.

This is true only with the split Ito integral where you know are evaluating the for a function at the left most point otherwise you know this half minus half times t might not appear, which is not true for in general for stochastic integral but it is true for an Ito integral that what we have it here. So this is what all about the Ito integral and its properties and when we the next lecture you know we will talk about the Ito's formula. Thank you, bye.