Mathematical Finance
Lecture 25: General Probability Spaces, Expectations, Change of Measure

Professor N. Selvaraju[1] and Professor Siddhartha Pratim Chakrabarty[1]

[1]*Department of Mathematics, Indian Institute of Technology Guwahati, India*

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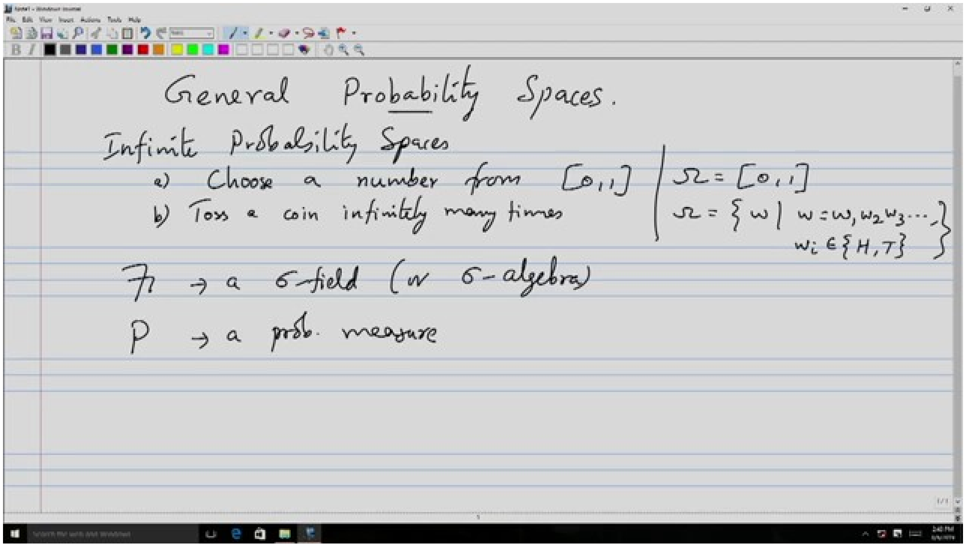

Hello everyone, welcome to the next lecture. where you know we are going to start the more difficult part of the course, which involving starting from general probability spaces and other materials, which we have seen in discrete time, but now we are move to the general one, which is essentially means the continuous time cases, we are going to encounter, but what we are going to do is that we will just recall some of the things, which you know and we might state most of the results as only results and without proof.

But the results are similar to what you might have seen in discrete time, but the proofs are a bit technical and since our objective is not essentially to master the stochastic calculus part, we will take the results as given and we will go ahead and we will try to apply those results into the financial field, which is what our objective is and hence many of the result that we will see in the next maybe about 6 lectures, what you would see is the results that are the required for application into the financial market models that we are going to deal with okay.

So, this part is what we would consider as stochastic calculus ofcourse one could have a full course on that itself, but we need it so we will just briefly take that input and then we will go ahead with it okay. So let us start with the basic idea of general probability spaces, specifically what this means is that we are considering since we have already considered discrete probability spaces, so what we actually considering is essentially the infinite probabilities spaces, where, what we try to model is a situation, wherein the random experiment results in an infinite number of outcomes, rather than being a finite or countable number of outcomes.

So some would say that it is uncountable infinite, but for us that would be sufficient. So basically, what we have in mind is this infinite probability spaces (is what we have). For example, if you pick a random experiment which is of choosing a number from $[0,1]$ is one such experiment that you can think of. And the other experiment is basically toss since we are so accustom to coin toss experiments. So, we can say toss a coin infinitely many times okay, so, this is what the two immediate examples that we can see. Of course, the first one is what we are going to consider for little bit more elaboration, the second one is long similar line. So, what we have now is suppose if we write down the sample space. For example, for these two cases, so in this particular case, this omega would be this which you see has an infinite number of elements in this omega.

And in this case the Ω is set of all ω such that this Ω can be expressed as $\{\omega_1, \omega_2, \omega_3, ...\}$ just like in the earlier case where each of these omega i's are either head or tail right, so this is what you would represent your that infinite coin toss sequences, it could be a you know infinite sequence of heads and tails, H and T is what the usual okay, so this is the sample space and we also have on this sample space a sigma field or a sigma algebra.

$$\Omega = \{\omega \mid \omega = \omega_1\omega_2\omega_3, ...; \omega_i \in \{H, T\}\}$$

So, the definition which we have already seen stands true, even in this particular case which is basically is a non-empty collection of subsets of the sample space omega, which is closed under countable Union and complementation right, that is what is a sigma field or sigma algebra which we have already seen. And on which we also define probability measure P, the definition stands again exactly the same as the previous one, just can recall that it is a mapping from $\mathcal{F}$ to $[0,1]$.

And the, if you take you know set is joined, mutually disjoint sets A_i z from omega, then the union of their probability, probability of their union is some of their probability, which is what we call countable additivity properties. So this is the probability measure which satisfies this okay. But now, if you recall in discrete time how we had defined or how we have specified the probability measure on the sigma field that we consider okay.

What we did is basically we took the elementary events or sample points from the sample and then we assigned non-negative numbers to that such that they sum to 1 and for any other event a we defined the probability of that event a to be the sum of the probabilities of the elements in that set finish right, but the same thing will not work here because you know you would see that for example, if you take this $\Omega = [0,1]$ right, so the sigma field you know you and a typical nature of this (continuity) continuous probability space or infinity probability spaces are that the each of these sample points carry a probability of 0, alright.

So, this is all you can recall from your probability theory idea and in such a situation there is no way that (you know) you can, you will assign probability such that this and then if and also, then if you want to extend that there is nothing like and uncountable sum, something like that you can talk about it. So, we must have a different way of assigning probability measure in such situations and so what is done is rather than assigning probabilities to individual elements the probabilities are assigned to certain sets and you assume or you ensure right that you know by defining that there are many subsets for which also the probability get assigned automatically by the properties of the probability measure. Say for example, right now we will take this simple example of this omega right.

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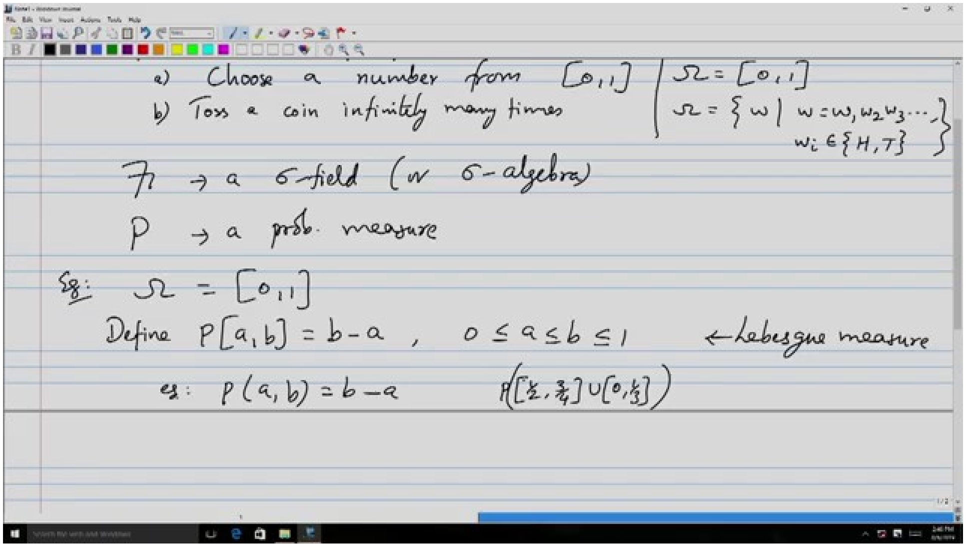

So, we will take $\Omega = [0,1]$ this is our sample space here. Now what we can do is that we can define probability of closed intervals in $[0,1]$ has given by its length for or a and b, so

$$P[a,b] = b - a,\ 0 \leq a \leq b \leq 1.$$

And those of you who are familiar you will also know that this is also called as the Lebesgue measure right.

Okay, so by defining this there are many subsets of Ω whose probability is determined by this P and the properties of this probability measure P right. Say for example, if you consider the open intervals for a, since open intervals can be written as unions of closed intervals right, so you can apply that and then the limiting continuity property of P would ensure that the open intervals also has the same probability as right.

So example cases, say probability of (a,b) would also mean, will be assigned a probability of b minus a, so what you are appealing? We are appealing to the properties of P and then we consider all possible closed intervals in $[0,1]$ which is what is of the form a to b for arbitrary a and b and then since you can write this, so you will get this.

Similarly, you would obtain probability of say for example, say some interval, suppose if you take say $[1/2, 3/4] \cup [0, 1/3]$ so you would see that this is again you can write it as probability of this first interval plus the second interval property of P then each one of them can be applied here and then you can apply in the probabilities here right.

So, this way there are many sets and for which the probabilities of all the, the probabilities gets assigned automatically okay. But what is this collection, right? What is this collection of sets for which the probabilities are getting assigned by this process?

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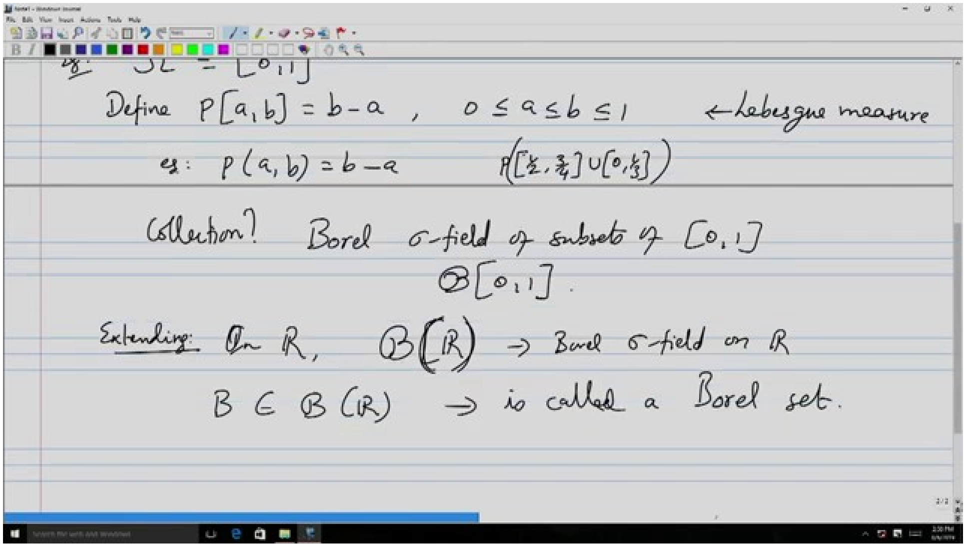

Okay, (that collection) so what is this collection of events? These are all event connected with this, so what is collection of events for which the probabilities of gets assigned in this manner? Which essentially means if I take closed interval a, b all possible closed intervals and I add whatever it is to make this this collection as a sigma field or sigma algebra that is this collection and this collection is what called as the Borel sigma field of subsets of [0,1] and it is also denoted as $B[0,1]$ because we started with the interval [0,1] took the class of all open intervals as generators what we call and formed the sigma algebra, which is what is called the Borel sigma field, Borel sigma algebra over 0, 1, which is sometimes written in this form (okay). Now, extending this idea suppose instead of 0, 1 on R, so what you will have? This is okay you can call it this way or this way, so this is what is the Borel sigma field on R essentially which is actually Borel sigma field of this. And a set B which is belonging to this is called a Borel set okay. Now in the whole of thing, so this is the sigma field with which we will be concerned with because this is what the sigma field with which we can assign probability measures on subsets of this Borel there are in measure theory those people were aware about measure theoretic idea.

There are non-Borel sets, but that is not of interest to us, so essentially we will be confined ourselves to when we say a subset of R in the whole course we mean a subset which is a Borel set right, so that is what anything that is what is we will be concerned with the case okay. Now so this is the collection on which this P is getting assigned.

Now I mean it is not possible to enumerate all the sets and all the probability measures that we have, but you know there are singleton is a Borel set, a finite set is a Borel set, countable set is a Borel set and uncountable in intervals is a Borel set, then open set is a Borel set, closed set is a Borel set, almost all what you can think about with respect to real experiments all such things will be there in the inside these Borel sigma fields, so we will be happy to deal with this itself, we will not look beyond this.

Okay, so going forward anything, any subset of R which we talk about is a Borel subset and the sigma field that we consider over R is the Borel sigma field right that is what you know we will have. Now, there is I mean as I said you know many things like we will just we call the results and we will be using it frequently the terminology, so the next terminology that we will going to see.

So, here in continuous probability spaces or infinite probability spaces you have this problem that you know, somehow if you take the an uncountable set say 0, 1 itself or 0 to half you see that each of these points have probability 0, but if you put together 0 to half it has probability half, I mean according to for example this is typical example that we have considered just now. So, this is basically you know a side kind of little bit uncomfortable some people might feel, but that is the nature of the subject okay.

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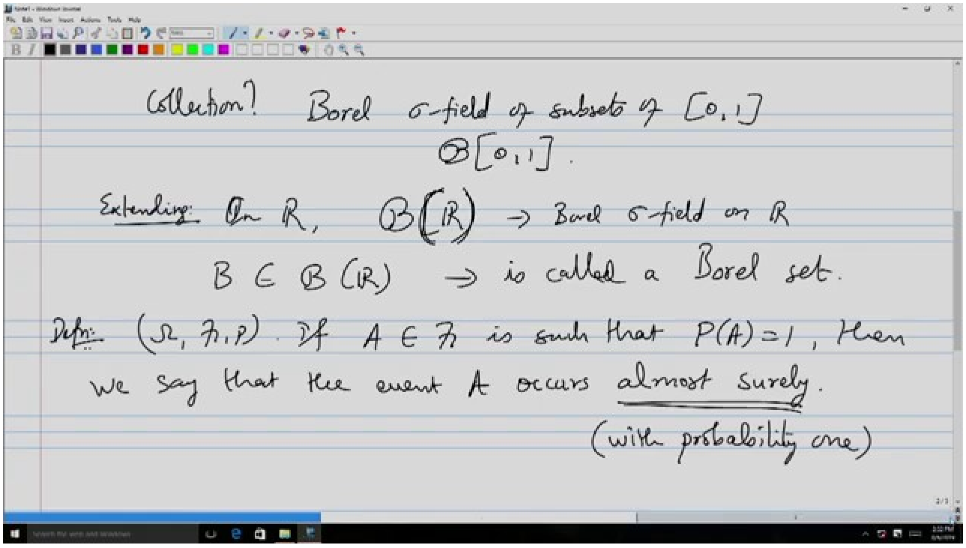

So, what we will do is that, we will define such things by you know doing some kind of terminology right, what we say is that one definition we have a probability space$(\omega, \mathcal{F}, P)$ as usual and if there is a set $A \in \mathcal{F}$ is such that $P(A) = 1$, then we say that the event A occurs almost surely. So this is the word that you will be using equivalently in the engineering literature you will also find the usage with probability 1, is what also used equivalently, so this what is the word almost surely, this is one form of.

So, what does that mean? So, which means that probability of A is 1, if there is an event A simply, a subset of you know omega, which is there in $\mathcal{F}$ for which this $P(A) = 1$, then the event A is set to occur almost surely or if this A represents some property, then we say that the property holds with probability one or the probability is true almost surely right, which means what? There are situations in which this is not true right, but those together carry a probability of 0 (right).

So, if you go back to the previous 0, 1 interval, you know that if you take a set of all (you know) rational numbers which is countable, so if you take their probabilities it will be 0 right, but that is what it means, so which means that this is what then you will have if this is true okay. So this is the definition that you will have with respect to probability.

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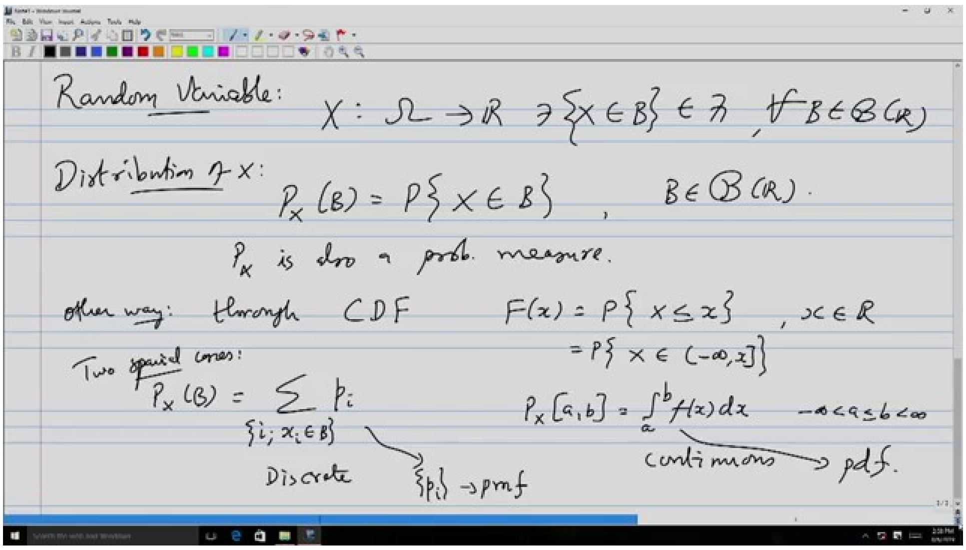

The next thing, that we will go is essentially the idea of random variable, we already have seen area of this random variable, what is that random variable? We have seen X it is a mapping from Ω to R such that $X \in B$ in $\mathcal{F}$ for all $B \in B(R)$, so this is the general definition where what is this B? B is the Borel set. So if this is true, a real valued function on omega such that its inverse image of Borel sets are in $\mathcal{F}$ is what then is a random variable.

Now associated with this you have what we call distribution of X right, distribution of X we define to be $P_x(B)$ which is essentially probability of X belonging to B again B is a Borel set, so this is what we call it as the probability measure, or I mean this probability distribution of X, this is different from the distribution function that you might have been aware, but this is probability distribution, so this is also a probability measure right.

P_X is also a probability measure which mean that it satisfy the axioms of probability measure right, so you can show. These two concepts are different random variable and distribution in finance that you need to keep this idea in mind, random variables and distributions are two different concepts, the random variable can have different distributions and there could you know be two random variables which have the same distribution, so these things you know you have to keep in mind.

One other way, in which the distribution is been represented on which you have been knowing is to record the distribution of X is the other way or the alternate way is through what you call cumulative distribution function right, this also records the probability distribution of X and what is that? You have already, those of you who are familiar recall that this is nothing but this function.

And remember this is nothing but if I write $X \leq x$, I can also write this as probability of X belonging to this quantity right, is what is X. So if you see minus infinity to X is a Borel set, so this is essentially same as this P_x, but we are taking a specific form of B, rather than taking for all Borel sets B as given here, we take for all X in R which means you extend X, you take only the form minus infinity to X and keep varying X. So you have one class of sets, this is also will generate the same Borel sigma field that you have it here okay.

Now, in two special cases in which the distribution can be recorded in more detail is one is this discrete, which is essentially meaning this is

$$P_X(B) = \sum_{i;x_i \in B} p_i$$

, for all Borel sets B okay. And what are these p_i? These p_i is what the collection is what the PMF of the random variable X.

The other case is essentially P_X of say a, to b if I take then this is this quantity for minus infinity less than a less than equal to b less than infinity. And in this case this is discrete and this is we call continuous random variable right and this f is what is called the probability density function and this is what call this p i as the probability mass function which you have seen already in discrete time. So these are the two cases in which you can record the details, these two special case in which you can record the distribution function in a much simpler form in more detail.

Of course, there are random variables which has a part discrete and part continuous which are called mixed random variables and there are random variables which have neither a probability mass function nor the discrete density function okay. Even in finance you will also like they do have application but that is much more advanced level. So elementary level what we will be concerned, just like in the discrete case we assume that there is a PMF, in the case that we are going to consider, we will assume that the random variable is continuous and it has a PDF and hence the probability measure would be equality described by the PDF that we maybe have okay.

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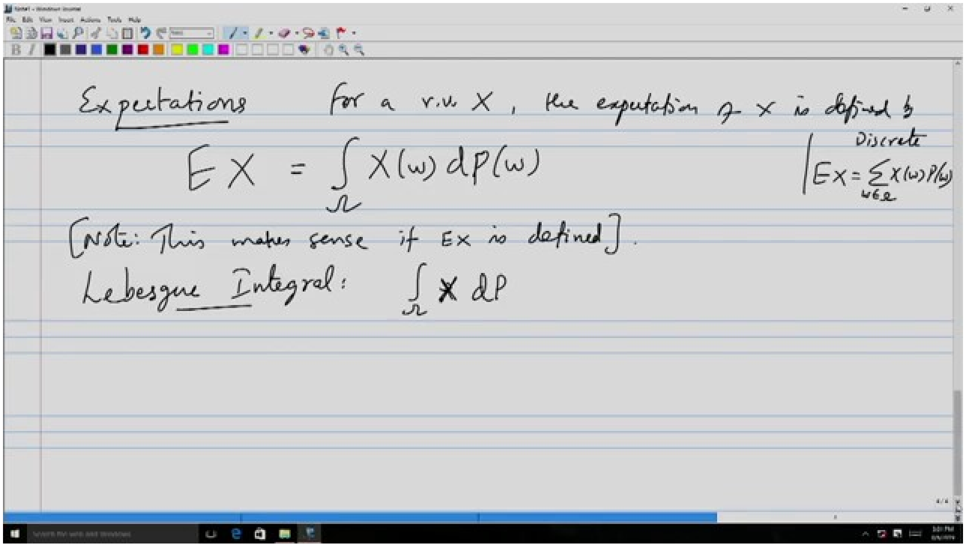

Now once we have these random variables, the next thing that we consider is the expectations right, we have seen in discrete also after this expectation. Now here, the $E(X)$ okay, so for any random variable X the

$$E(X) = \int_{\Omega} X(\omega) dP(\omega)$$

In discrete, what we have written?

$$E(X) = \sum_{\Omega} X(\omega) P(\omega)$$

So this is in a way much similar but there it is a countable number of omega, so there was a sum but here uncountable so this is, okay. But we need to make sense of what is this quantity? I will just come back in a moment. This definition would make sense provided this exists okay.

So note, I will just write, this makes sense if this is defined in a way okay. Now, what is that we will see in a moment okay. So, there is a particular case in which this quantity is not defined because this will not make sense okay. Now, what is this quantity? Okay, so this $\int_{\Omega} X(\omega) dP(\omega)$, is what is called or we are say writing this expectation of X in terms of this as, the in terms of Lebesgue integral right.

So what we have? So this is a form of Lebesgue integral, now those of you who are familiar with what is this Lebesgue integral okay can skip this part, I will quickly move to may be another 5 minutes later, but otherwise you know you can see what we mean by this okay. So, this is $X\ dP$ in short, $X(\omega)\ dP(\omega)$, is what it is. Now, what is this quantity or how do we define this quantity? That is what we are going to see and so what is this quantity? Is what that we are going to see.

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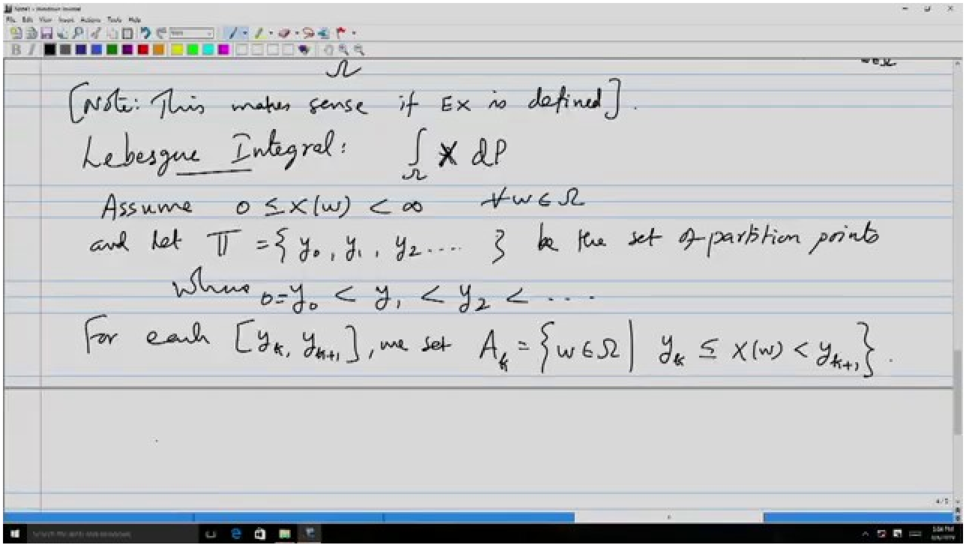

Assume first okay, assume that this whatever the X that we are considering is basically is a non-negative quantity, for all $\omega \in \Omega$ okay and let this I call it as p_i, which is a set of numbers $y_1, y_2, \ldots$ be the set of partition points okay. Where, what we have? Since this non-negative, so, we pick y_0 which is equal to 0 and then the next value is y_1, next value is y_2 and so on okay.

Now, for each sub interval, which is of the form y_k to $y_k + 1$, we set the set Ak to be the collection of all omega's, which gives value of this random variable lies between and including the left but not including the right side, right. So, this is set that we call okay, so this is what we have it here okay. So now, the Lebesgue integral is what we are talking about, people were familiar with Riemann integral, which is the ordinary integral, there what we do? How do we compute or how that is defined? You partition the X-axis which is the domain right and then just like here the partition this is and then you take the lower Riemann sum or upper Riemann sum or in general Riemann sum. And as the partition points are getting increased in the limit that lower Riemann sum or upper Riemann sum would converged to your value that is what you call the Riemann integral or simply integral that is what you will see.

Similar here, here if you look at you know this is what we have and then we are looking at not on the X-axis the partition but on the Y-axis, the reason is that X is a mapping from Ω to R, this omega could be any abstract set need not be a subset of R where you can make some altering properties there, so you can make a nice partition. But since Y-axis is real number, so we make partition Y-axis and then we will try to compute that is what the Lebesgue integral is right.

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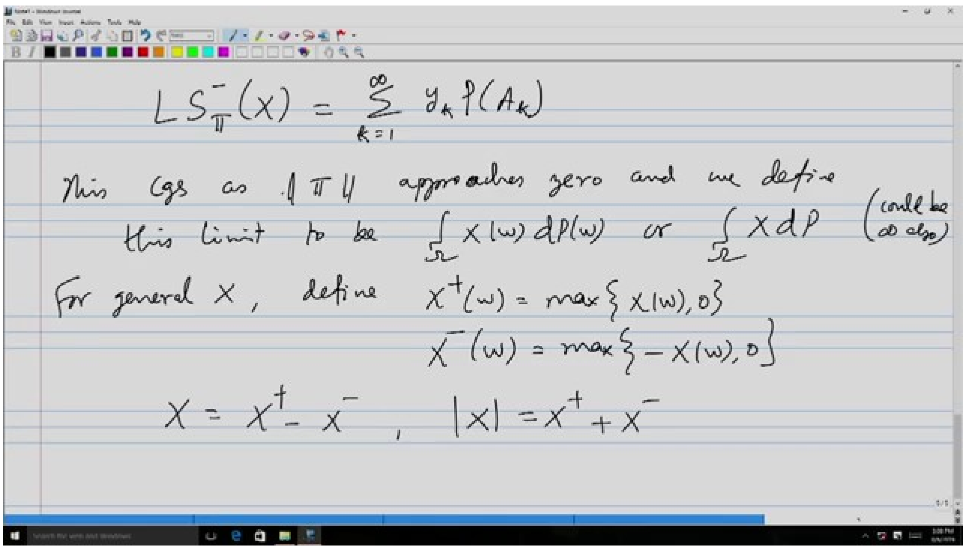

Now, we define the lower Riemann sum in this particular case with respect to partition the lower Lebesgue sum:

$$LS_{\pi}^{-} = \sum_{k=1}^{\infty} P(A_k)$$

And this converges as the norm of this partition approaches zero and we define the limit right, the conversion, the quantity which is this limit to be $\int_{\Omega} X(\omega)dP(\omega)$ or simply for simplicity we may always write in this form okay, so, this is what it is. We are assuming simple cases so there is not much issue, so we say that this lower Riemann you could again do similar thing like upper or you can use Riemann sum.

But here if you see you have to pick the y_k multiplied by $P(A_k)$ and this quantity converges, okay this converges as this norm of this partition, what is this norm of this partition mean? The length of the largest sub-interval of this partition (Y) a p_i, which is in terms of this y that you have here, okay. Now, note that here this could even be infinity right, so this limit or this Lebesgue integral we need not restrict X, so this sum could even be infinity we do allow okay.

So, this could be infinity also, this quantity right because this non-negative quantity and we are summing over some non-negative number multiplied by some proportions. Now, this is non-negative, now for general X, which takes both positive and negative values what we do, first we define two quantities, which is

$$X^{+} = max\{X(\omega,0)\}$$

$$X^{-} = min\{-X(\omega,0)\}$$

Also, observe that

$$X = X^{+} - X^{-},$$

$$|X| = X^{+} + X^{-}$$

Now, when you can write in this form you can see both X^+ plus and X^- is a non-negative quantity and the previous discussion of X being a non-negative and the definition of integral is right.

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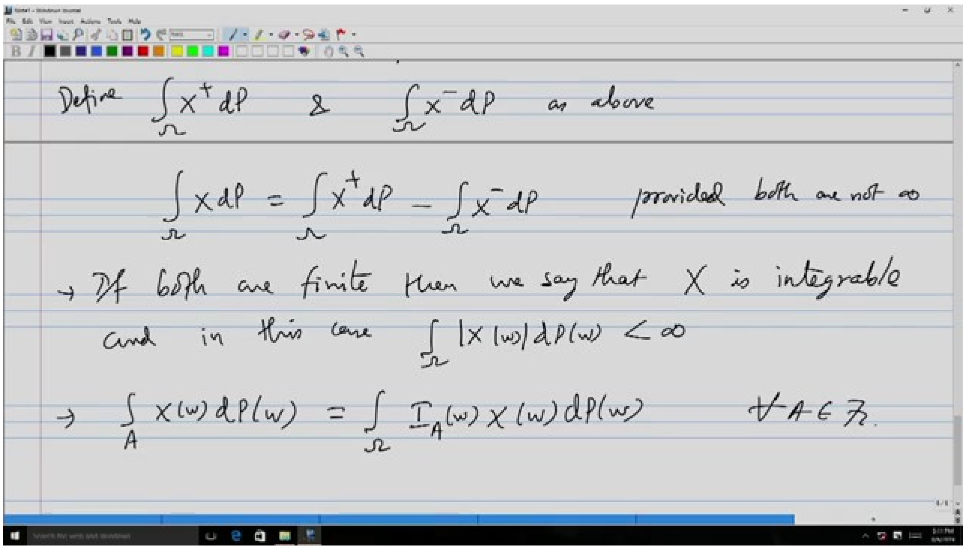

So, you can define

$$\int_{\Omega} X^{+} dP(\omega)$$

$$\int_{\Omega} X^{-} dP(\omega)$$

Now since we said that this is each one of them can take (you know) the infinity, the value infinity right and if provided both of them are not infinity you can define this

$$\int_{\Omega} X(\omega)dP(\omega) = \int_{\Omega} X^{+}dP(\omega) - \int_{\Omega} X^{-}dP(\omega)$$

provided both quantity in RHS are not infinity, right.

If the first one is finite, the second one is infinity then you can define this to be minus infinity, if the first one is infinity the second one is finite, then you can define this to be infinity but if both are the infinity then this is not defined. So this is what we mean when we define expectation of X provided this defined, we mean that this particular integral is defined for any X.

Now, we have a special name which we will did when which is the case of both are finite, what is this? If both are finite, then we say that X is integrable and in this case we can observe that $\int_{\Omega} XdP(\omega)$ would be finite because in this case $|X| = X^{+} + X^{-}$ So, this is what we mean in this case, so this is the first part.

Now, one more thing which you can, you have to note it down, is essentially instead of taking this Lebesgue integral over the whole set omega, we often might have to take over some subsets of omega, some Borel set something A, but you can see that this could be written in terms of the indicator function that we are aware already, so this is true for all $A \in \mathcal{F}$ that you know when we mean the Lebesgue integral over some subset of A, we mean we can also write in this form using the indicator function for competition.

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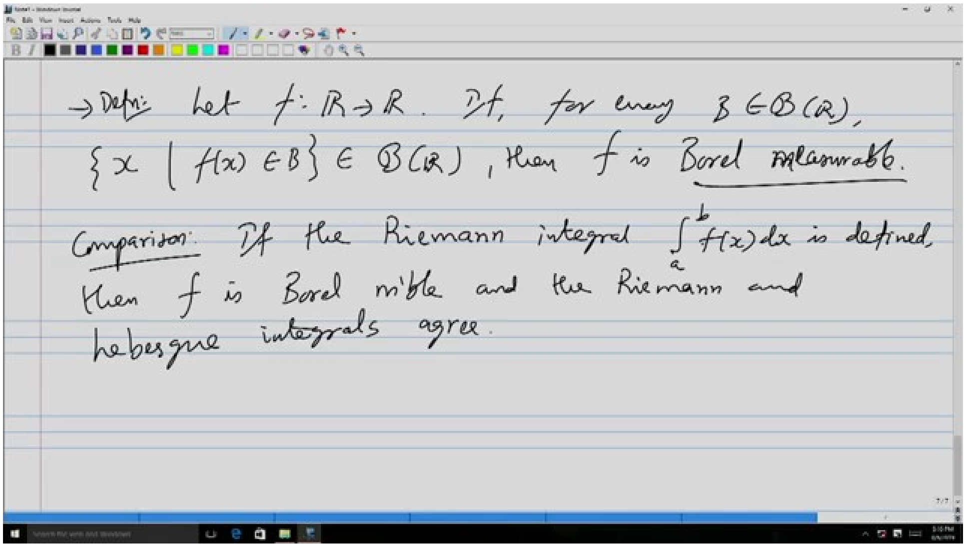

And another quantity that you know we will need is what we call the following. The definition essentially, so you have a function from R to R and if, for every Borel set this set of all x such that my $f(x)$ belonging to B right is also a Borel set, then f is, what we call Borel measurable okay, so this is what is the definition that we call.

Now, what are the Borel measurable functions? Normally, what you encounter the simple function that we always encounter, they are all Borel measurable functions say a continuous function is a Borel (measurable) function, step function, increasing function or you know these all of them are Borel measurable function that you normally we encounter.

So, again just like we said that any subset of a throughout we mean a Borel subset, any function of a real valued function that we are considering will be a Borel measurable function only from now onwards throughout the course. So, without any time repeating we always mean a function, a real valued function means a Borel measurable function which is in the sense, which means what? The inversely measures of Borel sets are Borel sets again right.

Now, why do we need this is basically when we are defined Lebesgue integrals but how this compares with the Riemann integral that we already know of okay. The comparison, which is what will help us in computations of this Lebesgue integral or expectations and we define to be this are whatever you already know we should also see that whether that is same as this one. So that is if the Riemann integral say $\int_a^b f(x)dx$ is defined, then what we can say? f is Borel measurable and the Riemann and Lebesgue integrals agree means they coincide. But there are situations where these will not agree, but in that case if the Riemann integral exist, if you put that condition, if the Riemann integral exist and is defined then this is that f is Borel measurable and the Lebesgue and Riemann integral are one and the same.

So, this gives as a way to (you know) write down the expectation which we defined in terms of, which we defined in terms of Lebesgue integral to also write in terms of the easier and more much more convenient form in terms of Riemann integral so that our computations can go with. So, this is something similar to what you might have seen in the discrete case.

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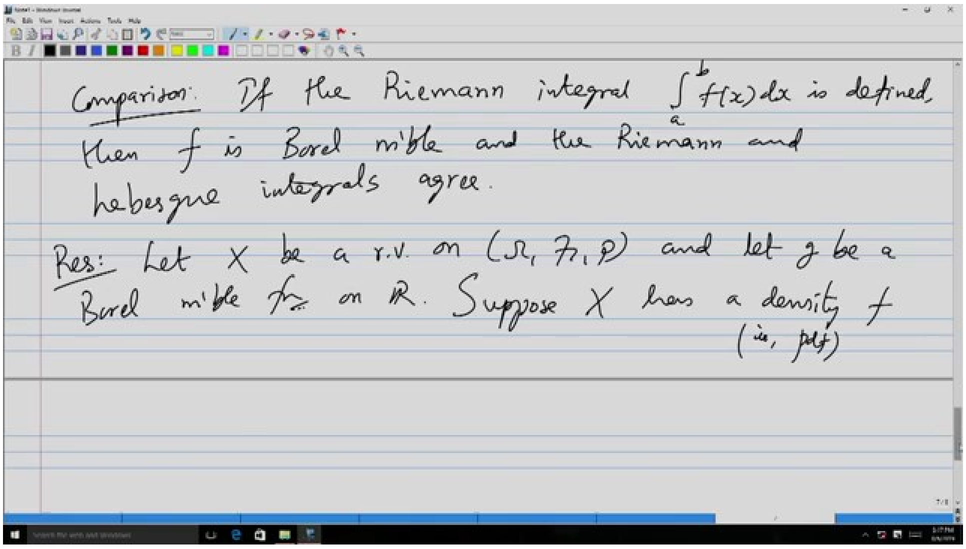

So, let us quickly see that, so what we have? The result is the following which is after a couple of steps you know you will obtain this result but since we are not worried about that, so let us take the result, that is what we have. So X be a random variable on some probability space $(\Omega, \mathcal{F}, P)$ and let g be a Borel measurable function on R. Also, suppose, this random variable X has density, which is essentially means that is PDF, okay f, right X has a density f.

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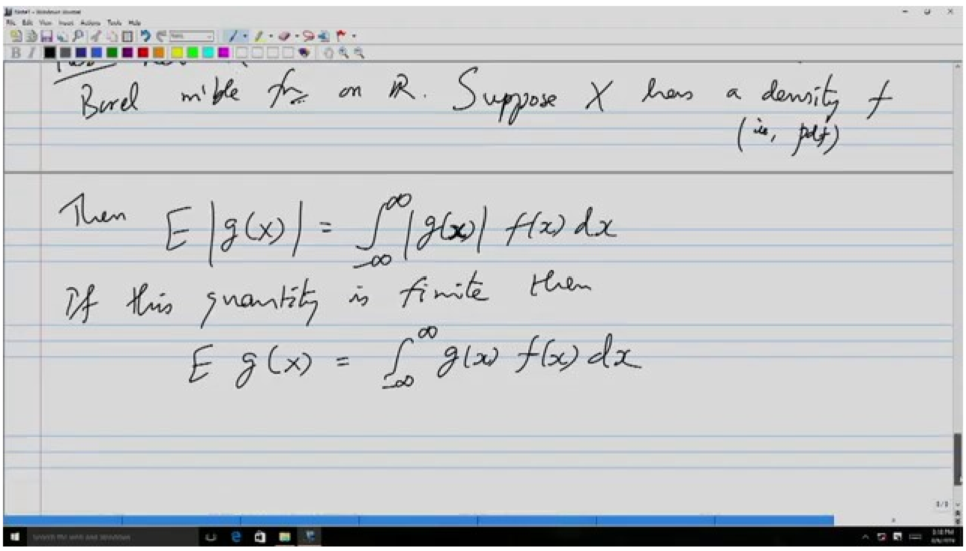

Then, so what we are claiming? Then expectation of this quantity is given by

$$E|g(X)| = \int_{-\infty}^{+\infty} |g(x)|f(x)dx$$

and if this quantity is finite then

$$E(g(X)) \int_{-\infty}^{+\infty} g(x)f(x)dx.$$

But, whenever this is defined which means this is defined right and then if this quantity is finite then only we define $E(g(X))$ to be this, which is what is the normal probability course you would have seen as $E(g(X))$ is defined or simply expectation of x defined in this case okay. It is also much similar to this is when x has a density f. Suppose in case of discrete what happens? You just replace this sum by you know, this integral by the sum, so you get your discrete case what we have defined as the expectation, this is the case, okay.

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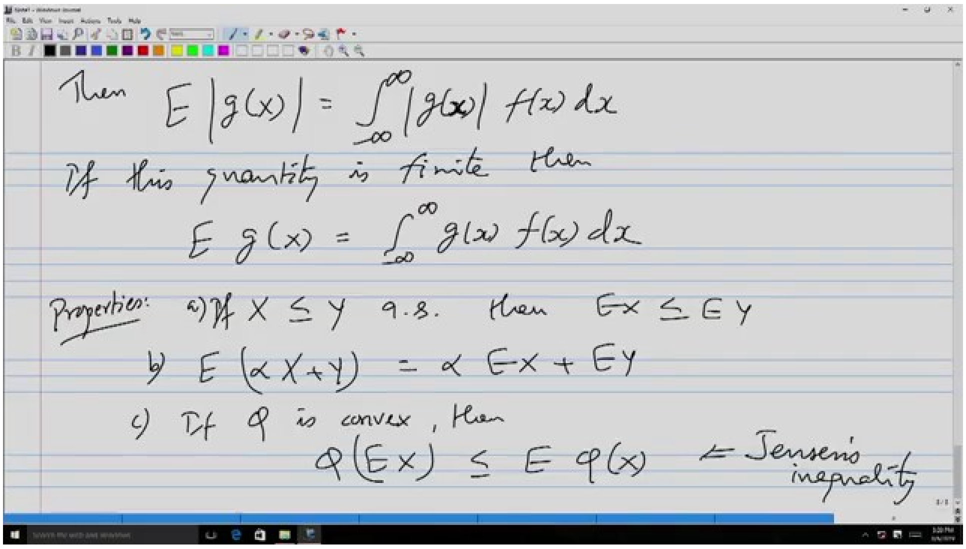

So, once we know about this Lebesgue I mean expectation of x, again we will just recall certain properties,

- $X \le Y$ a.s. then $E(X) \le E(Y)$.
- $E(\alpha X + Y) = \alpha E(X) + E(Y)$
- If ϕ is convex then $\phi(E(X)) \le E(\phi(X))$ (Jensen Inequality)

We have seen discrete also it is exactly same, it is true here, right. So like this you can list out property but this property if you apply this definition of Lebesgue integral sense like in this discrete case this is also follow very quickly. Many other statements that we might write, now we may take it in the almost surely sense, still the result will be true that is the beauty of this continuous time models that you might consider, okay.

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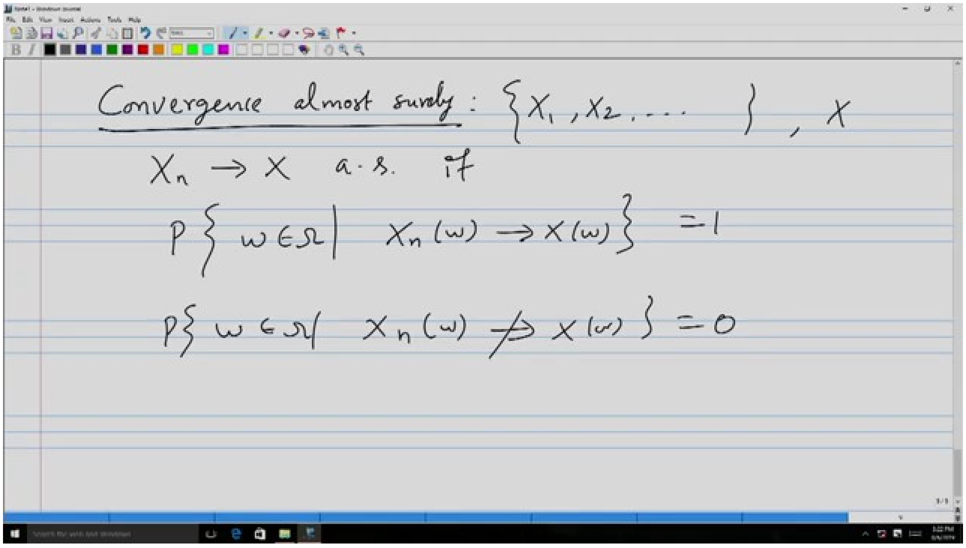

Now, we also have to talk about when we talk about random variables or sequence of random variables we may also have to talk about convergence of, but in this case what we are interested ofcourse in probability theory there are 4 type of convergence that you are aware. We will just recall what is convergence? Almost surely or convergence with probability 1, right. So what we have?

We have a sequence of random variables, sequence of random variables, so we say and there is another random variable X all defined on the same probability spaces and we say that X_n converges to X almost surely, if the

$$P\{\omega\ in\Omega|X_n \to X(\omega)\} = 1$$

Once you fix an omega right, then this sequence of random variables $X_1, X_2, ..., X_n$ are going to give a sequence of real numbers.

Now, what is that real number converse is what is given here, $X(\omega)$. So you collect all those omega's for which this convergence takes place and if this collection of omega carries a probability 1 then X_n convergence to X which is in a way equivalent to your point wise convergence that you talking about in normal sense, right. But there are omega's for which this Xn does not converged to this particular X, but together they carry probability. So the same thing you can also write $X_n(\omega)$ does not converges to this particular but this various probability 0 because this is the compliment okay.

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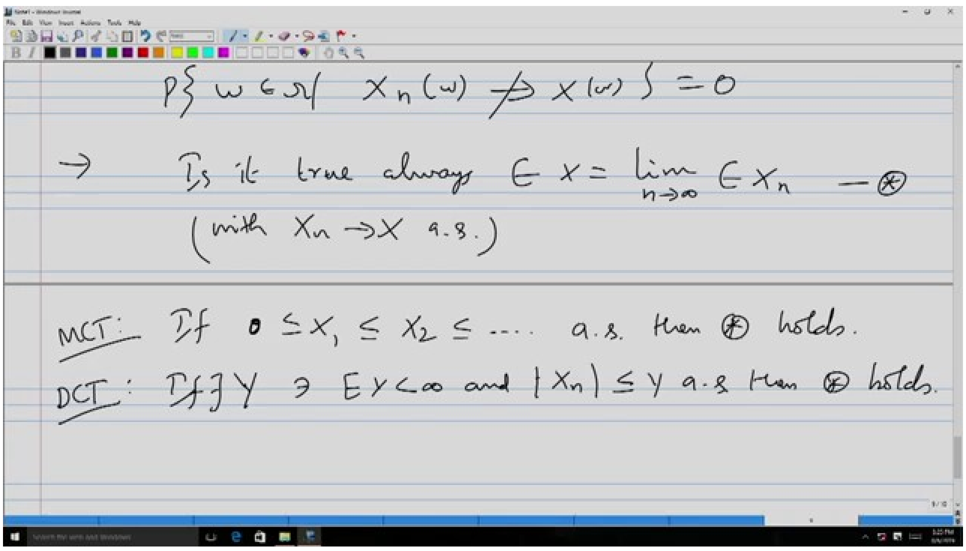

Now, the question will come which is what relevant to us in many situations is suppose if I have a sequence of random variables, the question is if this is sequence of random variables is there whether this convergence carries over to the expectations, right? So which means whether is it true always that expectation of X is given by limit of as n tends infinity of expectation of X_n, essentially you are saying expectation and limit operation you are trying to interchange right.

Two results which we would require is one is call Monotone convergence theorem (MCT), which means Monotone convergence theorem, when this is true? Suppose if these are all non-negative it is $X_1, X_2, \ldots$ and they are increasing almost surely then suppose if I call this as star, then start holds okay. The other case which is given by what we call dominated convergence theorem, if there exists another random variable Y such that your expectation of Y is finite and this X_n are bounded by Y almost surely, then star holds. okay.

So, we give two conditions, two different conditions under which you know the limit and expectation (operations) operators can be interchanged which essentially means that when X_n converges to X always, okay is it true always with condition is X_n converges to X almost surely right, then whether this is true? Then, if this is non-negative and increasing, then of course it holds, and if there is Y which bounds, so this is dominated convergence theorem and that is Monotone convergence theorem.

Under these two scenarios, you can show that these are the two famous results in measure theory which you know we might use whenever need a little later whenever we talk about a sequence of random quantities converging, right and this is what we will see, okay.

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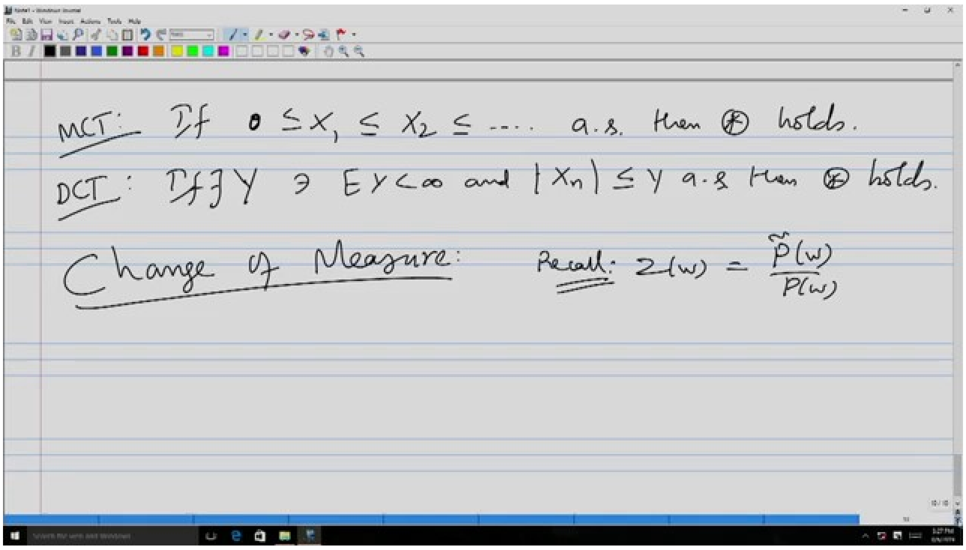

Now, what you have the next idea is basically more relevant to our field, which is the financial whatever we are going to deal with is this change of measure idea. You know recall, that in discrete also we had considered two probability measures P and $\tilde{P}$, right, so but how do these two quantities P and $\tilde{P}$, how are they related? Is via a quantity called

$$Z(\omega) = \frac{\tilde{P}(\omega)}{P(\omega)}$$

which we call as relevant equilibrium derivative, right. So you recall this quantity is we wrote right, so this is what it is discrete case via this Z okay.

So now, in continuous case ofcourse since P and $\tilde{P}$ here we could define and we could easily get it that both of them are strictly greater than 0, so you could define it in this manner. But in continuous case ofcourse for individual omega this is 0, and even if you write this equation as $\tilde{P}$ of omega as equal to Z of omega multiplied by P of omega, still both sides are 0, does not tell anything. But this gives the idea of how to get a $\tilde{P}$ out of P, I mean the idea is you have to suppose use this Z, right.

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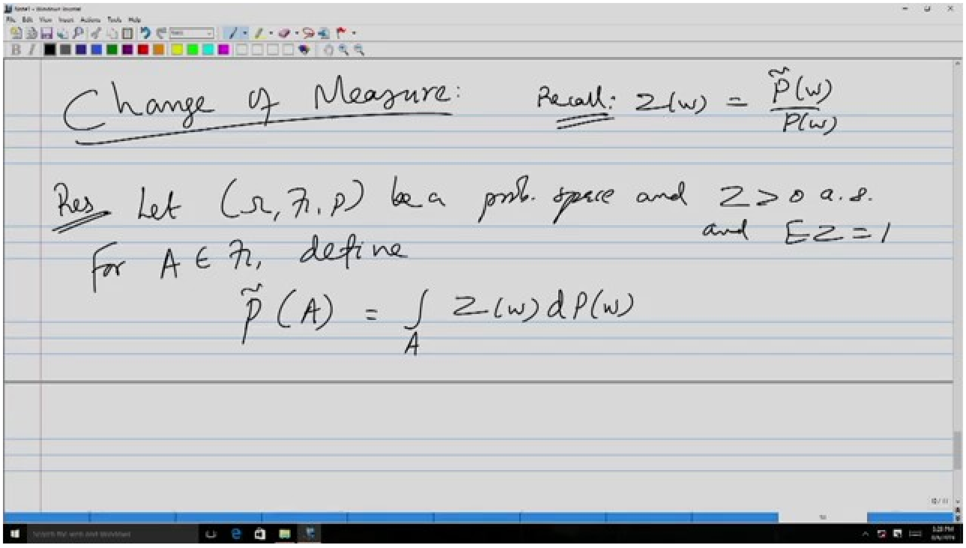

So, in which case what we have is this specific result which is what we will use. Let us quickly go through, so you have a probability space be probability space and you have, and $Z > 0$ almost surely and you also have $E(Z) = 1$ which means expectation Z is equal to under probability P this is 1.

Now, for A in f, define my $\tilde{P}$ of A which is equal to, so you define $\tilde{P}$ you see, if you rewrite the earlier discrete case, you could write

$$\tilde{P}(\omega) = \int_A Z(\omega)dP(\omega).$$

So this is inspirit captures that same idea, but since you cannot talk with respect to each of omega, each omega because both sides are equal to 0. So you talk in terms of events, in terms of sets in f, that is what the idea essentially right.

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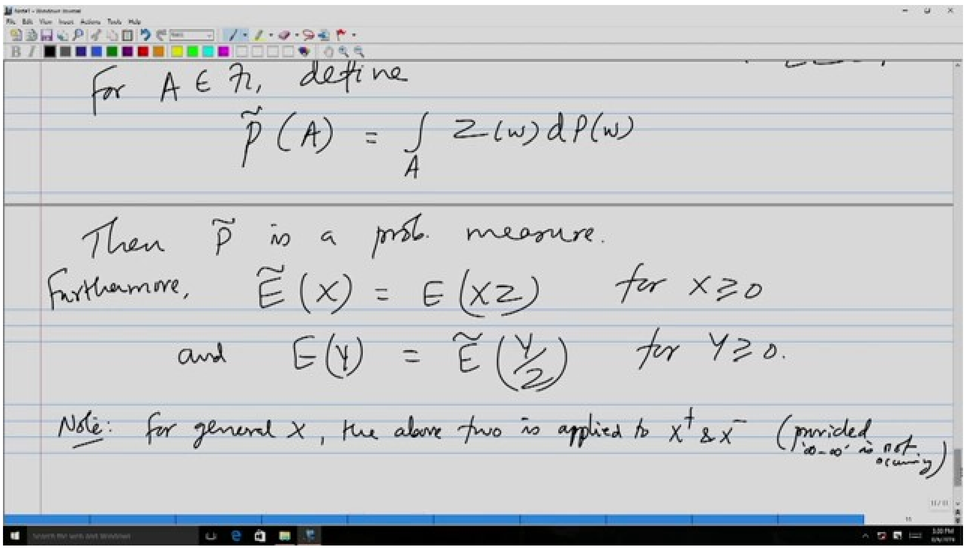

Then, what we have? This quantity is a probability measure, which is the first statement, furthermore, this relationship which you have also seen in discrete case XZ for X non-negative, in a moment you will know why we say X non-negative under this condition. So

$E(Y)$ is essentially $\tilde{E}(Y/Z)$ for Y non-negative okay. Now note, for general X the above two equalities connecting expectation under (E and) under P and $\tilde{P}$, the above two quantities is, the above two is applied to X plus and X minus and provided they will make sense, provided infinity minus infinity case is not occurring.

So, if that is the case of course these things are not defined so nothing you can talk about it, whenever we say that whenever this is defined which means that this infinity measure case is not occurring, in another case you can talk in a single way. But for the result mainly like you know you can consider X non-negative that is sufficient because in this particular case you will going up by individually, so this is what is the result.

What this says is it captured the spirit of what you done the discrete case, so pick a Z which is greater than 0 almost surely you need only almost surely. And $E(Z) = 1$, these are the two properties that you required for Z and this what is this required. And if you have that Z and in discrete also we have seen these two properties are satisfied and if you have that Z of course you will get this $\tilde{P}$.

And in fact, to get this $\tilde{P}$ even you do not need this Z, non-negative is sufficient, but for some other purpose we need Z strictly greater than 0, so we will keep that otherwise this last equation that he wrote here again you have to assume Z to be strictly positive, so we generally assumed Z to be positive case that you have. Now, this is the way you know your things are related in you are defining in this case.

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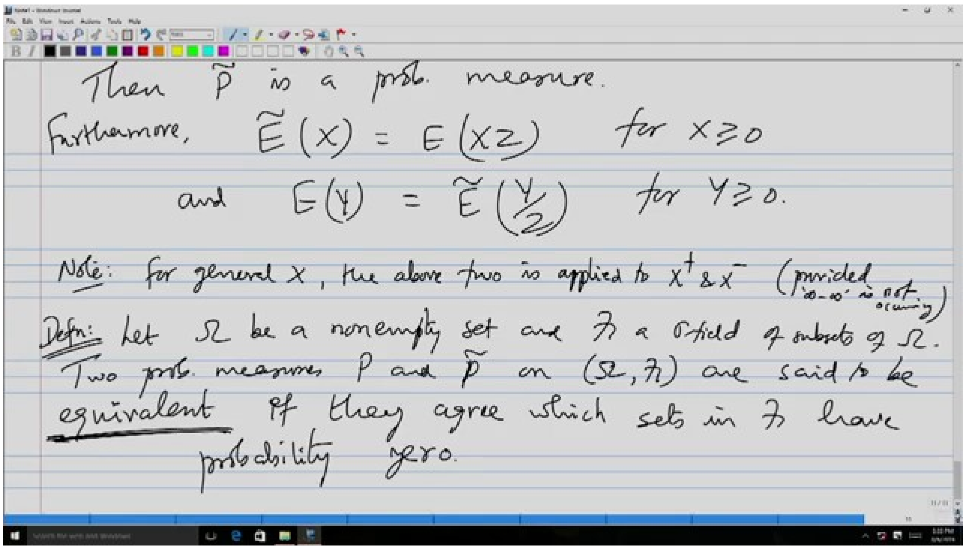

Now, this leads to my next definition which is after giving this definition then we will see why, and hence and what. So, you have omega be non-empty set and $\mathcal{F}$ sigma field of subsets of omega or a sigma field over omega. Then two probability measures say P and $\tilde{P}$ on this $(\Omega, \mathcal{F})$ which we call as a measurable space, again without this P attached to this, we are considering is a pair or said to be equivalent, remember this is the technical term here what we are defining.

If they agree, which sets in f have probability 0, so we say P and $\tilde{P}$, two probability measures defined on the same omega f is pair are said to be equivalent if they agree which sets in f have probability 0. Alternatively this would also imply that they also agree on which sets have probability 1, which means, which are the sets which are almost surely going to occur? Which are the sets which are almost surely not going to occur, right? So that is what the equivalent probability measures property that we have here.

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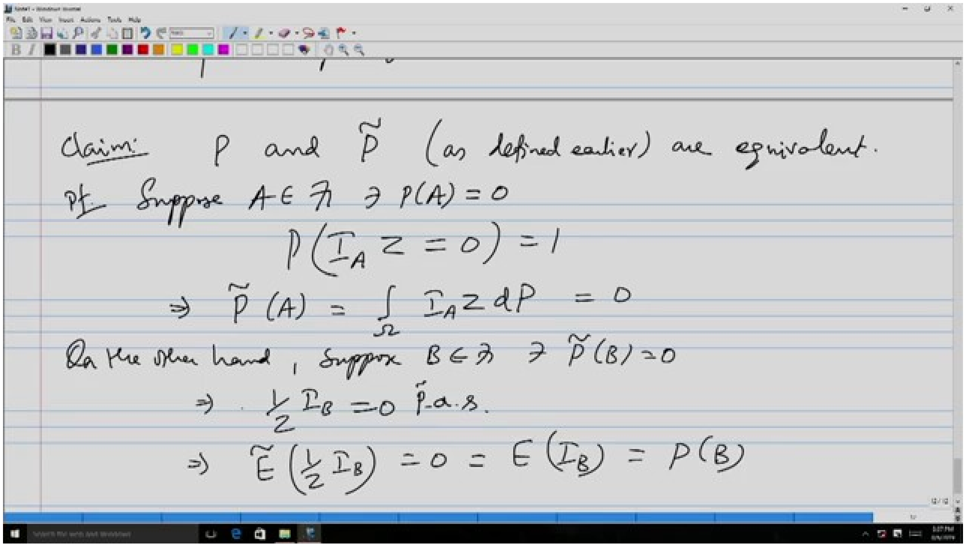

Claim: P and $\tilde{P}$ (as defined earlier) are equivalent.

Pf. Suppose $A \in \mathcal{F} \ni P(A) = 0$

$$P(I_A Z = 0) = 1$$

$$\Rightarrow \tilde{P}(A) = \int_\Omega I_A Z \, dP = 0$$

On the other hand, suppose $B \in \mathcal{F} \ni \tilde{P}(B) = 0$

$$\Rightarrow \frac{1}{Z} I_B = 0 \quad \tilde{P}\text{-a.s.}$$

$$\Rightarrow \tilde{E}\left(\frac{1}{Z} I_B\right) = 0 = E(I_B) = P(B)$$

Now, so once we define this right, (now our) what we state is that you know claim which we can verify is P and $\tilde{P}$ you know as defined area or above or equivalent okay with just the previous result that you recall we are defined $\tilde{P}$ using Z and that these two measures as under the condition that is specified in the result they are all equivalent, how do we see? This is a quick proof this in, this you required to know so we will just quickly do this.

Suppose you pick you know $A \in \mathcal{F}$ such that your $P(A) = 0$ is okay, Z as it is, Z is strictly greater than 0 with $E(Z) = 1$ and you are picking an A such that $P(A) = 0$. Now this should imply that $\tilde{P}$ of A is also 0 right. Now how do we see this? So since this is 0, you can see that the random variable indicator function of A and Z right, Z is strictly greater than 0, indicator function of A is 1 for omega in this one and otherwise this is 0 right, on a set of probability A this is 0 and Z is greater than 0 is what that you have.

You can see that this is equal to 0, in what sense? This sense, almost surely this is equal to 0, this quantity is almost surely equal to 0, so this implies my definition of $\tilde{P}$ A which we have here is essentially omega $I_A Z dP$ probability of A is integral over A of ZdP which is exactly equal to $I_A Z$ times dP, but $I_A Z$ is equal to 0 with probability 1, so this is equal to 0. So P of A equal to 0 implies $\tilde{P}$ of A equal to 0 okay.

On the other hand, suppose, you take a B in F such that your $\tilde{P}$ of B equal to 0, what does that mean? This means that this quantity 1 by Z indicator function of B this is equal to 0 almost surely, I am writing the same statement slightly differently okay, so what does this imply? This imply so almost surely under $\tilde{P}$.

So this means, that $\tilde{E}(1/ZI_B)$ equal to 0, which is equal to expectation of IB which is equal to probability of P, so probability of B. If I take $\tilde{P}$ of for a B for which $\tilde{P}$ gives probability 0 that also implies that P also assigns a value 0 so and hence this claim which is these two are equivalent is true. okay.

Now, as we said equivalent means that this is also saying that they also agree on which sets in f have probability 1 okay. Now, why do we require this here? Because we look for just like in the discrete case, we look for the equivalent probability measures is what we look for, so that just recall what we did in the discrete. We had a probable discrete time model for the financial market, their actual probability measure was P, we did the pricing under $\tilde{P}$ and we construct the hedge under $\tilde{P}$ and the hedge worked well in the real world, that is what we have seen, this is the cycle that you know you close, you start with P and then you come and close to the P.

Now, how did this work? Because the $\tilde{P}$ that we considered there was equivalent probability measure, so same thing here. So, we look for same thing also. We need look for an equivalent probability measure. So, we have a probability measure, which is P which is the real world probability measure and we considered an equivalent probability measure which is risk neutral measure, now that we can use the technology since you are aware about it, risk neutral measure and if this risk neutral measure is equivalent to risk this actual probability measure, a real-world property measure, then this risk neutral measure is you can use to price the derivative, construct hedge.

Now, since the hedge is going to work in risk neutral world in a sense or in almost surely sense, it will also work in real-world almost surely sense, if these two measures are equivalent. So what we look for in this model is again to construct an equivalent probability measure in this case okay.

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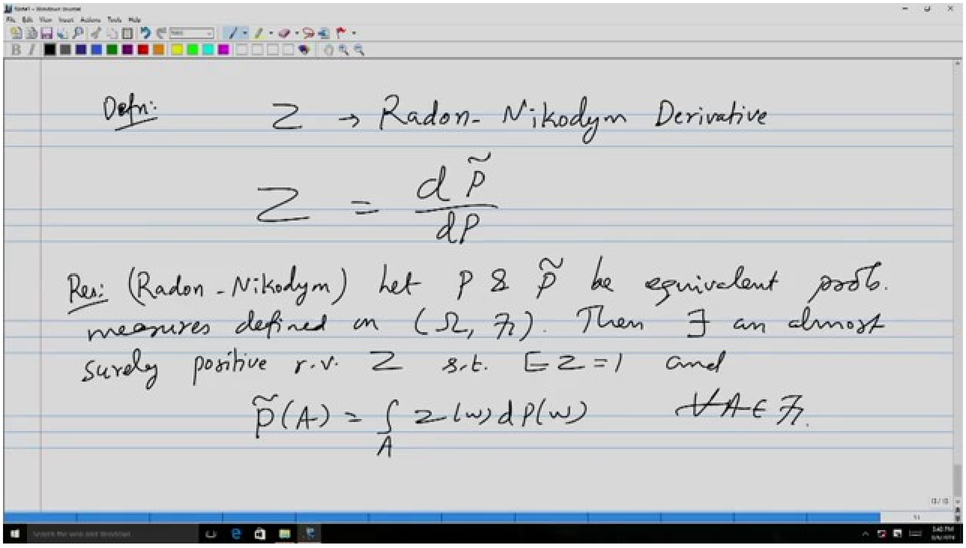

So what do we call this? I mean the definition is basically such an Z that we, you take all this thing omega F P be a probability measure and $\tilde{P}$ be another probability measure such that they are equivalent and let PR is a random variable with those properties right, so that is Z greater than 0 almost surely and expectation of Z is 1, then what is the Z is call? So that Z is what is called as Radon–Nikodym derivative.

And we write this $Z = \frac{d\tilde{P}}{dP}$ is what that we write, this is, so this is Z that we have. Now to construct an equivalent probability measure we need an Z. Now, suppose if I have to probability measures P and $\tilde{P}$ which are equivalent, now does there exists any Z? Only if I ensure that their exist an Z, I can construct that you know build that Z in some way or other right. So, that result is what guaranteed and the only way you can get the equivalent probability measure is the way that we are define in the previous result.

So, you need an Z, now does there exist an Z such that that is the there. The result is what called as the Radon–Nikodym derivative theorem which is what, the following. Now let P and $\tilde{P}$ be equivalent probability measures on which are defined on omega F then there exists an almost surely positive random variable Z, such that $E(Z) = 1$ and $\tilde{P}$ of A is essentially given by this, so this is what guarantees that their exists an Z.

So, what are the properties that you look for? Z greater than 0 almost surely and expectation of Z is 1, so if you can find out an Z such that, so this Radon–Nikodym is guarantees the existence, if there are equivalent probability measures then always there is an Z with the required properties, we just have to find what is that Z and that is what is call RN Radon–Nikodym derivative or simply RN derivative as we have seen in the discrete case the terminology is familiar, but this is the more general case that we have here, okay.

So, these are the basic ideas from the probability theory that from discrete and continuous when we want to move ahead with these things okay, yeah so we will continue with the remaining in the next lecture, bye.