Mathematical Finance
Risk-Neutral Pricing in Discrete-Time
Lecture 23: Risk-neutral Pricing of European Derivatives in Binomial Model

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Hello everyone, let us continue with the discussion on the martingales and subsequently within the context of binomial model, how this is being used to arrive at what we called the risk neutral pricing formula. In this context kindly recall the binomial model that you had considered few lectures ago and you did consider a general derivative which is obviously of European type derivative and you arrived at a Binomial pricing algorithm to price a derivative. For that matter, any derivative can be priced in the context of binomial model, that is what you had seen, so that is what we called it as binomial pricing algorithm through replication. Because through constructing hedging strategy you had priced or you arrived at quote unquote fair price and that fare price is what we call it as the no arbitrage price because the principle behind the fare price was this no arbitrage principle.

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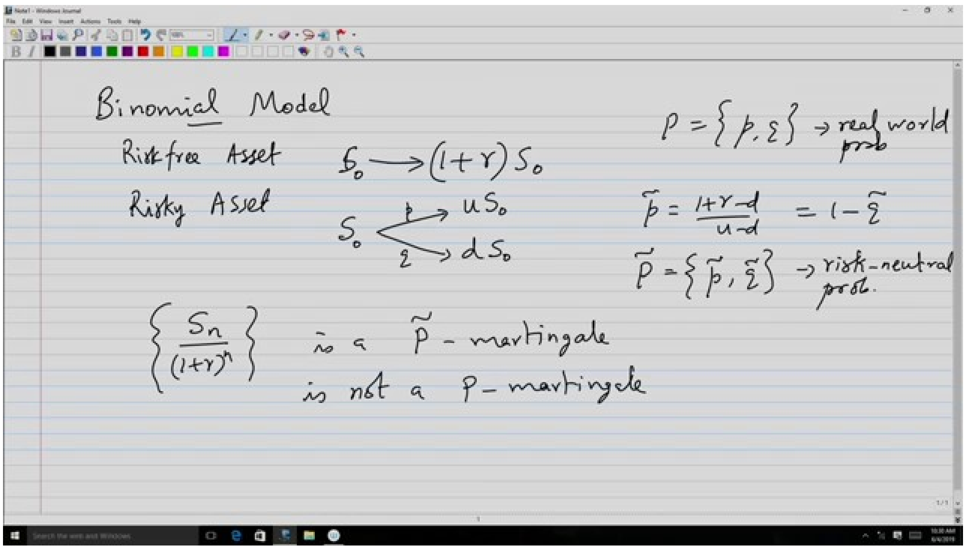

So, that binomial model is what we are going to take up again and we are going to rephrase the same pricing frame work using the risk neutral probability measure which, is what you already used it and using the same expression we will try to express in a different way through conditional expectation and that will then became the back bone when you go to continuous type model. Remember that in the binomial model, we had an N period binomial model that time was discrete, so you started from the last time point that is if it is an N period binomial model then you started at Nth period and you computed how much you require to hedge your positions from period $N-1$ to N.

And then once you determine all the $N-1$th period values then you went one step backward to $N-2$th period and so on you way went all the way up to V_0. But in continuous time model, of course there is natural naturally there is no time step, so you need to go in continuous, so this frame work that we are going to put up is what going to help us to write down and price in the general continuous type model which is the black souls model.

So, keeping that in mind, we are going to put the same thing that you had obtained earlier in a different using condition expectation, which will be generalized to the continuous type model when we actually consider it. It will recall what is the binomial model, so you have this useful assumption for this binomial model which were essentially you had assumed that there is an unlimited short selling of stock can happen and it is a friction less market which means that there is a unlimited borrowing and unlimited you know selling can happen and there is no transaction cost the buying price and the selling price of stocks as well as the risk free assets they are all equal.

These are not really the assumptions that you would encounter in a real world market but this way make it for a first cut mathematical approximation and later on you can expand or you can relax this assumptions to get to the bottom of the exact model. That is typically as you know the mathematical modelling frameworks.

And that is what you know you had assumed. So, now so that under the such assumptions and you had binomial model, the you had a Risk free asset, which could be you could call this as a either a bound or a money market whatever it is, it is risk free asset in the sense that 1 rupee invested at time 0, would become $1+r$, this is the risk free asset model.

And there is risky risky asset, what happens to this? (So, this) So if I multiply this by some value does not matter, so here, I can write S_0 becomes 1 plus r times S_0. So here S_0 in the Risky Asset, how we are modelling? This will become either u S_0 or this will become either d S_0, with say certain probability p an q, respectively. So, this P which is represented by this probability distribution p,q is what we call the real world probability, our real world's probability measure or real world probability.

So, this is the model that you have. So, you have a 1 risk free asset and 1 risky asset, risk free asset could be a bound or money market or anything of that sort, risky asset means you know you have this S_0 becoming u S_0 or d S_0 there is only 2 possibilities that we are assuming and hence this is name binomial model.

So, this is a set up that you have and p,q are the actual probabilities. We did define 2 other 2 other quantities $\tilde{p},\tilde{q}$, which we are forming again a probability distribution which, we called it as a risk neutral measure. Now, so which, is if we define say $\tilde{p} = \frac{1+r-d}{u-d} - 1 - \tilde{q}$.

Then this probability measure which is given by these 2 quantities is what, we called it as risk neutral probability measure or risk neutral probabilities. (And what we had observe with respect to that) so in real market, when we look at the martingales one of the examples that we used, is basically we looked at this, so, this is in one period what happen, suppose if, can extend this to an N period binomial model then (you have) you looked at this price process, this price process which, we called it as the discounted stock price process.

So, this price process, so for whatever time period suppose if you consider this as an N, periods are there for the in this binomial model then this process runs from 0 to N. And this was we are looking at this, now what is this really? This is basically, you can look at S_n as the time n-th value and $(1+r)^n$ you are discounting by that, so you are discounting back in time to time 0, so in a way.

But this is S_n is random variable. So, this $(1+r)^n$ you know, what is the growth factor up the corresponding risk free asset. So, essentially you are basically comparing the risky assets growth with respect to the risk free assets growth, that is what you know, this discounter stock price process would represent and we had observed that this is what the behaviour of this under $\tilde{p}$ or you know under p we can observe, what would happen to this. Now, so we said that this is a $\tilde{P}$ martingale of course, we will rigorously prove this in a moment, but this is not generally a P martingale. So, there is a (you know) property or the meaning that you have when we said that $\tilde{P}$ martingale which means, that the given some time n's value of this process. The value the process at time n plus 1 given the information up to time n is the best predictor is given by the conditional expectation, is nothing but the current value of the process itself. That is what, it would mean $\tilde{P}$ martingale and this is not in generally a P martingale but then in real world what we expect is that, so this would be, you can in a moment you can see like what you will expect in real world what it to be the case.

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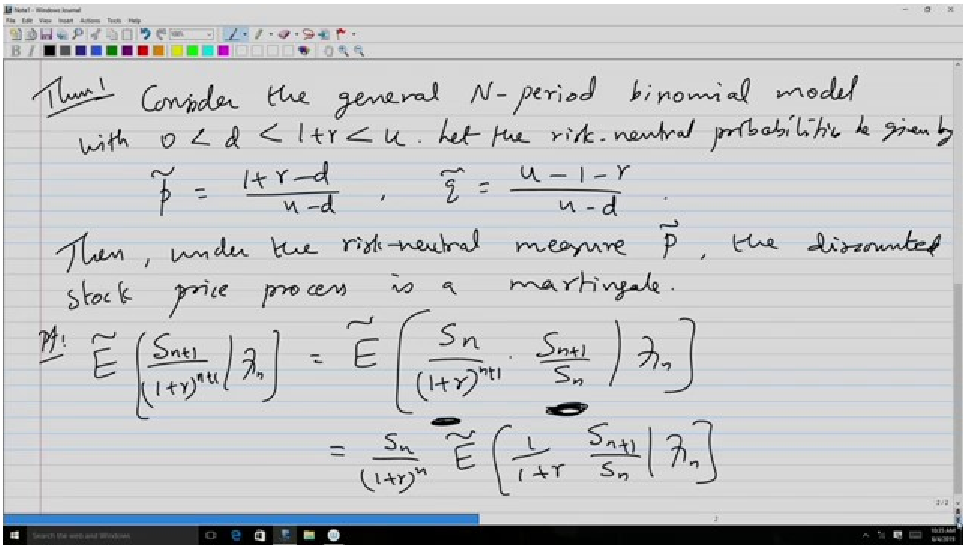

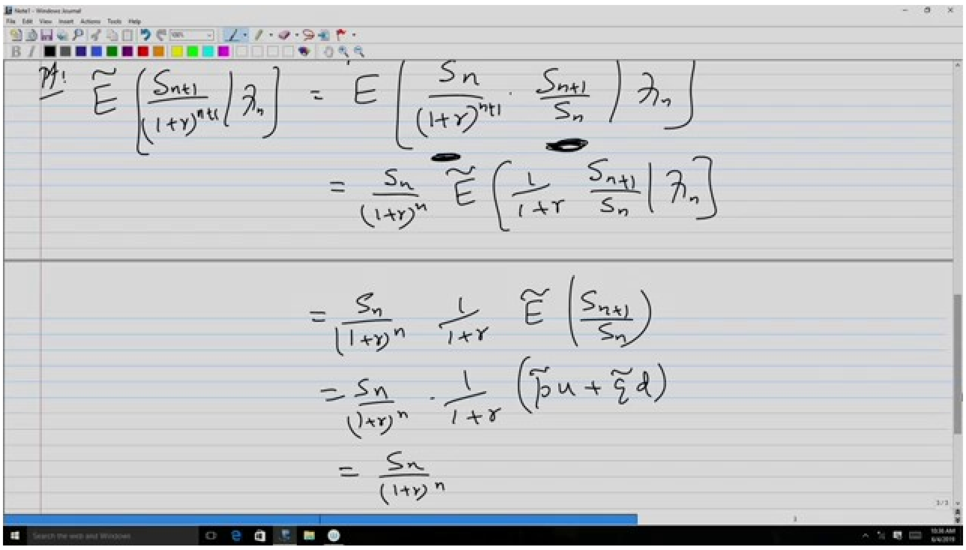

So, first let us prove the first fact which is basically we call it as some theorem, say 1 which a now let us consider the general N period binomial model with $0 < d < (1+r) < u$ and by now you know what this condition would mean, this is what the no arbitrage condition in this model. If, this is violated, then arbitrage will exist in the model. And how you can take advantage, you have seen it. By the way, this is binomial model is also in a way called as Cox Ross Rubinstein model or CRR model, so which is nothing but exact similar.

Now, let us define the usual risk neutral probability. Then, so this about is the claim, so let us look at what we wanted to show, so look at the $\tilde{E}[\frac{S_{n+1}}{(1+r)^n}|\mathcal{F}_n]$. Now you can easily derive the required quantity. Now if I look at this quantity, if I look at say this quantity which is $\frac{S_{n+1}}{S_n}$, this is nothing but what happens in the (n+1) of the toss.

Whether ,this toss is going to be head and tail accordingly whether (the stock price is going) price process is going to go up or down, so this takes value either u or d with the corresponding probabilities $\tilde{p}$ $\tilde{p}$, because this is another risk neutral measure. And you see that this is independent of the up to n coin tosses the n plus 1 coin toss is independent of any previous coin toss. So, this is essentially the independence and the other thing if you observe this quantity, the first one if you observe the $S_n/(1+r)^{n+1}$ and of course $(1+r)^{n+1}$ is a deterministic quantity and S_n is F_n measureable.

So, if you apply, those properties that first you apply the property that this is taking out what is known property then you can take out an S_n and the n terms of this $1+r$ out then write this as this and $\frac{S_{n+1}}{S_n}$, so you can may be let us write one more step here, so this is equal to S_n over $(1+r)^n$ and 1 by $1+r$. Since $\tilde{p}u+\tilde{q}d = 1+r$, we get the required result, so, what we have seen here is the discounted stock price process is a $\tilde{P}$ martingale is what, then we have seen here.

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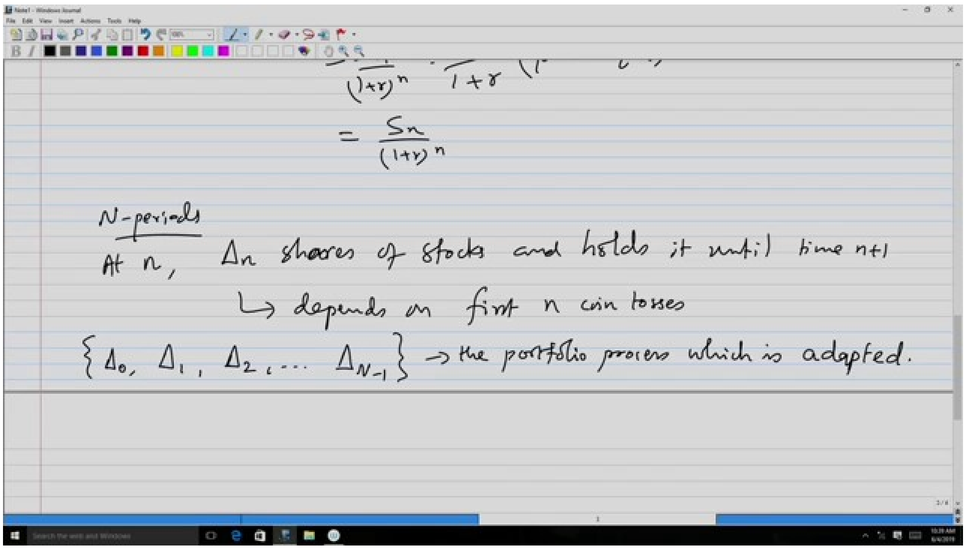

Now, this is once you have showed, now you assume that there is in this N period binomial model (there is an)we can associate with this N coin tosses, so in the N period and there is an investor of course, you assume that there is an investor who takes position delta n, so this is at n, so he takes position of this many shares of stocks and holds it until time $n+1$ and then he takes a new position of Δ_{N-1}.

Now, this portfolio rebalancing because you are doing it by borrowing or investing in the risk free asset. Now, this portfolio variable this delta n this generally depends on first n coin tosses or first n time period value, in order words what we are saying, is that suppose if I look at each n, so you start with a Δ_0 because at time 0 you take some position and you take at time 1 some position, at time 2 some position and at time $N-1$ in an N period model this is what you have.

So, this is nothing but the portfolio process. So, this is what we called the portfolio process, (which is nothing) which is an adopted process because depends on the first n coin tosses means Δ_n is $\mathcal{F}_n$ measurable assumed that you are useful complete filtration is there and so this portfolio process is an adopted process. So, what we require for a portfolio process to be it should be an adopted process that you have.

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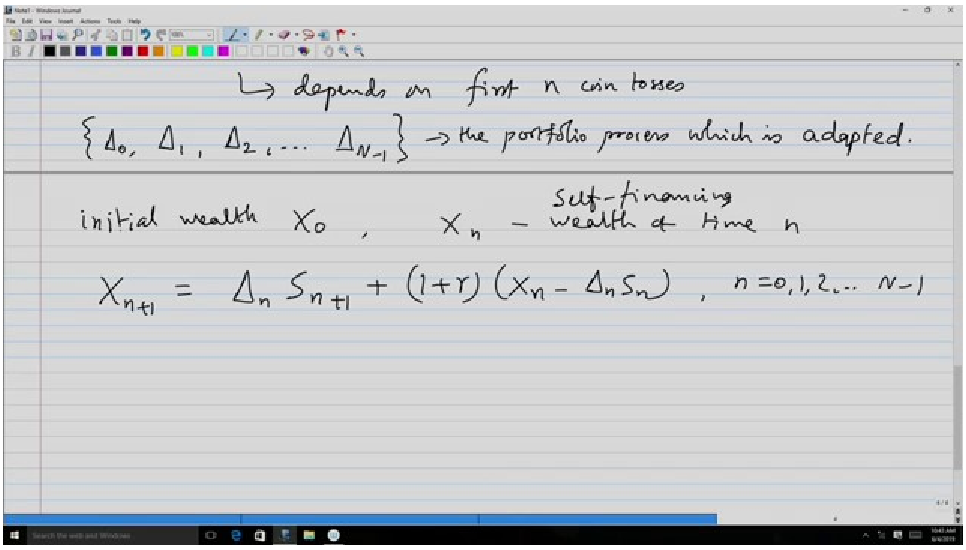

Now, if this investor begins with an initial wealth, say X_0 and if you adopts such a portfolio process then X_n is wealth at time n and this wealth process is a self-financing wealth process, because you are not taking out any money nor you are bringing any extra amount that you are investing in service.

Now, which you can recall the earlier chapter, how we have returned out this wealth process dynamics means how it involve from time n to time $n+1$ so this is the equation that you had encountered, so this is what is 0, 1, 2, $N-1$. And how did we write this equation? May be you may recall at time n you had a wealth of X_n now you decided to hold delta n number of stocks, which would require an amount of delta n time S_n.

So, this delta, so out of your X_n amount you invest $\Delta_n S_n$ in the stock and remaining $X_n - \Delta_n S_n$ in the risk free, it could be positive negative so you do the short selling or investing whichever is required as far as this risk free asset is concerned.

So, this the number of shares that you hold, so Δ_n is an adopted process. Now, at time n the value of your stock investment which is $\Delta_n S_n$ would grow to $\Delta_n S_{n+1}$ which is what the first term is and at the same time your investment whether you have borrowed it or you have in actually invested positive amount in the risk free of $X_n - \Delta_n S_n$ would have grown to $1+r$ times this quantity.

So, the total wealth at time S_{n+1} is given by this equation and this equation is what you have seen already and we are just repeating and recalling what you could have done in this case, once we have this. Now we may enquire about what is the average rate of growth of other investors wealth given any portfolio process but if it is if you are asking this question under the real world probability measure then the portfolio would matter.

That means, if you are asking this question of what would be average rate of growth of an investor under the real world probability measure are in actual market, then that would depend on what is the portfolio that we can hold and in fact generally since the stock has the higher rate of growth there then the risk free asset and if you are allowing unlimited short selling then literally like you can construct portfolio process which will give you theoretically speaking any rate of growth that you wanted. So, but such extremely leverage positons are also extremely risky so as usual like the risk will also be increasing along with such very highly leverage positions that you will take with respect to any portfolio process.

But which, means that when you generally talk, so you will not if you want to look at the average rate of growth of this wealth process then the probability under which you are talking about is matter. Now, but if you look at the rate of growth of this wealth process under the risk neutral measure then you would see that the portfolio that you hold daily does not matter, because the expected rate of growth of risky asset is the same or is equal to the rate of growth of risk free asset. So, no matter in whichever way you distribute your total wealth your expected rate of growth of this wealth process would be the same and technically what does that mean is that the discounter wealth process would also be martingale under the risk neutral probability measure.

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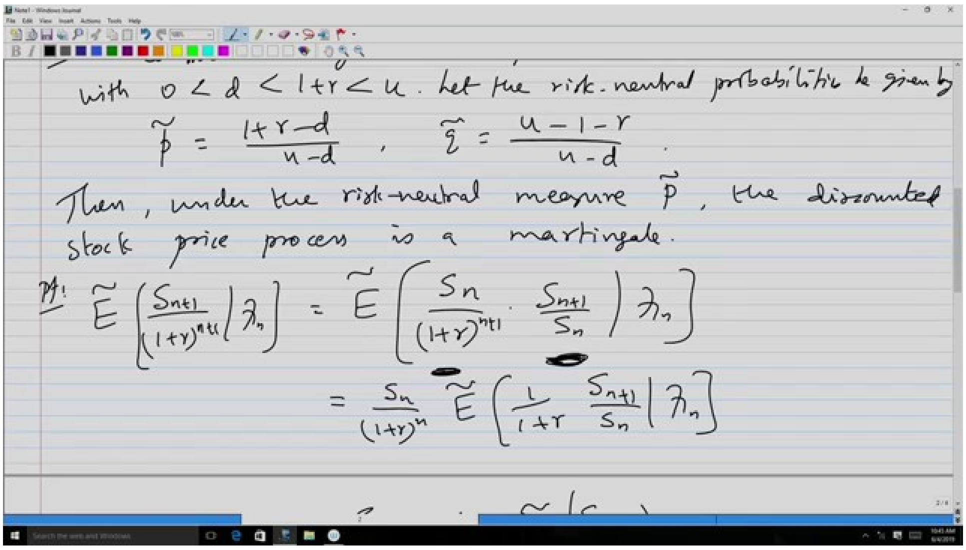

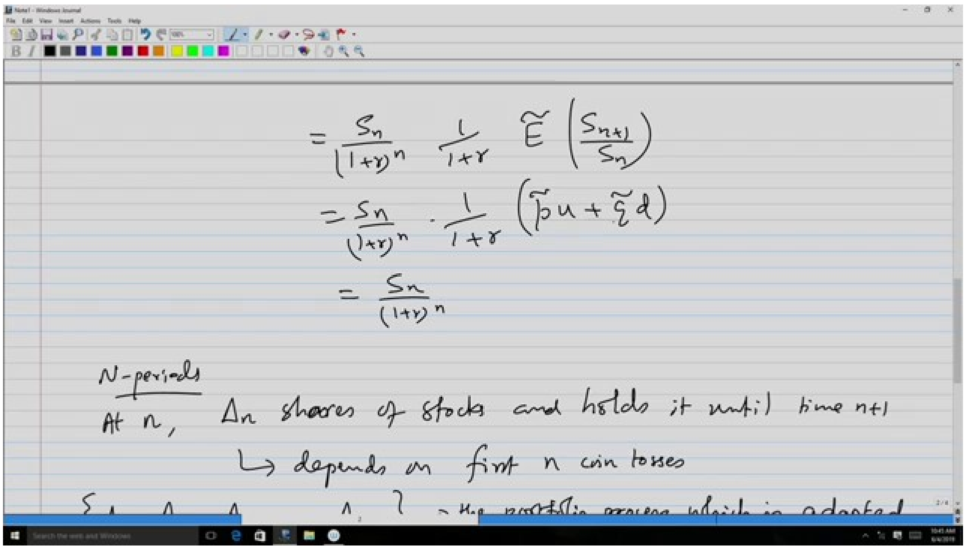

So, that is what now you would write and of course, in the real market earlier we have seen that this can go back to this example and you can see that we have climbed that this is under this, this is martingale under $\tilde{P}$ but then if you look at what would be this process under this P, since you expect generally, if you look at in this proof, so would see only in this place there will be difference.

When you write $\tilde{p}u + \tilde{q}d$, so in under real world probability measure this is $\tilde{p}u + \tilde{q}d$, which is the expected rate of growth. So, if that is the case then this quantity will not be equal to (1+r) but in general you expect this quantity to be greater than or equal to 1+r.

So, that would make this process as a sub martingale. So, that is what you know you generally expect in a real world scenario that the discounted stock price process are sub martingale but (in risk neutrals probability) under risk neutral probability measure so this will be become here martingale.

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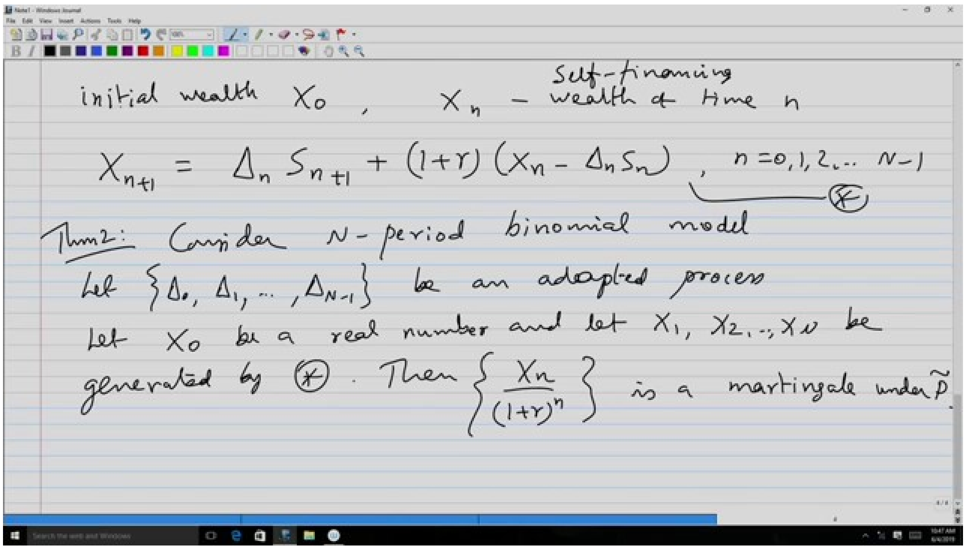

Now, so, what is our next theorem? this is theorem 2: consider the N period binomial model, N period binomial model, so this let $\{\Delta_0, \Delta_1, \ldots, \Delta_{N-1}\}$ be an adopted process. Let X_0 be a real number and let this one X_1, X_2 and so on X_n be, suppose if, I call this as star be generated by the recursive relation which is this 1 by the star.

Then, the discounted wealth process which is essentially $X_n/(1+r)^n$ is a martingale under $\tilde{p}$ or simply we would say it is a $\tilde{p}$ martingale. So, this what our claim is.

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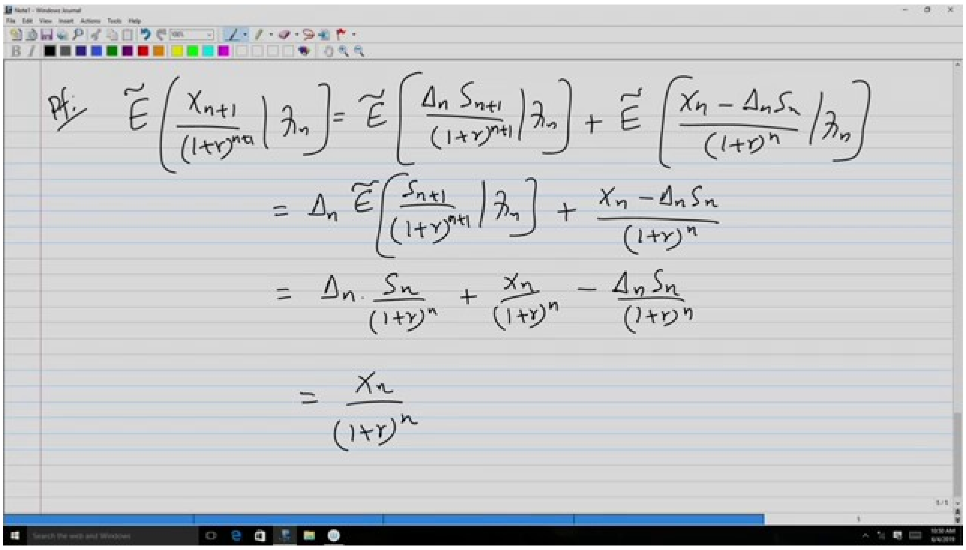

Let, us see how this is true, so what we have to show is essentially the other properties of show martingales are also to be real so we will look at the martingale property, the other property I mean is the integrability and the adopt adaptability properties, given Fn, so this is what we have to look at, this is equal to $(1+r)^n$.

So, this is essentially $\tilde{E}$ of $X_n + 1$ I know the expression, so if I plug in, so this is $\Delta_n S_{n+1}/(1+r)^n$ given $\mathcal{F}_n$ plus the expectation under of $(X_n - \Delta_n S_n)/(1+r)^n$ given $\mathcal{F}_n$. remember at each step you know how we can justify X_{n+1} expression we can simply plug in then I can use the linearity property of conditional expectation to write this step.

Now, this one could be equal to, so if I look at in this process Δ_n is adopted to $\mathcal{F}_n$ so by using the property taking out we can get the second line. Further simplifying, we get $\frac{X_n}{(1+r)^n}$.

So this is true for any n and for any node that you have, so this is what it is. So, what we have shown is basically the discountable process for which we use the fact that the discounted stock price process are martingale. So once we here and this is a $\tilde{p}$ martingale this is what we have seen.

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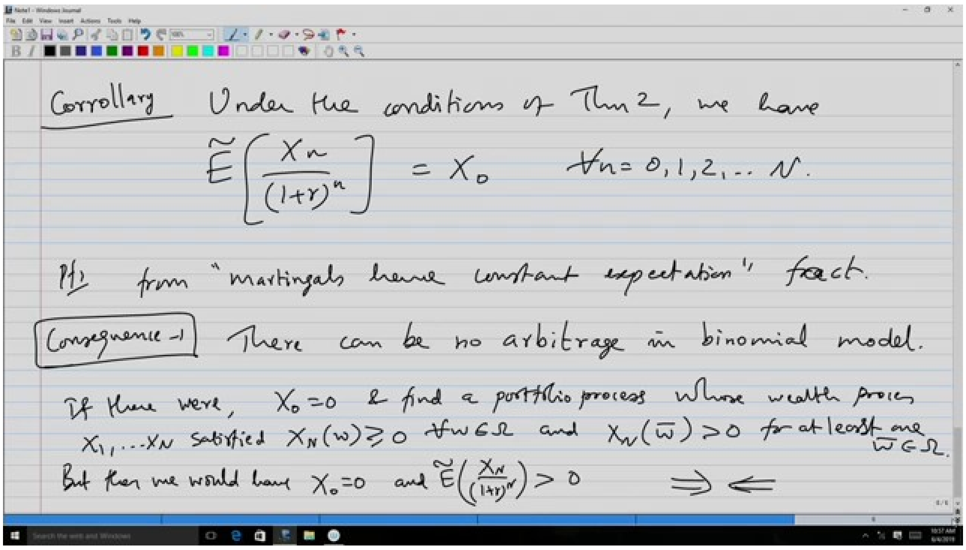

Now, this gives rise to a corollary, which says under the conditions of theorem 2, just we have seen, so we have

$$\tilde{E}\left[\frac{X_n}{(1+r)^n}\right] = X_0 \ \forall n = 0, 1, 2..., N$$

So, this proof is simple from the fact that the martingales have constant expectation, this follows immediately, so from this fact, applying this fact to this are even you can go back to the previous theorem and take again one more expectation on both sides.

Then X_{n+1} equal to X_n and you iterated up to X_0 and X_0 is non-random, so that is must be equal to 0, so you get this facts immediately. Now, the corollary has 2 important consequences that is what is important here. So, which is one is consequence 1 for this corollary is that the first consequence is there can be no arbitrage in binomial model.

Which by the arbitrage argument you have seen if you assume $d < 1 + r < u$, then you have seen that there could be no arbitrage, but this fact again establishes that you take any portfolio procedural time, remember in the previous theorem the portfolio process we generally define Δ_0, Δ_1 be any adopted portfolio process, any adopted process which is what we called as the portfolio process.

So, for if you take any Δ_n what you encounter this the corresponding self-financing wealth process is always a $\tilde{P}$ tilde martingale, so that is what is, so which means that this fact there can be no arbitrage in binomial model. Earlier you guys seen in one way, now we are try to interpret the same thing in a different way.

So, the fact that this discounted wealth process is a martingale, if it happens, so and this admits that there can be no arbitrage process. Now, you can see if there were arbitrage why and how this prevents the arbitrage. If there were arbitrage, then what we could do? We could begin with an initial wealth X_0 and find a portfolio process, find a portfolio process whose wealth process X_1 and so on X_N this satisfied X at N and N period binomial model is equal to non-negative for all omega, in omega and X_N of say some omega bar is strictly greater than 0 for at least 1 omega bar in omega.

It is non negative for all omega and this, but what would this that mean, if this is the case then we would have a scenario where X_0 equal to 0 and expectation of $X_N/(1+r)^N$ is greater than 0 which is a contradiction to the previous result. So, this says, that this cannot happen, if there were arbitrage, then does that mean that this is cannot be true.

So, that means that there can be no arbitrage. So, this is what we are going to take it to the continuous type model and in general what we can say is that, if there is a risk neutral measure, if we can find a risk neutral measure and what is a risk neutral measure by the way the property that we have seen here, it agrees with the real world probabilities on what is possible and what is not possible. And this makes the discounted the primary assets or that you have.

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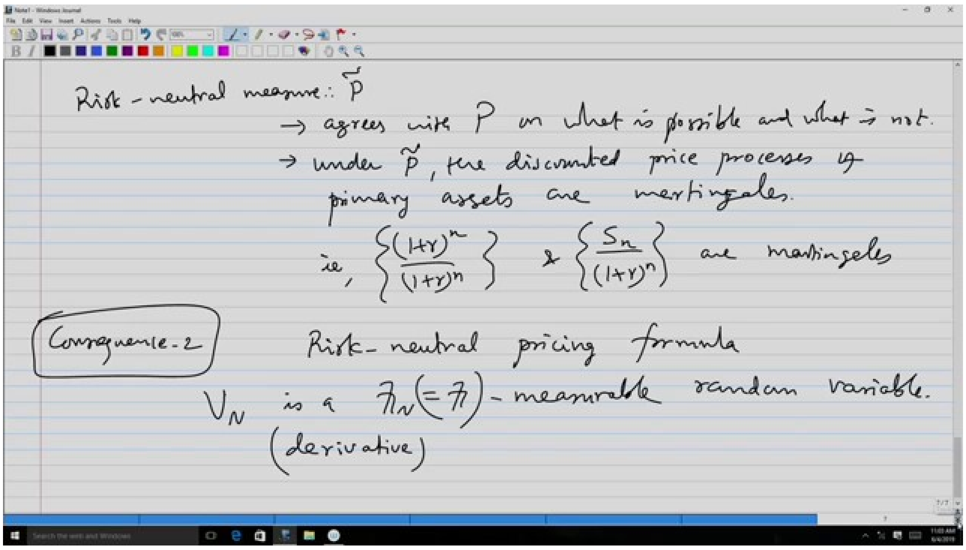

So, what is the risk neutral probability measure? So if you want to note it down. So, risk neutral measure is the one, this is agrees with P on what is possible and what is not. So, when we move to continuous type, we will technically talk about what this means. But in binomial model set up, what we mean? We have assumed that p and q both are strictly between 0 and 1.

Now, by under the assumption that $d < 1+r, u$ both your $\tilde{p}$ and $\tilde{q}$ is also strictly between 0 and 1. So, both assigns positive probability to both the branches of this binomial tree, that is what we say we mean when we say agrees with T and what is possible and what is not possible where they defer the possibility the probability of something is possible.

So, in one case the probability of head is p in the other case the probability of head is $\tilde{p}$, that is where they differ but both are strictly positive. Same thing with q and $\tilde{q}$, so that is about the first statement means. The second property of this is that under (so this is $\tilde{p}$ we are talking about) $\tilde{p}$ see the discounted price processes of primary assets are martingales.

So, here we really said only about the stock price, but the 2 primary assets if you look at in the binomial model which is basically the risk free asset and risky asset, but what this says is this. This process and $S_n/(1+r)^n$ this process, these are martingales. But the first one is obvious because this is a constant value, so that is why you know we did not bother to be when mention it but that is what it is, this is the risk neutral probability measure.

So, what this says is it in general, if we can find a risk neutral measure $\tilde{p}$ which has this 2 properties, then there can be no arbitrage in the market model that you have constructed and when you are building a market model the basic fact that you need to keep in mind is that you cannot formulate a model in which arbitrage exists.

But, then if you had formulate such a model than any result that you get is makes no sense at all, so you need to be careful on that, so under more much more general set up, so this how do we ensure. So, one way is that one sufficient condition is that if there is a residential measure, this is there can be no arbitrage in the model and that is what is later we will need to call this as the first fundamental theorem of a sub pricing or the first fundamental theorem of finance which is nothing but the risk neutral existence of risk neutral probabilities recruits the possibility of arbitrage.

So, this is the first consequence. So, the essence this is also what is called the we just stated what we called the first fundamental theorem of sub pricing. So, this which mean that you know since it has constant expectation on all these properties that you have seen so far pretty much include the other part.

Now, the second consequence, consequence 2 is basically what we call the risk neutral pricing formula, this is what the second consequence that what we have. So, you have an N period binomial model, so your V_N suppose if you peak, which is a $\mathcal{F}_n$ measurable random variable.

So, far as a derivative means simpler random variable which is measurable with respect to the corresponding sigma field. So, V_n suppose if it is N period binomial model so V_n is a derivate, so this is what is a derivative as far as we are concerned. Now, we know from the earlier mean few lectures before discussion on binomial model when we try to price this V_n through replication we know that there is an initial wealth X_0 and there is a portfolio process Δ_n, such that whose self-financing portfolio process would equal V_n that is what you know we had seen no matter what is the omega.

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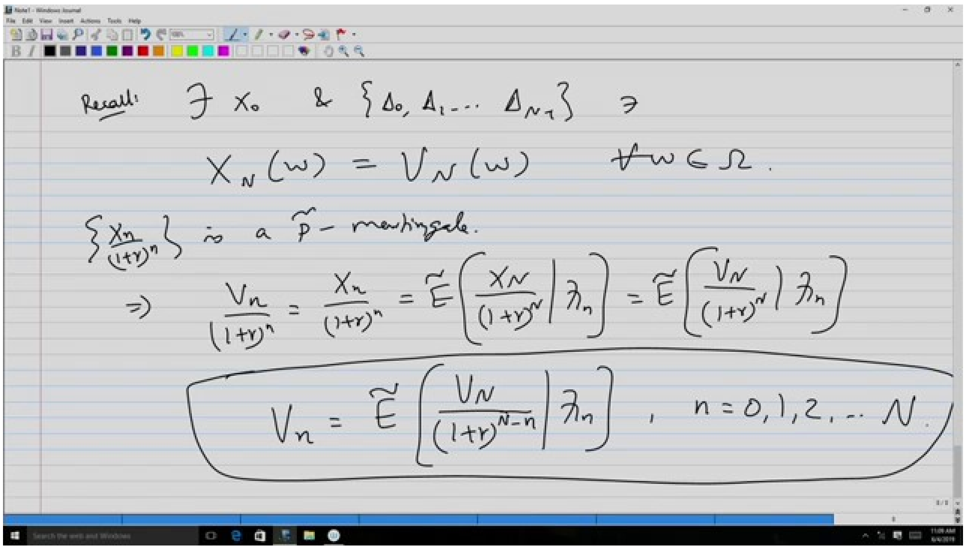

So, recall what is recall? There is an X_0 and a portfolio process Δ_0, Δ_1 and Δ_{N-1}, such that the self-financing wealth process generator by these quantities would be X_N of omega is V_N of omega for all omega in this. This is what you know we call as the property of binomial model and if for any simple European derivative type V_N if you can find and show that there is an X_0 and there is a portfolio process which will satisfy this property for every omega then if you can do for every V_N then that is what we call it as the model completeness.

Which mean, the model is said to be complete if this can be achieved that is the definition completeness of course, we will come back little later to do that. Now, (what we have observed) this is what you know we have seen no matter what is the omega or what is the coin toss or which node you end up the N period model, so you are going to have the sufficient wealth to pay off the derivative which is given by the V_N.

Now, we have observed that the discounted wealth price process is a $\tilde{P}$ martingale this implies that my $X_N/(1+r)^N$ would be equal to expectation of that quantity given $\mathcal{F}_n$, how do we write this, this is the really the one step martingale property is extended to multi step martingale property and (since X_N would equal sorry) $\tilde{E}$ this is, $\tilde{E}$ of $V_N/(1+r)^N$ given $\mathcal{F}_n$, since X_N would equal V_N.

Now, if the wealth at time small n is X_n and this Xn satisfies this property which mean, this is the wealth that is required at time N to hedge your position from small n until N and hence this X_n is nothing but then the no arbitrage of this derivative but which is V_n.

And hence, this is nothing but my this quantity recall X_n is any generic wealth process which we call V_n if this is a replicating wealth process of some V_N. So, this is what then you have, so if you write this equivalently, what you are end up with V_n is essentially $\tilde{E}$ of $V_N/(1+r)^{N-n}$ given $\mathcal{F}_n$ for n 0, 1, 2 up to N, (so you can) so this quantity, what is called the risk neutral pricing formula that is what you know you have here, so this is the risk neutral pricing formula that you obtained.

So, you can show that this V_N that is generated by this is exactly same as the one that you have generated earlier using the algorithm, but what we write here is that this V_N is given by this and this is the reason, why I said that the conditional expectation place an important role, because now you see ultimately there is neutral pricing formula is given in terms of the conditional expectation.

And if you can evaluate this conditional expectation then this V_n is going to be the no arbitrage price or quote unquote what we call a fair price which in our sense is no arbitrage price of the derivative V capital N or if you do this is you can simply call it some random variable V_n, which is F_n measurable.

So, this is what the risk neutral pricing formula. So, this formula one can show without and with not much difficulty that this pricing formula generates the same value V_n's as you had generated earlier through the binomial pricing algorithm.

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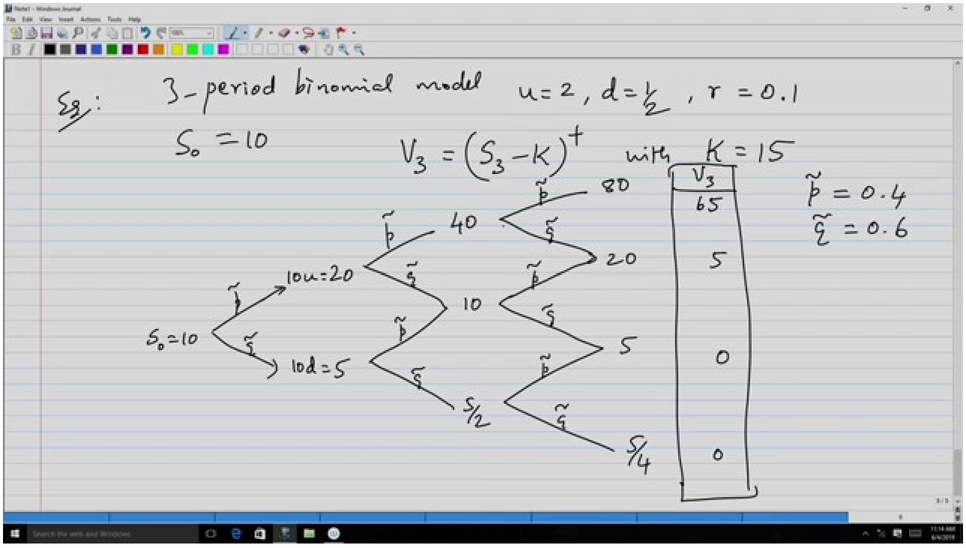

Now, we will see an example, to see that that is really the case were I will just compute using this expectation and I leave it to you to compute using the algorithm that you had earlier. So, let us look that an example of how you can compute, so you have considerd a 3 period binomial model, 3 period binomial model and you with the parameters $u = 2$, $d = 1/2$, u times d is generally taken to the worth which is may not be the case always and my $r = 0.1$.

So, this is means you are assuming the 10 percent interest rate is what you are assuming for a period generally we assume that the any r that is given is per period interest rate so that is what you know have it here and you also assume an initial stock price of S_0 to be 10 and you consider a derivative V_3, which is $(S_3 - K)^+$ which is nothing but an European call option on the underlying asset S with strike as 15.

So, now if this is the case then your, if you simply draw the binomial models so your S_0 is 10 and $u = 2, d = 0.5$, so if I write this S_u at time 1 which is equal to now it will be 20 and this will be 10 times d which is equal to now 5 is that 1 period and at period 2 what you will be having is this as 40, this as 10 and this as 5 by 2.

So, this you could say that this is 80, this is 20 and this is 5 and this is 5 by 4 or the underlying stock price values, now if I compute for under this parameter quantities is, if I compute my $\tilde{p}$, $\tilde{p}$ will turn out to be 0.4 and hence $\tilde{q}$ will turn out to be 0.6, I just have to plug in the values of r d and u to get this $\tilde{p}$ and $\tilde{q}$.

So, this is what the stock price process will look like the corresponding probabilities, if I marked it here under this probability measure $\tilde{p}$ $\tilde{q}$ then I will have $\tilde{p}$ here this is $\tilde{q}$, $\tilde{p}$ $\tilde{q}$ $\tilde{p}$ $\tilde{q}$. So, we can then calculate the prices of this derivative $(S_3 - K)$, you can easily compute what is the value of V at this 4 or V_3 in this 4 nodes.

So, corresponding to 80, I will have 65 and corresponding to 20, I will have 5 and this will give me 0 0 or the values of this and I know this in this is 3 period model, so this would corresponds to 1 omega 65 would corresponds to 1 omega and this 5 will corresponds to 3 omegas, which are head head tail head tail head tail head head, so all this 3 would land at a same value of 20 so you will get this 5 and so on. So, this is the simpler model, so you get this only 4 values here.

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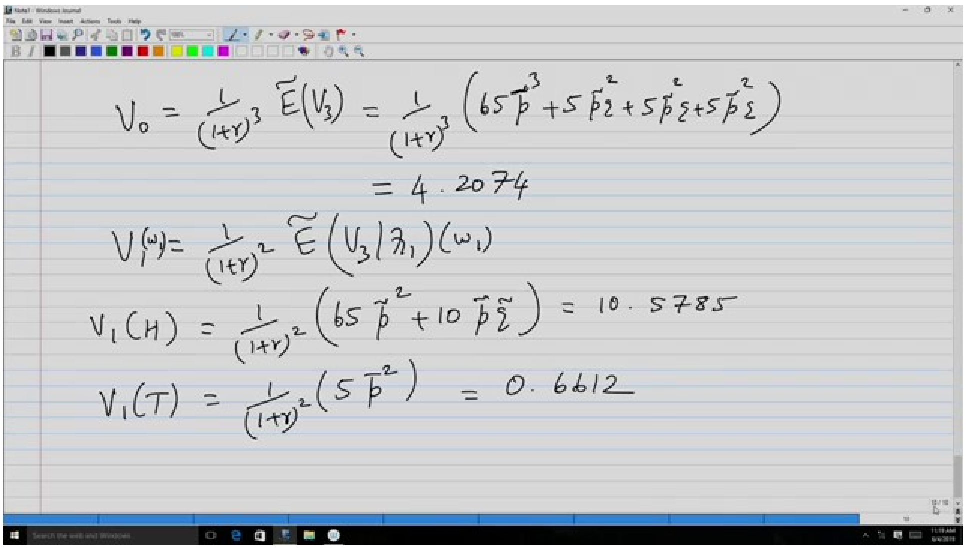

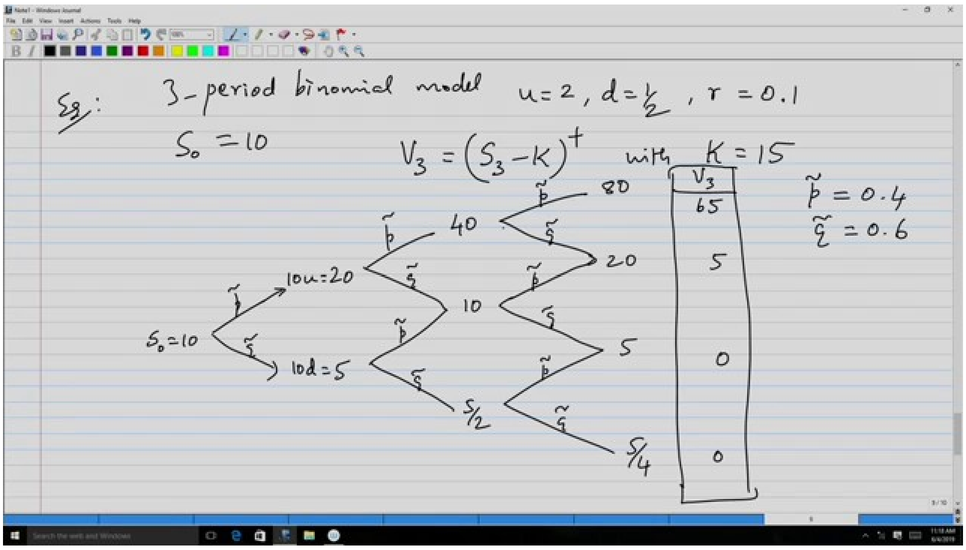

Now, once I have this 4 values you can compute now my conditional expectations under expectations V_0, now V_0 would then now be equal to 1 by 1 plus r to the power 3 $\tilde{E}$ of V_3, so this $\tilde{E}$ V_3 given $\mathcal{F}_0$ which is nothing but your $\tilde{E}$ V_3 which if you look at the picture, so you would see that this is essentially 65 with the corresponding probability of $\tilde{p}q+5$ times p^2q and this comes 3 times, so this is essentially 5 times $\tilde{p}$ square q plus another 5 times $\tilde{p}$ square q plus another 5 time $\tilde{p}$ square q plus rest of all are 0s.

If I compute that then I will get this as 4.2074, so this is simple expectation you can compute it. But, now when it comes to V_1, V_1 is 1 by $(1+r)^2$ $\tilde{E}$ of V_3 given $\mathcal{F}_1$, now this will be if I write omega 1 it will depend on T omega 1 whether it is head or tail. And, now to compute this quantity we can simply appeal to the reduced formula that we have written down for the conditional expectation of X given $\mathcal{F}_n$ in the binomial model, you can appeal to that easily and you can compute V_1H as 1 by $(1+r)^2$ and this will be 65 $\tilde{p}^2+10$ $\tilde{p}$ $\tilde{q}$ because 5 plus 5 there will be 2 of those this will give me.

So, how did we write this if we look at in the graph so we are in the tree, so we are sitting now here we won head and from here a possibilities are this, are this, are this, are this, so you will have this are the possibilities, so this means that the atom is A H as far as this $\mathcal{F}_n$ is concerned.

So, we are trying to look at the atom on A H that is what we will give you your V_1 H, V_1 H is essentially this and this if you compute this will turn out to be 10.5785 and similarly, if I am sitting on the down node I will end up this with 5 times p bar square, so if I go back here just the previous one, so I am sitting it here, so form here I will get only this 1 path starting from this 5 along this path if I move from 5 to 10 and 10 to 20 then only I will end up a 5.

In another process this is going to be 0. So, starting from here, so essentially you can think of as if you are looking at from this part to the complete part, that is how you can easily make up the computations. So, this would then be equal to 0.6612 V_1 H and V_1 T.

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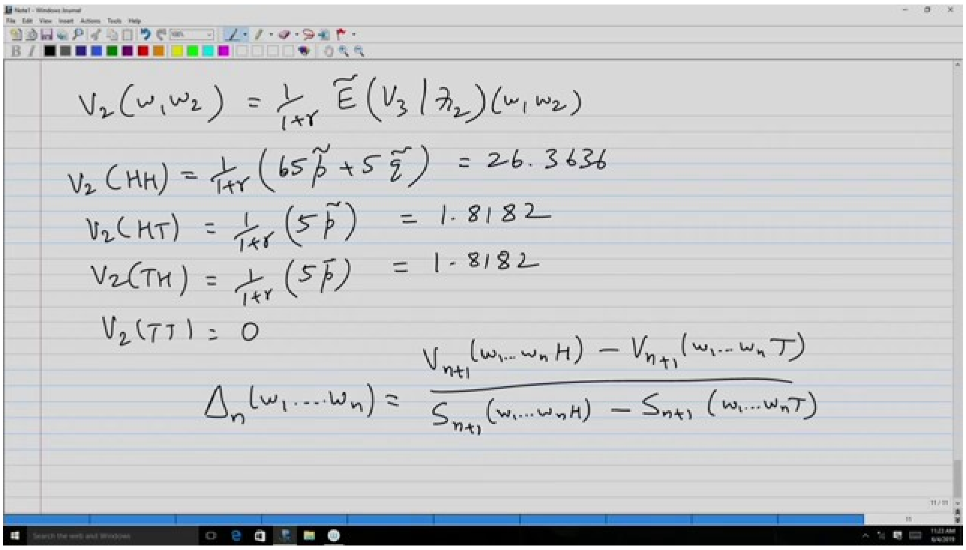

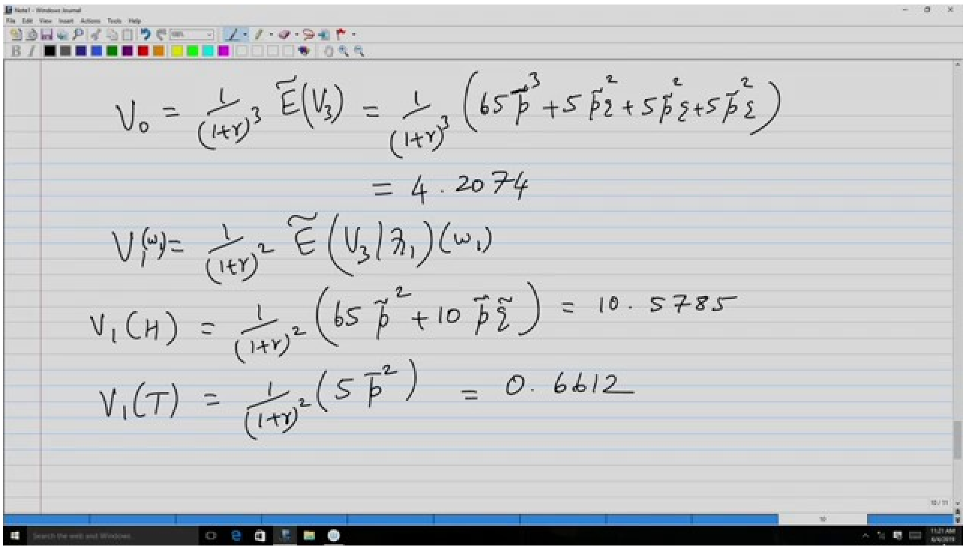

And along similar line, now my V_2 of omega 1, omega 2 would then be equal to 1 by 1 plus r $\tilde{E}$ of V_3 given F2 of omega 1, omega 2, is what then you will have and this expression you can easily see that is nothing but the 1 step recursive step that you have written in the basic binomial pricing algorithm earlier. This of course if you want to compute, so sitting at the node HH you can look at what is going to be the case. So, this is 1 by 1 plus r 65 times $\tilde{p}$ plus 5 times $\tilde{q}$ which will give me 26.3636. And similarly for this, this is $\frac{1}{1+r}5\tilde{p}$ will give me 1.8182 and V_2 of TH is again $\frac{1}{1+r}5\tilde{p}$ is equal to 1.8182 and V_2 of TT is 0.

Now, you see here V_2 of HT and V_2 of TH , the values are same because this is a recombining binomial tree and you expect that this values to be same. So, you have computed all the prices now the difference is that really in this step is exactly same as your basic binomial pricing algorithm step.

But, this particular steps in computation of V_0 and V_1 we directly employ the conditional expectation formula to get to this. So, now these 2 values whether you use the basic binomial pricing algorithms or the conditional expectations and evaluating the conditional expectation directly would result in the same value.

So, that you know simple martingale theory that one can also proof it. Now, once we have this now you know the generic formula for my delta n of omega 1 to omega n would be that, what will be that? This is essentially V_{n+1} of omega 1 to omega n head minus V_{n+1} omega 1 to omega n tail by in this particular case, you can write generally omega n H minus S_{n+1}.

So, we know, that this is the portfolio process that you get. So, if I have this V_2 then I can compute what is the portfolio process that I need to take in our to hedge this particular derivative, this will give me the complete values if I have to compute this, this formula reminds the same as the earlier case that you have here.

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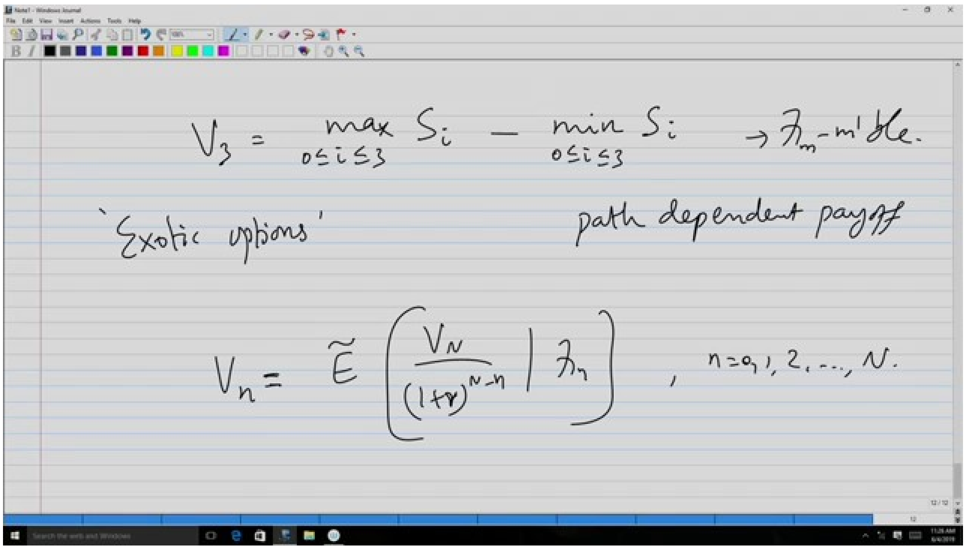

Now, this $V_3 = \max_{0\le i\le 3} S_i - \min_{0\le i\le 3} S_i$. It could in general be any path dependent derivatives say something . So, this is again you see that V_3 is an $\mathcal{F}_3$ measureable random variable, but you are, now this are all belong to the class what we called Exotic options, the simple European call and put and American call and put are call simply as plain vanilla options and anything other than that we generally called this exotic options.

And specially here the payoff is basically a path dependent payoff. So, this of course once you determine V_3 than of course you know you will compute, so this as you saw even the binomial model case, any such V_3 measureable random variable, so this is $\mathcal{F}_3$ measureable. So, $\mathcal{F}_3$ measureable random variable any of 3 measureable random variable this is true, so only thing is the evaluation or the evolution of this conditional expectation might become little cumbersome then you do this path dependent or when you have more complicated payoff and that is where you know you try to appeal to various properties of conditional expectation and how one can evaluate depending upon the situation or you can adopt the alternative methods of computations. So, with this we say the main thing that we ended up is that the binomial pricing algorithm give you the price and this you know we have written down this particular expression which is what is going to be the main result as far as this is concerned and which will generally take this to the continuous type model as well by the same idea.

So, this is the risk neutral pricing formula, so give at any random variable V_n, which is $\mathcal{F}_n$ measureable, which in our case is nothing but the derivative, then in the binomial model we can define $\tilde{p}$ $\tilde{q}$. But as you go to continuous type model we cannot simply define we have to in a way indirectly define what is $\tilde{p}$ measure. So, here it is obviously it is in terms of the parameters, we can immediately write down what is $\tilde{p}$ $\tilde{q}$, and under that risk neutral measure this is the pricing formula. For this pricing formula you do not need the actual or real world probability measure to compute the prices what you need is the quote-unquote some abstract probability measure which is their risk neutral pricing measure.

So, that is why which called this neutral pricing measure are simply a pricing measure some type it is being called. That is what, we need to price the derivative we do not really need the real world p and q and once you have this formula this is what then the prices are given by. Now, this can be extended suppose if there is single payment if you have a cash flow of payment or anything of that sort, so this formula can be used you know to construct any such prices that you will have it here. So, this they will conclude the discussion on our, the binomial model pricing for the European type of derivatives. Thank you.