Mathematical Finance
Risk-Neutral Pricing in Discrete-Time
Lecture 20: Filtrations and Conditional Expectations

Professor N. Selvaraju[1] and Professor Siddhartha Pratim Chakrabarty[1]

[1]*Department of Mathematics, Indian Institute of Technology Guwahati, India*

Hello everyone. In the previous lecture we have seen about the Discrete Probability Spaces and we have refreshed our knowledge on discrete probability spaces. In this lecture we will move on to the next concept which is related to the sigma field first and then we will move to conditional expectation in the later part. But, before we go in to the filtration idea, so let us review over again one more review about our, what we know about the binomial model that we are going to stick to from now onwards when we deal with the discrete time models.

In this simple binomial model, what we are assuming is that the stock prices are going to follow the simple binomial model that we are going to describe. What we are assuming is that at each time step, the stock price is going to go up by a factor u or it is going to go down by a factor d. Now, as you see since this is the outcome is binary we can associate with this a simple experiment as if there is a coin toss takes place and it results in head then the stock price is going to go up and if the coin toss results in tail the stock price is going to go down.

This is a simpler model. You can associate since this outcome is binary outcome, you can associate a coin toss to each of this movements. At each step we will associate a coin toss. So, the whole movements over different time periods can now be associated with more and more number of coin tosses and the resulting outcomes. That is the model that we have in mind, which we have seen earlier in the lectures, the binomial model for asset pricing.

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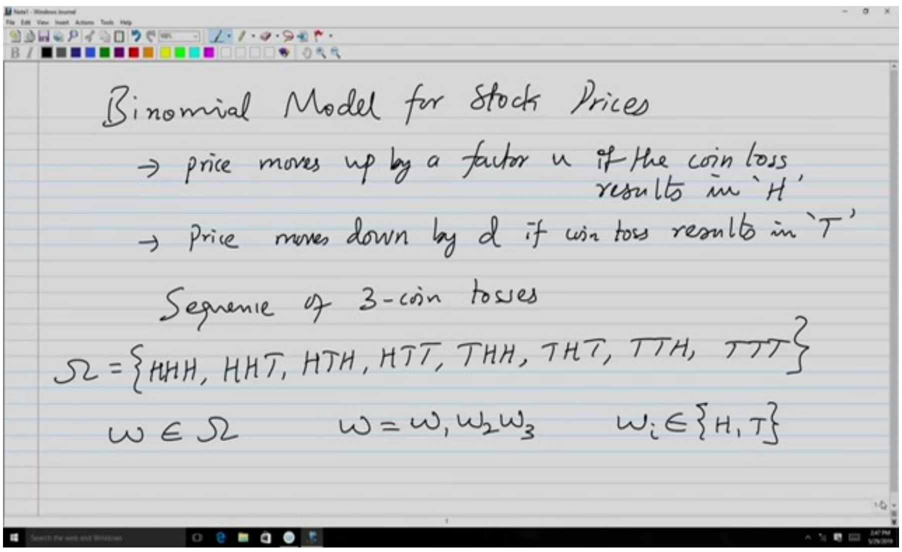

So, what we have is the binomial model for stock prices, where we say that the price moves up by a factor u if the coin toss results in head or the price moves down by d if the coin toss results in tail.

Which is denote by H and T. Note that we are not specifying the probability of up or down movements. We are just looking at the values of the stock prices, which is what you have it here.

Now, consider for our simplistic setting for simple example, you assume that there is a sequence of 3 coin tosses. Then the collection of all possible outcomes of this three coin tosses can be enumerated by

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

so, this sequence of this 3 coin tosses will result in any one of these eight possible outcomes. So, this is our sample space for this experiment and these are the eight possible outcomes that you have for this. So, a typical sequence of this would be denoted by a ω which will belonging to Ω, where this ω can further be written as $\omega_1\omega_2\omega_3$. Where each of this ω_i's would be either head or tail. Now, whenever we require, we could use the full ω or sometimes if a part of it would also be used when we are considering certain random variables.

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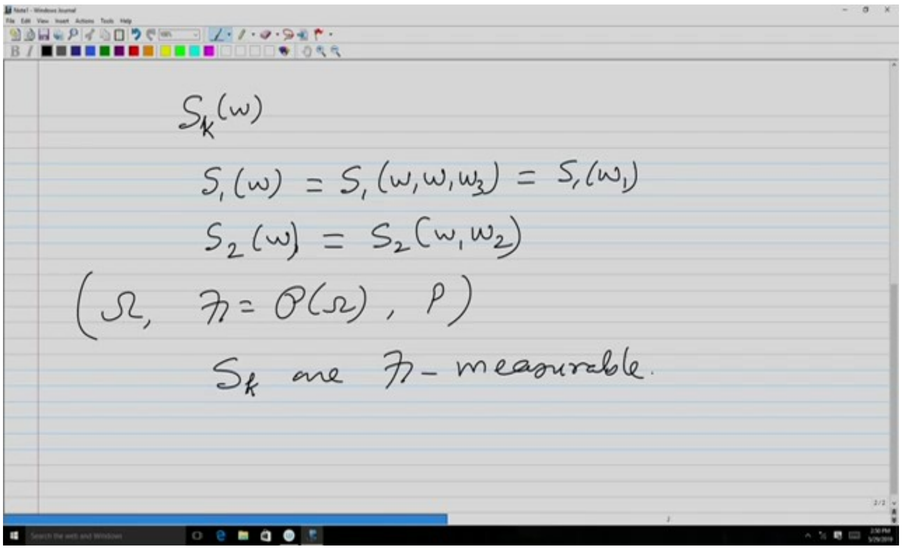

Say for example, if you denote $S_k(\omega)$ as the stock price at time k for the sequence ω. So, this ω typically depends on all 3 in this particular 3 coin toss otherwise if you go for it, you will have more.

So, you could see that we could write in variably like $S_1(\omega)$, which will be the same as $S_1(\omega_1\omega_2\omega_3)$ which we could also write it as $S_1(\omega_1)$ because, the stock price at time 1 is really going to depend only on the first toss, second and third toss will not pick an impact.

So, in such situation, we can also write this. Whether we write $S_1(\omega)$or $S_1(\omega_1\omega_2\omega_3)$ or $S_1(\omega_1)$ for a particular ω would be clear from the context, and all three will be used interchangeably. Similarly, your $S_2(\omega)$ could also be written as $S_2(\omega_1\omega_2)$. Because, to determine stock price at time 2 at least 2 outcomes are needed, third outcome is relevant because for any of those ω_3 this is going to result in the same value. In that context we will use this notation.

Now, you can see that if I take this Ω and my $\mathcal{F} = \mathcal{P}(\Omega)$ (power set of Ω), I could see this S_k which is the stock price at time k is a random variable if I define, then all this random variables S_k, they are all random variables or depend on this along with whatever the probability measure that you define. They are all this $\mathcal{F}$ measurable, later you might see, you will see that this are in fact measurable with respect to your smaller sigma field.

Now, if I look at this particular example of 3 coin tosses, then my $\mathcal{F}$ is power set of Ω which is going to have 2^8 elements which is 256 elements. So, this is with respect to that sigma field that you know you can see that this is measurable.

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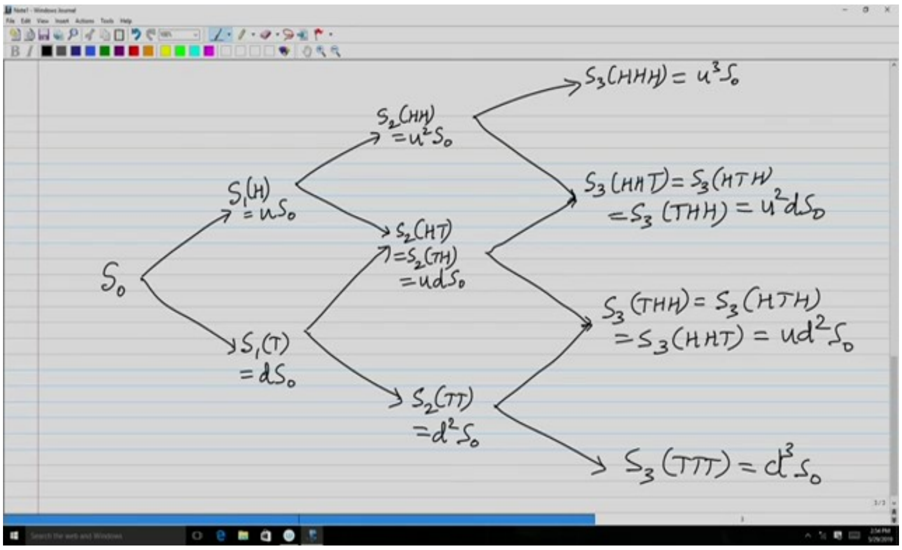

Once we have this example in mind, if you are so comfortable you can even draw out a picture starting from some initials time period which starts with a value of S_0 and at time 1 this can become either $S_1(H)$ which is going to be uS_0 or $S_1(T)$, which is going to be dS_0. So, this could be any one these two values and in this, this is a time 1 and at time 2, I would have $S_2(HH)$ which is value is going to be u^2S_0. Or this can become my $S_2(HT)$ which incidentally, if I draw out, this will also be of $S_2(TH)$.

This will be udS_0 and this one this node would be $S_2(TT)$ of which is value is going to be d^2S_0 so this is at time 2. So, the stock prices, if I denote so then this would further become: $S_3(HHH)$ would be u^3S_0, $S_3(HHT)$ of this which is equal to $S_3(HTH)$ which is equal to $S_3(THH) = u^2dS_0$; so, this will be the evolution of the stock prices for the 3 time periods. It may start from S_0 it may go up by a factor u which becomes, it may be become uS_0 or it may be down by a factor d which means it would become dS_0 which we would say that it is S_1 with head.

As we denoted, you can say $S_1(HHH)$ is a same as $S1(HHT)$. Like that if you can innumerate all eight possibilities all of them would result in the same value of uS_0. So, we simply denote it by $S_1(H)$. Similarly, for $S_1(T)$ and $S_2(H)$ would be this value and $S_2(HTTH)$ will be equal to udS_0 and $S_2(TT)$ will d^2S_0 and so on for the third time period where you would see that the essentially 4 possible values that we have.

So, what we are essentially saying is stock price in 4 time periods become 1 value becomes one of this 4 values. So, that is what this binomial model basically describes, how the stock prices evolve. So, let us please keep this figure in mind because for at least another three or four lectures, so when we keep referring to the binomial model we mean this set up and every time instead of maybe we are writing this we may just draw out this picture then it is be clear that what we need at each node or what we meant by the values of the stock price of each node would be clear from this particular figure.

So, this you can remember this figure in mind so that it is easy to understand what is happening and what we are trying to do with respect to this or in this binomial model when we are trying to do at least in the complete discrete time asset pricing models. Till that time we are going to and this can be extended to any number of time periods. Only thing is this is this binomial tree, this is called as a recombining binomial tree. So, this binomial tree will keep expanding in each.

And if you have n periods then it is going to have $n+1$ values at d frag end, that is the idea that you will have and the number of possibilities coin toss if you have to access then if you have n time periods then you are going to have 2^n values or 2^n possibilities. Some of them will result in the same stock prices as happened here. Now, with this binomial model in background we keep referring to this, we will come back this for each example.

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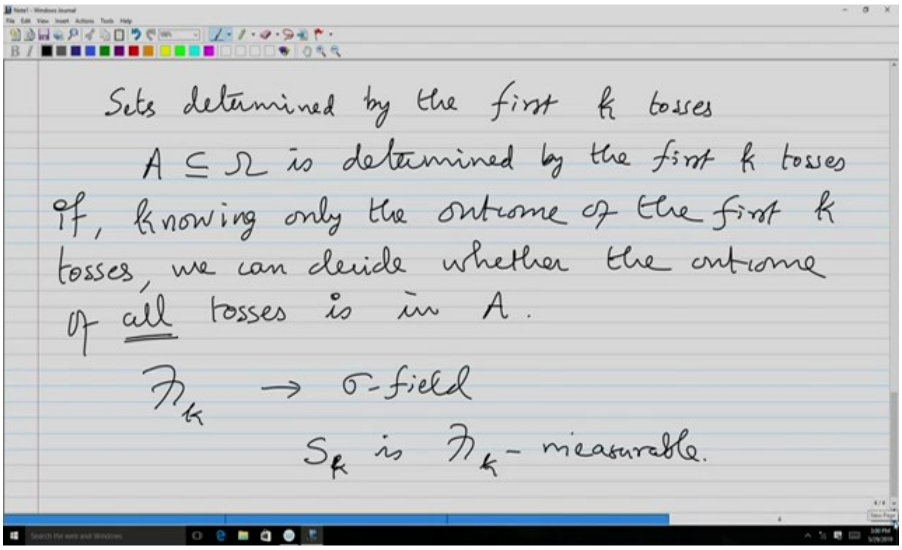

We will now next look at the idea of; in this binomial model set up, sets determine by the first k tosses. So, this is the idea that we look at now that the sets determine the first k tosses. Now, what we mean, when we say that asset $A \subseteq \Omega$, is you say that this is determined by the first k tosses if, knowing only the outcome of the first k tosses, we can divide whether the outcome of all tosses is in A. So, we see that if the subset of Ω if we call it A, we say that this is determined by the first k tosses, if knowing only the outcome of the first k tosses, we can decide whether the outcome of all tosses is in A or not.

So, given an ω which is in outcome of all tosses. Now, if this set is given and if we can safe or sure that whether that ω will belong to this A or not. If we can say that then that set is determined the first k tosses. So, this is the first k tosses idea that you have it here.

Now, if we collect all those sets that determined by the first k tosses and we call that collection as $\mathcal{F}_k$ then, it is easy to check that this collection is a sigma field. Which means that it is having the property of for the sigma field that this is non empty collection which is closed under complementation and countable union. So, in this case there will be fine x so of course the finite union will 2. So, if you collect all the sets that are determined by the first k tosses and the collection if we call that as $\mathcal{F}_k$ then in this script $\mathcal{F}_k$ would be a sigma field and you would also observe that the S_k that we talked about the stock price at time k is also $\mathcal{F}_k$ measurable. Now, be clear once we write down some of this $\mathcal{F}_k$ that we think at how this actually follows here.

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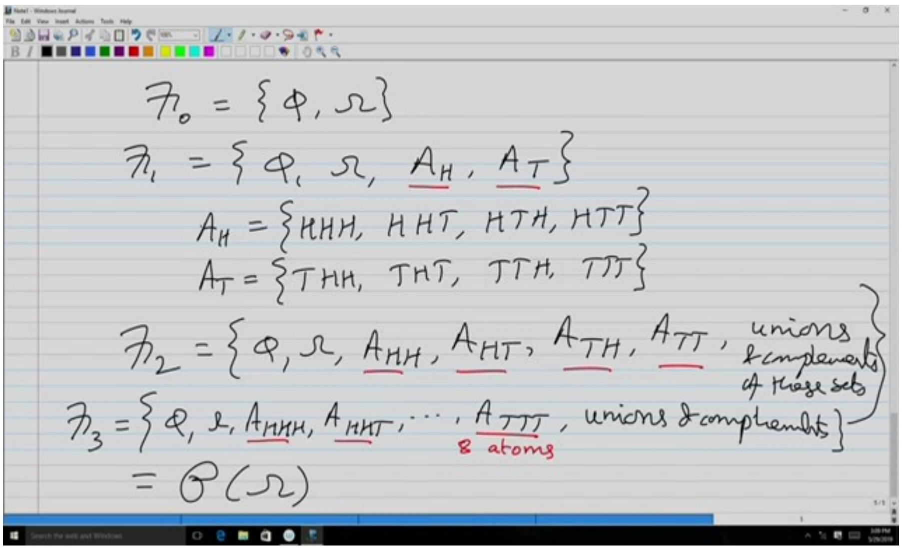

Now, let us start this. Now, when as I said $\mathcal{F}_0 = \{\phi, \Omega\}$ in the earlier lecture that this is the trivial sigma field which we denote by $\mathcal{F}_0$. When that coin tosses has not started so this is what it is. Now, given any omega we can say, whether this belongs to each of this set. So, these are the set determined by no information this is no information. Now, we are coming back to the idea of sigma fields viewed as information content. That is what we are aiming to go.

So, in the case of three coin toss example that we have said are the three period binomial model whichever way equivalently you think about it. Let us look at what is $\mathcal{F}_1$ consist of. So, what is $\mathcal{F}_1$? So, $\mathcal{F}_1$ it is the set of the sets determined by the first 1 toss. So, this consist of the following 4 sets: ϕ, Ω and A_H and A_T where my A_H would be all those ω of which is there in the sample space and that starts with the first coin toss as head. Which essentially mean in the 3 time coin toss experiment.

This and A_T would be the events that start with first coin toss as tail. So these are the four sets that are determined by the first 1 toss. Now, if I pick any ω in the sample space, I could say for sure with each respect to each one of this four sets where the particular ω will be in or out. And hence these four are the sets determined by the first 1 toss. And, if you notice in this particular case these are the 2 atoms for this particular sigma field $\mathcal{F}_1$. If we recall the notion of this because, these are disjoint and there union is Ω.

And, this generates this sigma field $\mathcal{F}_1$. So, this is $\mathcal{F}_1$ and similarly if I have to look for $\mathcal{F}_2$ which is the sigma field generated by the first two tosses, we will extend this notation of A_H and A_T in a similar way so that will be ϕ, Ω A_{HH}, A_{HT},A_{TH},A_{TT}. So, as you see these four sets forms the atoms for this particular sigma field $\mathcal{F}_2$. But, then, this is itself is not $\mathcal{F}_2$ because, this is not a sigma field. So, what you have to do? You have to add all the sets that are required to make this as sigma field. And, all those sets would be forming part of this $\mathcal{F}_2$. So, instead of enumerating all this so I would simply right that as unions and complements of these sets. If you take then this would form a sigma field and you can observe that this four there are four atoms, are the four sets which partition is Ω. This sigma field then is going to have 2^4 which is 16 elements. Similarly, if I write the full as far as the 3 coin tosses experiment is concerned, the sigma field generated by the all the sets which are determined by the first 3 coin tosses, which is sincerely the all 3 coin tosses then I will have ϕ,Ω, A_{HHH}, A_{HHT} and so on.

So, which essentially A_{HH} in this particular case would mean that it is just that HHH alone. So, since we are following notation from A_{HH} means anything up to start it. Similarly, A_{HH} means essentially HHH, and HHT is forming this A_{HH} and so on for A_{HT} and A_{TH},A_{TT}. And similar way if you expand, so this will be this 8 basic sets and unions and complements of this sets and as you see so this are there are 8 sets of this.

There are 8 atoms that you have here and you would see that this is nothing but, the power set of Ω if our Ω is the three coin toss experiment. And, this is going to have 2^8 elements which is 256 elements and this what this sets determined by the first 3 tosses. Suppose, if you have more number of periods in the binomial model then this would still remain as sets determine by the first 3 tosses and in that case our A_{HH} would be expanded to include more number of elements possible.

Now, this sets determined by this now if you observe these, these four sigma fields. This is $\mathcal{F}_0$ which has been associate with no information case, $\mathcal{F}_1$ Which is after knowing the first toss what is going to be the outcome or the even the sets determined by the first toss which forms $\mathcal{F}_1$ which is sigma field $\mathcal{F}_1$ and $\mathcal{F}_2$ are collection of sets determine by the first two tosses. $\mathcal{F}_3$ is the collection of sets determine by the first 3 tosses and so on. if we have more number of time periods.

And, this essentially if you look at these four sigma fields at least we can easily recognize the following that $\mathcal{F}_0$ is contained in $\mathcal{F}_1$, $\mathcal{F}_1$ contained in $\mathcal{F}_2$ and $\mathcal{F}_2$ is contained in $\mathcal{F}_3$ and these are which essentially means, that you have not lost any if I go back to the information interpretation of sigma fields, I have not lost any information, I have only added the information. To $\mathcal{F}_0$, I have added the information about the first toss. To $\mathcal{F}_1$, I have also added the information about the second toss and to $\mathcal{F}_2$, I have added the information about the third toss and so on.

So, I have only added the information about the further tosses or if I have to stock in terms of the stock price movements, up movement or down movement what has happened for that particular time point. I have added that information and I formed this sigma fields. So, this sigma fields if you observe that the previous one is contained in the next one and this behavior is continued in this manner.

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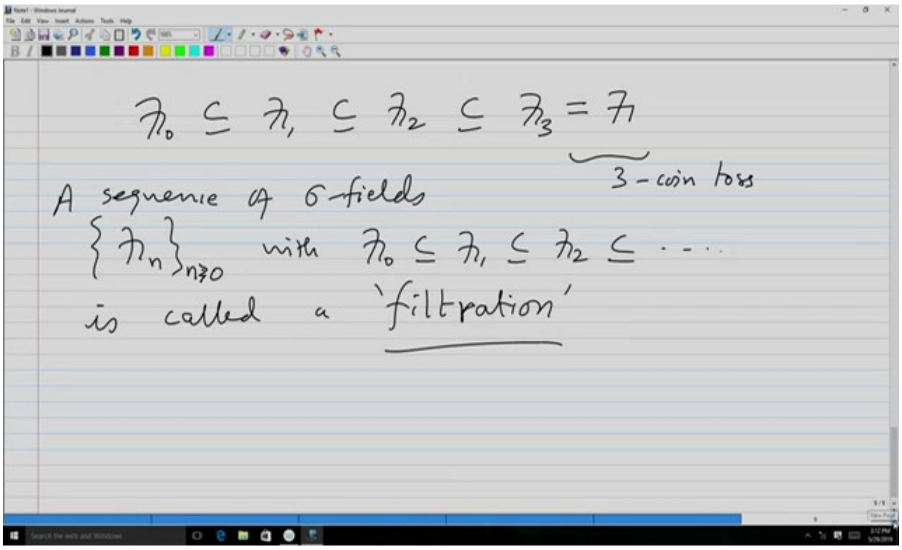

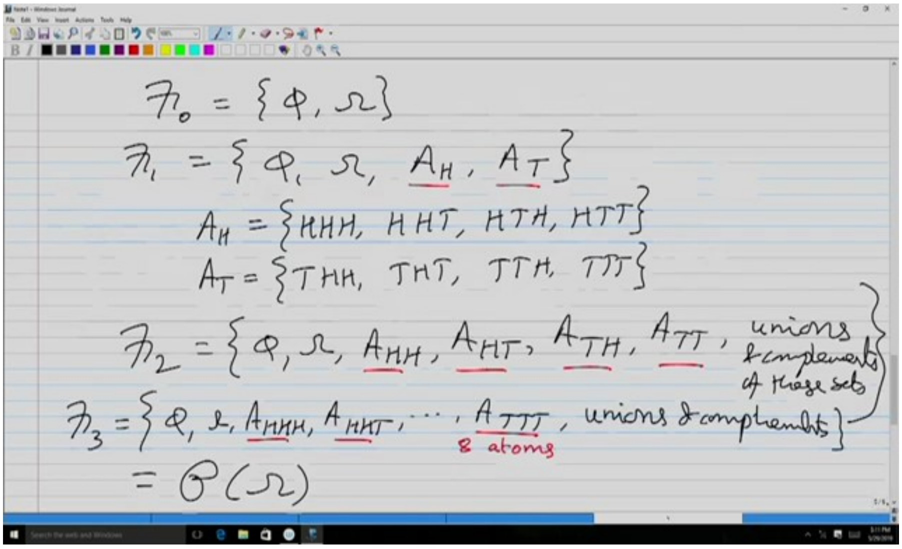

So, we have scenario where this is in this particular case in 3 period model. So, this is we are saying in a 3 coin toss situation, you have your observing this. Now, in general so this is what you are observing. In general, you could have a collection of 0 to say some other finite number or a collection which goes on. This is with so which means that now a sequence of sigma fields $\mathcal{F}_n$ with the property is called a filtration.

So, in mathematical finance this filtration plays important role because this is the modelling element for information content that you have and everything you determined based on information available and information available would be given in terms of this filtration. Where, as the time accumulates, as the time progress, the information accumulates and no information is last is the scenario that we are having. So, this is what is we called a filtration.

This is what in this simple set up you have easily observed that $\mathcal{F}_0$ contained in $\mathcal{F}_1$, contained in $\mathcal{F}_2$ and contained in $\mathcal{F}_3$ and so on. So, in general such as sequence of sigma field is what you called a filtration. And when we talk about this binomial model, we will say the filtration, the full complete filtration we mean that this $\mathcal{F}_2$, $\mathcal{F}_3$ as we have defined here are in a similar manner. If you are extending this to further time points. Now, so this is the only filtration with respect to the binomial model that we may consider; not necessary.

So, we could also consider filtration could be in a different way. So, one other way in which we will encounter filtration is in terms of the information carried by the random variables. So, this is about the complete coin tossing. Now, we could also see in the contest of this binomial model that the random variables S_k denotes the stock price at time k. They also carry some information. Now, how do we construct a filtration in those information that we highlight.

Now, for which if I go back to the previous lectures that talk about the sigma field generated by a random variable X which is $\sigma(X)$ or the smallest sigma field with respect to which X is measureable. So, if we go back to that idea, you would see here with respect to $\mathcal{F}_0$ that I have already used it. So, S_0, S_1 would not make much difference. But, let me take directly the S_2 and look at the sets determined by the random variable S_2.

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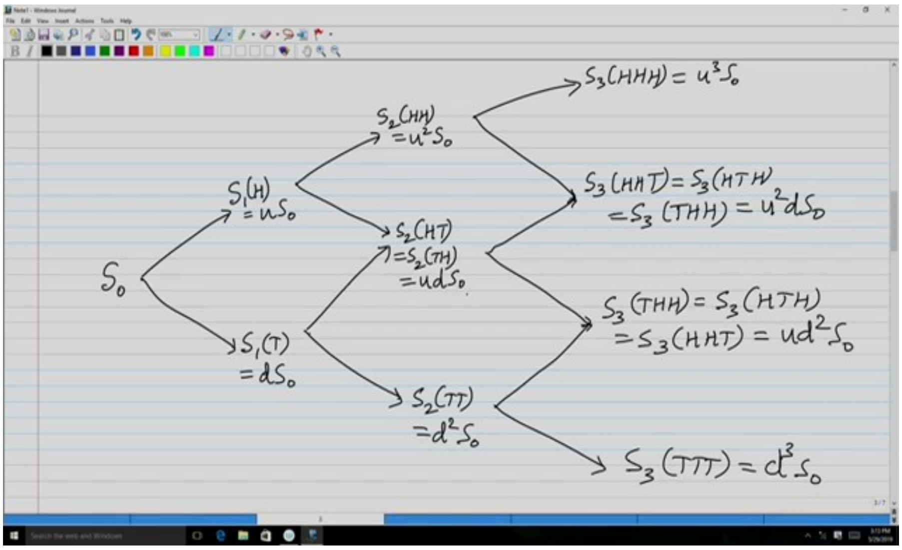

The reason is if I go back and look at the this binomial model you would see that S_0 single value $S_1(H)$ and $S_1(T)$ values whereas when you come to time 2, your HT and TH collapse to form one single node which is udS_0, so that will really help us to understand better what is this point here.

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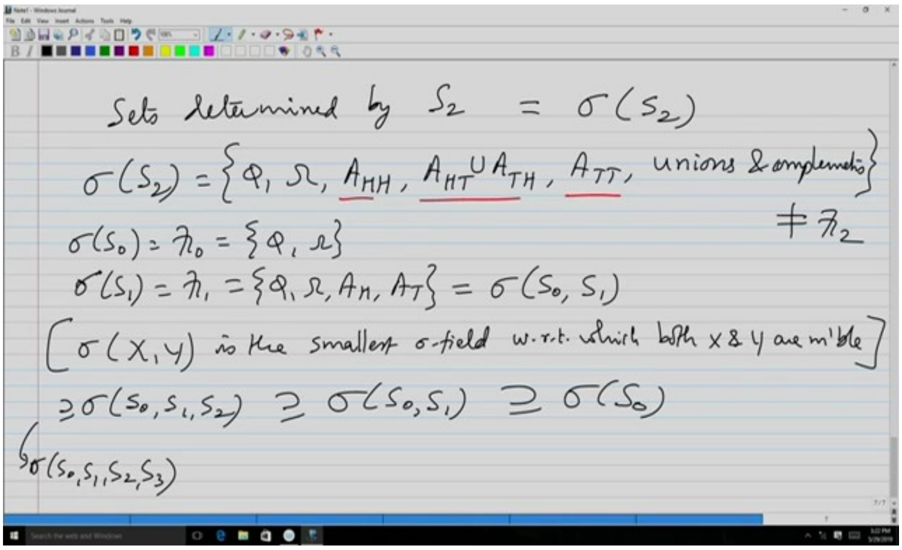

So, let us look at sets determined by S_2 in a similar fashion as the sets determined by the first two tosses. So, this is basically the sigma algebra generated by S_2. So, which essentially means that if you know only the value what are the set determined by S_2, if I extend the same idea. If you know only the value of S_2, then we can determine whether a particular outcome is inside that set or outside of that for sure.

So, if I look at this $\sigma(S_2)$ in this particular case, so you would see that this will consist of this two obviously ϕ,Ω, then A_{HH} obviously, then $A_{HT} \cup A_{TH}$, the next element is what is the difference A_{TT} and; so you see unions and complements we write but generally if we take all possible unions of these atoms, automatically this thing will the sigma field. Now, look at here what we have underlined. So, this is the sigma field generated by the random variable S_2 or the sets determined by S_2. Because, in the node HT or TH is stock price is simply udS_0 and in A_{HH} it is u^2S_0 and A_{TT} is d^2S_0.

So, this knowing the value of this random variable can know only then this part, which is essentially what is condensed is that there is 1 head and there is 1 tail. Which one occurred first is a material as far as that particular node is concerned. So, this collapses into a single. So, if I compare this with $\mathcal{F}_2$. $\mathcal{F}_2$ had 4 atoms $A_{HH}, A_{HT}, A_{TH}, A_{TT}$ but, here this f $\sigma(S_2)$ has these 3 atoms A_{HH} is one element, A_{HH} is another element whereas this union of this two elements would form another atom and this is what the sigma field generated by $\sigma(S_2)$.

But, now if I look at same thing $\sigma(S_0)$ which is the trivial sigma of field which will be same as $\mathcal{F}_0$, which is basically ϕ,Ω and my $\sigma(S_1)$ would be same as $\mathcal{F}_1$ and this will be ϕ,Ω,A_H,A_T. Because, by knowing the value of S_1 is equal to knowing the outcome of the first toss. Whereas the knowing value of S_2 is not exactly equivalent to knowing the value of two coin tosses, first two coin tosses because here we just collapsed information into single.

But, $\sigma(S_2)$ if I look at it, this is not equal to $\mathcal{F}_2$. But, whereas $\mathcal{F}_0$ and $\mathcal{F}_1$ are exactly same as $\sigma(S_0)$ and $\sigma(S_2)$. And, similarly if you go beyond the $\sigma(S_3)$ is also not going to be equal to $\mathcal{F}_3$. And, that is not all. So, this is the sigma field generated by this. But, here you see that $\sigma(S_0)$ by virtue of that being equal to $\mathcal{F}_0$ and $\sigma(S_1)$ is equal to $\mathcal{F}_1$, so this is $\mathcal{F}_0$ is contained in F_1. But, if I look at F_1 and $\sigma(S_2)$ which is F_1 is not contained in $\sigma(S_2)$. So, this will not form $\sigma(S_0)$ $\sigma(S_1)$ and so on will not form a natural filtration.

So, now to get a filtration what we do is, we slightly extend this concept of sigma field generated by rather than a single random variable, it can be extended to multiple random variables 2, 3 and so on. So, now but easily you can see that in this particular case this is also, we also can write this S_0, S_1. And, what is the meaning of this $\sigma(X,Y)$ is the smallest sigma field with respect to which both X,Y are measurable. Just extend the idea $\sigma(X)$ is the smallest sigma field with respect to which X is measurable.

$\sigma(Y)$, so $\sigma(XY)$ is essentially what we are saying is, take the union and that union of two sigma fields is not necessarily a sigma field. So, add whatever sets that are required to make it a sigma field and get that sigma field and that is what is exactly equal to $\sigma(XY)$. So, this is smallest sigma field with respect to which both $\sigma(X)$ and $\sigma(Y)$ are measurable. Means, this $\sigma(XY)$ contains, $\sigma(X)$ as well as $\sigma(Y)$. And, it is the smallest to contain this two, so that is what it means.

So, now this $\sigma(S_0,S_1)$ would be this and along the similar line, if I take my $\sigma(S_0,S_1,S_2)$ is the smallest sigma field with respect to which S_0,S_1,S_2 all three are measurable. And, which obviously will contain $\sigma(S_0)$, $\sigma(S_1)$ and $\sigma(S_2)$ and some more sets to make this as a sigma field. So, this is smallest one. So, but if I take this, since this would already contain $\sigma(S_0)$, $\sigma(S_1)$ and hence $\sigma(S_0,S_1)$. So, this would contain my $\sigma(S_0,S_1)$ and this would contain $\sigma(S_0)$. And this would further I mean expanded by including a S_3.

So, which I will write it in this position, so this essentially $\sigma(S_0,S_1,S_2,S_3)$ and so on. So, if I look at now this $\sigma(S_0)$, $\sigma(S_0,S_1)$, $\sigma(S_0,S_1,S_2)$ and $\sigma(S_0,S_1,S_2,S_3)$ this form a filtration. So, when we say this sets determined by this stock prices, so this is another way in which you could get a filtration in this model either you could take the complete information case which will form in a filtration complete or filtration generated by the stock prices. That is what we will see generated by the stock prices or the information content is exactly same as what you would get by observing the stock prices.

Remember, if I say the $\sigma(S_2)$ and $\mathcal{F}_2$, if I compare this $\sigma(S_2)$ and my $\mathcal{F}_2$ here which is these two are not equal let say the information content in $\mathcal{F}_2$ is more. Because, there you have the information about whether the first toss is a head or the second toss is a head. Whereas, when you come to $\sigma(S_2)$, if you see that there is you have only one information which is that one toss is a head one toss is tail. You do not know which one is that.

So, that is where you lose some information and then you get a something smaller. So, as you expand generally you have more information. That's what generally happen. So, this filtration is basically then the information accumulation. The information accumulation is what we are modelling through a filtration. So, the filtration would be a complete filtration as we saw in the $\mathcal{F}_0$, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ case or a filtration could be determine by this sequence of random variables S_0, in this particular binomial separation model case.

It is basically the stock prices of various time points and the filtration generated by them is in this field. So, this is another filtration. Now, we might what with either of this two filtrations, as we go along both in discrete as well as in continuous model. So, but the basic understanding of filtration is this that we have here. So, with this you know we will come to this filtration idea with close. Now, once we have the discrete probability spaces and the filtration idea.

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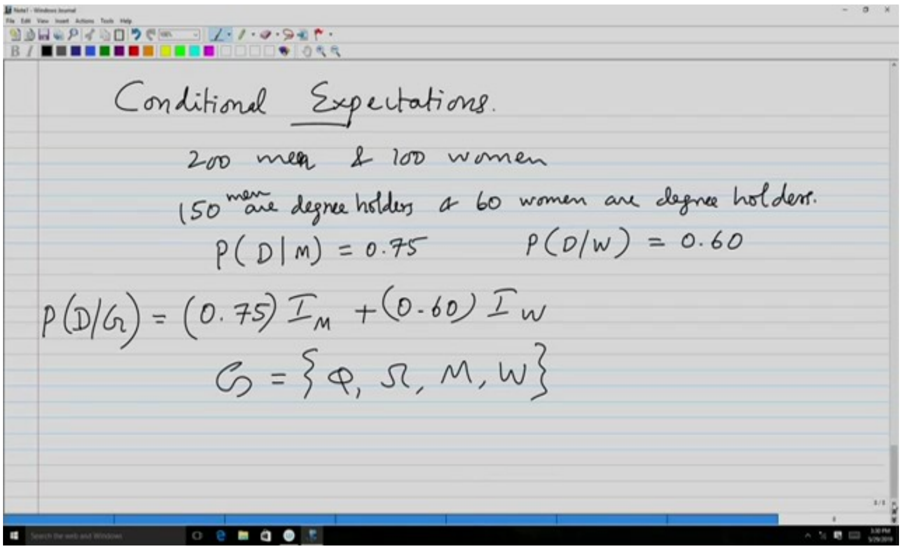

The next thing that we will do is basically the idea of conditional expectations is what then the idea that we are going to explore. And, as far as mathematical finance is concerned, this is one crucial topic which you need to master, it may be little tough but once you understand the basic ideas with a little bit of effort it is going to be easy for you to follow the remaining parts. So, I would suggest that you spend some time on this conditional expectation idea explore a bit in the discrete time it is very easy to follow.

But, in the continuous time when we reach we may not able to see what is happening. But, we will follow whatever happen in discrete time expect to follow in continuous time and we will go in. So, there we will be quickly move through but in this discrete time, please spend some time to understand the notion of this conditional expectations. To start the notion of this conditional expectations let us take the ideas from the conditional probabilities and try to take it forward to reach to the conditional expectation.

Recall in the previous lecture we define at the end towards the end the conditional probability and the conditional expectation of random variable X given as set A. Same idea that we are going to follow here also. Now, to put the two give little bit of ideas to where we are heading.

Let us assume that in some locality there are 200 men and 100 women. And, out of which see 150 are degree holders out of men, 150 men are degree holders and 60 women are degree holders. If a person is chosen at random then the conditional probability that the person is the degree holder given that it is a man. So, which we called it as the conditional probability of a degree holder. Suppose if we call it as $\mathsf{P}(D|M)$ is basically 150 divided by 200 which is going to be 0.75 and the conditional $\mathsf{P}(D|M)$ is going to be 0.60.

Now, what we want? We want to encompass both this information as a single entity. The way to do this is that make this conditional probabilities as random variables by associating indicator functions as you normally do. So, what we are doing? What we are making? We are making the conditional probabilities as random variables.

Now, suppose if this is what is the case what we introduce is basically this random variable, $(0.75)I_M + (0.60)I_W$, where M and W denotes respectively men and women and D denotes degree holder. Then this is what, so, on the set M it takes the value 0.75 this particular quantity that we have define. On the set M this particular quantity takes the value 0.75 and on the set W this particular quantity will take the value 0.6.

Now, we need to give this random variable a name. So, what we do is, we let some $\mathcal{G}$ to be this sigma field m and w suppose. Because, your omega is 300 people and then men and women are the only two groups. So, you just say this two this is $\mathcal{G}$, then you can call this random variable as $\mathsf{P}(D|\mathcal{G})$. So, essentially what we are seeing is that$\mathsf{P}(D|\mathcal{G})$ is equal to this. So, this now $\mathsf{P}(D|\mathcal{G})$ is a random variable. If ω that random person that you are picking is a man, then this particular $\mathsf{P}(D|\mathcal{G})$ will take value 0.75.

And, if a person picked randomly is women then this will take the value 0.06. So, this is what so essentially what you wanted to do, like basically the conditional probabilities is thus specifying individually we want to represent that as a single entity. How do we that: we define it is a random variable and under random variable is define through this particular $\mathcal{G}$ and then this $\mathsf{P}(D|\mathcal{G})$ gives us the required quantity.

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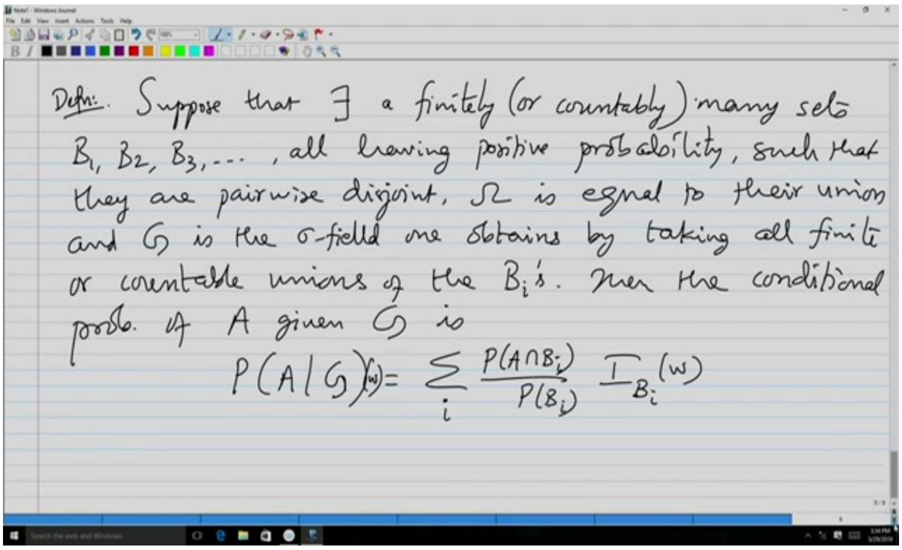

So, this now let us define preciously definition. Suppose, that there exists a finitely many sets which we call $B_1, B_2, B_3, \ldots$ all having positive probability. And, these sets are such that they are pair wise disjoint and Ω is equal to their union. Because, they are pairwise disjoint and their union is equal to Ω. And, $\mathcal{G}$ is the sigma field one obtains by taking all finite or countable unions of the B_i. So, this is the setting, there is a finite or countable number of sets disjoint and their union is ω. And, $\mathcal{G}$ is the sigma field generated by this finite or countable number of sets.

Then the conditional probability, $\mathsf{P}(A|\mathcal{G})$, is defined as follows:

$$\mathsf{P}(A|\mathcal{G})(\omega) = \sum_i \frac{\mathsf{P}(A \cap B_i)}{\mathsf{P}(B_i)} I_{B_i}(\omega),$$

So, what this means? Go back to the previous example, we will say that on each set B_i this is nothing but the conditional probability of A given B_i, $\mathsf{P}(A|B_i)$. And, this whole quantity is $\mathsf{P}(A|\mathcal{G})$, and this is a random variable.

Now, let us look at an example in this set up. It is recall our $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in the 3 coin toss settings where $\mathcal{F}_0$ is the information content of the first one toss and $\mathcal{F}_2$ is information content of the first two toss, and the sets that we written down as the atoms for that sigma field which is nothing but the B_i is that we are talking about here is what that partition of this Ω are the atoms that generate that sigma field is what this B_i is mean.

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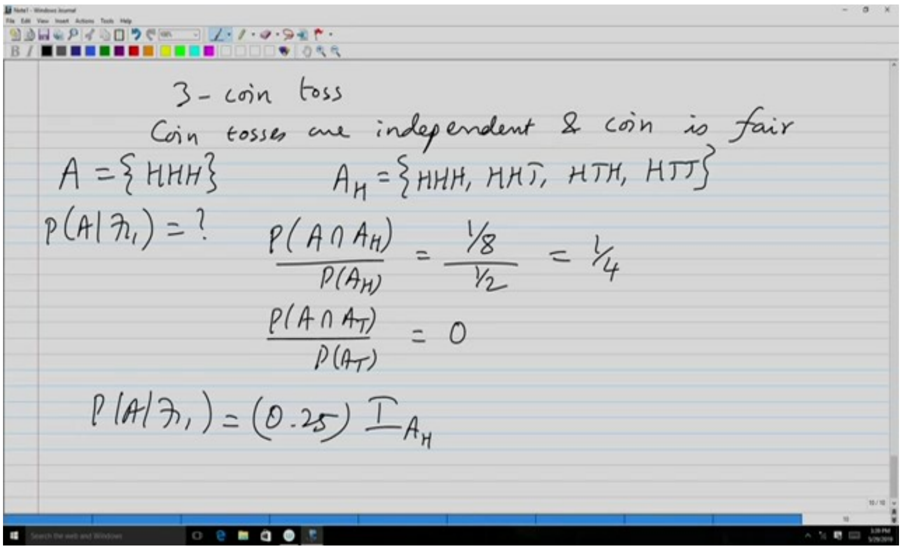

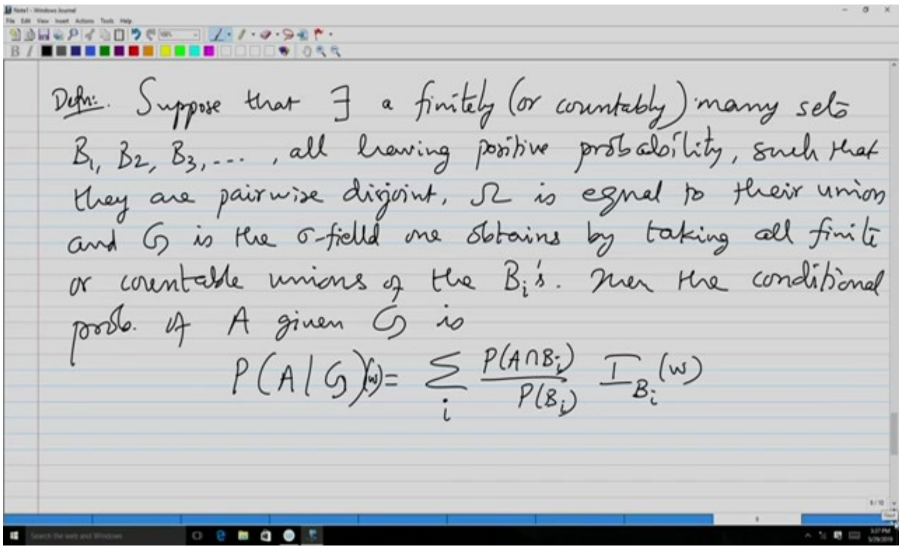

Now, in that set up, we also assume that the concept of 3 coin toss or 3 period binomial model set up. And assumed that the we toss the coin independently. And, the coin is also a fair coin, which mean that probability of head and probability of tail both are equal to half. So, in this set up suppose if I pick my $A = HHH$. Now, I want to compute $\mathsf{P}(A|\mathcal{F}_1)$, $\mathsf{P}(A|\mathcal{F}_2)$, $\mathsf{P}(A|\mathcal{F}_3)$ and whether that makes our intuitive sense as we know about those things.

So, let us first take $\mathsf{P}(A|\mathcal{F}_1)$. We know $\mathcal{F}_1$ the atoms are A_H and A_T, where A_H consist of 4 ω which are starting with head. $A_H = \{HHH, HHT, HTH, HTT\}$. A_T is just the complement of this and with this complete Ω with the 8 element set that we have. Now, this is my A_H and this is my A. So, we want to see $\mathsf{P}(A|\mathcal{F}_1)$.

Now, as per this definition, so what we need to see is that first we have to compute $\mathsf{P}(A|B_i)$ and this indicator function will give you that. Now, what is probability of $\frac{\mathsf{P}(A\cap A_H)}{\mathsf{P}(A_H)}$? We need to compute this quantity. This is now $\mathsf{P}(A|A_H)$. Since the coin toss are independent and the coin is fair, what does that mean is: each of this 8 elementary evens in this Ω are the sample points in the sample space as an equal probability of $1/8$.

So, that means $A \cap A_H$ which will be only the element HHH that have an occurrence of $1/8$, and A_H which will have occurrence of $1/2$. Hence, $\mathsf{P}(A|A_H) = 1/4 = 0/25$. And, Similarly $\mathsf{P}(A|A_T) = 0$ as $A \cap A_T$ this is a null set, so probability of this is going to be 0. So, my $\mathsf{P}(A|\mathcal{F}_1) = (0.25)(I_{A_H})$. So, this is what $\mathsf{P}(A|\mathcal{F}_1)$.

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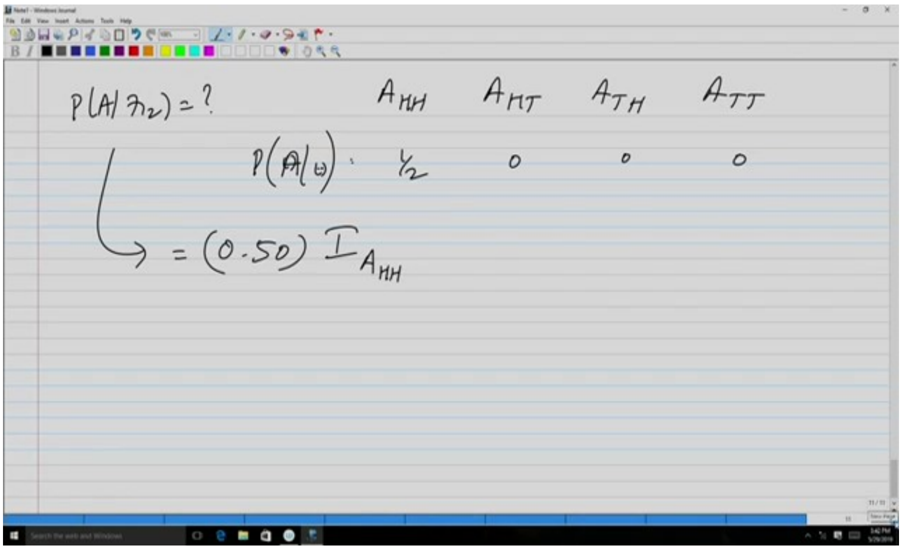

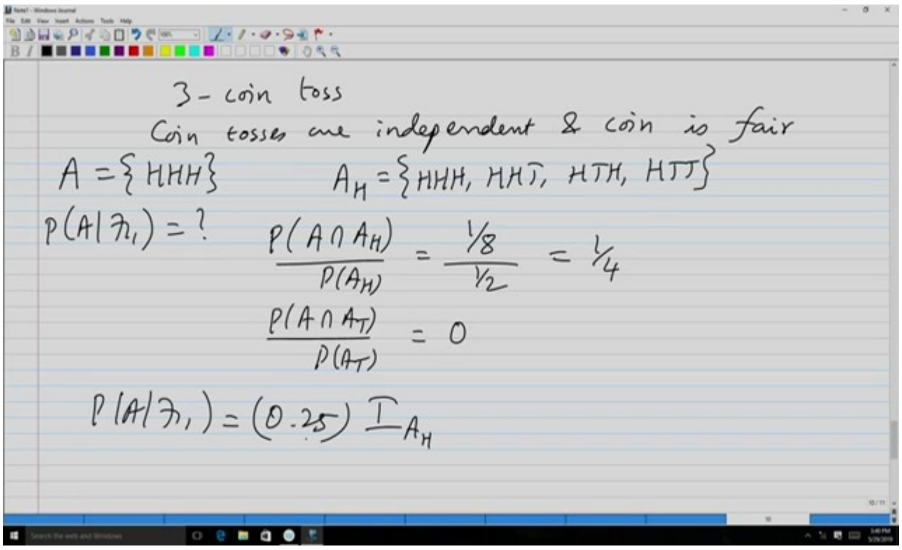

And, similarly, if I take F_2, and if want to ask the similar question: what is $\mathcal{F}_2$? And would you see, let us go back and see that whether this makes intuitive sense. So, what this says is that: you should have felt that given the first occurrence is a head, the probability of 3 heads occurring is essentially $1/4$. In the second and third toss only matters toss are independent. So, you get $1/4$, so 0.25 is worth and you will get here.

Similarly, $\mathsf{P}(A|\mathcal{F}_2)$, you can compute in a similar way. Because my $\mathcal{F}_2$ now has basically 4 sets: $A_{HH}, A_{HT}, A_{TH} and A_{TT}$ are the 4 atoms or the B's. The B_1, B_2, B_3, B_4 that we can think about with respect to this $\mathcal{F}_2$. These are essentially 4 sets and A is $\{HHH\}$. I can extend the same idea here to compute what are the elements and with respect to this probability of any of these sets. Now It is seen that $\mathsf{P}(A|A_{HH}) = 0.5$, $\mathsf{P}(A|A_{HT}) = \mathsf{P}(A|A_{TH}) = \mathsf{P}(A|A_{TT}) = 0$.

So,

$$\mathsf{P}(A|\mathcal{F}_2) = (0.5)I_{A_{HH}}$$

When you are given $\mathcal{F}_2$ the information of first two tosses and the probability that you are going to get three heads in three coin tosses given the first two are both are heads is half and at least one tail any of those is 0 is what you get. So, this is what you represent as this probability of this. Now, once we have this idea, probability of A given $\mathcal{G}$ idea, we can now define along similar lines as the $E(X|\mathcal{G})$.

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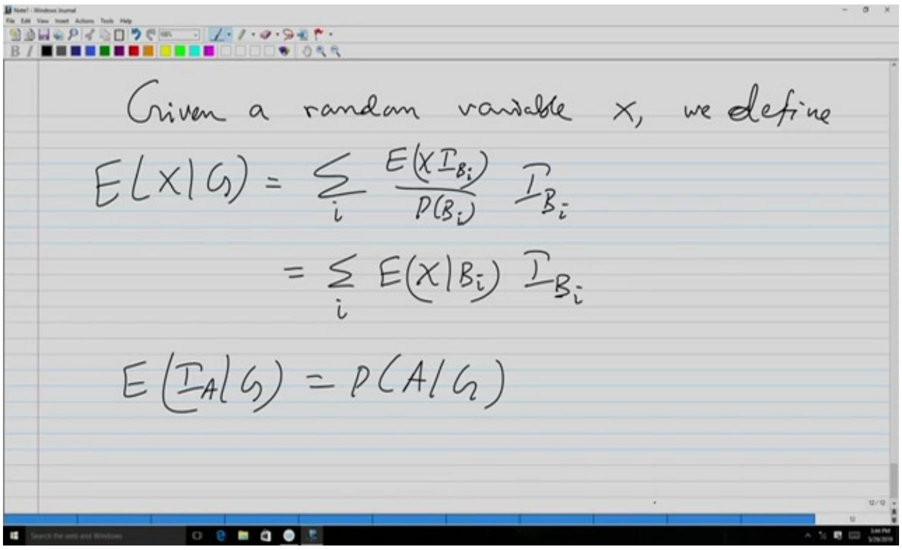

Now, recall so first we will defined given a random variable X, we define

$$\begin{aligned} E(X|\mathcal{G}) &= \sum_i \frac{E(XI_{B_i})}{\mathsf{P}(B_i)} I_{B_i} \\ &= \sum_i E(X|B_i) I_{B_i} \end{aligned}$$

So, this is the obvious definition and if this agrees with what we had before because you could notice that

$$E(I_A|\mathcal{G}) = \mathsf{P}(A|\mathcal{G}).$$

So, this is the definition when we extend: So, what we had $\mathsf{P}(A|B)$ and $E(X|A)$ rather in the simple case. Now, when we extend, we are just extending $\mathsf{P}(A|\mathcal{G})$ and $E(X|\mathcal{G})$ in a similar fashion into this case.

So, once we define this we will now have to study about certain properties and characterization that going to play as crucial role in the future when we deal with this model. So, that will take it up in the next lecture. Good bye.