Mathematical Finance
Lecture 24: Relation between Actual and Risk-Neutral Probabilities, Markov Process, Pricing of American Options

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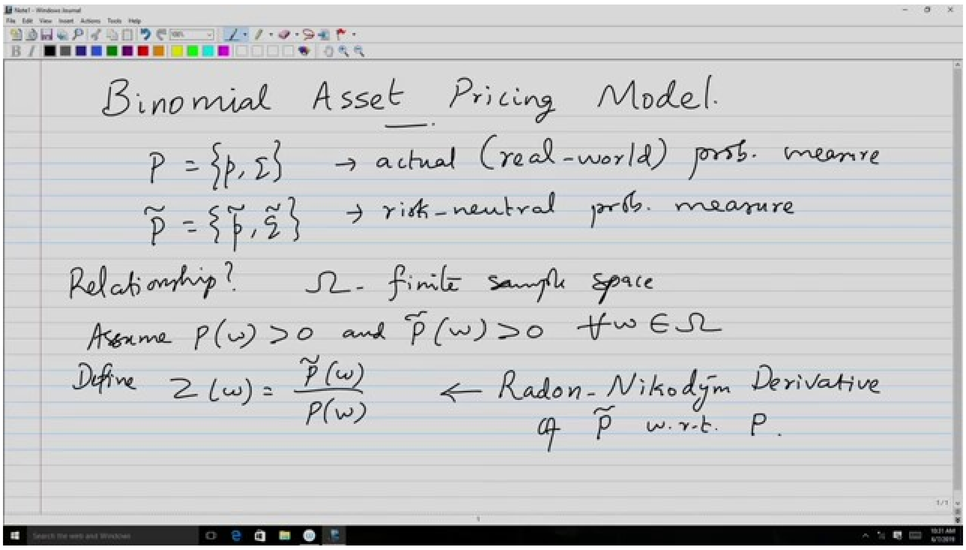

Hello everyone. Let us continue our discussion on binomial asset pricing model. In this binomial asset pricing model or CRR model, as also in the Continuous Time Black Scholes modern model, you would (you will) encounter or you have encountered at least in this binomial case you already seen that there are two probability measures with which you play around, one was the actual probability measure which, we gave in terms of this. So this is what we call actual or real world probability measure and the other one which was helping us in expressing the price of the derivatives in a neat form is this probability measure $\tilde{P} = \{\tilde{p}, \tilde{q}\}$, which we call it as risk neutral probability measure (okay) and you also know from where the word risk neutral came from and what does it mean, okay. Now in general, as I said in this binomial pricing model as well as in the Continuous Time Black Scholes modern model, you will encounter these 2 probability measures, which will play around.

One is the actual with which you try to use your statistical techniques to estimate the parameters, to fit the model or whatever you want to do. If you want to specify the model specifically and the other one is the usual construct that we have constructed in terms of $\tilde{p}$, $\tilde{q}$ and we gave expression of for $\tilde{p}$, $\tilde{q}$ in terms of the binomial pricing model parameters, right. That was helpful in to us in expressing the results, which is basically the prices of various derivatives in this binomial asset pricing model, okay.

Now the question will come like, you know, what is the relationship between these two? Is there any relationship between these two ? So, how do we move from one to the other, right? is the question basically. So this means, we are asking, what is the relationship between these two probability measures, right? That is what we are going to look at that now and in this idea is simple and you could see how exactly this follows but in the continuous time model, this will exactly be followed in a similar fashion but there as I said and repeatedly many times. Things will not be that easy to understand really, what is going on behind, but in case of discrete models, you know, you can easily feel and you can easily understand step by step how exactly the results follows, okay. Let us go ahead, so the general case what we have the scenario is, we have a finite space and assume that there are 2 probability measures, right, P and $\tilde{P}$ and both give positive probability for each of these omega in the sample space, okay.

So, this is what then your assumption and if you look at this P and Q we are assumed that P and Q both are strictly between 0 and 1 and under the no arbitrage condition of the model both $\tilde{p}, \tilde{q}$ are also between critically between 0 and 1, okay. In the case of Binomial model but this is you can say in general for any finite sample space situation. Now you will define a new quantity, which we call it as $Z(\omega)$ as the ratio, okay. This is what in this setup, it is actually the ratio, right, but when you move to the continuous model, this is really the derivative of one probability measure with respect to the other probability measure and there is a name to this particular quantity Z and that is nothing but Radon-Nikodym derivative or simply we will call from now onwards RN derivative of $\tilde{P}$ with respect to P.

So, this is what will be called as the Radon-Nikodym derivative and this is one useful quantity through which you know you connect different probability measures, I mean general (probability) that is what you know you will see in the continuous type. So here we have you independently constructed P and $\tilde{P}$. P is given, $\tilde{P}$ we are constructed and we are trying to connect Z. In continuous model we will use P, we will take P and use an z to get to $\tilde{P}$ that is way you know, we will define contest time

So the understanding this would be helpful when we move on to that stage, okay. So, this is really a ratio in the case of discrete type model but it is actually the derivative in general so we call that as the Radon-Nikodym derivative or RN derivative of $\tilde{P}$ with respect to P.

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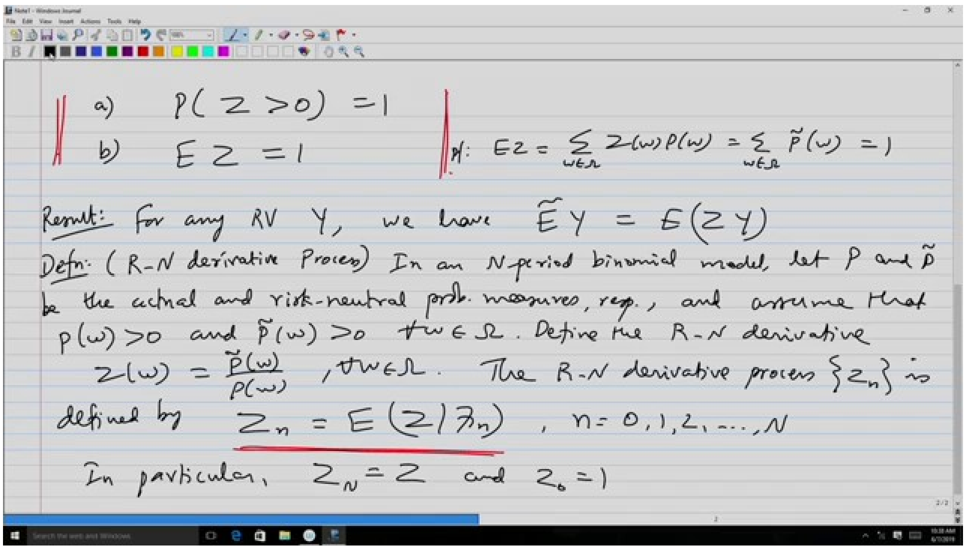

Now, there are certain properties of this is Z that which you can easily observe. A, $P(Z > 0) = 1$ and in fact, this required that Z greater than 0 with probability 1, that is what this statement says. But in this particular case, since we have assumed for each omega both gives positive probabilities. So, it is actually just greater than 0, almost surely, right, even I mean, for all omega right, which is more than this but this is sufficient, this you can observe easily from the definition itself and B, which is expectation of Z is 1, which is also from the definition Z you have defined it as your $\tilde{P}$ over P.

And this expectation is since we did not put $\tilde{P}$ it simply E, so it means it is expectation under P. So simple by definition so this follows immediately, right. To note expectation of Z is summation of omega over omega Z of omega, P of omega. Now substitute, you will n up with $\tilde{P}$ of omega and this is 1, okay. So really you will see later that these two properties are what going to be helpful in picking an Z. When, we actually want to move from 1 to the other or to pick an Z such that satisfy these 2 properties.

So, these 2 are very important properties that you will recall even when you look at the continuous T model in fact, that the Radon-Nikodym derivative is the one that satisfies these 2 properties how and you take it, okay. Now, there is a small result which is also follows from the definition itself, you take any other random variable, for any random variable say, we call that as Y, we will have $\tilde{E}$ of Y is given by E, Z Y. Number $\tilde{E}$ Y means we are taking expectation of the random variable Y under the probability measure $\tilde{P}$ and $E(ZY)$ means we are taking the expectation of the random variable ZY under the probability measure P. Now this relates, right, the expectation of certain random variable under 2 different measures P and $\tilde{P}$ and again, you will start with the right hand side, you know, you apply the definition just like in the first line that we have, this is a proof for that part that we have.

So, apply the same way you will be able to notice that this is true, right. So there is nothing much that you could do. Now let us define, so this is the Z actually it is suppose if you look at in the n period of binomial context, right. So this is, so far it is fine that for any finite probability space this is true but if you look at actually the binomial model setup, so this Z depends on all n periods in an n period binomial model. Now, we also like to see the corresponding quantity which are dependent on are which are measurable with respect to fewer than and n tosses, or fewer than n period of things right. So to do that we need the following definition, which we call as RN derivative process (okay), right. So what do we have? So in an n period binomial model let P and $\tilde{P}$ be the actual and risk neutral probability measures respectively and also assume that both this gives positive probability for each of those omega.

Then, define the RN derivative Z omega by the definition, usual definition that you do, then the RN derivative process, is defined by Z_n equals $E(Z|\mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$. You can see that in particular Z at the n is the same as the Z that we have because this is the n period binomial model and Z at 0 will be 1, right, because $E(Z) = 1$, that is what you know, okay. So, this is what is the case, that we call it as an RN derivative process. you are given two probability measure and (given) you define the RN derivative with respect to one measure with respect to the other. Then the derivative processes is given by this.

You know, recall that this kind of things we have seen when we see Martingales that most of the Martingales that we normally encounter are of this form. Now you see that RN derivative processes is one such for such random variable Z, right and this is what we have seen and since, as we just mentioned.

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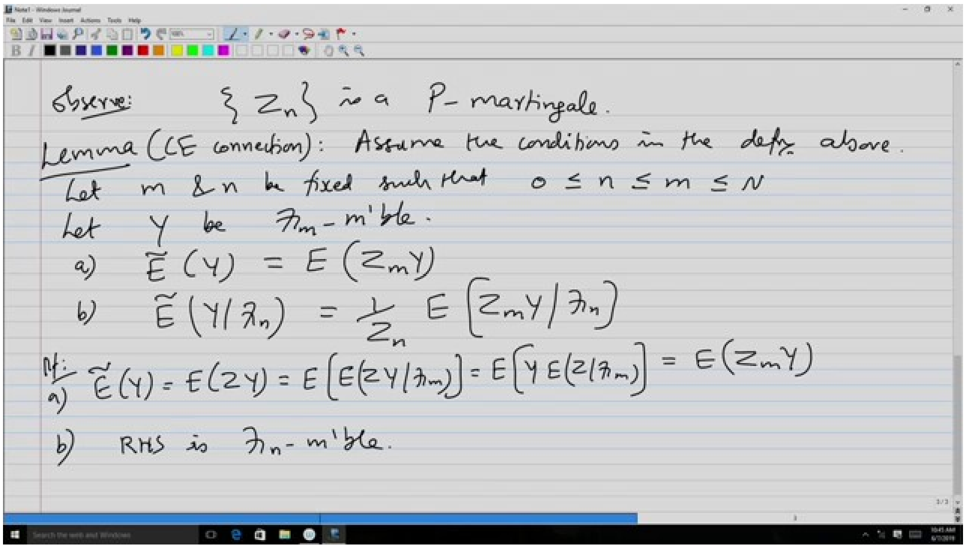

We can easily observe, that this is nothing, but. So you can see this Z_n, the process Z_n is a P martingale. So the RN derivative process starting from 1 at time 0, it is a P Martingale that gives you the Z at the end, which is what the RN derivative process, RN derivative is all above, okay. Now, there is a lemma, (which is what is actually gives us the) you know, we have already seen that how the simple expectations are connected between these two through RN derivative.

Now, the RN, through RN derivative process we want to connect the two probability measures in terms of the expectations, okay. So, this is lemma result or whatever you want to call. So this is basically the conditional expectation connection, is what then we will see. So, we will assume, the whatever we have written the conditions in the definition above okay. Now,(if you have) you will also fix 2 quantities, let m and n be fixed for the moment such that 0 less than or equal to n, less than or equal to m and which is also less than or equal to n, okay.

We will also take a random variable Y, let Y this be $\mathcal{F}_m$ measurable, okay. So, this is what you have. So you have whatever condition that is written in the definition and you are fixing two quantities m and n such that $n \leq m$ in the binomial model and we pick a random variable Y, which is measurable with respect to the, the higher index which is small m in this case, okay.

Now, part a of this lemma is the simple connection, which is much like the previous case which you can easily see Z_mY okay. So this is the simple relationship or simple expectation relationship with respect to the quantity SZ term and quantity Y which is $\mathcal{F}_m$ measurable, okay. This and then the next one, which is what really more general ofcourse from which the first part is a special case, Y given $\mathcal{F}_n$ is nothing, but 1 over $(1/Z_n)E(Z_mY|\mathcal{F}_m)$, right.

So, this is you know, as you see like the second part is more generic in nature and the first part if you take F, if you take the n to be equal to 0 then you get really the first part. So, that is not really anything big but this is the second part is what is most genetic, which is what you know you need to understand and do. So, to the proof the first part is very simple, okay.

Let us quickly see that this is actually expectation of. So, if Y, Z is the RN derivative then for any random variable, we have already seen that this is what it is, right, E and $\tilde{E}$ are connected through Z in this manner, Y is $\mathcal{F}_m$ measurable and hence Y is $\mathcal{F}_n$ measurable, right. So that means, it is there in the full space so and hence this result is true as we are seeing this part is earlier itself.

Now, this by the tower property, right, you can write it as expectation of, $E(ZY|\mathcal{F}_m)$, right, which is then since Y is $\mathcal{F}_m$ measurable. So you can write this $Y = E(Z|\mathcal{F}_m)$, right. So we are using the Y is F m measurable property that you are using and then you are taking this out. First is tower property, the second is the measurability property or taking out what is known, right every condition take note what is known property and once you are.

Now what is this $Y = E(Z|\mathcal{F}_m)$ as per our definition? Is simply Z_m times Y, right, so this is the first part, part A, very simple, okay. Now, for part B first thing is, what do you want to show? You want to show that in the part B part $\tilde{E}$ of Y given $\mathcal{F}_m$ right, is nothing but given the quality given by the right hand side, right. So, we will use the characterization to show that something, A is conditional expectation of some (know) some random variable given sigma field under some probability measure then you have to show the 2 properties of condition 1 is measurable with respect to that sigma field and the partial averaging properties should satisfy.

So you use that argument here and if you look at here So, what you would see the right, right hand side is what you have to show that $\tilde{E}$ of Y given $\mathcal{F}_m$ and then the right hand side should be a $\mathcal{F}_n$ measurable, right, you see Z_n is a $\mathcal{F}_n$ measurable. So this $1/Z_n$ is a $\mathcal{F}_n$ measurable and $E(Z_mY|\mathcal{F}_n)$, right. Whatever be this quantitative it is given $\mathcal{F}_n$ so this part is $\mathcal{F}_n$ measurable and hence, the RHS is essentially you can see is $\mathcal{F}_n$ measurable, okay.

Now this part we will utilize it, right so this is what you know we have seen that the RHS is really the $\mathcal{F}_n$ measurable quantity, right. Now there is partial averaging property is what we have to check, right.

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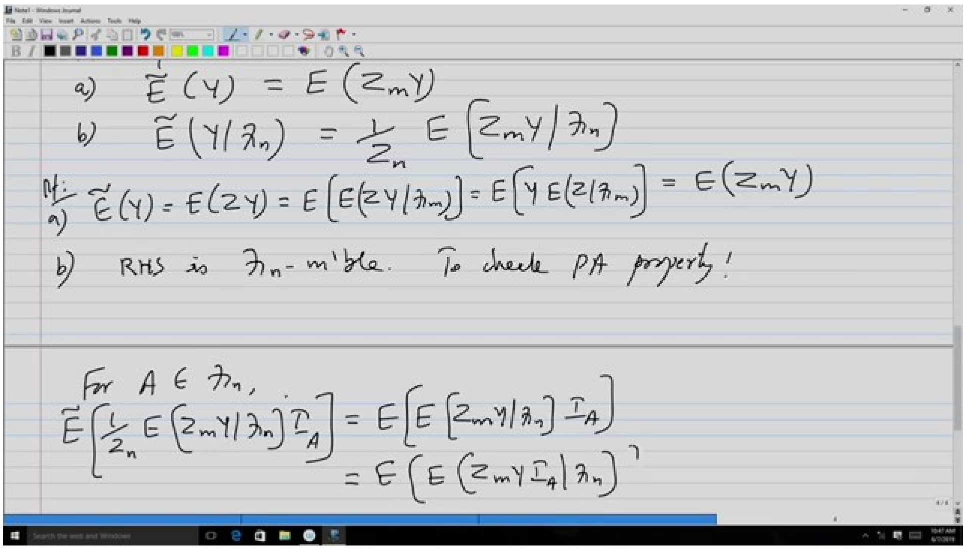

So to check PA property, right partial averaging property is what then we are check, right. So you pick $A \in \mathcal{F}_n$ and you consider the right hand side quantity, this is $1/Z_n$, $E(Z_m Y|\mathcal{F}_n)I_A$.

So, you end up with this quantity, right, is what then you have to consider in order to look at that. Now, if I use the part A of this lemma for an $\mathcal{F}_n$ measurable random variable, right. This quantity is $\mathcal{F}_n$ measurable random variable. So this quantity if I use, this is this whole thing if I consider as if the Y in the part A with for $\mathcal{F}_n$ measurable random variable because that is true for any m, right $\mathcal{F}_m$ measurable and variable then it will be $\tilde{E}$ of that quality would be equal to $E(Z)$ at that point, the RN derivative process at that point, multiplied by this Y.

Now here Z_n multiplied by so the whole quantity be multiplied by Z, so this would then take the form as $E[E(Z_m Y|\mathcal{F}_n)I_A]$. So it is basically we are utilizing the part A of the result from part A of this lemma itself, okay. Ideally you can define it as a 2 different results and you can use it but I thought it, does it matter too much, okay. Now, this quantity since my A, $I_A \in \mathcal{F}_m$.

So, whether this is taking out what is known property. Now, we are bringing inside. So this can be written as $E[E(Z_m Y I_A|\mathcal{F}_n)]$.

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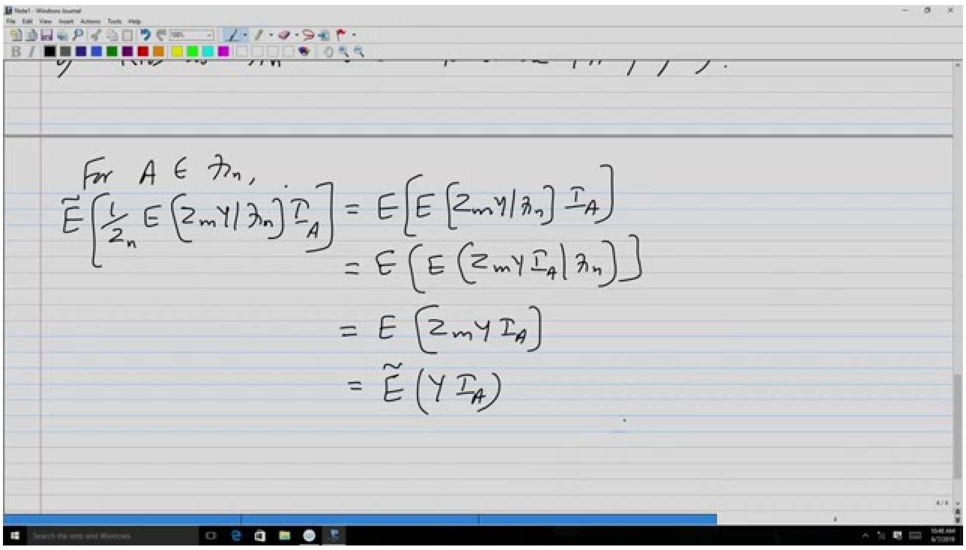

Now this, right, expectation of this if I take the average of all those condition expectation , what you will end up with? $E(E(ZmYI_A))$, which again if I use the property, right. So this will be equal to $\tilde{E}$ of Y, indicator function of A again sub part $\mathcal{F}$ or $\mathcal{F}_m$ measurable random variable, right. So, what we are shown here is that the quantity that we have defined on the left hand side under the $\tilde{E}$, right. So, without this I_A the quality that we have here is nothing but the $E(YI_A|\mathcal{F}_n)$ under the measure $\tilde{P}$. So, that is what we are shown here.

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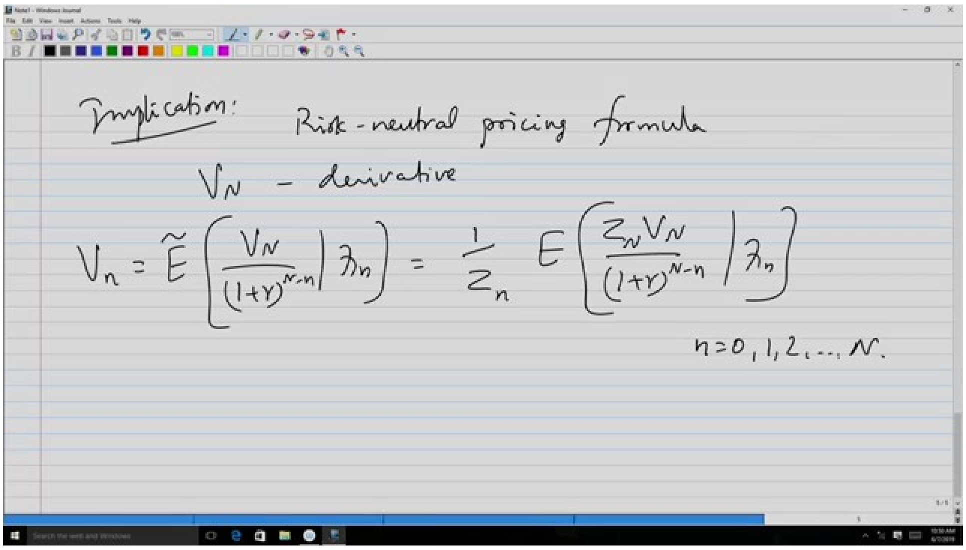

Now, what is the implication? Now, how this is helping us to do anything in the thing? The implication is basically on the risk neutral pricing formula is what we have set, right, with a formula.

So what we did? We have a derivative, right, this is n period binomial model that we have in mind. Now we know that $V_n = \tilde{E}(V_N/(1+r)^{N-n}|\mathcal{F}_n)$ right. Now, this is what? If I use the part B of the lemma for this conditional expectation of some random variable given $\mathcal{F}_n$ under $\tilde{E}$, if I transform this to E then what you would get is $(1/Zn)E(Z_N V_N/(1+r)^{N-n}|\mathcal{F}_n)$ for $n = 0, \ldots m$.

So, you see here there is no true pricing formula. Now, you can express it in terms of the real world probability measure also, right. So this would be same as this, that is what you have established, but now, you need to know what is the Radon derivative, RN derivative process. If you know RN derivative process then you can use the risk neutral probability, instead of risk neutral probability measure, you can use really the actual real world probability measure itself to compute the prices of the derivatives at times which is prior to maturity, right.

So, there is more one can say that you can consider the process $Z_n/(1+r)^N$ and that is called a state price density process and so on which if you are interested you can go through it, right. So, this is what is the connection between that we have established between the 2 probability measures P and $\tilde{P}$, okay.

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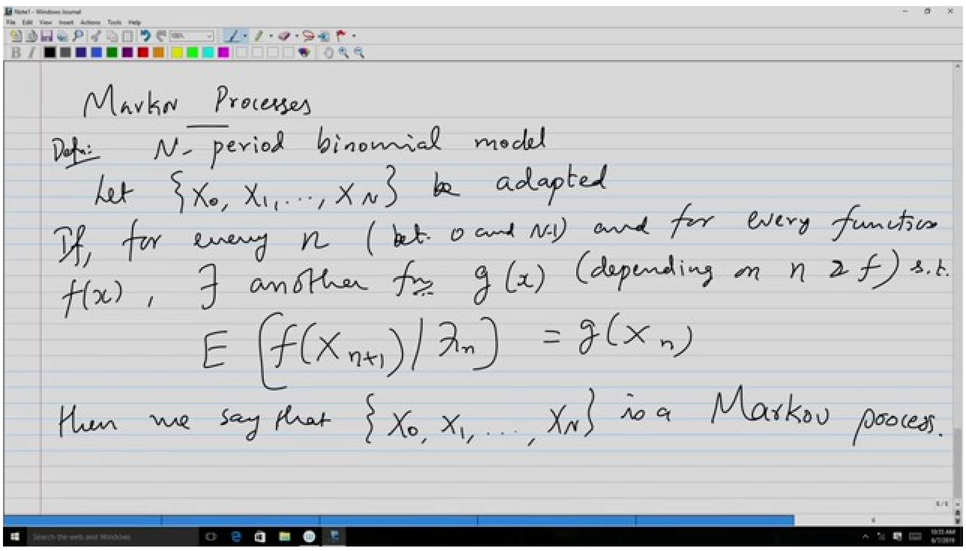

Now, let us will be coming back to the last part of the binomial asset pricing model consideration and the next what we will consider is something called as Markov process, okay. So essentially this is to determine in the binomial asset pricing model or in general to determine what is relevant and what is not relevant. That is what is expressed in terms of the Markov process. So let us define quickly the in binomial model what is the definition. So what we are considering? You know we are restricting our attention to the binomial asset pricing model, but with this definition is general and then period and n period binominal model is what then you are considering, in which you consider a process X_0, X_1 and so on X_n be an adapted process of course, you need the adapted process.

Now, what is the definition? If for every n which is between 0 and $N-1$, and for every function say $f(X)$, there exists or there is another function say you call $g(x)$ which might depend on n and f. So, for every f and for every n, right, there is another function g such that $E(f(X_{n+1})|\mathcal{F}_n) = g(X_n)$, then, we say that this process $X_0, X_1 \ldots, X_n$ is Markov process, right.

So, you might have seen Markov process definition in some other ways, does not matter? So, this is also an equivalent way which we defining in terms of conditional expectations, what really it says is essentially the same that if you had an idea about what is a Markov process. Look at this conditional expectations definition. You know some function of the process at $m+1$, given the information all the way up to n, right, which is what the best estimate is what this conditional expectation gives on the left hand side is can be expressed through a function of the process value at time n, right. So here essentially, saying the evolution from n to n+1 for the process to describe the process evolution from n to n+1 time period, all that information that is required from the past is the current state of the process which is given by X_n. So $\mathcal{F}_n$ is the complete information, X_n is the value of the process of time n, right. So this conditional expectation if this is a function of $g(X_n)$, we have already seen such examples like expectation of recall, the example that we did it for expectation of S one given S_2, which is the function of some such function that g, right.

So what we need? For every n and for every $\mathcal{F}$ if there is a g such that this is satisfies this then the process is Markov, right. So, which essentially means that the condition expectation is depends on $\mathcal{F}_n$ only through the value of X_n, right that is what we say, okay.

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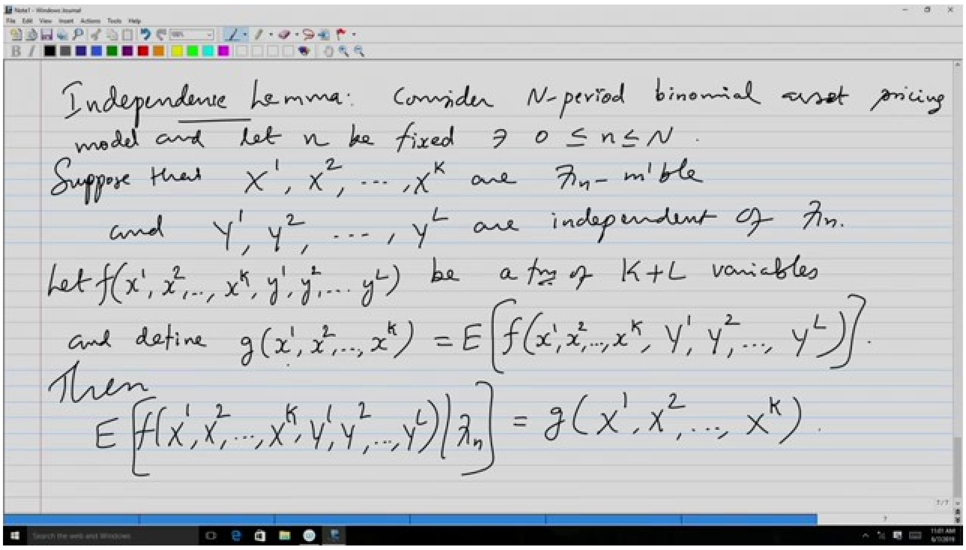

Now, there is a result, which we will use frequently even later to which will help us which gives us a tool to verify whether a given process is a Markov process or not, okay. Since this we might require this even later. So we will state in generality right, in the case of, consider n period binomial model itself. So we will just consider n period binomial asset pricing model and let n be fixed such that between 0 and n.

Now, suppose, that there are k random variables that you are picking which we call $X^1, X^2, \ldots X^k$, this is not the square but this is just the super script which we are using X^1 to X^k or suppose that these are

$\mathcal{F}_n$ measurable and you consider another some l number of random variables again this is our only super script, this is not squares Yat L which are independent of $\mathcal{F}_n$, okay. Now let f be a function of all, some function of this $k+l$ random variables, right. So this is f of $X^1, X^2, ..., X^k$, $Y^1, Y^2, ..., Y^l$ be a function of $k+l$ variables okay which are given by some function of some $k+l$ variables, okay.

And okay so what are the variables? It is essentially $X^1, X^2, ..., X^k$, $Y^1, Y^2, ..., Y^k$ to these other variables that you have. And you define a function $g(X_1, X_2)$ again this is super script, this is not power maybe we could use a better notation but does not matter understanding is clear, okay. Now this is expectation of f with these values and these arguments Y^1 to Y^L now are replaced by this random variables $Y^1, Y^2, ..., Y^L$ okay then, then what you have? given $E(f(X^1, X^2, .., X^k, Y^1, ...Y^l)|\mathcal{F}_n) = g(X^1, X^2, ..., X^l)$ this is what this is, okay. Let us try to understand what this tells, okay and so you have some k random variables which are $\mathcal{F}_n$ measurable some l random variables which are independent of $\mathcal{F}_n$, okay.

Now, ultimately, what you want to find? You want to find the conditional expectation of a function of this k plus L random variables, K of them are measurable with respect to the sigma field that you are considering and a L of them are independent of this is $\mathcal{F}_n$, then what this says is this conditional expectation is simply given by a function G of these random variables X1 to Xk which are measurable with respect to $\mathcal{F}_n$ and what is this function? This function is simply the expectation of some f (of this) you pick a function of this f that you have here, right.

You keep this $X^1, X^2, ..., X^k$ as the dummy variables small $x^1, x^2, ..., x^k$ and replace this functions capital keep this Y^1 and Y^2 as random variable and compute the expectation. Remember this is E, so it is we are talking about any (gen prob) general probability measure. So, in the binomial model case of course, if you want whichever measure that you want to consider that you know you will put accordingly. So (the) you compute the expectation of this function of this random variables remember.

Now, this part expectation of this part if I look at it, so, this part if I look at it, it is a function of these dummy variables X^1 to X^k and the random variables y^1 to y^l. So, you compute the expectation with respect to this L random variables by keeping this as fixed then you get this function which in the case of dummy variable x^1 to x^k and replace the dummy variable by the corresponding random variable. So, that is what this conditional expectation is, right. So this is what it gives, okay.

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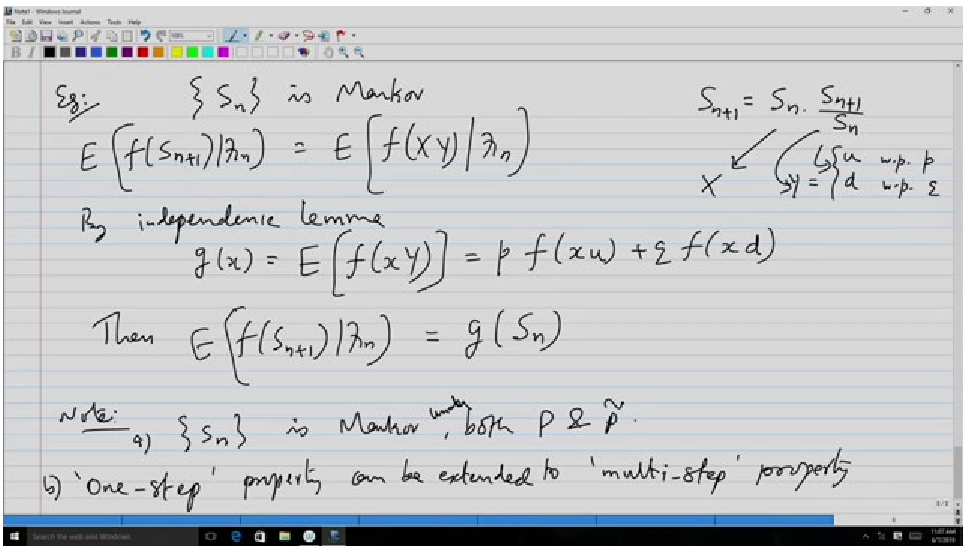

So now, this will try to use we have already seen it in the case of, earlier case itself. So in the binomial model, we will say that, we will see that how this helps to identify that this process is a Markov process, okay. So what we have to show? So this process is a Markov process so far every n and for every f there is a g such that expectation of f given S_{n+1} given $\mathcal{F}_n$ equal to some function $g(S_n)$ that is what we want to show.

Let us see how we can show. So we consider expectation of some function of S_{n+1} given $\mathcal{F}_n$ but now you see that you are writing it as some function S_{n+1}, which is neither measurable with respect to $\mathcal{F}_n$

nor independent of your $\mathcal{F}_n$, right. That is clear, this is first $n+1$ kind does it depends the stock price at M_{n+1}. So it is not independent of your $\mathcal{F}_n$ and also it is not $\mathcal{F}_n$ measurable right this function of S_{n+1}.

So, we need to figure out a way how to do that. So this trick is what generally you will use, so you can write S_{n+1}, okay. So you can write my S_{n+1} as $(S_n S_{n+1})/S_n$ where, what is this random variable? This takes value either u or d with probability P and with probability Q, right. So this is S_{n+1}/S_n is just the result of the $n+1$ of the kind task which is either head or tail which gives the stock price process either up or down with corresponding probability is P and Q and S_n is a quantity.

So now, you can see that if I pick my this quantity as my X and this quantity as Y which is equal to this, right. Now you can see that this one is essentially $E(f(XY)|\mathcal{F}_n)$, right. So of course this is a function of xy. So this is some function of some $f(X,Y)$ you can write which is actually given as $f(xy)$, right. So this is a function of x and y, right, now by independence lemma, because now we get to express this quantity as in terms of two random variables X and Y, one of them is measurable with respect $\mathcal{F}_n$, the other is independent of your $\mathcal{F}_n$, right.

So, now I can apply the independence lemma to get what is the G of X function,, which is expectation of f x y, right. So this quantity which if I remember, when Y is independent of f and expectation of f Y given I mean $\mathcal{F}_n$ would be simply expectation corresponding to the random variable itself because independent of $\mathcal{F}_n$. So that is what from where this come. Now what is this function? This is $pf(xu) + qf(xd)$, right. This is the function of G. Then $E(f(S_{n+1})|\mathcal{F}_n)$ is simply $g(S_n)$, right. So but how we have shown? We have shown that sn is Markov, we just you took the definition we wanted to say and then use the independence lemma.

So, whenever you encounter random variables, which are neither measurable with respect to a particular sigma fields are independent then you try to find the random variables some X_1, X_2, X_3 and so on some X_k, Y_1, Y_2, Y_l such that some of them are measurable with respect to $\mathcal{F}_n$ and some of them are independent of $\mathcal{F}_n$ and you try to express that particular random variable into consideration as a function of these k random variable then apply independence lemma to actually compute.

So, it is essentially what we are doing is we are completing the condition interpretation in this manner, right. So, that is what we are doing it here right. So this is Markov under this. Note, (we did not), we took E and P Q which is, if you are considering the binomial pricing model, this is actually the, the actual probability measure but S_n is Markov irrespective of under both P and $\tilde{P}$, okay. And this property what we call another note A, this is note B the one step property can be extended or it can imply or it can be extended easily to multi step property, that is also you can answer. So, because if you take once one more step right, with respect to n minus 2, suppose if you take and this then of course, you will this. So this is also true that the multistep property also it will imply in the one step property.

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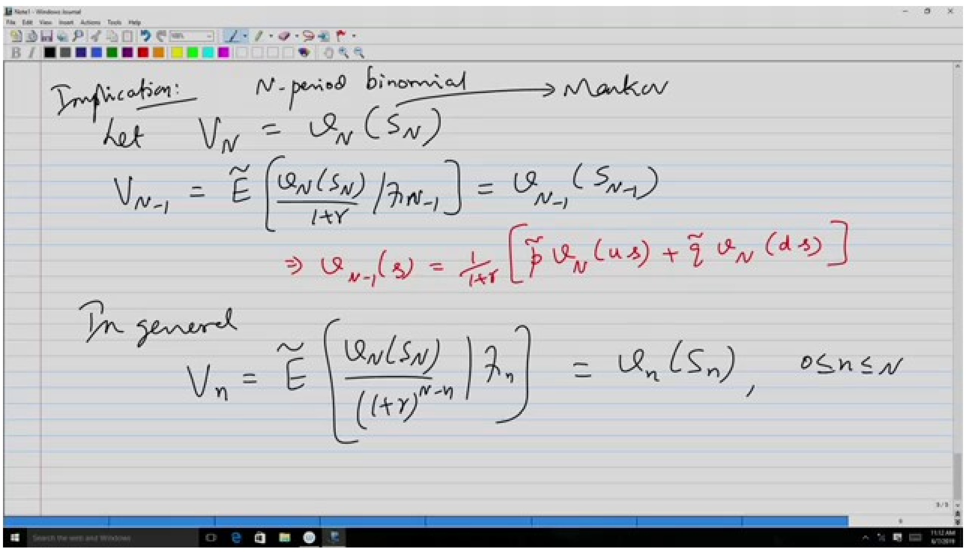

Now, what is the implication? Which is what is relevant for our discussion we have this N period binomial model only in mind, okay. Now in this case what we had. So, we had suppose if, you take a random a derivative V_N, which I call it V_N this is a function, some function of S_n, right. What are these derivatives? We can take simple European Carl, European put and so on, but this will not be a path dependent option if you take then this is not true, but we are looking at the simpler case where this is a function of Sn, right.

So, what this says is that if you have a derivative, which is a function of S_N alone means you are expressing a European Carl, European put forward contract all such derivatives will fall between this. So any path independent derivative is what, then my V_{N-1} right, would be $\tilde{E}$ of this quantity by 1+r given $\mathcal{F}_{N-1}$, right. Would now be if so, $V_n(S_n)$, right. So, here as S_n is basically a Markov process. In general even if you consider path dependent derivative then you have to find the corresponding Markov Process and express the derivative in terms of the Markov Process a function of Markov Process, but here we are taking simpler case Sn is Markov, which we have already shown.

So, we are taking this 1. So, this is some function of you know Markov process given some $\mathcal{F}$ say $n-1$. Now we know that for every n and for every function there is another function which is true because your S_N is Markov here. So which means that there is another function V_{N-1} and this is a function of V and S_{n-1}, right. So, this is what you know you will see, so this actually if you look at a little bit more closely, so this if I can write if I fixing the value, right.

So, my this particular step right the right hand side equality, if I fixed for some S then I can even write what is my B n of Sn. So that is useful. So let us write that part. So this implies that my V_{n-1} at S for some is $1/1+r$, $\tilde{P}$. V_{n+1}, in general if I write. This is n and $n-1$ we have so this is I am writing for any genetic step, plus $\tilde{q}$, okay. So I can write, why to write? So, I will write only for this topology.

So this is true and this is V_N of dS, right. So, we know what this function is and so on, right. So this is essentially, it gives this relationship between V_{N-1} and V_N and this is also true for any 1 step that you will look at it here, okay. (Now, in general you can also see that my V_N would be.)In general since you see the left hand side V_N equal $\tilde{E}(V_N(S_N)/(1+r)^{N-n}|\mathcal{F}_n)$ is then would be by the multi-step property of this Markov process, right, multistep Markov property.

Then this would be (some function at a time n,) at a time small n will be a some function of S at n, right. So, this is what will turn into. So, this is what is important step that you have that mean if your derivative is expressed in terms of Markov process then at any time prior to its maturity, if you look at its value, the price of the derivative then that is some function of the Markov process at that point of time, that is what really you get here.

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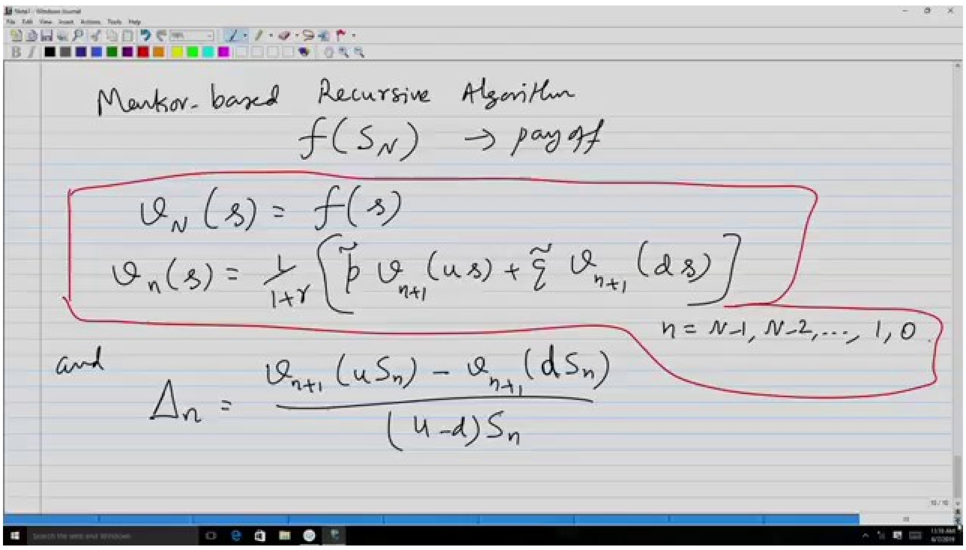

So this results, now we have the algorithm for computing the binomial the price of derivatives in the binomial asset pricing model. Now we can look at the Markov based recursive algorithm, right. So, which is basically what you have? You have a European derivative with pay off as this, that is what you have, where my S_N is Markov. So what will be the algorithm now? So my $V_N(S)$ is $f(S)$, right and my V at n is express as above.

And accordingly you can also write the delta hedging strategy in this particular case as above. So the corresponding delta strategy you have. So, what we have this is the algorithm step that you have. So this is what is the Markov based recursive algorithm. Now, if you want to computationally implement the binomial pricing algorithm efficiently then one would always look for a Markov based version of the basic binomial pricing algorithm and this is essentially that, that will give us, okay.

So, this is the implication of this Markov process once you identify that then of course you can simplify the computations, you can simplify the algorithm and you can do an efficient implementation because binominal asset pricing model is actually practically used with the large number of periods. So, when you go for large number of periods such an efficient algorithm implementation is what is required to be done if you have to really implement that theory in practice efficiently, right. Now, with this idea of Markov based thing here.

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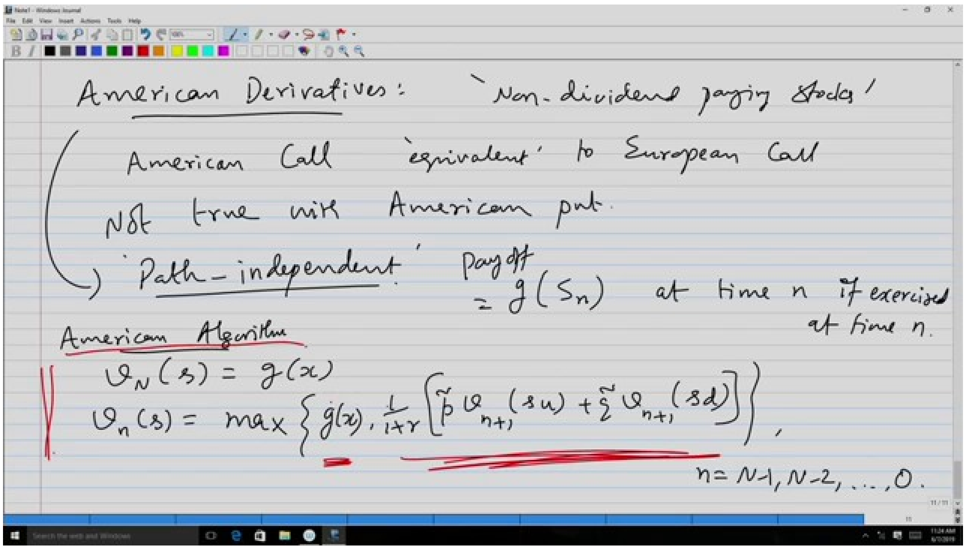

Now let us consider the American Options or American derivatives, okay. So you can make it American derivatives what is that? You know we have already seen the, you might be the calling, what is an American derivative and European derivative mean, right? So in the case of European derivative, the difference is only on the exercise states, right.

In the case of, suppose if you consider the n period binomial model we will restrict our attention towards that, then what you have is the European case derivative means it is exercised at only had the maturity time which is N but in the case of an American derivative, it can be exercised at any time including the maturity or any time prior to maturity also. So, which means the they are in a maturity time and exercise time or one and the same but here maturity time is N and exercise times are 0, 1, 2 and so to all the way up to N, that is what, right.

So, that what the main difference is that the feature additional feature that you have, is the early exercise option, now does it demand a premium? And which cases it is? Okay, so simple case since (we all) we have also considered only the cases with where in the underlying stock does not pay a dividend. So, we will confine ourselves to that discussion on that because binomial models are whatever we have considered we did not consider the case of dividend paying stocks it is only that non dividend paying cases that they we have consider.

So, keeping that in mind we have already seen in case of at least through one derivative that (American call option, right, we have seen in the basic idea) American call option is in a way equivalent to European call because we have non-dividend paying stocks, okay. So, European call, that means in this particular case, right, what you have is that the early exercise does not demand a premium, because it is always in case of this call it is always optimal in a way quote, unquote optimal to exercise our d maturity.

If it is the non-dividend paying stocks and hence you have this American call in some sense in quality European call but the same is not true with American put options, So, we will now generally consider American derivative which is path independent versions only, right, path of the dependent version one can do similarly but you know, we will not be handling that part.

So, path independent case is what then you have which means what you have a derivative which pays off $g(S_n)$ at time n this is which means that the payoff is equal to g at time n. If exercised our time n that n could be any time between 0 and N in the binomial set up. So what you have is this setup, right. So you have a path independent option, so let us take you know this $g(S_n)$ some function where the g is actually the payoff function at any time n.

If you do not exercise ofcourse then you got to hedge into the future, right. Now earlier you recall that how did we determine the prices? If you go back to the arbitrage lesson theory argument itself that you started at the last step was last time point n which is what is the payoff you determine at each state then you determine the amount required at the previous time point in our to hedge from that point to the next point. And you went this backward in time all the way up to 0.

Now, what you have to incorporate? You have to incorporate the additional feature which is that not just that you have to consider that the option is going to be not access at time n and it is going to be exercised later which is what the European case is. Now you also have to consider additional feature that the option can be excessive at time n. So, the amount or the amount that you require at each time should be sufficient enough either to pay off if it is excess at time n or to hedge if it is not exercise at time n, okay.

So, this suggest our previous algorithm that we have returned the Markov based one to modify this as follows, okay. So, the American algorithm, American derivative pricing algorithm would then be equal to my V, I am taking with respect to this Markov $g(S_n)$. So it is Markov based. So, at any time this will be the case and so, all these ideas that you are taking it here, so this step would remain at this. Now V_n at S would be let us write that particular step which we return earlier, which is this part right.

So, this is what you had earlier in the European case, this is what the algorithm, right. But now as we (express) explained, that this particular quantity is required if you if it is not exercised at time n and if it is to be hedged for the future. So, the amount that is required given that the V_{n+1} is you know is what given by this expression, but now, you have the additional feature that you know this can be exercised at time and now if exercised then how much you need? You need $g(S)$.

So, this Vn should be at least $g(S)$ and in V_n should be at least this much. So that suggest that, this quantity be the maximum of these two for $n-1$ and all the way up to 0, right. So this is then the American algorithm, where what you are looking at here is that it is a maximum of this is the exercise at time n part and this core part is if it is not exercised at time n and at each time, then what you need to have is this at least either this or that provided both options you have to consider and accordingly you have to come to the place and hence this is what it is, okay.

So, this is what is the American pricing algorithm in case of path independent operation exactly the same if even if you have your engineering derivative instead of $g(S_n)$ if I take any $\mathcal{F}_n$ measurable variable as the payoff time n then I just have to replace this $g(x)$ by appropriate that quantity random variable and that is what we will we will do that, okay.

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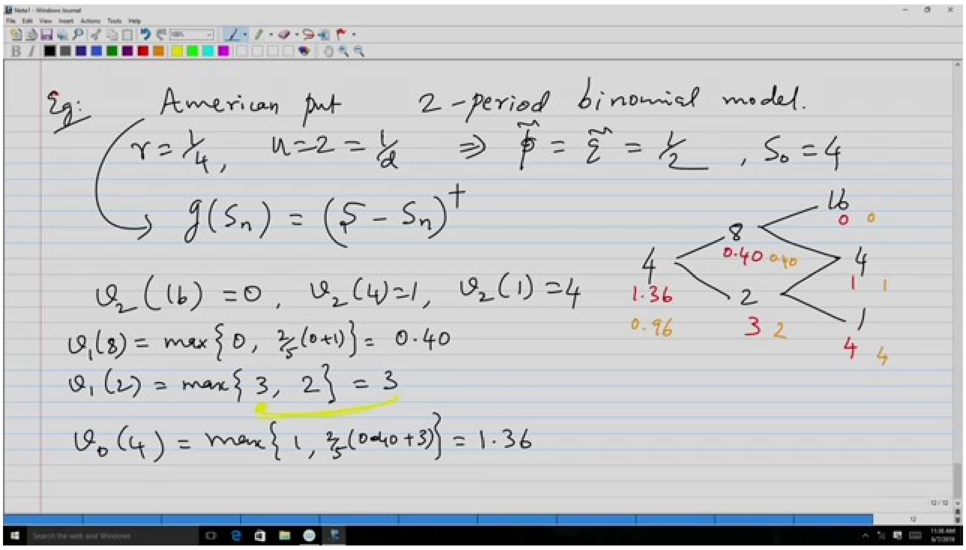

So now let us take the example, let us take the example of an American put in a 2-period binomial model, So what you have? You are, your usual things that you assume is equal to 1 by 4 u is equal to 2 which is equal to 1 by D and this gives my $\tilde{p} = \tilde{q} = 1/2$. Now American put so the payoff. So, my $g(S_n)$ if I had to pay off at the time a S_n would be $(5 - S_n)^+$ in the positive part, is what the payoff that you will have.

Suppose which means that this is American put with the strike 5, right. So then if I simplify, if I simplify the R you can write down the algorithm step in a simplified manner and then do the computation or otherwise you can directly do the computation with respect to each of these 2-period binomial model. So 2 periods are the period, the (period) binomial model would be so you are starting so I have not provided the S_0. So S_0 is my 4. So s knot becomes either 8 or 2 and this becomes 16 and this becomes 4 and this becomes 1, is in 2 period, this is a setup that you have, this is the underlying stock price process at various notes and for various tosses. So this is, since it is a Markov based and we have incorporated American within that. So these prices at time 2 you can easily compute at each of these nodes, this is going to be 0, and V_2 at 4 is going to be 1 and V_2 at 1 it is going to be 4, right.

So, if I look at the prices, so this is going to be 0 and this is going to be 1 and this is going to be 4 is the payoffs that you need at each of these nodes, okay. Now let us go back to the previous step, we want at 8 will turn out to be maximum of which is 8 here suppose if I look at here, it is 5 minus 8 the positive part which is 0 or the other part, recursive part, okay.

So, this is maximum off it will turn out to be 0 and if I compute 1 step from the 2 values 0 and 1 and if I take the $1/(1+r)$, $\tilde{p} = \tilde{q}$ both are equal to half so, this is 4 by 5 times or you can write say 2 by 5 times 0 plus 1 which is equal to now 0.40 and similarly, V_1 at 2 will be max of now 5 minus 2 which is 3 and this part again will turn out to be 2 between 1 and 4 if I sum it up and then multiply by 2 by 5, then I will get this 2 which is 3.

So, and okay, next step also will compute and then I will see. So V_0 at 4 for which is the time 0 price will be the maximum of 1 and the discounted value of 2 by 5 into 0.40 plus 3 would be equal to 1.36, so if I really write down the prices are to each nodes, so, this is going to be point 4, 0 and this is going to be 3 and this is going to be 1.36, right. So this is what will turn out to be the prices of the American put option in this particular case, right. Right? so this is how you compute and once you have this of course, we are already given the delta formulation remain the same, there is not there is no difference between once you compute the V_n then according then you will compute the your delta hedging strategies in the usual manner, there is no difference here, okay. Now, if I look at now here, if I compare with the reset the corresponding the American call on a non-dividend paying stock is the price is the same as the corresponding European collection, but which is not the case in American put option.

Now, if I compare this, so what would have been the price of the you know, the corresponding European call option prices, put option prices? If this is not where American, right. If I look at that, now let me put with the different color the corresponding values. So, that remains 0 here, 1 here, 4 here, because there is no difference in that. Now between 0 and 1, since I do not have the what? I do not have the early exercise feature then wherever I wrote this function maximum of something (comma) something the second quantity, I need to worry only about the second quantity because the first quantity, because it is not possible in the corresponding European case, right.

So, the second quality and if I look at V_1 at 8 and the maximum is attained through the second quantity, so that is going to remain as 0.40 there is no change but whereas in the case of V_1 at 2 the maximum is attained through the first quality, which is the intrinsic value, what we call with the early exercise option gives 3, but if you do not exercise, you know you need to, right. So but in this case, the corresponding European value is going to be 2, And now once this is changed, of course, this 1.36 is not going to be, so in place of when you compute V_0 of sigma, you will complete the second quality. But now with 3 replaced by 2, if I compute, you will end up with 0.96, right. So if I look at the time 0 price of this 2 period American put color, American put option, then the European put version the price is going to be 0.96 and American case it is going to be 1.36.

And at the first time, in the up node, both prices are the same and the down node the prices are different and so on, you are observing, right. Now these observations has some meaning, right. Now, this also tells you when to exercise the American put option, right. So, if I look at the down node at time 1, what this tells is that the American value is 3 but the corresponding European values 2, right.

So, that means if I, if I am the holder of this option, if I have a long position in the option, then it is optimal for me to exercise at time 1, if these stock prices down at time 1, right. If I do not do it, which means what? If I do it I will get the value 3, if I do not do it, right, then I will lose that value and at time 2, I will get either 1 or 4, okay which on an expectation on an average value even if whatever probability that occurring to the you know in this case we will do so, it is you know, will be less than and what is the r here ?

It is one fourth, so 1 period, but whereas if I take 3 if I had done this, I would have a fixed a quantity of 3.75 and if it is random payoff, then it is 1 and 4. So, right is what then. If it is possible that I may get 4 but it may possible that I will get the bit 1, right. So, but whether if you compare this, these 2 then obviously one would prefer 3.75 over the random pay off for 0.4. So, obviously, this at time 1 then the down node you have an optimal exercise point. And that tells you wherever if we look back the V_1 at competition of V_1 at 2 in the case of American put. So this optimal is basically this is (correct) connecting to this part, right. This 3 is obtained through the first quantity. So those are the time points in which it is optimal to exercise because to continue hedge, okay. If you will not do what will happen to the other part? The shark position holder, what he will do? He knows, if you did not do this, then he will need only (you know) he can consume 1 you can compute and see (you can) he can consume 1 and he can hedge the remaining 2 because that is what the corresponding European case is.

He can hedge for the future to reach 1 or 4 you only need 2 at a time 1 to get to the value either 1 or 4 in time 2, right and that is the hedging value that you have. So he has the benefit in this particular case, if you let go and opportunity to exercise at an optimal time then the shark position holder will gain advantage. So, in this case, the option holder, if option holder if you face to exercise at time 1, if when the stock price is down then he look he lost an opportunity.

So, this American pricing algorithm or American pricing problem is associated with also the question of when to exercise optimally, okay. Which we will not explore in detail, but ofcourse, this is a very complex theory behind when if you want to go further into that, but in simple binomial cases I already explained how one can detect, whenever the maximum is obtained to the first quantity you will always have an optimal exercise time points at those nodes, right. And which you can express also in terms of the other properties, okay.

So, this is with this you know, we end our discussion on this American options, we have considered only the path independent payoffs in case of American put, American cases, which we have a slightly modified the European case to account for the early exercise and then we are priced it, in fact American put we can any path dependent function then this is similar way and you can also identify what is the corresponding European and when it will be accessed optimal to exercise the American option.

So, with this, we end the discussion on this binomial pricing model which you have considered both you know the European case and American case and we have considered the pricing problem in both these cases and which prices main result was this risk neutral pricing formula, which he gave us the prices and the algorithm can be used to compute the prices if you want to implement this okay. So, we will take up this with this we end the discussion and will take up the other issues later. Thank you.