Mathematical Finance

Lecture 27: Brownian Motion and its Properties

Professor N. Selvaraju[1] and Professor Siddhartha Pratim Chakrabarty[1]

[1]*Department of Mathematics, Indian Institute of Technology Guwahati, India*

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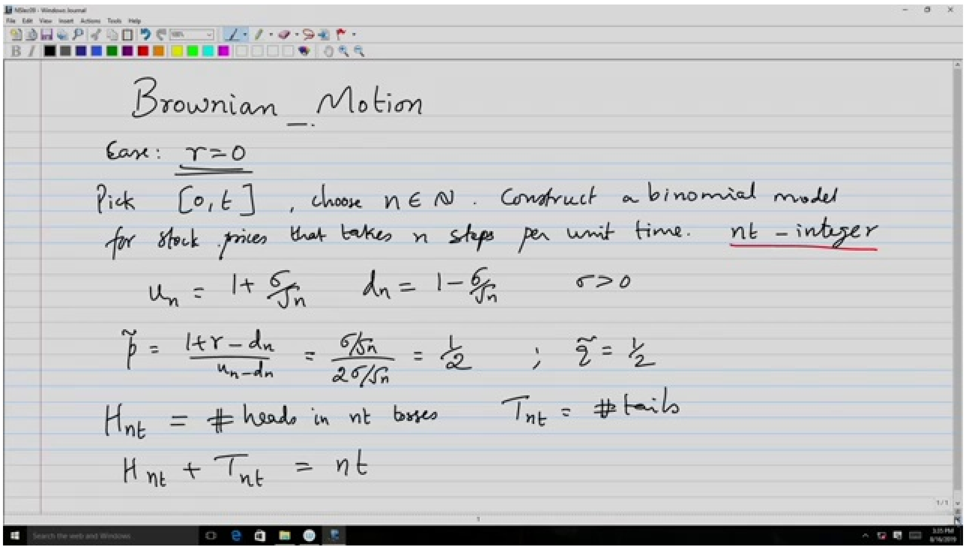

So, before we actually define one can straight away start with defining and then looking at its properties because that is what we need as we go ahead for you seeing in stochastic calculus as well as, as a modeling for financial asset prices. So let us see how this browning, why this particular thing Brownian motion came into Πcture. So this, the intuition behind is what we will try to give and so that is also, I mean from another angle, if you look at it, it also show that such a stochastic process exists. Now to give this intuition, let us consider and go back to our binomial asset-pricing model where what we did. So, we consider a binominal sub pricing model wherein the risky asset follows a sort of a binomial tree as time progresses. So that is what to be keep in mind, you recall.

We will now see that if you scale that binomial model in a particular way, then you are going to end up with a model, in which involves the Brownian motion, so and hence we will be able to relate these by binominal model to the geometric Brownian motion model, which we are going to use it as the model for the asset prices in continuous time finance.

So, let us see how this comes. Now the case that we consider is basically $r = 0$ but $r > 0$ can also be done in a similar way. For simplicity we just take the case $r = 0$. And what we do, you Πck an interval, say 0 to t and choose an integer and construct your binomial model for stock prices that takes n steps per unit time, so, recall binominal model we set time 0 time 1 time 2. So, now what we are going to do either between 0 and 1, we are going to have n time points and each of this would now become the new time points.

As if you look at the original binomial model. And we are going to increase this number of steps in the to the limit so that you know we are looking at the stock prices at every time point on the time access rather than it is 0,1 and so on. So how do we make this continuous time analog is that you divide stop looking at 0 and 1, you look at 0 half and 1, 0 and 0.1. 0.2, 0.3, 0.4 and 1 and then you keep on increasing the number of points. Then you get to the limit, the continuous time code, so that is way, we approached any modeling from discrete to continuous when we make a movement.

So, that is precisely what we will do. So now what we could consider, we could consider an up factor for which we would make it as a dependent on n,

$$u_n = 1 + (\sigma/\sqrt{n})$$

and the down factor

$$d_n = 1 - (\sigma/\sqrt{n})$$

and r we have already assumed to be 0. So, this satisfies the no arbitrage condition for the binomial model. Now, once we are given this, then the risk neutral probabilities, $\tilde{p} = \frac{1+r-d_n}{u_n-d_n} = 1/2$ So that, implies, my $\tilde{q} = 1-(1/2) = 1/2$.

Now, recall the asset price in a binomial model. The asset price time at n is determined in terms of the initial stock price S_0. And the number of up moments and number of down moments in the stocks, that essentially means the result of first in rn tosses in if it is n period model, now so this n and t, we also have this additional condition that we Πck nt to be an integer. So, if not anyway there is nothing is going to be lost but this for convenient that we Πcked in this way.

So, we will take far easy understanding that did the nt to be an integer to be Πck n and t has that this is true. Now, that has defined few quantities. So let us call so basically if you are looking at t time unit, 0 to t and each time we need has the n time points, then total number of time units are number of tosses that you will associate with the such binomial model would be nt number of tosses. So out of those nt number of tosses, this H_{nt} defines the number of heads in (nt tosses) nt number of tosses and equivalently T_{nt} will be the number of tails. So, what you have

$$H_{nt} + T_{nt} = nt$$

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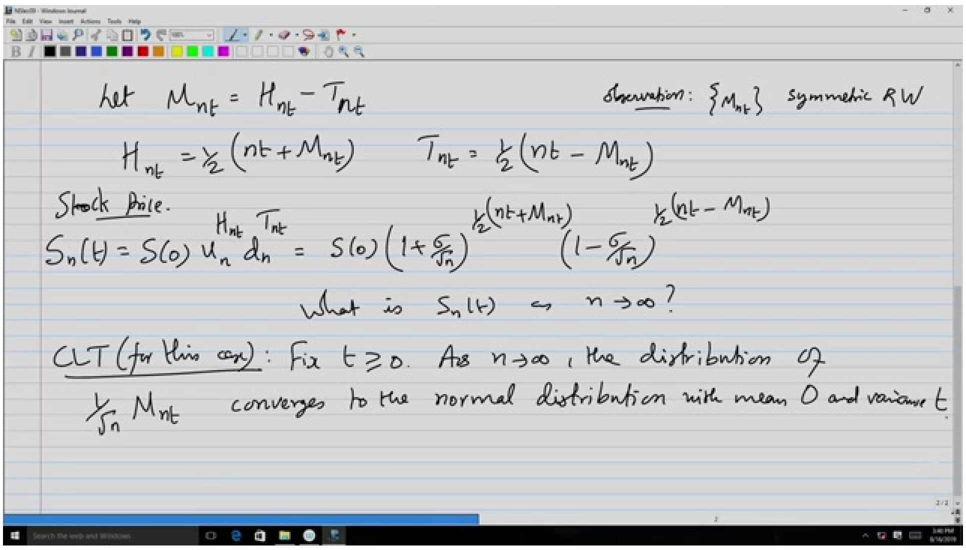

This is what the new root end up with this, also let some M_{nt} to be the differences $H_{nt} - T_{nt}$. One can also identify the M_{nt} to be a symmetric random work process. This is you know, observation, which if you associate with an up moment of unit size one for each of these heads and tails for minus 1, then you can observe that this is essentially M_{nt} in general, you can associate a symmetric random work, people who do not know what this is, it is another simple stochastic process but we do not need that as such, but we just need this to be some quantity this.

Then, I can see by solving these two equations, I can write my H_{nt} I can write in terms of nt plus M_{nt} and T_{nt} to be half nt minus M_{nt}. Now, if I look at the stock price, which we write in general, S_{nt} to be s at 0 and un to the power H_{nt}, dn to the power T_{nt}. So this is initial price multiplied by the un to the power number of heads and our up moments and dn the number of down moments multiplied this is what would be the stock price.

Now un, vn I know H_{nt} we know, so we can express this as s0 times 1 plus sigma over root n to the power half nt plus M_{nt} and 1 minus sigma over root n half nt minus in M_{nt}. Now the question is, like, if you want to look at the limiting cases of this stock prices as n terms infinity, we were looking at what is this quantity is going to be as and n tends to infinity. So, that is what the case here.

Now, for this to look at what is S_{nt} as n tends to infinity, you got the question. Now to get the answer, so we need a result from probability the theory, which we already said about the n central limit theorem and CLT for this case with appropriately defined a M_{nt} that is the following. You fix t now as n tends to infinity the distribution of 1 over root n, M_{nt}. So the remember this we are looking at it as if we are assigning an addition to this process, having 1 for every up moment and minus 1 for every down moment. Then you are now scaling it to this. That is what we are looking at it.

So, it is called the scaled symmetric random work in this particular case, so the distribution of this, which is a scaled symmetric random work converges to the normal distribution or Gaussian distribution, whichever way you go with mean 0 and variance t. So this is what the central limit theorem, a specialized to this particular sequence of a random variables that we have here a M_{nt}. So, I mean, you are taking it that this is what is happening of what appropriately defined M_{nt}, whichever way you will look at it. So this is the result that here.

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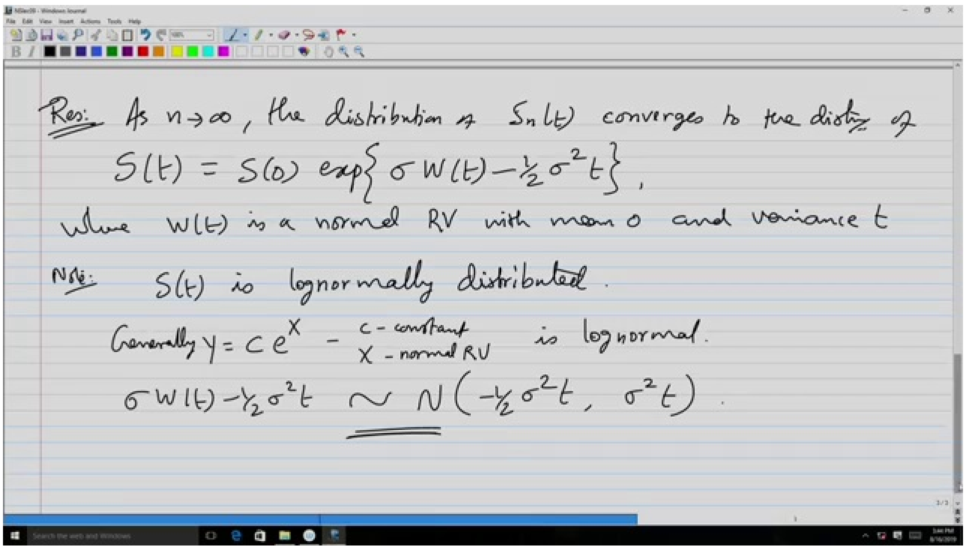

Now, if you use this result, if you use this result then we have the, the result for questioning that what happened to this S_{nt} as n tends to infinity is the following. Now as n tends to infinity, so the distribution of S_{nt}, (S_{nt} as given above,) converges to the distribution of S of t, which is

$$S(t) = S_0 exp\{\sigma W(t) - (1/2)\sigma^2 t\}.$$

where this $W(t)$ is normal random variable with mean 0 and variance t. (So this is what we can show and this is what is called as the) that means that S_t is what we say is log normally distributed, so then the limit, what we get is that the distribution, (this distribution limit) distribution of this convergence to the distribution of this S_t.

Now this S_t is given by this expression, which is in terms of this w where w is normal, so anything of this form is what is called the log normal. If you want to see generally, so any random variable after form, some ce^X, where c constant and x normal random variable is Log normal which means this y in this form. So if I take my log of y, that will be, given x, so x is normal. So log of y is normal and hence y is log normal that is how you know we look at here. So in this particular case, the quantity in the exponent, the $\sigma W(t) - (1/2)\sigma^2 t$. This is a normal distribution with mean this and variance as described above. So will use this notation to denote some random variables, which is distributed as normal. The first quantity would be mean. The second quantity would be the variance of the normal random variable.

Now, while going forward so we will need some properties because now this normal t as come this $W(t)$ is the normal and $W(t)$ is what is going to be the Brownian motion component that we have. And this is what we call it as the xe to the power something. This is what is called as geometric Brownian motion. So, S_t is given by a geometry Brownian motion is what our continuous time model would be. Now while working with that so we need again some note, so what we call this mgf of normal random variable which convenient in many places in our calculation that we will use. So lets us give it us as an expression. If x follows normal with mean μ and variance σ^2, then the mgf of X, which we call it as the $M_X(t)$, which is equal to $E(e^{tX})$. This is one of the you know, important expectations formulas that we might use and plus $(1/2)\sigma^2 t^2$ for all t and r, is what an mgf may not exist as we know.

And in this particular case, the mgf will exist for all t. So this is what it is. You know, what is the meaning of mgf that it is if x presents a power series in terms of t square, the coefficient of the I can obtained the moments of this particular random variable. So this expression, for any random variable X with normal μ-σ^2 square, the $E(e^{tX})$is equal to this expression. This expression, we may use it multiple times in many situations. So let us give it as this. So now you see this w_t that they are getting as the limit of this S_{Mt} is what basically to be our Brownian motion.

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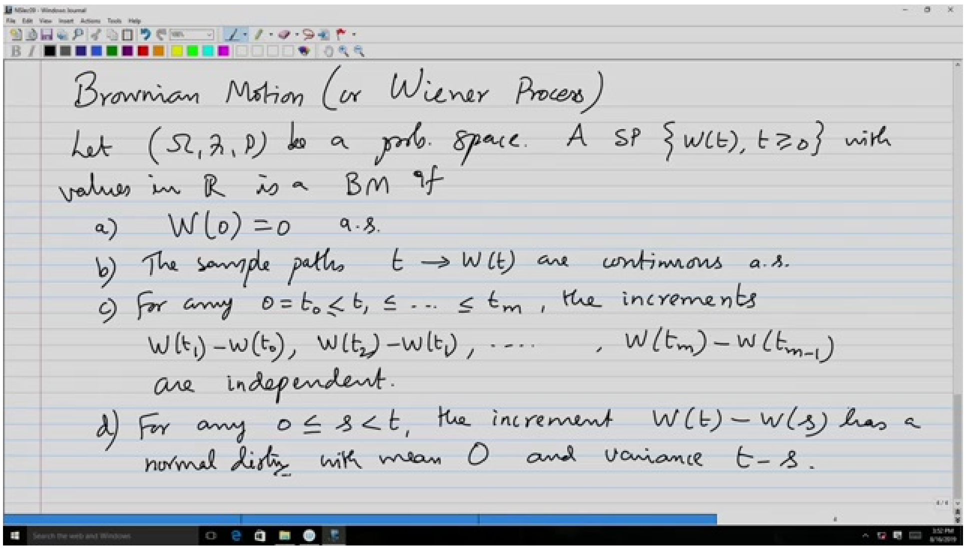

So, its properties also again from this risk random work place like properties also it inherits and those properties defined Brownian motion. So now le us precisely define what we call, A Brownian motion or another name is wiener process, though it is Robert Brown who observed this process and discovered as stated by Einstein. It was Wiener, who gave or derived the mathematical properties. So it is also equivalent called as Wiener process or Brownian motion both would mean as we are as far we are concerned one and the same.

Now, what did this process, so let us precisely define. So like any random process that you have so you have n underlying probability space on which the random variables are defined how stochastic process which we call $w(t)$ here with values in r which means the state space is r, is a Brownian motion, shortly we may call it as in short by bm. If it satisfies these 4 conditions that we are going to list.

a) $W(0) = 0$ a.s. W at 0 is 0 almost you listens even if you do not write almost you listen, it would be so fine if you understand what this means. It is also that little bit, probability one this is true.

b) Now the sample paths, which is as a function of t. If I look at this process or continuous, almost surely again.

c) for any time points that you Πck here, the increments, so, the following are, what we call it as the increments of the process over the sub interval, which means the state change process, in the sub interval points, is what then call the increments. So these are independent. So we know when we say some n random variables are independent, it is precisely that quantity.

d) last property for any $0 \leq s \leq t$, The increment, which is $W(t) - W(s)$ has normal distribution with mean 0 and variance $t - s$. So this essentially means that if I take increments in this process, the increments would also be what we call stationary increments, in only it depends at the length of the interval, not on where you place this in interval s to t.

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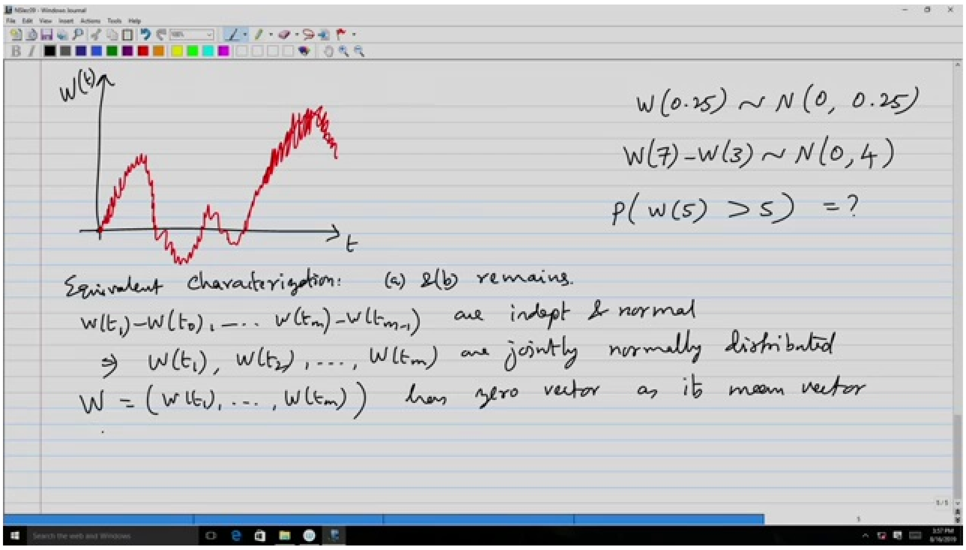

Now, (if I look at) actually we talked about sample paths, so if I look at a tyΠcal sample path, so where in the x axis I have t and then the y axis I have $W(t)$ (so I not yet used so these things). it is w of t it should start at 0. So, let me start with 0 and it will at each time point it will go up or down, but tyΠcal process, you know if we look at it, will have these kind of behaviors, is what then you would observe.

So, these oscillations that you know it can be a little bit more into this form also, so this is how it will look like into the tyΠcal sample path. So, this is what we call the sample path is many as time goes, what the realization before time t you know it is random but when it comes to time t, then you know what is the value of the process and hence you have this realization, other sample paths that you have here.

Now this definition also gives us the various quantities that you can compute with respect to this Brownian motion. If you are interested in those, say for example, if I look at W at 0.05, if you want to look at its distribution, you can see that it is 0.25 is the variable. Suppose if, I look at $W(7) - W(3)$, I know that this is distributed as normal 0 and with variance 4, (so this is what) and any calculation with respect to this W since you know it is normal distributor say bias down to the calculations connected with the normal distribution as far as the distribution properties, probabilities with respect to Brownian motion is concerned.

If you want to know what are the probability that at time 2, say for example, this quantity suppose if I want to see a W_5 is greater than 5, suppose how do it compute because I know its distribution so I can compute these probabilities and so on. So all these calculations you can do (with respect to) by using the normal properties that you here.

Now for our convenience and the ease of use a little later. So we can also have and equivalent way of defining our equivalent in terms of characterization. It is not equivalent characterization really it is characterization. So it is also equal and to that definition characterization and in this path a and b remains true. Now the c and d could be replaced by the following because the increments or say $W(t_1) - W(t_0))$ and so on, $W(t_m) - W(t_{m-1})$ are independent and normal, this implies that this quantity that random variables, which are Wt_1, $W(t_2)$ and so on $W(t_m)$ now these are independent and normal.

Now you can see Wt_1 the same as this.Wt_2 If I have to add, I can express this in terms of the first 2 in those a previous line that we have written as the increments. So you can express it as a linear combination of these independent normal. (So this will be) these are then jointly normally distributed and this vector w which is essentially Wt_1 and so on,Wt_m has 0 vector because each of the random variable has 0 mean as its mean vector.

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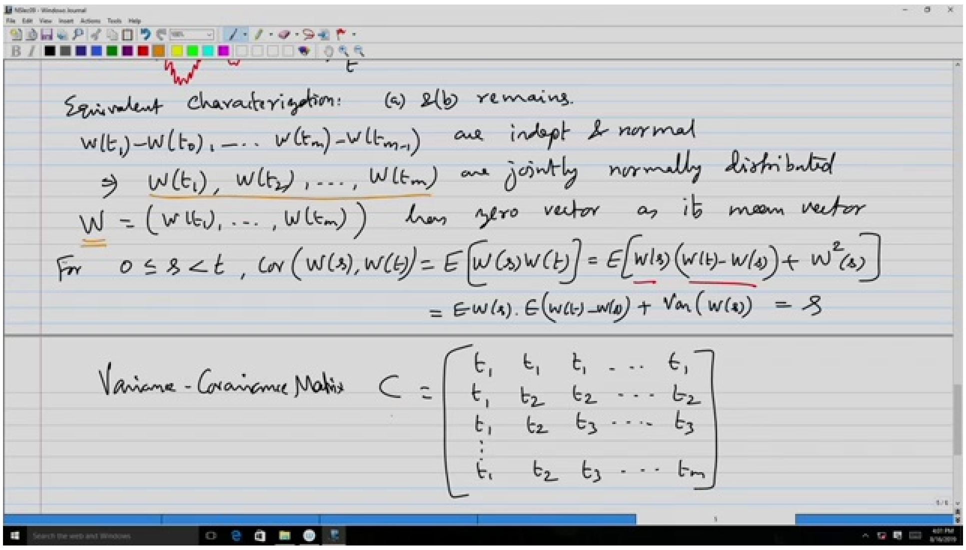

Now, we only need to know the variance covariance metrics, for which we just compute, you Πck $s \leq t$ and look at covariance of these two quantities. W at s and W at t, which is because mean, is 0. It bias down to this quantity because this is what is now the covariance because means are 0. (Now this you can write it as). So, these are not independent and so on. So you cannot, you really have to evaluate, but in a convenient way you can write this as ws and wt minus ws plus w square s.

So what I do, add and subtract ws to this wt and then leave this. Now since expectation operation is linear, I can separate this and the first quantity if I look at it, this two, so this and this are independent. So this means that I can write this quantity as expectation of ws into expectation of wt minus expectation ws plus expectation of the ws square is again, because w expectation ws is 0. This is nothing but the variance of this. So which is essentially variance of ws and these quantities are 0. So this quantity is s, so this is s. Hence, for $s \leq tt$

$$Cov(W(s), W(t)) = s$$

Which means the minimum if for arbitrary s and t, $Cov(W(s), W(t)) = min\{s, t\}$. So, if I use that now by variance covariance metrics, which is recall c as per the earlier, so this is nothing but the matrix described above.

So, which means that what is the equivalent characterization now without writing a and b remain which means process starting at 0, process has continuous paths we have, instead of saying that the increments are independent and the stationary and it has now distributed. You can say that these particular random variables, this set of random variables are this vector W has 0 mean vector and variance covariance metrics as given by this C, then that process is a Brownian motion process. This is equivalent characterization which we might use in some places.

Of course, we may have some more equivalent characterization which will come will later depending upon the other property. But this interchangeably one can use, if you want to prove some processes Brownian motion, either you can use the basic definition or this equivalent characterization. In the same way, any other equivalent characterization if it is there that can also be used to prove that this process is Brownian motion.

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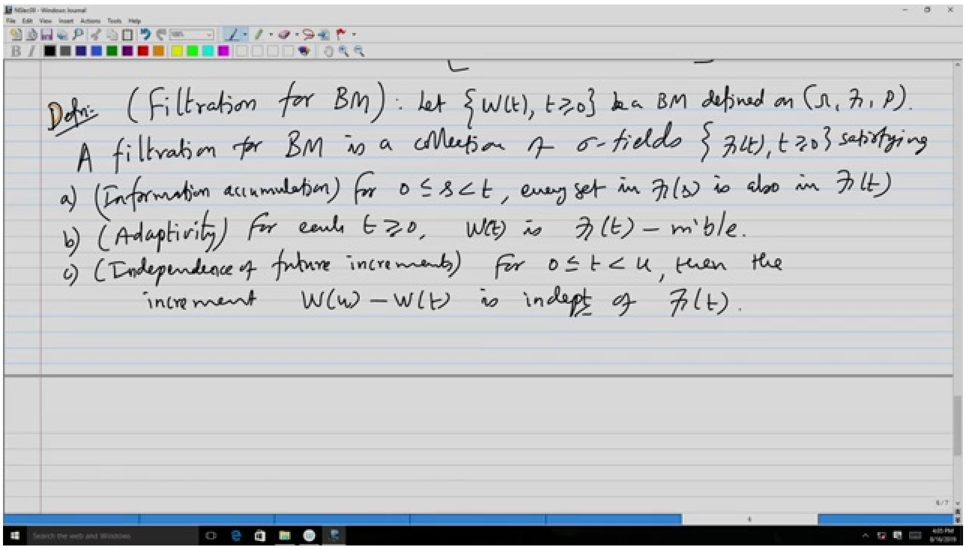

Now, connected with this Brownian motion, as we go along, we will also need another concept, which is called the filtration for Brownian motion. So, this is slightly different from the general filtration that we talk about. In a moment you will see what is it. So you have w of t, which is a Brownian motion, which is a defined on some probability space. Now a filtration for Brownian motion is a collection of sigma fields say $\mathcal{F}_t$, for $t \geq 0$ and that satisfies 3 properties.

A) the first one is called information accumulation the usual first condition that you require here, which is for $0 \leq s \leq t$, every set in $\mathcal{F}(s)$ is in $\mathcal{F}(t)$, which means that at time s and time t if you look at this as information, then the time t information is at least as much as what you had at the information at time s. nothing is lost from when you move from s to t, information only accumulates.

B) The second is adaptivity, which means that what, for each t, $W(t)$ is not yet measurable, which means the information in $\mathcal{F}_t$ is such that we can determine what is the value of the process at time t, so that is one $W(t)$ is $\mathcal{F}_t$ measurable (this is also should be this notation).

C) Independence of future increment property. That means what, you Πck you know, t and u such that it is greater than t. Then the increment, which is $W(u) - W(t)$ is independent of f of t, means the information in the filtration and information especially a time t the sigma field, which is the $\mathcal{F}_t$ should give no clue about any future value of the process w, which is the Brownian motion. So which means this increment should be independent of this ft.

So, in the normal, in general, filtration only partly you might call simply as filtration. But whenever we say a filtration for Brownian motion means that it is connected with a Brownian motion in some way or other, then that means these three properties would be satisfied. That is what you call a Brownian motion a filtration for a Brownian motion.

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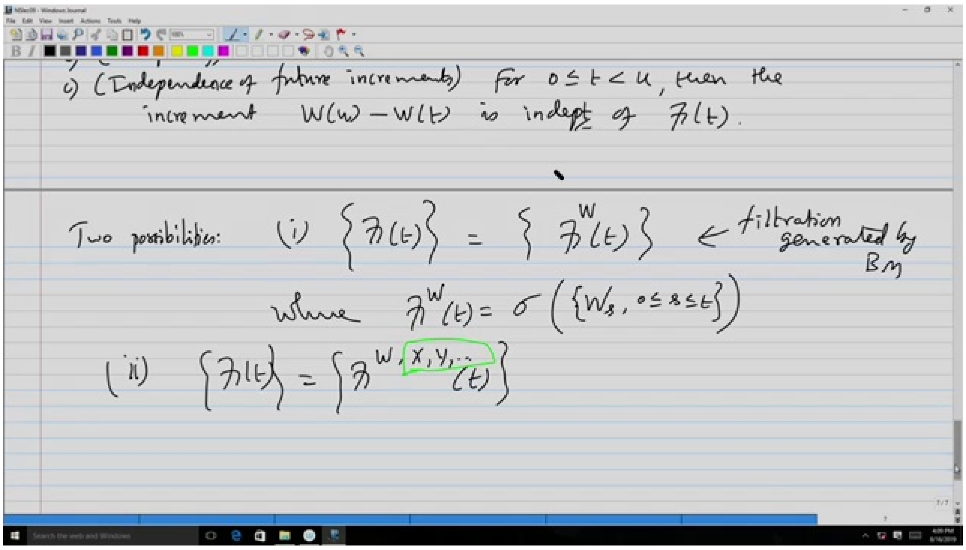

Now, if I Πck any, a random process, Δ_t and if I say that that process is adapted to this filtration, which means that process is measurable with the respect $\mathcal{F}_t$. So that is also true with whatever you have it here. Now how do we construct such a filtration? There are two possibilities, which normally happens. One is, you know, you take this at a time are this filtration to be the filtration generated by the process itself. Where you know what we mean when we say $\mathcal{F}(t)$ of W, we call the $\mathcal{F}_t$ is the sigma field generated by some ws this collection, this is what we have.

So, this means that the filtration has exactly the same information as that of the one that you might have by absorbing the Brownian motion up to time t. In this case we say that the filtration is generated by the Brownian motion, so this is $\mathcal{F}^W(t)$ is nothing but the filtration, generated by the Brownian motion is what then you have it here.

Or the second case could be you could have $\mathcal{F}$ a general filtration with these properties. Something like, you know, the filtration could be not just W, X, Y and so on. So many processes are also might be there to generate this, but they should satisfy the condition that we have, which means that this filtration, when we take this as a filtration for w the additional quantities that you know you have here, that this x, y and these things again should give no clue about the future behavior of this w then only this will become filtration.

So this filtration could be simply generated by W or more than one or more of the processes apart from w also can generate such a process, but what you have to keep that in mind you the additional process that you are using in general while having this Brownian motion for the filtration, for this Brownian motion, again should not give a clue about the future behavior of this w that is what you know, one is the keep it this case.

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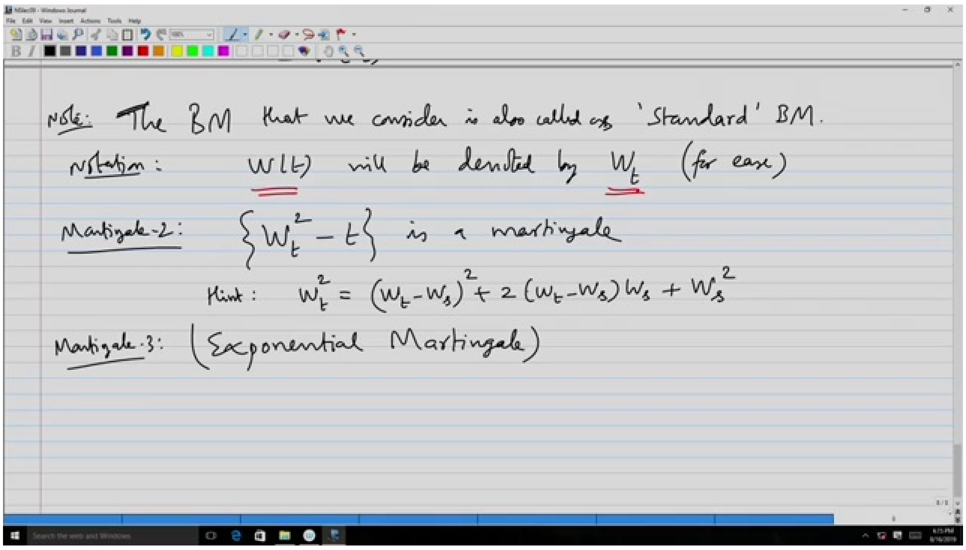

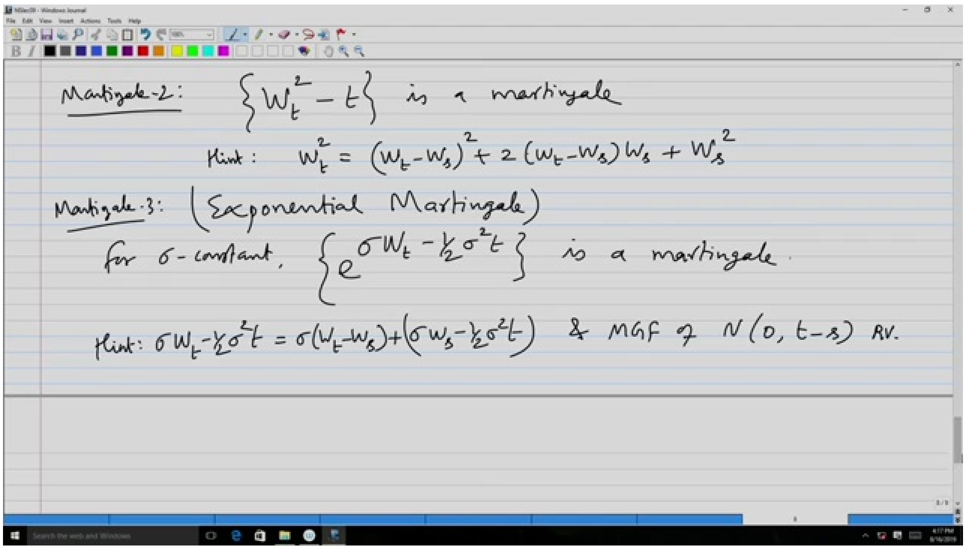

Now once we have the definitions, then we look at certain properties of this, Brownian motion and especially the two properties or other three properties have this Brownian motion which are important one is the martingale property, the other is the path properties which we had a glance as in the quadratic variation property, third is the Markov property.

Now let us take first case as the martingale property or martingales connected with this. The result is that the Brownian motion is a martingale. So the Brownian motion means, like what we have used so far is what this part is essentially. Now, let what you have to prove, so you have to prove, if you Πck these quantities be given, then the Martingale property, then you have $E(W(t)|\mathcal{F}_s)$, again, you see when you look at this, $W(t)$ is in neither measurable with respect to $\mathcal{F}_s$ nor independent of $\mathcal{F}_s$.

So really you have to evaluate this so what you can do, you can add and subtract something as some up manipulations you can do so that you know, you can use the properties of conditional expectation to simplify this process. So, what we do is, you subtract and add given $\mathcal{F}_s$ now so this quantity you can see is independent.

First of all, you can use the linearly property to extract this as the two conditional expectations some of two condition expectation. The second property that you would use is that this $W(t) - W(s)$ is rather, you know, independent of $\mathcal{F}_s$ and $W(s)$ is $\mathcal{F}_s$ measurable. So you will use all the properties to end up with simply thing, this is independent of that. So this is simply wt minus ws plus this is measurable. So this is $W(s)$ and this quantity is 0. So we end up with ws. So this is true for arbitrary s and t.

So, what is the property that we have shown. This is the Brownian motion, sorry martingale property or Brownian motion is a martingale property. Of course, for martingale, other properties are there, which are you know the things who takes pay extract and testing. So you have this filtration and everything so the other conditions are to be satisfied, only the conditional property we need to verify, which we have verified and which we have seen that this is true. So Brownian motion is a martingale. So what we have here. By the way, before we go further, there are notes. So, the Brownian motion that we consider is sometimes also called as (inconveniently) standard Brownian motion.

So, that is what we mean whenever we say simply Brownian motion without getting objective. Of course this essentially mean it contained with the standard normally in some sense 0 1 instead of that, if I put some other then I am going to get a shifted and scaled Brownian motion or with drift or in other cases that you see so that whenever we call, will call that specifically. The other thing is the standard Brownian motion.

The other thing is a notation wise, again, you know we will use from the further you know for this lecture purpose alone. So W_t will be denoted by w, I use subscript t for ease, there is nothing else other than this but it is actually, it should be mean that this, so that you remember you know, little neat when we are writing this. That is only reason we are doing this other than that, there is nothing.

Now the simple Brownian motion, a simple standard Brownian motion is a martingale that we are shown and there are two other martingale get connected with this, which importance to us. Another one so this is a martingale 2 is what $W_t^2 - t$. This process is a martingale.

Now, if you apply your, this property, then you will be able to show that immediately this is a martingale. The other martingale, which is martingale 3, which is also known as exponential martingale, a form of this is already you have seen earlier. This is 1 martingale, the other martingale is exponential martingale is basically if I pick a constant σ, if I look at this process $e^{\sigma W_t - (1/2)\sigma^2 t}$ e to the power sigma of wt minus half sigma square t, if I look at this, this is a martingale.

Now you see this whenever in you want to show this martingale property, this will be independent of $\mathcal{F}_s$, this is $\mathcal{F}_s$ measurable you can pick it up and then this will become simple expectation e^{σ} of this, which again you can use the MGF of a normal 0 t minus s random variable. We have already seen and use that expression then you will be able to show that this is a martingale too. So this experiential martingale also will play a crucial role in the analysis which when you go forward. So this is what is the first group of property which we call the martingale properties.

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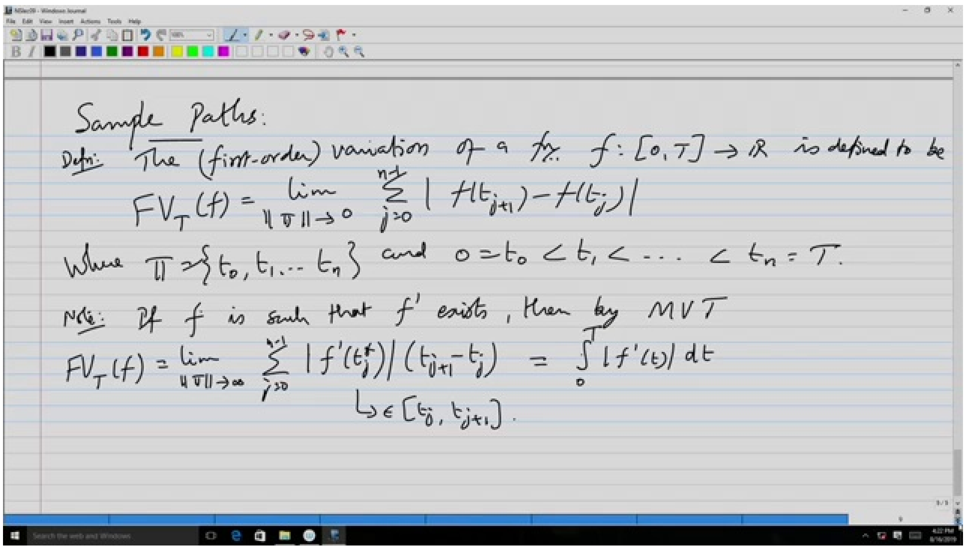

And now the go will go to the second group of properties, which we will call the sample path look at this sample paths. For any stochastic process the sample paths means a realization, what you have actually observed. Now for this, now let us look at the properties for which we start with a simple definition, which we call the variation or the first order variation. simply sometimes it is called as simple variation, but in this particular case we really mean the first order variation of a function say f, (which we are defining in) is defined to be because this has some first order variation of the function in 0 to t of the function fs, limit tends to 0 of J is equal to some n minus 1, $|f(t_{j+1}) - f(t_j)|$, where this Π means this partition of this in trouble and this actually speaking like instead of this limit none of Π tends to 0. One should say that it limsup but anyway for the some should be the exact things so which we can still take lim here and then we can describe it rather than making it a little easier that is it.

Now, this is what we defined to be the variation. Now if we imagine what this tells me is, that if you look at here you are having interval 0 to t you are partitioning this interval into some endpoints or end sub intervals and in each of these sub interval you are actually looking at the absolute value of the change in the function value, which means how much the function has moved, so you are making absolute value not up moment alone or down moment alone or you are neither subtracting.

You are just adding the changes that happens in the function. Now as you increase the number of partitions where the normal partition means that the maximum length of this, any of this sub interval tending to 0. This norm of Π means exactly that as we seen earlier. So as you know, you increase the number of points more and more. So, it is really going to evaluate the function, looking at the function and it is how much this has you know made moments the amount of moments that you have, amount of variations of f during $[0,T]$ is what this variation function we will give.

Now, you can note easily that, if the function is such that the derivative exists then by MVT which you would have observed in ordinary calculus theory. That is the mean value theorem, by using the mean value theorem, you can see that this particular variation can also be j_0 to $n-1$ this could be written as some $|f'(t_j^*)|(t_{j+1} - t_j)$, star is a point in the sub interval $[t_j, t_{j+1}]$. This essentially in the limit, this will be equal to, because this exists to so this will be equal to $f'(t)dt$ is what then you get so where, this belonging to $[t_j, t_{j+1}]$. This is what you would see. Say essentially first order variation is nothing but this quantity is what the you can observe.

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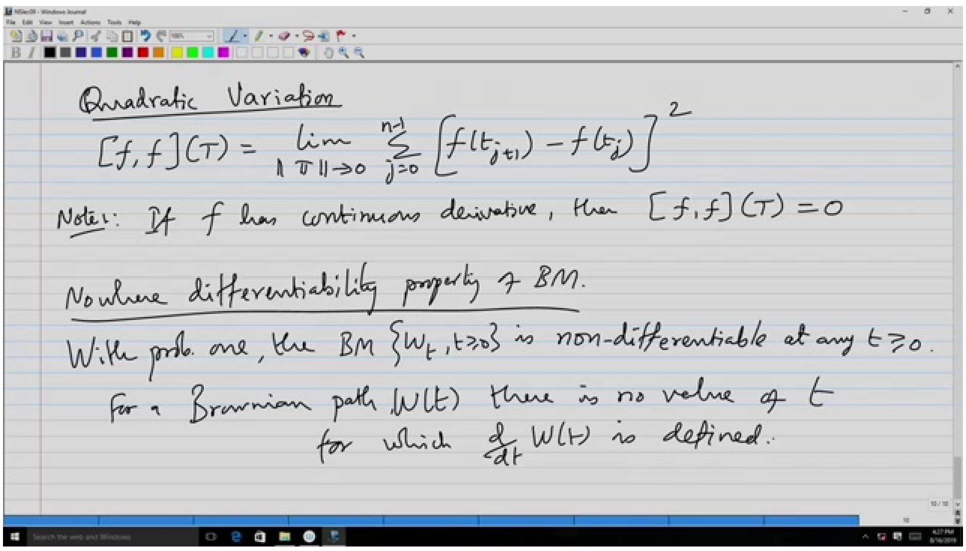

Now, (there is another) this is the first order variation and similar to that you can also define, what we call, a quadratic variation. Let me go to the next page itself. What we call quadratic variation, as suppose to the first order variation, we also deal with, everything remained the same the function and the partition and everything. Only thing is the quantity that we defined, is this one. Whatever condition that we had earlier for first order condition that remain the same, but this quantity now will be given by f of tj square first order variation, we limit absolute value of this function and the difference in the function, at the subsequent points in the partition.

Now, we are looking at the square difference. So this is what the quadratic variation. And it can be shown easily that if f has continuous derivative, which means, in the first case f' exist we assume, now f' is also, in addition, continuous, then one can show that this quantity is 0. And this is for functions that we encounter in the ordinary calculus. Most of the time it so happens that you know the function has continuous derivative and the second order variation is a quadratic variation, is the second order variation, is a quadratic variation 0.

And hence, we really did not bother too much about this quantity, the simple ordinary calculus. But of course there are also, you know, you would at sometimes want to look at that. But in stochastic calculus this plays a major role. precisely because what we are going to see next is a property. and that is because of, that is quadratic variation is important in the case of stochastic calculus because the function that we have now taken up is the Brownian motion and the Brownian motion, one important property is the nowhere differentiability property of Brownian motion, which we take without any proof because the proof is involved.

What we say is the, with probability one the Brownian motion, which is W_t is non-differentiable at any $t \geq 0$. So what this mean is that if I take a path of the Brownian motion, you know we can go back and see the sample path that we had. We had just this up and down moments, very sharp moments that we have. So all those sharp points, as you know, like for example, $|x|$at the 0 is non-differentiable so because it has a sharp edge. In similar way, these are all have you know, sharp edge paths, path is full of sharp edges, which make no point of time you know, you will be able to differentiate. That is what it means, which means that for Brownian path say w of t there is no value of t. We are saying all in the almost surely since. So for which this is defined.

So, this is done. so, which means the first order variation, second order variation that we have noticed that f' exists here, for W_t that does not exist and then no question of continuous everywhere property also, So this whole things will break in the ordinary calculus thing. And this will break in this stochastic, which was there in the ordinary calculus. you break in the main property is this nowhere differentiability of the Brownian motion paths, which you are taking it granted because the poof is requires a lot of effort and of course those who are interested you can always look it up any book is not a major issue at all.

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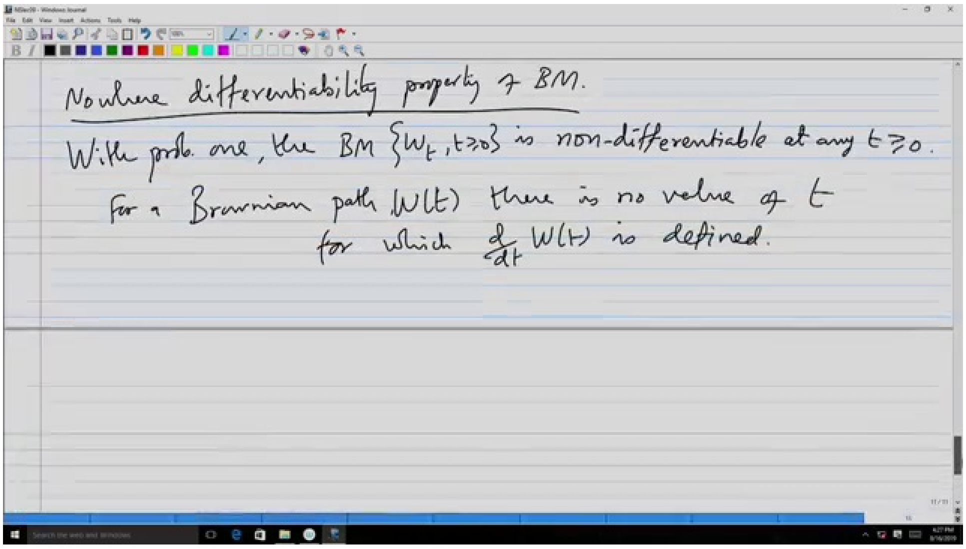

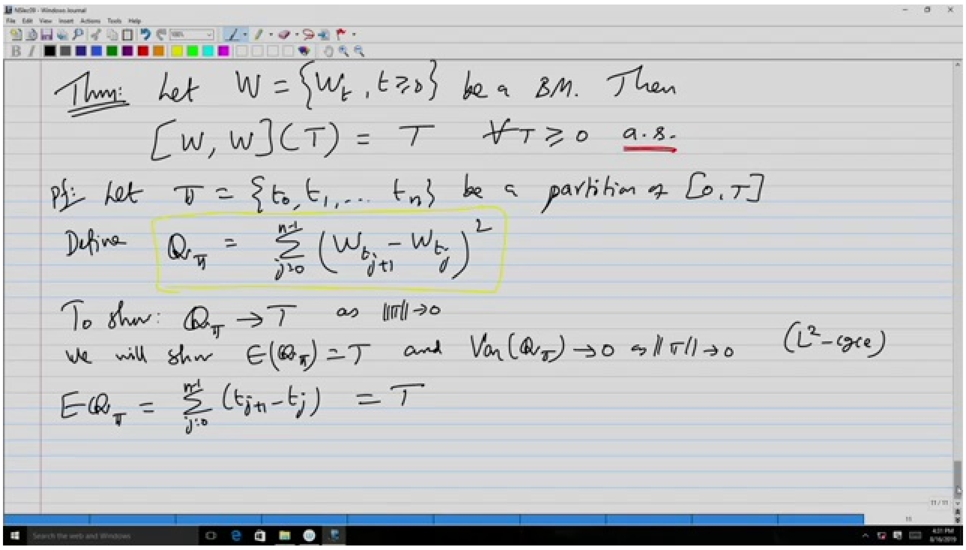

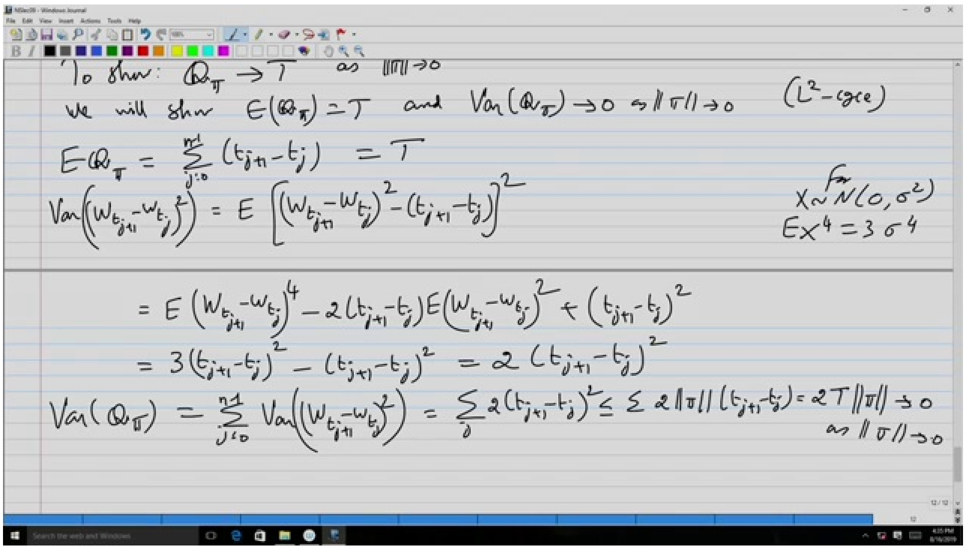

Now, what is the main research that we have, before we look at the first order property, we look at the quadratic variation property. That is what the major property of this Brownian motion paths as a sample path property. Now let w as usual, this is what it is, be a Brownian motion then t for all T non-negative almost surely.

So, the sense that we talk here is almost surely but you know in a moment you will see like we are actually not proving this but we are proving something slightly different but this can also be proved using advanced results stochastic calculus. Now so basically what we want to show is the quadratic variation process and the quadratic variation of this, is equal to t. Now let us take a partition t_0, t_1 and so on t_n be a partition of the interval $[0, T]$ and we also define the sampled with respect to this partition the sampled quadratic variation for this.

Now, what we want to show, to show $Q_\Pi \to T$, almost surely. That is what we want to show. Now, how do we show ? This is basically as tends to 0. this is in the almost surely sense we want to show and this is tends to 0. Now what we will show, we will show the expectation of Q_Π is t and variance of Q_Π is tends to 0 as norm of this tends to 0, which is actually L^2 convergence, but a technicality apart.

It can also be shown that this is true. So whenever this convergence takes place there is a subsequent along which the convergence almost surely and hence, you know it can infer from that. So we will not worry about that we just show this, you know, we will say this, essentially what we are proving is to L^2 convergence. so this is L^2 convergence, this is for only technically people oriented, otherwise you do not worry about that, so you see that it is accumulated in this manner.

Now first we will show what the $E[Q_\Pi] = T$. , If you look at it, it is what. so if I take expectation of that, that means the variance of this. So that means this is 0 to $n-1$, the variance of each of these quantities are all independent. so expectation of this Q is simply this, expectation of this one only we are not variance that you are liking about. So this expectation of this, which is variance of the increment, which is $t_{j+1} - t_j$, because you know the length of the interval and if you sum you will get T quickly.

So, there is nothing a great other thing complex to show that this part is true. But for the other part, which is the variance part, you know first we will look at the one quantity inside the sum, which is essentially variance of $[W_{t_{j+1}} - W_{t_j}]^2$. This is what we look at it, so this means, expectation of $[W_{t_{j+1}} - W_{t_j}]^2$ minus its mean and the square, is what this quantity. Now you can again expand, expand and write this as expectation of W_{t_j} plus 1 minus W_{t_j} to the power 4 minus twice $t_{j+1} - t_j$, W_{t_j} plus 1 minus W_{t_j} to this square plus tj plus 1 minus tj, the square. Now here we we are using an important property for normal x follows normal 0 sigma square. Your expectation of x to the power 4 is 3 times sigma to the power 4, this is one of the properties of normal random variable.

If you use that, then you get this as 3 times tj plus 1 minus tj square minus this is again 2 times this is 1 more time t minus. This is plus so again, I would say that this 2 could to quantities simplified we will give you this, which ultimately resulted in 2tj plus 1 minus 2tj and if I look at variance of QΠ as summation j equal to n tends to minus 1 variance of the quantity here because each of them is independent there is nothing more than the resulting here.

So, this is essentially summation over j twice tj plus 1, (sorry this is square here),tj square. Now this is less than or equal to, instead of one of this tj minus 1 minus tj, I replaced by norm Π, the rest remaining, I leave it as it is. So then it will end up with summation twice, norm Π times, tj plus 1 minus tj, which is again equal to 2 times t and norm Π, which tends to 0 as norm Π tends to 0. So on, hence we have shown that the quadratic variation of this is true, what is given by this. So this is the proof. We have shown it in almost L2 convergence but actually this is true in almost surely sense.

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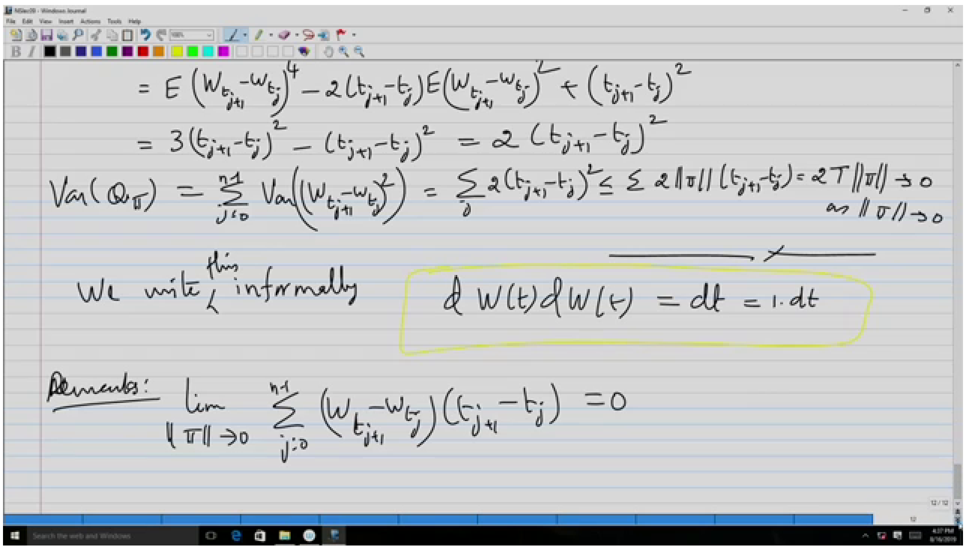

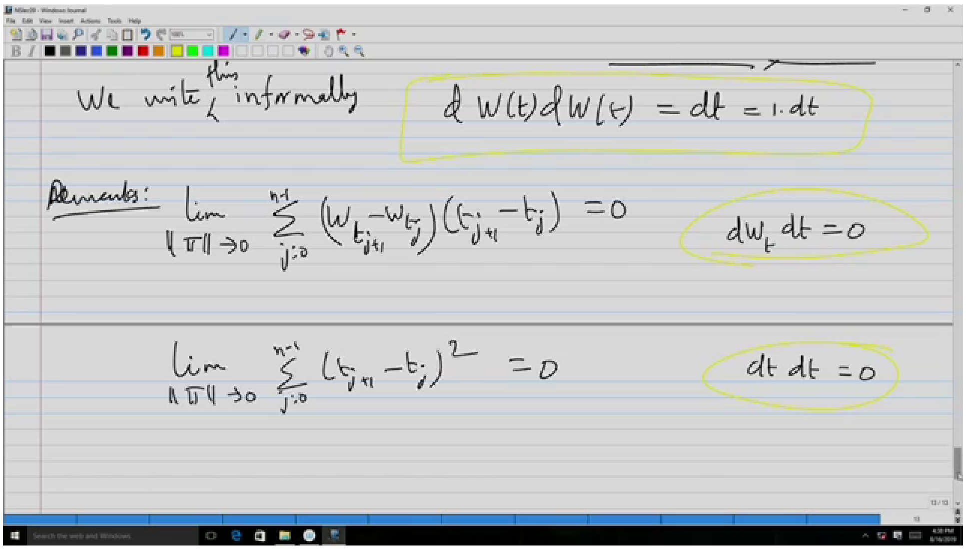

Now we write this, that what is important. We write this informally in the following form: dw of t, dw of t is equal to dt, this actually equal to 1 times t is what, and you know you would see that. This is what you know you use. So this is what is the form that you will write as twt, which essentially means Brownian motion accumulates quadratic variation at the rate one per unit of time, that is what it means. Now this quadratic variation is, what is trouble, turn out to be the source of variability, volatility in at the asset prices driven by the Brownian motion based models. So that's what will come out to be the case that we have here.

Now, in the along similar lines, along similar lines, you can also show 2 other quantities. So this is a quadratic variation of a w with w, which we what we call the further results, or remarks along similar lines. Similar processes by using, You can also show this quantity of W_{t_j} plus 1 minus W_{t_j}, tj plus 1 minus W_{t_j} equal to 0, which we write informally again, which we write informally in the, in this form dw t dt equals to 0.

And one more limit to 0 summation, $[t_{j+1} - t_j]^2$ square, which will also turn out to be equal to 0, which we write informally as this. S,o these are the 3 basic results, basic formulas that we will, again you can show it in along similar lines that we have seen. I mean this is much easier here because one of them is a normal function and only w, is that we are encounter here so we can show simply using the simple properties so these two. So these are the 3 bread and butter formulas that is going to be for all the pen and paper calculation that we are going to use in the stochastic calculus a quantity that we have.

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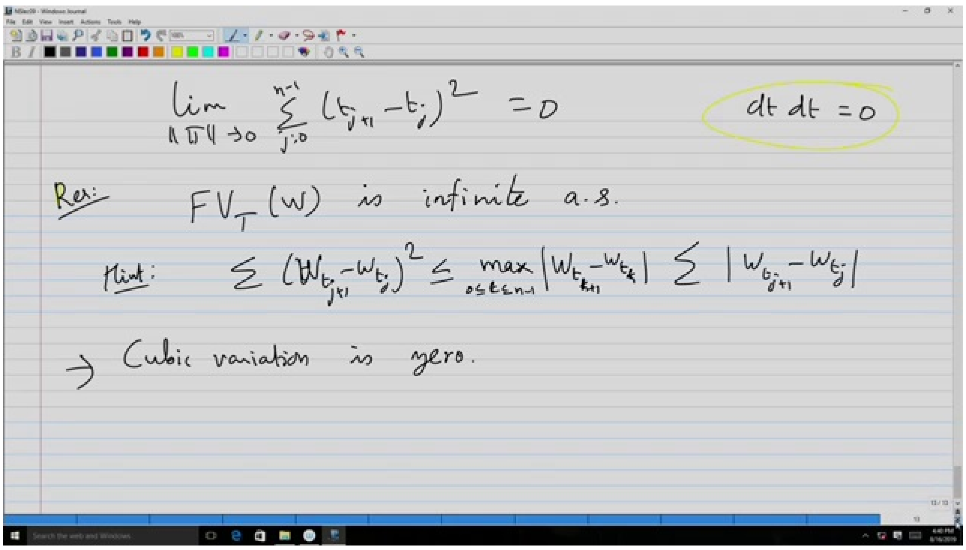

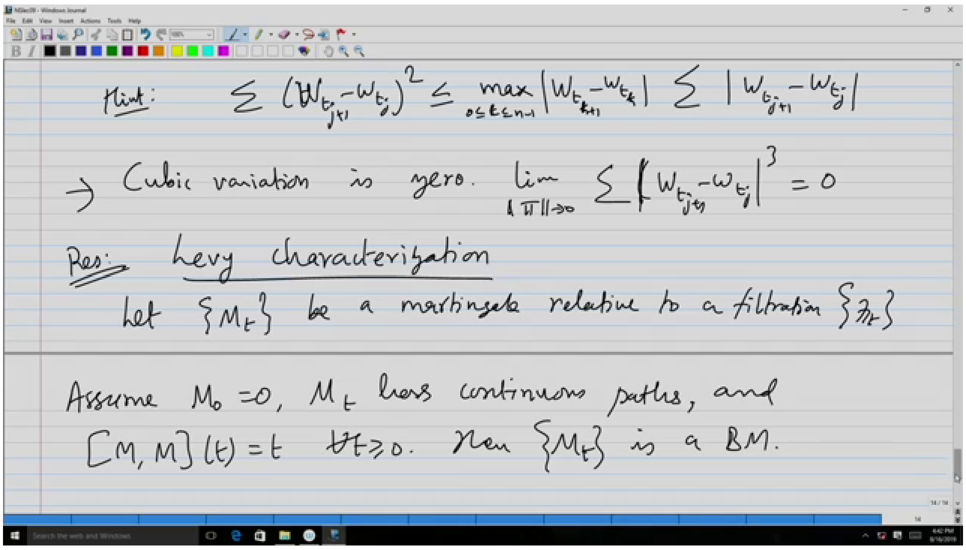

Now, by using this, one can also show by using the quadratic relation property of this w the first order variation of this Brownian motion is infinite in the almost surely sense. The hint for that is essentially you look at this W_{t_j} plus 1 minus W_{t_j} sqaure is less than or equal to some maximum of a wt some k plus 1 minus wtk into summation of W_{t_j} plus one minus W_{t_j}.

So, this is k less than or equal to n for all, so you have this n minus 1, so this is what you have. so this is the hint, you know, they can show that this is the case. Similarly, the higher order variation, cubic variation or any other higher order variation is 0, for the Brownian motion. First order variation is infinite and hence only like we are ending up with this quadratic variation being positive and quadratic variation is positive and for finite interval it is also finite and the cubic variation are any third, fourth and higher order variations, they are all 0. So these properties we will use later. So we will just mean in this particular case, this mean as you say is this summation of a W_{t_j} plus 1 minus W_{t_j}. This is absolute value to the power 3 is 0, that is what we mean when we say it cubic variation is 0.

So, this also leaves us with the result cubic once we have this a quadratic variation property called as Levy's characterization of the Brownian motion. So, what is that, so you take a martingale, levy's martingale characterization be a martingale relative to a filtration $\mathcal{F}_t$. Now you, assume this process is also starts at 0 and M_t has continuous paths and this is equal to t for all t. Then M_t is a Brownian motion. Again, you see $M_0 = 0$, M_t is just here and there also, second and third and fourth properties is replaced by being martingale and the quadratic variation property that we have here in this case, so this is what we have with respect to the second group of property.

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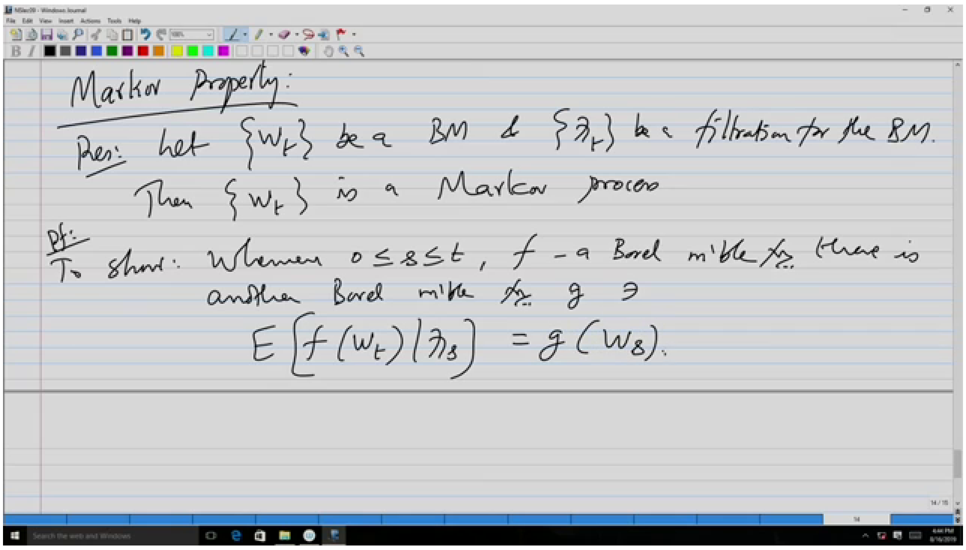

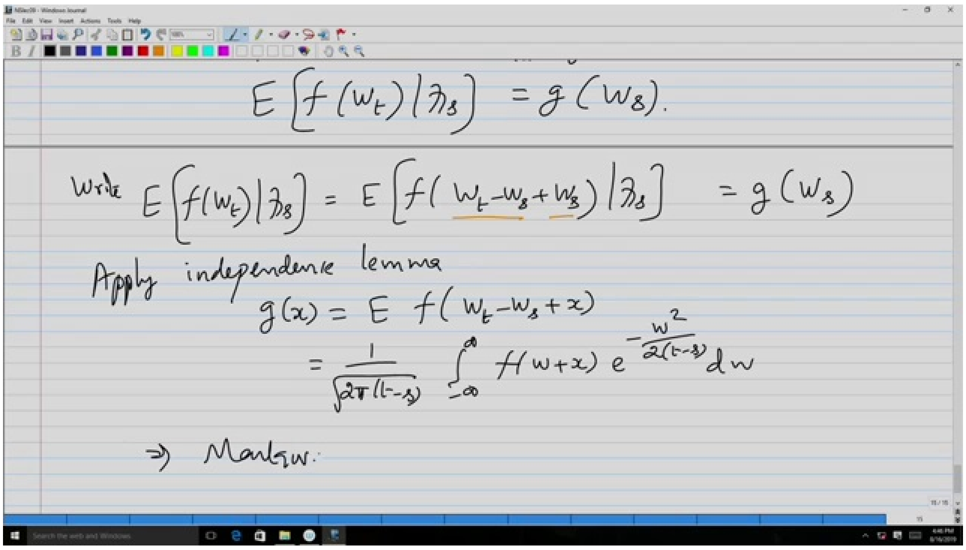

One final group of property that we would see is this Markov property, which we say is the following, let as usual w to be a Brownian motion and ft be a filtration for the Brownian motion, it not be generated by Brownian motion itself. It could be any filtration for the Brownian motion.

Then, wt is a Markov process. we can quickly see how this is true and then to show what we have to show. So proof to show what we have to show is a following, whenever your 0 is less than or equal to s, less than or equal to t and f a Borel measurable function, there is another Borel measurable function g such that expectation of f of wt given if fs should be g of ws. So to show that w is Markov, we need to show only this part.

Now, write this left side quantity $E(f(W_t)|\mathcal{F}_s)$. Now you see W_t and $\mathcal{F}_s$, if you see the connection, it is measurable with respect to $\mathcal{F}_s$ nor independent of $\mathcal{F}_s$, but you can write this as $E(w_t - w_s + w_s|\mathcal{F}_s)$. Now if I call this quantity as one random variable, this quantity as another random variable. And you see the first one is independent of fs. The second one is a you know, measurable with respect to fs, then you can apply your independence Lemma, many are times that is what you use to show that some paths of this Markov also to compute the conditional expectation in both cases that is what being used.

So, in this case you can simply use. so what did the relevant $g(x)$ function, the $g(x)$ function is now for $E[f(W_t - W_s + x)|\mathcal{F}_s] = g(W_s)$, After simplification, you get the quantity here and this is depends only on W_s and hence this is Markov.

So, we can show that easily. This is a Markov so basically what it means that if it evaluate a function of Brownian motion at time t given a filtration for the Brownian motion, then what you need to remember is only ws and its progress from s to t is what you would require to evaluate this.so that is what to real implication. So these are the properties that Brownian motion that we will need, we will use in our further when we go ahead with stochastic calculus. We will see, in the next lecture what the stochastic plus point. Thank you.

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