Mathematical Finance
Module 6: Risk-Neutral Pricing in Discrete-Time
Lecture 19: Discrete Probability Spaces

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Hello everyone. Today's lecture, we will start with reviewing the probability theory that you might be, you might have studied already. Still we will refresh our knowledge on your elementary probability theory with little bit more emphasis on whatever we wanted to do. So, let us confine ourselves the beginning to discrete probability spaces because that is what we are going to need in order to put the binomial asset pricing model into this framework.

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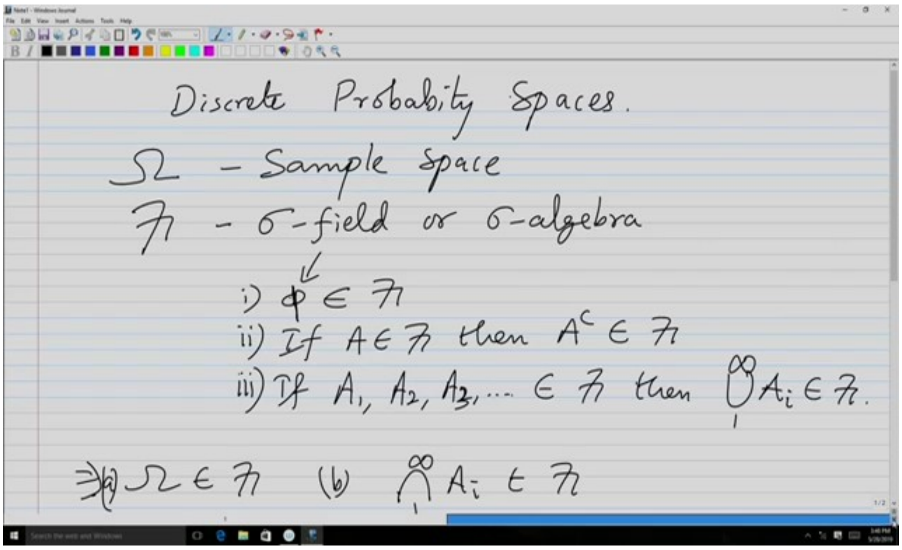

So, we will restrict ourselves to what we call Discrete Probability Spaces. As you know for any probabilistic or stochastic model, the underlying object is the probability space which is what we are going to describe now. The first element of this probability space is sample space, which is nothing but the set of all possible outcomes of random experiment, which basically describes the complete enumeration of all possible choices that might occur in the probabilistic model. So, we will denote the sample space by Ω , which is what we will be going to call this as sample space.

Now, on this sample space we are going to consider what we are calling it as an event space, which mathematically, if we have to describe, is given by a script letter $\mathcal{F}$, called sigma field or sigma algebra.

The language that those of you who might be familiar with the measured theory would be aware, but in probability more often you know we may also use the word sigma field. Now, what is this? This is what is the sigma field here. If we have to simply describe, it is a non-empty collection of objects, subsets of this Ω and that is closed under complementation and countable union. That means that It should have 3 properties:

(i) $\phi \in \mathcal{F}$, where ϕ is a null set;

(ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;

(iii) If $A_1, A_2, A_3, \cdots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

These 3 properties will immediately imply two other immediate consequences for this property:

$$(a)\ \Omega \in \mathcal{F},\ (b) \cap_{i=1}^{\infty} A_i \in \mathcal{F},$$

which is basically my $\Omega \in \mathcal{F}$ because $\phi \in \mathcal{F}$. Since $\phi \in \mathcal{F}$, $\phi^c = \Omega \in \mathcal{F}$ (from property (ii)). This is part (a).

Part b is basically, if you have $A_1, A_2, A_3, \ldots \in F$ then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$'s, which we can see easily. If you have aware about, simply if you use $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ then part (b) follows from the property (ii) and (iii). Why do we need these two? Because these two will help us to easily construct the sigma field in our cases.

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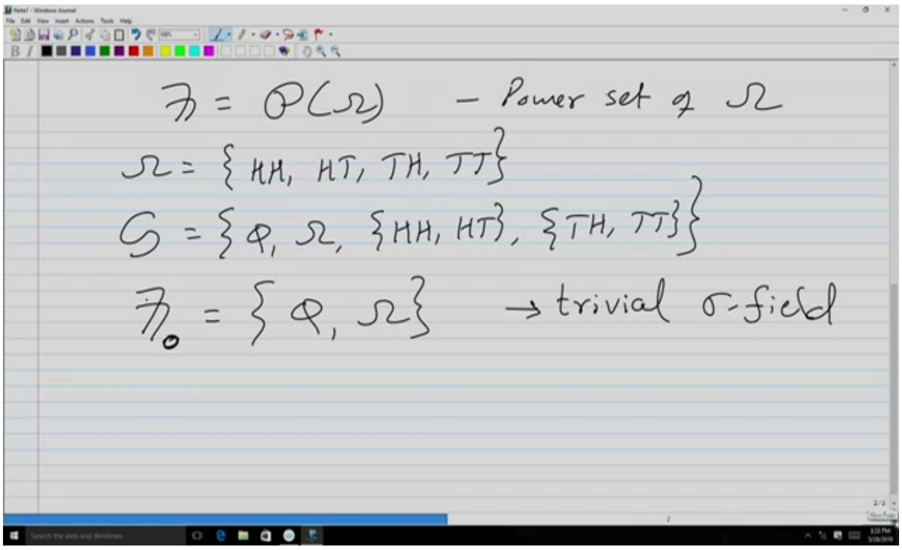

Now, typically in elementary probability theory, we will take this $\mathcal{F}$ to be the set of all possible subsets of a particular set, which in this particular case mean that most often you might take this to be, what we call the power set of Ω, denoted by $\mathcal{P}(\Omega)$. Whenever if Ω is finite or if it is countable, this is most convenient to take $\mathcal{P}(\Omega)$ as the sigma field, which will cover all possible events that we might think about with respect to this random experiment.

But in our case, in financial mathematics, we need to distinguish between various sigma fields so we will be specific to certain sigma fields, and we will need to distinguish various sigma fields, and we view the sigma fields as if the information content. So, as we go along, you will understand what they mean. So, we need to distinguish various sigma fields that is the first thing that we need.

Now, in this particular case, suppose let us take a concrete example. So, let us take the experiment of tossing a coin two times. In this particular experiment of tossing a coin two times, my Ω would then be consisting of these 4 possible outcomes:

$$\Omega = \{HH, HT, TH, TT\}$$

This is what we call as the enumeration of all possible outcomes of a random experiment which is what the sample space.

So, in this case the sample space would be this particular term. Now, in this case if I take the power set of on this particular Ω which will have $2^4 = 16$ elements.

Now, there could be a multiple sigma fields with respect to this particular Ω. We just need to ensure that it satisfies the definition or it has the properties that we have specified when we define as particular sigma field. So, in this case what we have suppose, let us take another sigma field:

$$\mathcal{G} = \{\phi, \Omega, HH, HT, TH, TT\}.$$

Now, if you look at this $\mathcal{G}$, this is also sigma field you can verify easily that this satisfies all the properties of the definition that we have specified. So, this is also a sigma field right. Like this with respect to this Ω you can define so many sigma fields, of which the simplest one is what we call some $\mathcal{F}_0$.

$$\mathcal{F}_0 = \{\phi, \Omega\}.$$

This is what we call it as trivial sigma field.

Now, as far as our information content is concerned like this does not have any information, that is what we mean. If I look at $\mathcal{G}$, then it tells you about what events you know, what events you do not know, right. So, what events you know by observing $\mathcal{G}$? If you look at this particular sigma field $\mathcal{G}$ this takes this $\{HH, HT\}$ as one event and this is its complement event. So, this set is a sigma field.

Now, if I look at these two, this tells me that this is an event that the first toss is a head and this is an event the first toss is tail. So, essentially this $\mathcal{G}$ is the information content of the first toss, whereas if I take $\mathcal{F}$ as the complete sigma field which is the power set of Ω then it is the information content of both the tosses of this particular coin toss experiment. So, more about this will come to know later. Now, let us move on after having seen this sigma field and our view as sigma field as this is information content.

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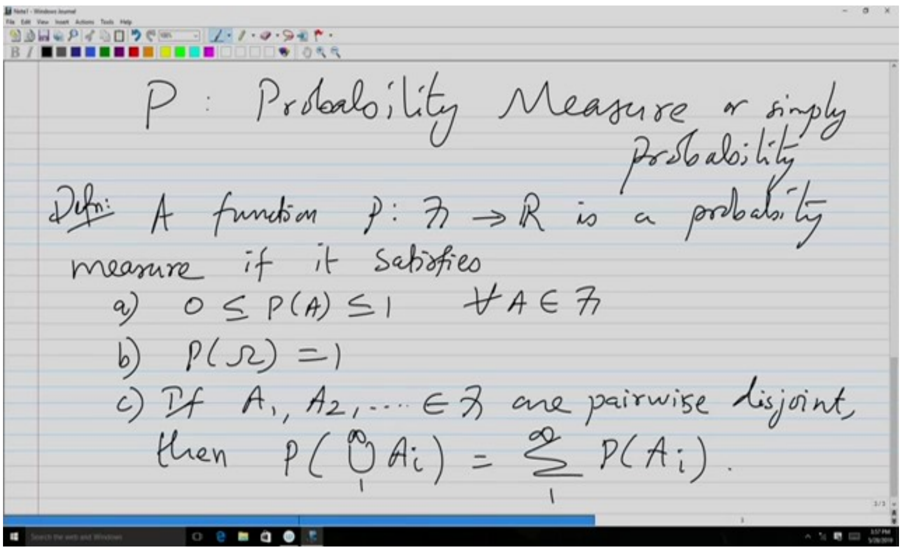

Now, we move on to the next part of this probability space definition called probability measure. It is denoted by P. It is the third ingredient of a probability space. Now, how do we define this probability measure or simply probability? If you are not comfortable with the word measure, you can just call the probability function of simply as a probability.

Now, what is a probability function? So, how do we define? So, we define a function P from $\mathcal{F}$ to $\mathbb{R}$.i.e.,

$$\mathsf{P} : \mathcal{F} \to \mathbb{R}.$$

Here, $\mathbb{R}$ is the set of all the real numbers. Here P is a probability measure or simply probability if it satisfies the following properties:

(a) $0 \leq \mathsf{P}(A) \leq 1 \; \forall \, A \in \mathcal{F}$;

(b) $\mathsf{P}(\Omega) = 1$;

(c) If $A_1, A_2, \cdots \in \mathcal{F}$ are pairwise disjoint, then $\mathsf{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathsf{P}(A_i)$.

Property (a) is that it is non-negative for every event. Property (b) is it is normalized to 1. And property (c) or the third property is if you have sets or events $A_1, A_2, \cdots \in \mathcal{F}$ and if they are pair wise disjoint which means that $A_i \cap A_j = \phi$ for $i \neq j$. Then probability of union of A_i's is equal to the sum of probability of A_i's, i.e., $\mathsf{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathsf{P}(A_i)$. Any function P since it defined on the class of sets, so P is actually a set function. The function P which we called is a probability measure, if it satisfies these 3 properties.

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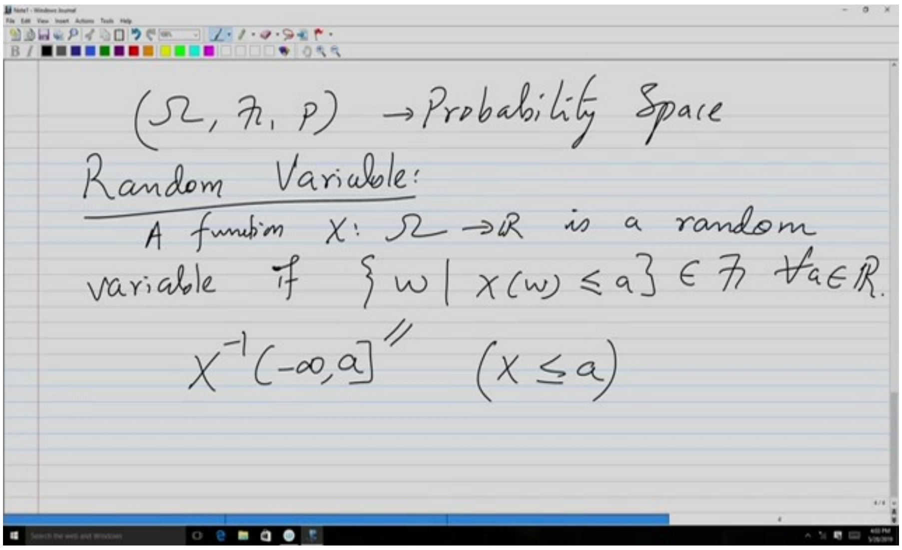

So, there are number of conclusions which one can draw immediately from this definition and which you know are given, you might have be aware in the elementary probability idea so that we will not go into that. Now, someone familiar with measure theory, you will know that this probability measure is nothing but a measure with of total mass 1, that's it. So, what if we have these 3 elements then we have the complete triplet $(\Omega, \mathcal{F}, \mathsf{P})$ which is what we call it as a probability space.

So, far any probabilistic models or stochastic models there is always an underlying probability space sometimes you may be able to describe completely sometimes this is hidden underneath, and you are happy to work with something which is defined on a particular probability space. Now, once we have this probability space. Next we define the notion of what we call a random variable. Now, simply saying a random variable is a map of real value function on Ω. This would work as long as the sigma field that you are taking is the power set of Omega but since we need to distinguish various sigma fields and we might consider sometimes sigma field which is not the power set of omega when we define a random variable.

So, if we have to define in general which will also work across even a continuous time cases when Ω becomes an uncountable sample space then we need to define X as a measurable function. What do we mean by that? So, to be precise, we say a function X from Ω to $\mathbb{R}$ is a random variable, if, simply saying, X is $\mathcal{F}$ measurable which is equivalent to say that

$$\{\omega | X(\omega) \leq a\} \in \mathcal{F} \; \forall \; a \in \mathbb{R}.$$

For all real numbers a, if this is true, of course you can define this in various ways but we will take this particular way of writing. So, this is a function define on the sample space is a random variable, if this particular object that we have defined belongs to $\mathcal{F}$ for all a. The reason why we want this because we know already that, say for sets belonging to $\mathcal{F}$ we have defined P so when later when we want to measure the size of this X so we can immediately find the probability of this.

Now, this particular set can be written in many ways, sometimes you know for convenience in a we will not elaborately write something like this: $\{\omega | X(\omega) \leq a\}$, we might conveniently write something like $X^{-1}(-\infty, a]$. This would mean the same as this set or simply $X \leq a$. So, this means that the set of all ω such that $X(\omega) \leq a$ or $X^{-1}(-\infty, a]$.

Therefore, $X : \Omega \to \mathbb{R}$ is a random variable if

$$X^{-1}(-\infty, a] \in \mathcal{F} \; \forall \, a \in \mathbb{R}.$$

So, this $X^{-1}(-\infty, a]$ is what would help when we write in general even in general probability space. So, this is what is the measurability of this function means, and $\mathcal{F}$ measurable real valued function defined on Ω is what we called as a random variable and, note that when we actually defined the random variable the P does not come into picture only ω and $\mathcal{F}$ plays a role. So, this notion of measurability is a very simple definition but it is very subtle.

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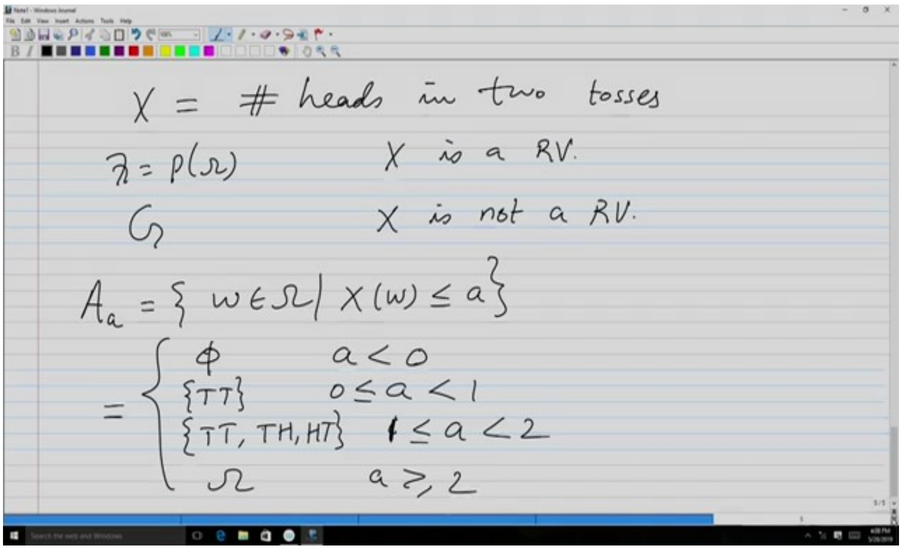

Now, let us take a simple example in this particular case and see like how we can define and how we can relate to the sigma field that we have it here. Now, let us take the same example of tossing a coin twice and in that particular case, let X denote the number of heads in two tosses. Now whether X is a random variable or not would depend on what $\mathcal{F}$ I pick. If I pick my $\mathcal{F}$ to be the power set of Ω or $\mathcal{F} = \mathcal{P}(\Omega)$, then X is a random variable, so let us write as RV for the random variable. If I pick $\mathcal{G} = \{\phi, \Omega, HH, HT, TH, TT\}$ that we defined earlier as the sigma field then X is not a random variable with respect to that $\mathcal{G}$. So, to be a little bit more precise, let us see how and why this X-$\mathcal{G}$ is or X is not a random variable with respect to this $\mathcal{G}$. What we said that X should be measurable with respect to $\mathcal{F}$.

Suppose, if I take this $\mathcal{G}$ as the underlying sigma field. Then consider this particular set

$$A_a = \{\omega | X(\omega) \le a\} = \begin{cases} \phi & a < 0 \\ \{TT\} & 0 \le a < 1 \\ \{TT, TH, HT\} & 1 \le a < 2 \\ \Omega & a \ge 2 \end{cases}$$

We wanted that for every a this set A_a should belongs to $\mathcal{G}$ if X is a $\mathcal{G}$ measurable.

Now, let us look at what are these events for various kinds of a. This will be null set for all $a < 0$. X counts the number of heads in the two tosses so it can take possible values as 0, 1 and 2 because this is a two coin toss experiment. So, for any $a < 0$ this will be the case. And for any $0 \le a < 1$, this event would be $\{TT\}$.

And going forward, for any a which is greater than or equal to 1 but strictly less than 2 you will get this set A_a as $\{TT, TH, HT\}$ and Ω for $a \ge 2$. Now, if X has an easy random variable with respect to this particular $\mathcal{G}$, then these sets must belong to $\mathcal{G}$. But you obviously know, you can notice that $\{TT\}$ or $\{TT, TH, HT\}$ none of them belongs to $\mathcal{G}$.

So, at least for at least one real number a, the test fails and hence this X is not a random variable with respect to this particular $\mathcal{G}$ but X is a random variable with respect to this $\mathcal{F}$ because $\mathcal{F}$ contains all these sets. So, so this is what you need to understand. And by taking some $a = 1/2$ is equal to half, then I will get this set as $\{TT\}$, the Singleton element, which is not there in $\mathcal{G}$ and hence X is not a random variable with respect to $\mathcal{G}$. So, X is not $\mathcal{G}$ measurable but X is $\mathcal{F}$ measurable. Now, since we were working on discrete probability spaces so what do we say as you say discrete random variable.

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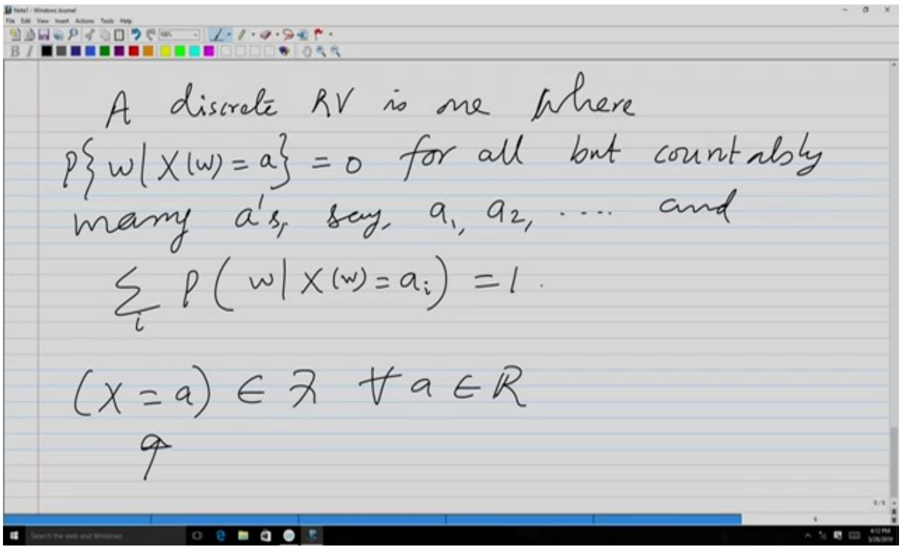

So, a discrete random variable is one where or for which this particular ω such that my $X(\omega)$ is some value. So, a discrete random variable is one for which this particular probability $\mathsf{P}(\{\omega|X(\omega) = a\}) = 0$ except for at most a countable number of a's say $a_1, a_2, \ldots$ such that $\sum_i \mathsf{P}(\{\omega|X(\omega) = a_i\}) = 1$. The probabilities are this is the total mass of 1 is distributed on a countable set of points a_i such that the total mass is 1. So, the sum is over i.

So, when you see here, you know we are using, I mean without much difficulty, like we can save a little bit of abuse of this notation flower braces as well as this braces but you know this is fine. Now, in discrete case especially we said that $X(\omega) \leq a_i$ what you need to check. To check the measurability of a particular random variable with respect to your sigma field. But in discrete case to check measurability with respect sigma field, it is enough if you consider $(X = a) \in \mathcal{F}\ \forall a \in \mathbb{R}$. This is either flower braces $\{X = a\}$ or these braces $(X = a)$ you can use. So, if you check for a discrete random variable it is enough that if you check for $\{X = a\} \in \mathcal{F}\ \forall a \in \mathbb{R}$. Then X becomes $\mathcal{F}$ measurable.

So, the reason is that you can easily observe that if you know $(X = a)$ all these sets then you can determine $(X \leq a)$. Since a only for at most a countable number of a's we are going to have so they can be written as union of countable number of such sets with equality. And hence this is sufficient. To check for the discrete case this one, $(X = a) \in \mathcal{F}\ \forall a \in \mathbb{R}$.

Now, there is associated notion of the sigma field generated by the random variable X or the smallest sigma field with respect to which X is measurable. We said $\mathcal{G}$ is not the sigma field with respect to which X is measurable but with respect to $\mathcal{F}$, X is measurable. So, what is the smallest sigma field with respect to which X is measurable?

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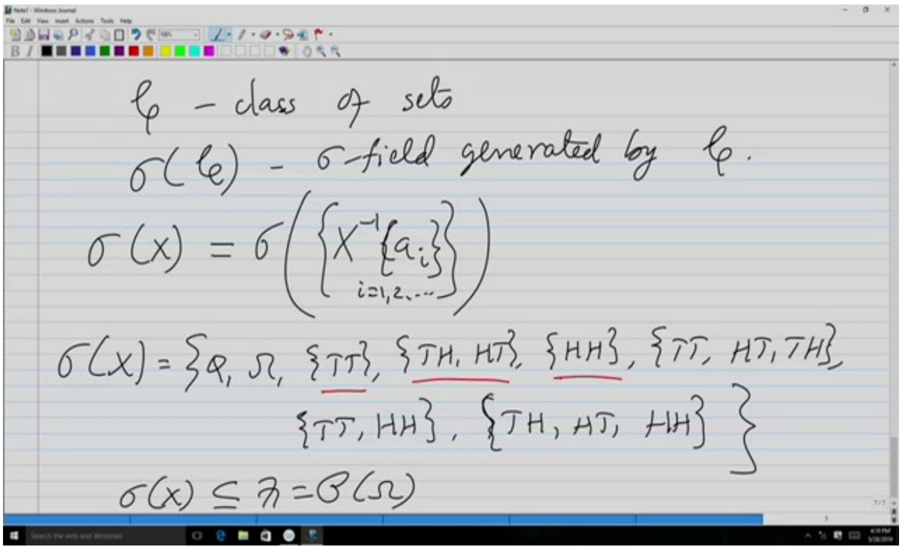

That to define that suppose if I have a class of sets and if we denote my sigma of this class $\mathcal{C}$ as the sigma field generated by this class $\mathcal{C}$, $\sigma(\mathcal{C})$. Suppose if this has some 2, 3, 4 whatever number of element that we have. Now, by taking all possible combinations of unions and complementation, you form a sigma field which will contain that.

So, that is what you call sigma field generated by the class. It is just notation if $\mathcal{C}$ is a class of sets of this then $\sigma(\mathcal{C})$, is a sigma field generated by $\mathcal{C}$. Now, in the same way we can also define the $\sigma(X)$, which is the sigma field generated by this random variable X.

Remember we are restricting currently to a discrete random variable. So, this is then would be given by in one way is the sigma field generated by sets of this class. Suppose if I call this as class so the sigma field generated by sets of this form where my X inverse of a_i, so i is equal to 1, 2 and so on, i.e,

$$\sigma(X) = \sigma\Bigg(\{X^{-1}\{a_i\}\}\Bigg),\ i = 1,2,\ldots$$

So, in this particular example that we have seen X as the number of heads in two coin tosses so we can easily see that this sigma field that I am writing, so this that I have written down here the sigma field is the sigma field generated by the random variable X and this is the smallest sigma field with respect to which this X is measurable.

Now, if you see the $\mathcal{F}$ that we have considered earlier which is the power set of omega contains this $\sigma(X)$, so you can see that this $\sigma(X)$ is contained in this $\mathcal{F}$, which is the power set of Ω. But this $\sigma(X)$ is not contained in $\mathcal{G}$ that we have seen because this and in this you would see that actually if I look at these 3 sets, the red underlined ones so these 3, if you look at it, have some things we need to say about a little bit.

So, these 3 if I look at X inverse of 0, I will get TT. X inverse of 1 because the X takes value 0, 1, 2. So X inverse of 1 if I take, I get TH and HT. X inverse of 2 if I take, I get HH. In fact if I take the union of these $\{TT\},\{TH,HT\},\{HH\}$, I get my Ω. So, these 3 sets are what we call a partition of this Ω.

They are disjoint and their union is Ω. Such sets are called partition of Ω, and these 3 is what we called atoms for the sigma field. So, these are the 3 co-exist partition that you could have with respect to this Ω which generates this particular sigma field sigma X.

So, this is what we called atoms for the sigma field, of course if you are not familiar with the terminology it is fine, you do not need that but what information this is? This also tells an information. Again we come back to that point why we wrote the $\sigma(X)$ is that sigma fields are information content for us.

And this particular sigma field what is the information that it tells is that, if I look at these basic sets so this is either I mean the partition sets are the atoms, if I look at it I will get I know exactly what is the information that this particular a particular sigma field, how that matter in a sigma field will tell.

In this particular case, if I look at $\{TT\}$ this is X is equal to 0 case and this is X equal to 1, this is a H. So, what is the information that it tells is that it has happened either 0 head or 1 head, 2 head. This is the information content. And all associated things suppose if I look at this $\{TT, HTTH\}$ that is basically tells in at least 1 tail or at most 1 head when whichever way you want to describe the events. So, it is all depends on the number of heads and number of tails is what then you get this information content.

This is if I look at with respect to $\mathcal{G}$ which tells about the information about the first toss where the first toss is a head or first toss is tail. In this case it is now talks about the number of tosses. So, this is the crucial point when we model in the quantities various underlying states of the economy in financial markets that the sigma field will be always viewed as information content and this is what it is.

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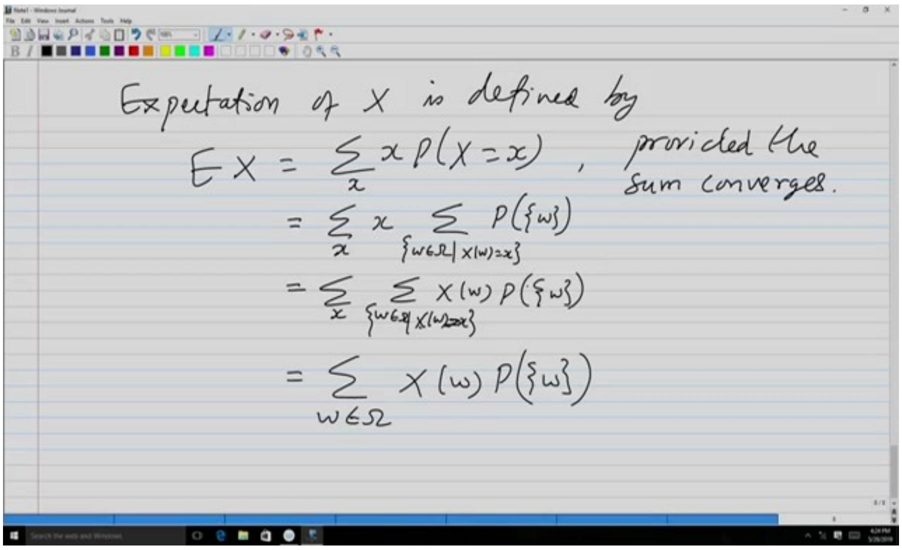

So, given a discrete random variable, we also know another quantity that we defined which is basically the expectation of X, which is

$$E(X) = \sum_x x\mathsf{P}(X = x).$$

So, we define expectation of a discrete random variable by this definition provided the sum converges.

Whenever X takes a finite number of values, of course this convergence of this is guaranteed so we will not worry too much about that but whenever this X takes a countable number of values, of course you need to worry about the convergence and then in the elementary probability theory like you say that this is provided the sum converges, you say that expectation of X is exist and is defined by or given by this particular quantity.

There is an alternate definition which is equivalent in the discrete setting which we know we will just expand this and then we will see, we are about reach that particular definition. So, let us see this is equal to;

$$\begin{aligned} E(X) &= \sum_x x\mathsf{P}(X = x) \\ &= \sum_x x \sum_{\{\omega\in\Omega | X(\omega)=x\}} \mathsf{P}(\{\omega\}) \\ &= \sum_x \sum_{\{\omega\in\Omega | X(\omega)=x\}} X(\omega)\mathsf{P}(\{\omega\}) \\ &= \sum_{\omega\in\Omega} X(\omega)\mathsf{P}(\omega) \end{aligned}$$

I can write it in this form because this is for some omegas gives rise to the same value of x so which I am separating and I am writing it in this form.

Now, this particular X because for all those omegas which gives the same value of X so I can push this X inside, I can write it as $X(\omega)$ because I am constraining my $X(\omega)$ to be x only for all these sets. So, you can write it in this form. Now, what we have here this is equal so what we have now is that you pick a particular value of X and you have written some inside and you are summed over all possibly x. So, if you relax what you would get is essentially this, $E(X) = \sum_{\omega\in\Omega} X(\omega)\mathsf{P}(\omega)$.

So, this so expectation of X which is the elementary probability theory you write it in the as in the first line and defined by, it is also equivalent to something which we have written on the last line there. We are writing it as in terms of ω, where $\omega \in \Omega$.

So, this is the same and the advantage of this is would be clear as you go along and certain properties of X say for example if we want to show expectation of X plus Y equal to expectation of X plus expectation of Y. If I use the original definition it will be some distance before I prove that but whereas if I use the last part it is immediately clear. So, like that we will have an advantage and also notationally when we use it later you will see that the second definition is convenient for us to use.

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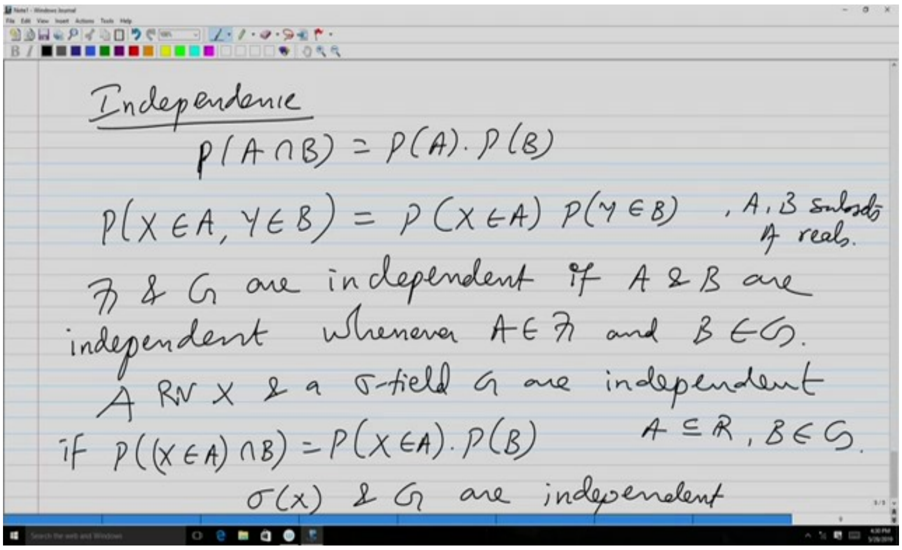

Now, once we define expectation now we also recall the notion of independence of two events. We know that two events are independent if this satisfies this. But remember when we say independent we mean probably strictly independent or stochastically independent is what we call, not our intuition or what we might think that something should be independent of this. We talk in terms of probabilities. So, independence of two events in terms of probability is basically this which is what you know.

Now, similarly we also say two random variables are independent. So, what do we mean by that? We also know that $\mathsf{P}(X \in A)$ and $\mathsf{P}(Y \in B)$. If you can write $\mathsf{P}(X \in A, Y \in B) = \mathsf{P}(X \in A)\mathsf{P}(X \in B)$. Of course this is not precisely correct to write subsets of I will not write this way. Not all subsets but in this case discrete it is fine that you can write. So we assume that they are discrete random variables, so X and Y are independent if this is true.

Similarly, now we can extend this idea or we can write generally this that two sigma fields being independent of each other. So, we can say $\mathcal{F}$ and $\mathcal{G}$ are independent if A and B are independent whenever $A \in \mathcal{F}$ and $B \in \mathcal{G}$. So, for every $A \in \mathcal{F}$ and for every $B \in \mathcal{G}$ if A and B are independent then we can say that these two sigma fields $\mathcal{F}$ and $\mathcal{G}$ are independent.

Similarly, you can talk about a random variable X and a sigma field $\mathcal{G}$ being independent. We mean exactly similar what they mean is that suppose, if we have to say a random variable X and a sigma field $\mathcal{G}$ are independent if $\mathsf{P}((X \in A) \cup B) = \mathsf{P}(X \in A)\mathsf{P}(B)$, where $A \subseteq \mathbb{R}$ and $B \in \mathcal{G}$.

So, this is same as saying that a random variable X and sigma field $\mathcal{G}$ are independent is the same as saying that $\sigma(X)$ which is the sigma field generated by the random variable X or the smaller sigma field with respect to which X is measurable, this and $\mathcal{G}$ are independent.

Because if you look at $\sigma(X)$, the sets inside the $\sigma(X)$ is nothing but X belongs to A. So, it is essentially that is what we are saying. Similarly, when you say 2 random variables are independent what really mean is the underlying sigma fields generated by the each of the random variables they are independent. So, this just the notion, the same notion, that you are aware but we are just slightly putting it in a little general framework.

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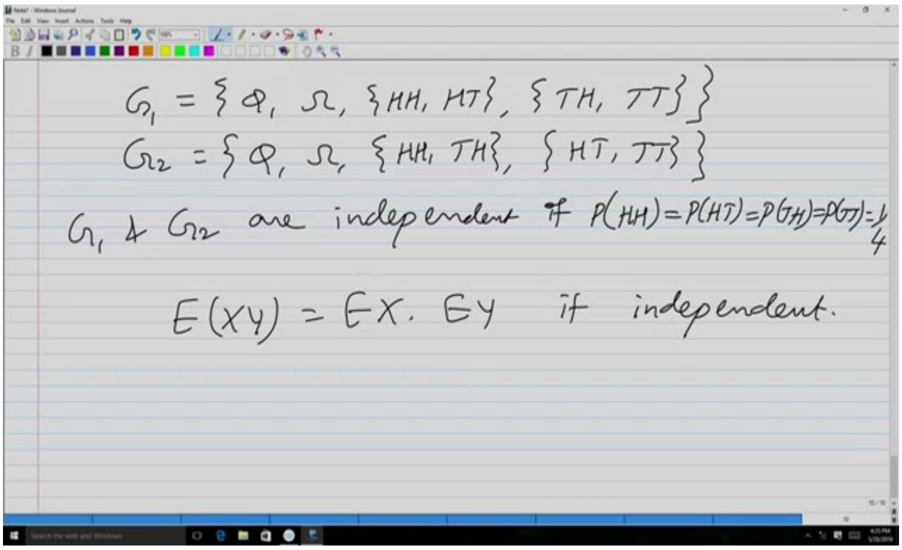

Now, to see an example of sigma field being independent, suppose you consider the same 2 coin toss experiment and let us consider now two sigma fields which is this:

$$\mathcal{G}_1 = \{\phi, \Omega, \{HH, HT\}, \{TH, TT\}\}$$

$$\mathcal{G}_1 = \{\phi, \Omega, \{HH, TH\}, \{HT, TT\}\}$$

We look at these two sigma fields $\mathcal{G}_1$ and $\mathcal{G}_2$. I mean if I go back to the notion of information and you can observe that the $\mathcal{G}_1$ is the sigma field which tells about the result of the first toss and $\mathcal{G}_2$ is a sigma field which tells about the result of the second toss.

Now, these two $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent if say for example, if I assign $\mathsf{P}(HH) = \mathsf{P}(HT) = \mathsf{P}(TH) = \mathsf{P}(TT) = 1/4$. Suppose, if I assign because as I said sigma fields we can write down but the sigma fields being independent or dependent will be decided by the probability measure because we talk independence in terms of probability. If I assign this one, you will see that this is these 2 sigma fields are independent.

But, since I can assign arbitrary manner for these 4 elementary events any value, any nonnegative will such that they sum to 1, if I assign in some other way you might see that this may not be independent. So, that is why you need to keep in mind that first coin toss independent of the second coin toss.

So, that would essentially mean this are you can even write in terms of in general some rules generically you can write but does not matter. This is the simplest example that you can take. Suppose, if I assign an arbitrarily, say for example for $\mathsf{P}(HH) = \mathsf{P}(HT) = \mathsf{P}(TH) = 1/10$ each and $\mathsf{P}(TT) = 7/10$ with the total probability 1.

Then you would immediately see that these two sigma fields are not independent because you can take any one appropriate set from each of these and then you see that it does not satisfy the required equality property. So, that is what you no need to remember. And whenever they are independent you have certain advantages. So, you can say that for example one immediate consequence that you would find is that this quantity is would be equal to this if they are independent. So, expectation we have defined. Now, similarly you can also define other moments like variance and higher moments as and when we need we made.

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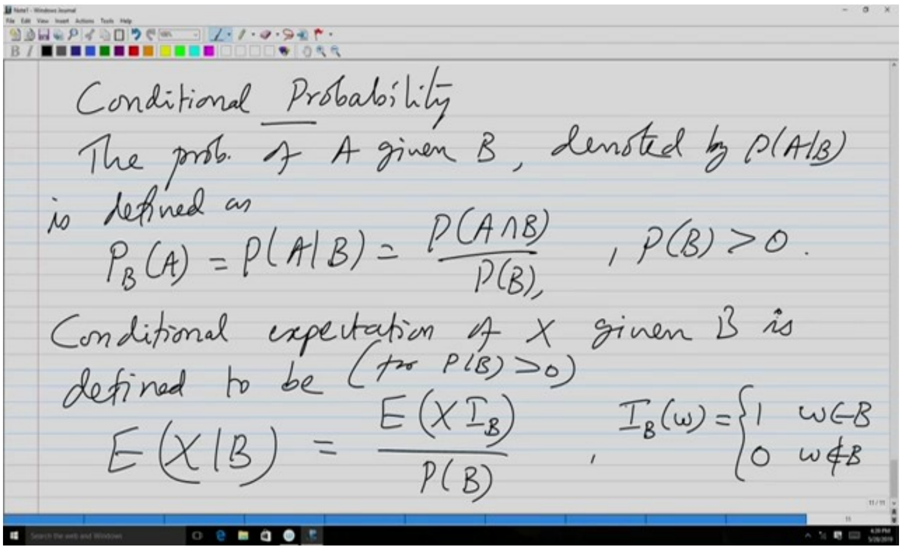

Now, when we come to the final part of it, we close this review with the notion of conditional probability. So, the probability of A given B, denoted by $\mathsf{P}(A|B)$. You can say that

$$\mathsf{P}(A|B) = \frac{\mathsf{P}(A\cap B)}{\mathsf{P}(B)}$$

provided $\mathsf{P}(B) > 0$. So, this is sometimes you can also write as if I write $\mathsf{P}_B(A)$ to mean the conditional probability of A given B.

So, you can also observe that that also defines indecisive probability measure. But what is relevant for us is the following. This probability measure would also induce when we define a random variable on this, we can also define the conditional expectation of a random variable X given B is defined to be this

$$E(X|B) = \frac{E(XI_B)}{\mathsf{P}(B)}$$

which is equal to which we define to be $E(XI_B)$ divided by $\mathsf{P}(B)$ where, what is this I_B? It is an indicator function of ω which is equal to 1 if ω belongs to B and 0, ω does not belongs to B.

So, it just gives indicator function of this. So, this quantity if we go further is what we define to be obviously, when we redefine this we also know that this is defined to be for $\mathsf{P}(B) > 0$, if not we cannot define this. So, $E(X|B)$ is just like a $\mathsf{P}(A|B)$ we defined in this manner.

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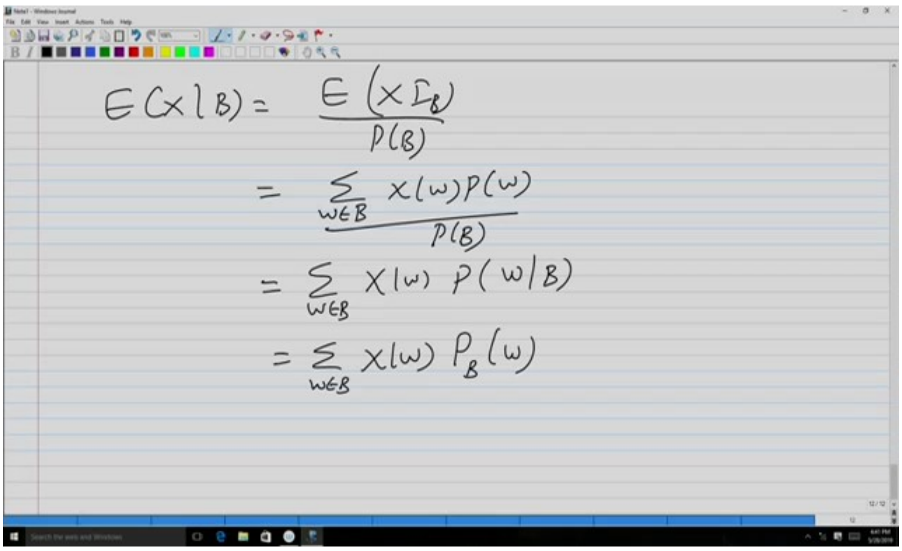

The conditional expectation $E(X|B)$ is given to be this which you can easily see that what we wrote here is expectation of $E(XI_B)/\mathsf{P}(B)$. Now, $E(XI_B) = \sum_{\omega \in B} X(\omega)\mathsf{P}(\omega)$, you can write this if I write, you can write in this in this manner because X indicator function of B, if I write; if I apply the usual expectation formula and if I notice that I_B gives value 1 for $\omega \in B$ and 0 for $\omega \in B^c$. So this is: $\sum_{\omega \in B} X(\omega)\mathsf{P}(\omega|B) = \sum_{\omega \in B} X(\omega)\mathsf{P}_B(\omega)$

So, this conditional expectation of X given B is nothing but the expectation in the normal way when we define but with respect to the induced probability measure P_B which is the conditional probability measure that you have it here. So, let us summarize this lecture what we have done.

We have just refreshed our idea of finding probability spaces, discrete probability spaces especially, finite probability spaces. We define what is a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, the triplet. And in the process in this sigma field our view was that we view it as it was the information content and we also defined the notion of a random variable not just as simply as a real valued function but is a measurable function with respect to some sigma field.

So, if you take any general sigma field $\mathcal{G}$ or $\mathcal{F}$ then a measurable function with respect to that is what to called it as a random variable. And once we have that then the independent notion again in terms of sigma fields and conditional probability and additionally finally we ended up with conditional expectation of X given an event B. So, here we just finished with an event B in this lecture. So, we will expand these ideas in the following lectures. Bye.