Mathematical Finance

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Module 5: Derivative Pricing by Replication in Binominal Model
Lecture 3: Derivative Pricing in Binomial Model and Path Dependent Options

Hello viewers! Welcome to this course on Mathematical Finance. You would recall that in the previous lecture, we talked about replications strategy being used by the seller of the option and then accordingly, we had ascertain what is going to be the price of an option in the binomial model framework and we had extended the case of single step binomial model to a generalize case where we had basically n number of steps in the binomial model.

And we had concluded the lecture with a statement of a theorem which basically connected, which basically gave a precise setup of how exactly the valuation is going to be done in terms of starting with the contingent claim at the final time point and then basically recursively looking at the price of the option at different time points by making use of a risk neutral valuation after we had introduced the risk neutral probabilities namely $\tilde{p}$ and $\tilde{q}$.

In todays class, we will proceed with the proof of the theorem and then we will talk about the notion of path dependent option, which is generalized extension of the European option that we have already seen and we will look at an example of how this replication strategy and the risk neutral pricing in case of with the binomial model can be extended in case of a path dependent option through an example pertaining to what is known as a look-back option.

So, accordingly we will start off today's lecture. So, just to do a brief recap of what we had previously seen, remember that in the binomial model for asset pricing, we have started off with the stock price S_0 and we had assumed that at time, at a subsequent time point it could go up to $S_0(u)$, which you denoted by $S_1(H)$ and $S_0 d$ which you have denoted by $S_1(T)$ and then we had extended this to N time periods.

Now, we had the condition that $d < u$ and $1 + r$, where r is the risk free rate for that single period is sandwiched between d and u and we had in the course of the discussion introduced p tilde which is the probability of the upward movement in the risk neutral world and it was defined as $\frac{1+r-d}{u-d}$ and consequently $\tilde{q}$ was obtained to be $\frac{u-1+r}{u-d}$.

Now, in a N period model we had taken V_n to be essentially the payoff for the derivative, in this case option at the end of n period and we had recursively defined V_n to be ω_1 to ω_n of which were basically n number of tosses and this was given by the discounted value of two possible values V_{n+1} in case, the first one was $V_{n+1}(\omega_1, \dots \omega_n H)$, which was the value if a head followed the first n number of tosses and $V_{n+1}(\omega_1, \dots, \omega_n)$ if a tail followed the first N tosses and with respective probability $\tilde{p}$ and $\tilde{q}$.

Also the replication strategy in terms of the number of stocks that you guy at the end time interval in the binomial set up which was given by $\Delta_1(\omega_1, \dots, \omega_n)$ was given by $\frac{V_{n+1}(\omega_1,\dots,\omega_n H)-V_{n+1}(\omega_1,\dots,\omega_n T)}{S_{n+1}(\omega_1,\dots,\omega_n H)-S_{n+1}(\omega_1,\dots,\omega_n T)}$.

Alright, so we now give the proof for the theorem that we stated last night and we will prove it by a fairly the standard approach which is the forward induction and this induction will be done on N so that is we prove that $X_n(\omega_1, \dots, \omega_N) = V_n(\omega_1, \dots, \omega_N)$ for all $\omega_1, \dots, \omega_N$. Now, where this $n = 0, 1, \dots, N$.

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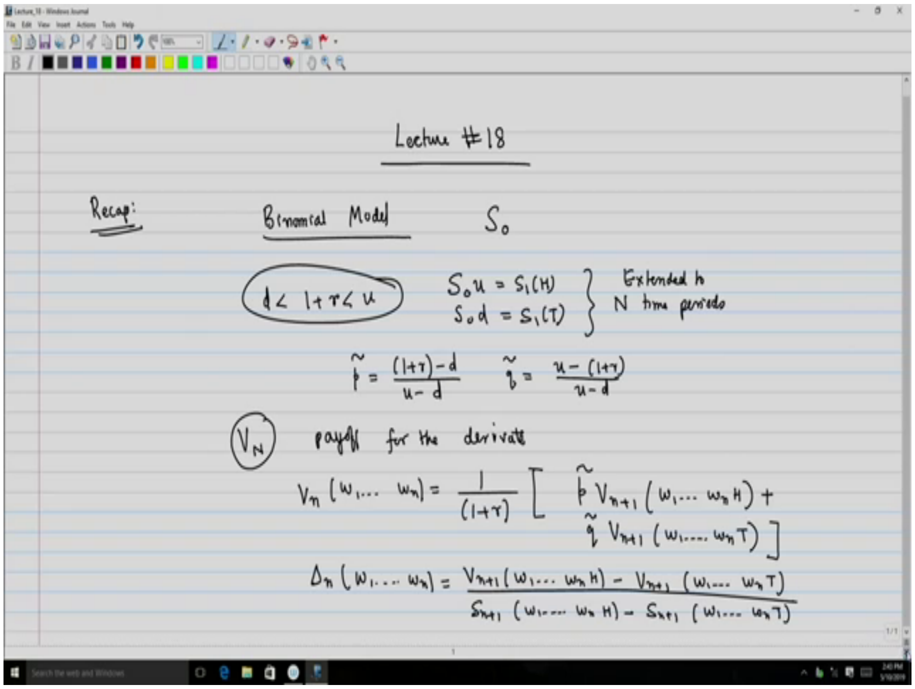

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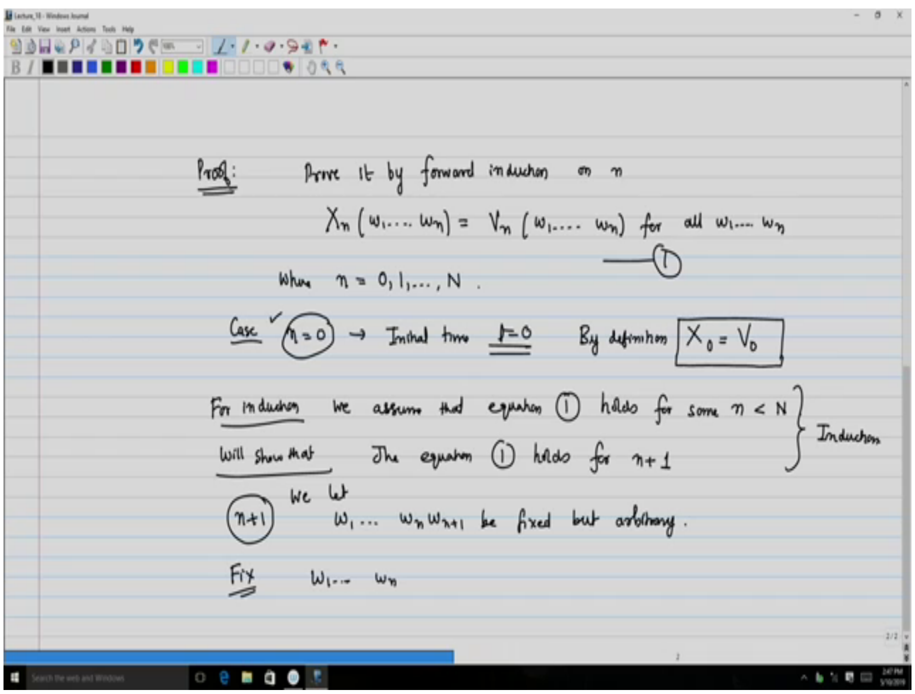

Now, let us first start off with the case that $N = 0$. The case $N = 0$, means that you are at the initial time $t = 0$, so obviously by definition remember that X_0 was the amount that you invest in order to replicate the portfolio and then this must be exactly same as V_0, which was the price of the derivative, so by the way the definition is setup, $X_0 = V_0$, so that means this result holds for the case $N = 0$.

Alright, so now for induction, so let us call this say equation 1, so for induction we assume that equation 1 holds for some n which is less than N and we will show that the equation 1 holds for $n + 1$. So, this is just the principle of induction that you are all familiar with, something we do in school.

Alright, so now, so when I am talking about, so when I talk about $n + 1$, so obviously I am looking at $\omega_1, \dots, \omega_n$ and then ω_{n+1}, right? So, we assume that, so we assume or rather we let this $\omega_1, \dots, \omega_{n+1}$, this be fixed but arbitrary. So, now, so first of all, what you do is that, we first of all start off with a fixed $\omega_1, \dots, \omega_n$, remember $\omega_1, \dots, \omega_n$ is nothing but a sequence of n number of coin tosses or rather the results of the coin tosses.

So, here $\omega_1, \dots, \omega_n$ is a string of H and T of length n. Now, there could be various possible combinations of this H and T of this length n, this particular string. So, first of all we fix this $\omega_1, \dots, \omega_n$ and whatever argument we hold for this fixed $\omega_1, \dots, \omega_n$, this will, the same argument will automatically hold in case of all other possible combinations of heads and tails that appear in the first n.

So, basically once you have fixed $\omega_1, \dots, \omega_n$ whatever argument we are using it to prove the induction that same argument will hold in case of all the remaining possible combination of H's and T's all the way to, from $1, 2, \dots, n$. Right, so now, so here what we do is that, we have fix this $\omega_1, \dots, \omega_n$, so this is known that means you are at time n.

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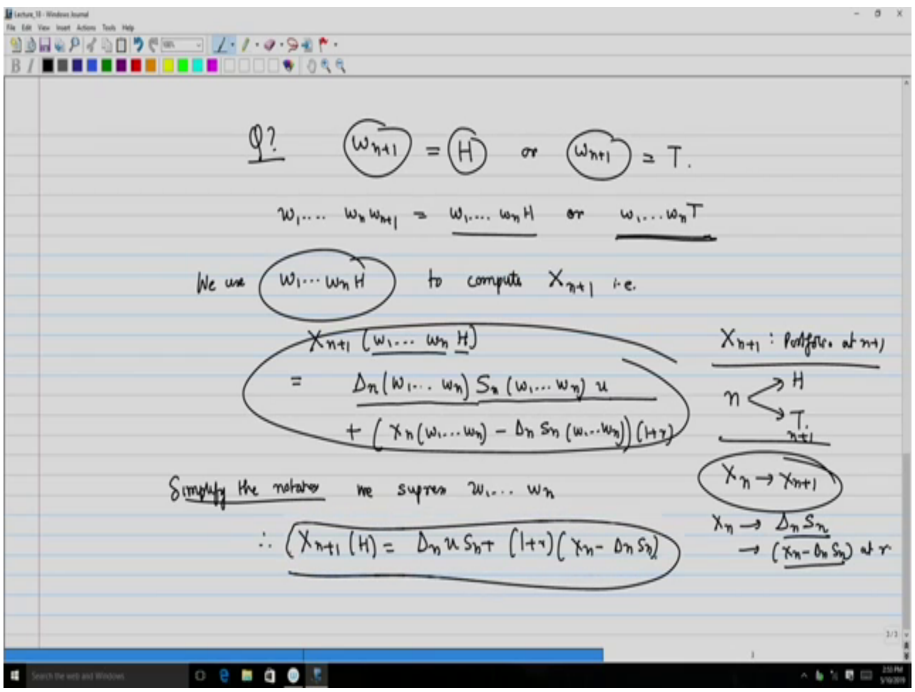

So, now the thing is that what is going to be ω_{n+1}. So, ω_{n+1}, we don't know, so ω_{n+1} is either going to be equal to H or ω_{n+1} is going to be equal to T. So, this means this string $\omega_1, \dots, \omega_{n+1}$, this is going to be nothing but $\omega_1, \dots, \omega_n H$ or $\omega_1, \dots, \omega_n T$.

Okay, so accordingly let us first of all start off with this and automatically an identical proof will work for the second case, so we use $\omega_1, \dots, \omega_n H$, right? So, accordingly we take, we compute, we use this to compute our X_{n+1}, that is in this case we will calculate $X_{n+1}(\omega_1, \dots, \omega_n H)$, so what does this become, so this term then becomes.

So, now recollect that X_{n+1} is going to be the value of the portfolio at $n + 1$. Now, remember that when you are at the level n, right, it can be followed by a head or a tail, so this mean that, so now, so this is one observation that I want to make, now second case is that you know that when you are moving from X_n to X_{n+1}, so what you do is basically you are replicating portfolios be that, from the X_n amount you invest in Δ_n number of stock with the total cost being $\Delta_n S_n$.

And the remaining amount that you have which is $X_n - \Delta_n S_n$, this you invest at rate r. So, the value of the portfolio X_{n+1} at that step $n + 1$, what it is going to be, it is going to be the value of the number of stocks that you hold, what is the number of stocks that you had actually purchased at time n, you have purchased Δ_n number of stocks at time n, remember this Δ_n was dependent on $\omega_1, \dots, \omega_n$.

And multiplied by the present value of this stock which is going to be $S_n(\omega_1, \ldots, \omega_n)u$. Why am I multiplying this by u, because as I am moving from $\omega_1, \ldots, \omega_n$ followed by an upward movement that means the stock price $S_n(\omega_1, \ldots, \omega_n)$ now becomes $S_n(\omega_1, \ldots, \omega_n)E$. So, this is the stock price that results from an upward movement or an equivalently a head occurring at time $n + 1$.

So, this is going to be the valuation of the portfolio as far as from the component of Δ_n number of stocks, plus u remember that you had invested the remaining amount $X_n - \Delta_n S_n$, so $X_n(\omega_1, \ldots, \omega_n) - \Delta_n S_n(\omega_1, \ldots, \omega_n)$, this now grows by a factor of $1 + r$ during the intervening period of n and $n + 1$.

Now, in order to simplify notation what we will do is that we suppress for the time being $\omega_1, \ldots, \omega_n$, so that means this, so therefore we can rewrite the above expression, this expression here, as $X_{n+1}(H) = \Delta_n u S_n + (1 + r)(X_n - \Delta_n S_n)$. Alright, so you can make use of this. So, from here we obtain, so we have obtained what is going to be my $X_{n+1}(H)$.

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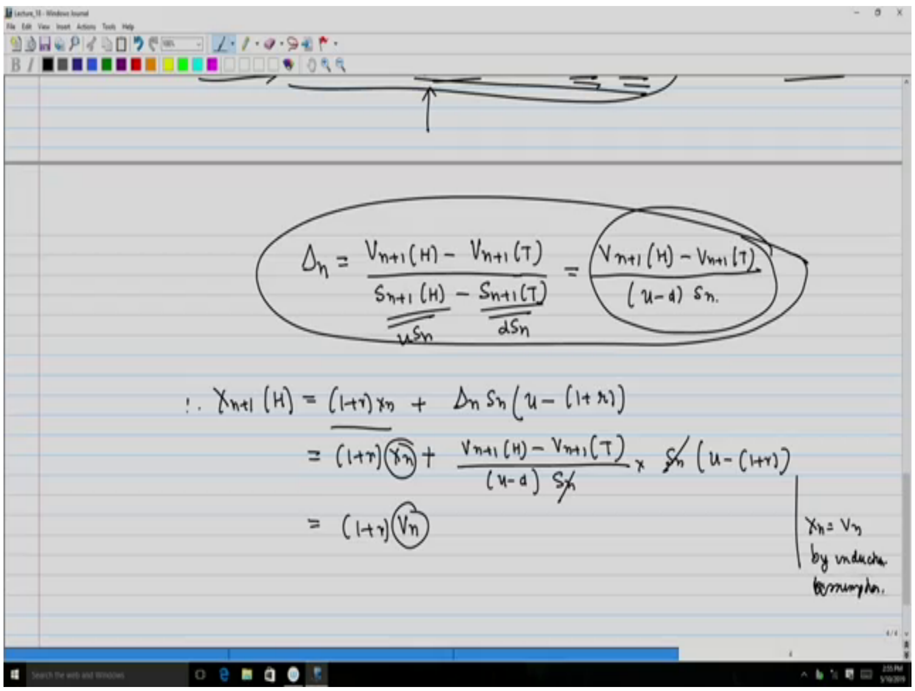


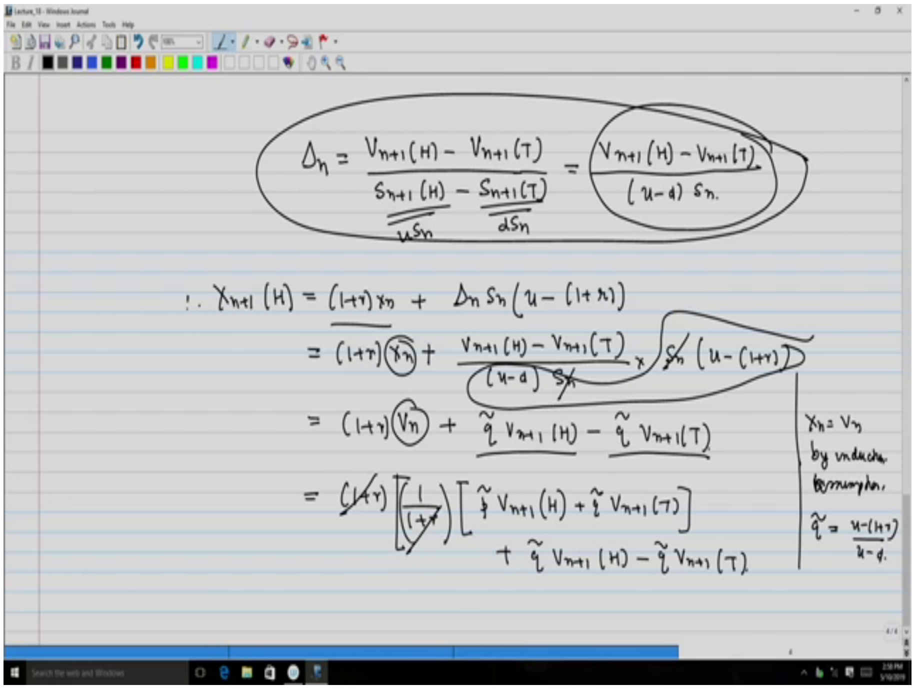
So likewise we can obtain our delta n after suppressing $\omega_1, \ldots, \omega_n$ as $\frac{V_{n+1}H - V_{n+1}T}{S_{n+1}(H) - S_{n+1}(T)}$. Now, this can be rewritten as, so remember so the numerator is $V_{n+1}H - V_{n+1}(T)$ and $S_{n+1}(H)$ is going to be nothing but uS_n and $S_{n+1}(T)$, so this is going to uS_n and this is going to be dS_n, so the denominator can actually be written now as $(u - d)S_n$.

Alright, so I will do now is that we have this expression for the portfolio value at $n + 1$, which includes a Δ_n, which can now be written in this particular form. So, what we do is that, so therefore, $X_{n+1}(H)$ is going to be, so I can look at this expression so I will take the $(1 + r)X_n$ term out, plus we have $\Delta_n S_n$, so this X_n term is already taken care of and these are two terms which involved $\Delta_n S_n$, so this term becomes $\Delta_n S_n[(u - (1 + r)]$.

So, now this can be now rewritten as $(1 + r)X_n$ plus we can substitute this expression for delta n so this becomes $\frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$, right. Multiplied by $S_n(u - 1 + r)$. So, the S_n's will cancel, we know S_n is non zero. So accordingly what do we get, I get $1 + r$, remember by induction the $X_n = V_n$, right. So, here we have $X_n = V_n$ by induction hypothesis or induction assumption.

So, remember what I am doing here, we assume that the equation holds for some n, so making use of that I am taking $X_n = V_n$. Now, plus I have this term here. So, if I look carefully, so this can be rewritten as V_{n+1}, so I can actually write this as $\tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T)$.

Now, how do I do this, for this I recall that my $\tilde{q} = \frac{u-1+r}{u-t}$. So, this term here, this entire expression, this



is nothing but $\tilde{q}$. So, accordingly I have $\tilde{p}V_{n+1}H - \tilde{q}V_{n+1}T$. So, what can I write? Now, remember that what is going to be my V_n. Remember that V_n is going to be equal to, the way V_n has been defined, so remember we had defined our V_n in this particular fashion.

V_n is nothing but the discounting of the expected value of V_{n+1}, so I am going to make use of that result, so what do I get? So, I get $V_n = \frac{1}{1+r}[\tilde{p}V_{n+1}]$, which V_{n+1}, the one where you had moved from n-th level to 1 plus 1th level, after a resulting head plus $\tilde{q}V_{n+1}(T) + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T)$. So, this term and this term, this factor cancels out.

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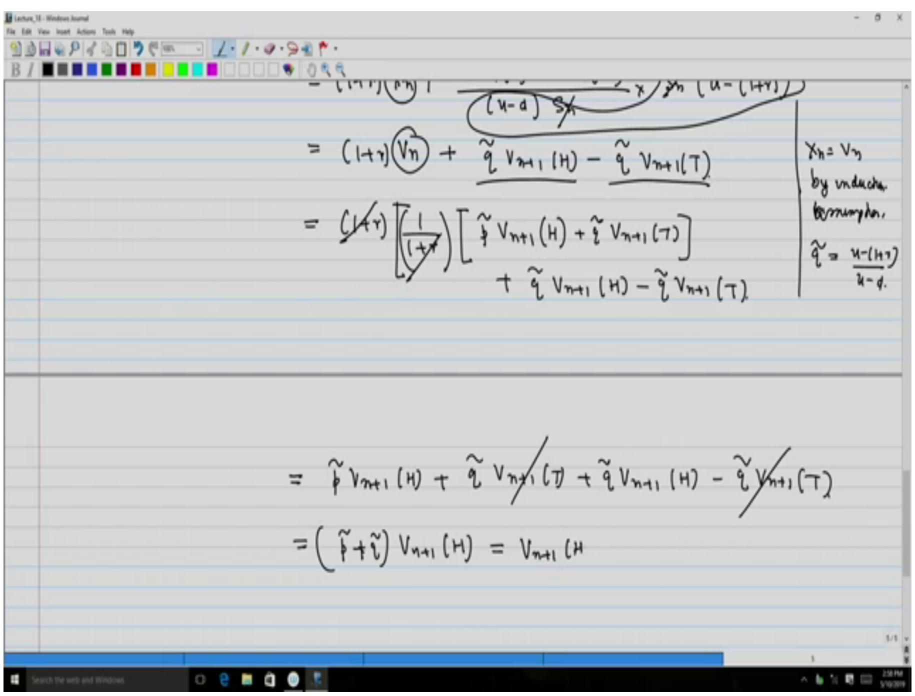

So, accordingly what do we get, we get the resulting expression of $\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T)$. So, these two terms cancel out and you get $(\tilde{p} + \tilde{q})V_{n+1}(H)$, recall that $\tilde{p} + \tilde{q}$, where there is neutral measure which sums up to 1, so this becomes $V_{n+1}(H)$.

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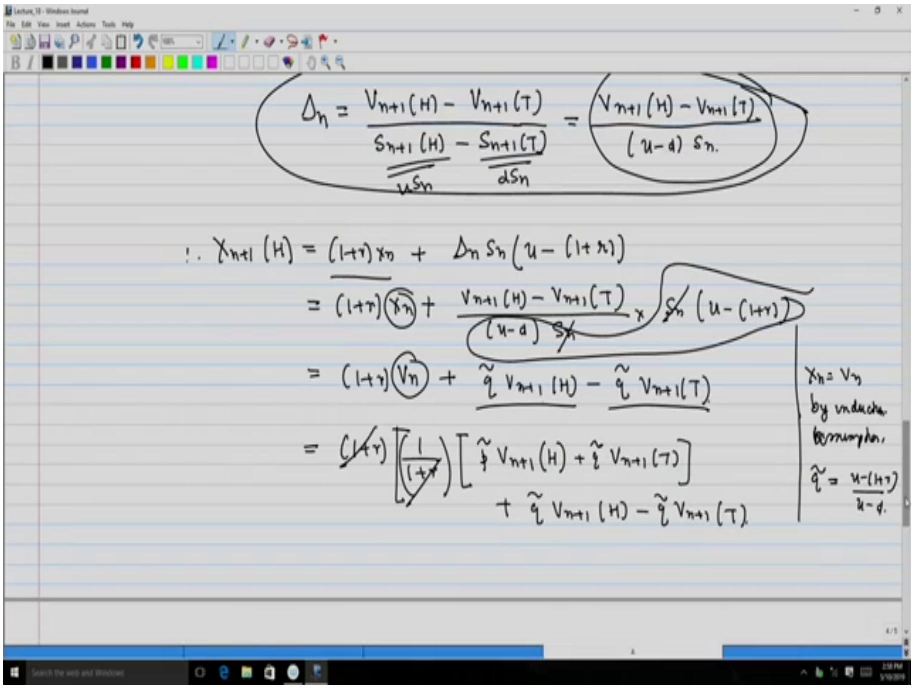

So, here you see we have made use of all the previous definitions, namely, $\tilde{q}$ and the definition of V_n and we have made use of the induction at this, induction assumption at this step. So, therefore you get that $V_{n+1}(H)$ is nothing but equal to $V_{n+1}(H)$.

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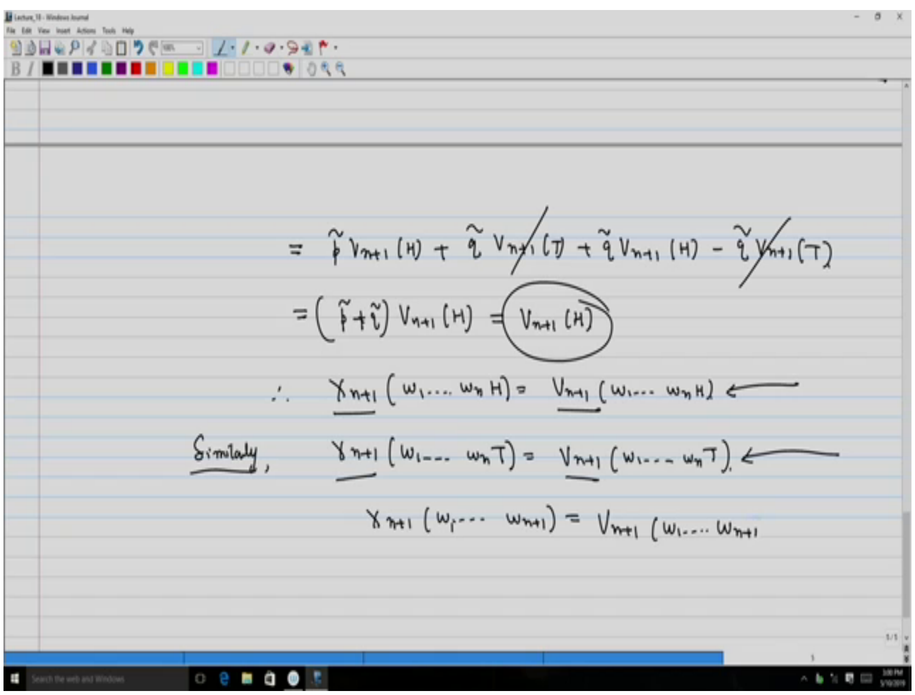

So, therefore, X_{n+1}, so restoring the ω_1 to ω_n, $X_{n+1}(\omega_1, \ldots, \omega_n H)$ is going to be equal to $V_{n+1}(\omega_1, \ldots, \omega_n H)$. Similarly, and exactly the same fashion we can prove that $X_{n+1}(\omega_1, \ldots, \omega_n T)$ is going to be $V_{n+1}(\omega_1, \ldots, \omega_n T)$. So, this means that in both the circumstances namely a head occurring at the step $n + 1$, which is this case and a tail occurring at step $n + 1$ which is a second case, this relation of $X_{n+1} = V_{n+1}$ holds. So, in general now we can make the statement that $X_{n+1}(\omega_1, \ldots, \omega_n)$, because you have shown that this holds for both

ω_{n+1} values namely a head and tail so therefore now we can conclude that $X_{n+1}(\omega_1, \ldots, \omega_n)$ will be the same as $V_{n+1}(\omega_1, \ldots, \omega_{n+1})$. So, this concludes the proof for this particular theorem, so we just come to one last example which will introduce us to the concept of path dependent options and how we can actually use this particular theorem and the overall concept of risk neutral valuation to determine the price of particular path dependent option as an illustration.

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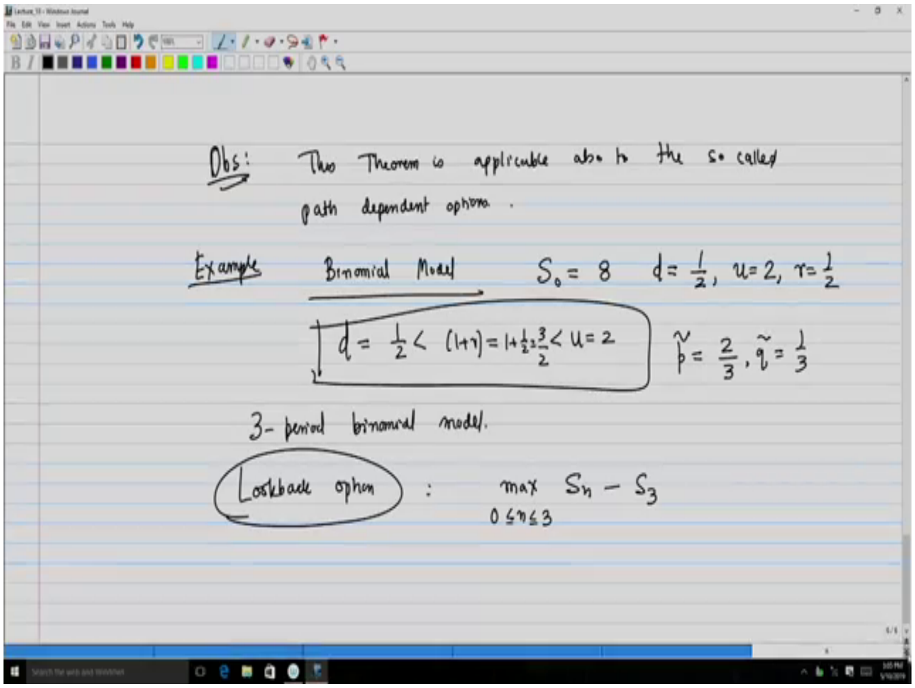

So, for this, so I just want to make an observation first, the observations that this theorem is applicable also to the so called path dependent options, so what I am saying here is the following, that so far when you are looking at a European call or a put option, the contingent claim or the payoff is just dependent on what the stock price is at the final time T or in the binomial model it is dependent on what is going to be the stock price at the step N.

However, there are certain other options where the payoff is not really just dependent on what happens to the stock price at the N time point or the final time point but also what is the pattern of the movement of stock price at all intervening time points and consequently because it depends on the entire path of movement of the stock prices so the nomenclature path dependent option is obviously applicable in this case.

So, we will explain this option pricing mechanism in the context of the theorem by a particular example of what is known as a lookback option. The name lookback option as it suggest is something that will basically look back at the entire history of the evolution of the price of the underlying stock for that particular option.

So, we start of this example and the parameters, so again I am assuming that in the background I have the binomial model which is driving the asset prices, so the initial time point S_0 is taken to be 8 as before as the previous example, at d is taken to be equal to half, u is taken to be equal to 2 and r is taken to be equal to half. So, here d is equal to half is strictly less than u is equal to 2, alright.

And these sandwiches $1+r$, which is $\frac{3}{2}$. So, this no-arbitrage condition is satisfies as you have previously seen, also in this case we have seen that the risk neutral measure $\tilde{p}$ turned out to be equal to $\frac{2}{3}$ and $\tilde{q}$ had turned out to be equal to $\frac{1}{3}$. Alright, so what you are going to do is here we will take a three period binomial model and the option is what is known as a lookback option.

And here the payoff is simply going to be maximum of S_n for $0 \leq n \leq 3$, that means $-S_3$, so the payoff is going to be nothing but the maximum stock price of among all the stock prices from $n = 0, 1, 2, 3$ minus the stock price at final time S_3.

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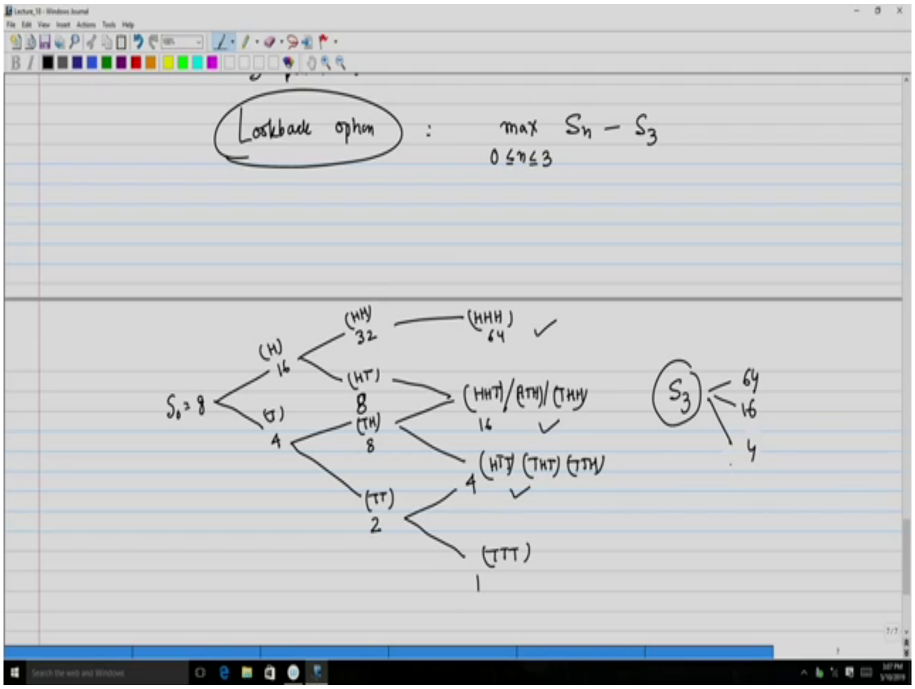

Alright, so let us actually draw the binomial tree first, so the binomial tree what do we have, we first have $S_0 = 8$. Now, in the event of a head this goes up to 16, because $u = 2$ and d is equal to half so this means that in the event of a tail this will go down to 4.

Now, if the head is followed by a head this goes up to 32, but if the head is followed by a tail then the price is halved, then that means it comes down to 16, so rather 8. Likewise, if the head, if this tail, if the second step is followed by a head then 4 goes up to 8, but if it is followed by a tail then the stock price 4 now goes down to 2.

Finally, at the last step you have HHH is one possibility, in which case the stock price is 64, the other possibilities are that you either get HHT or HTH or THH, so in this case the stock price is going to be 16, then you could have a head tail tail, tail head tail, and tail tail head in which case the stock price is going to be 4 and finally if we have all tails then the stock price is going to be 1.

So, this means that S_3, there are four possible values, namely, 64 or 16 or 4 or 1. Okay, now let us see how we can actually ascertain what is going to be the price of this lookback option, so for this you noticed that we have to look at what is going to be the maximum of S_n from the time 0 to 3 and also take into account what is S_3 and see what is going to be equal to the difference between two in each of the possible paths.

Now, you see when you want to start off the pricing using a particular theorem you need to know what is the value at N, which in this case is equal to 3, so I need to calculate V_3 under all possible circumstances of ω_1, ω_2 and ω_3. Now, what is ω_1, ω_2, ω_3?

These are just nothing just a sequence of three coin tosses so that means the sequence of heads and tails of length 3, so there are 8 possible outcomes that can happen, so you can arrive at the final stock price S_3 in 8 possible different ways which will enumerate one by one.

So, what are the different ways you can go about it, you will have head head head, you will have head head tail, you can have a head tail head, you can have a head tail tail, tail head head, tail head tail, tail tail head, and tail tail tail. So, these are the 8 possibilities, so I am interested in finding out what is my V_3 which then I will use to find V_2, V_1 and V_0 recursively using the theorem.

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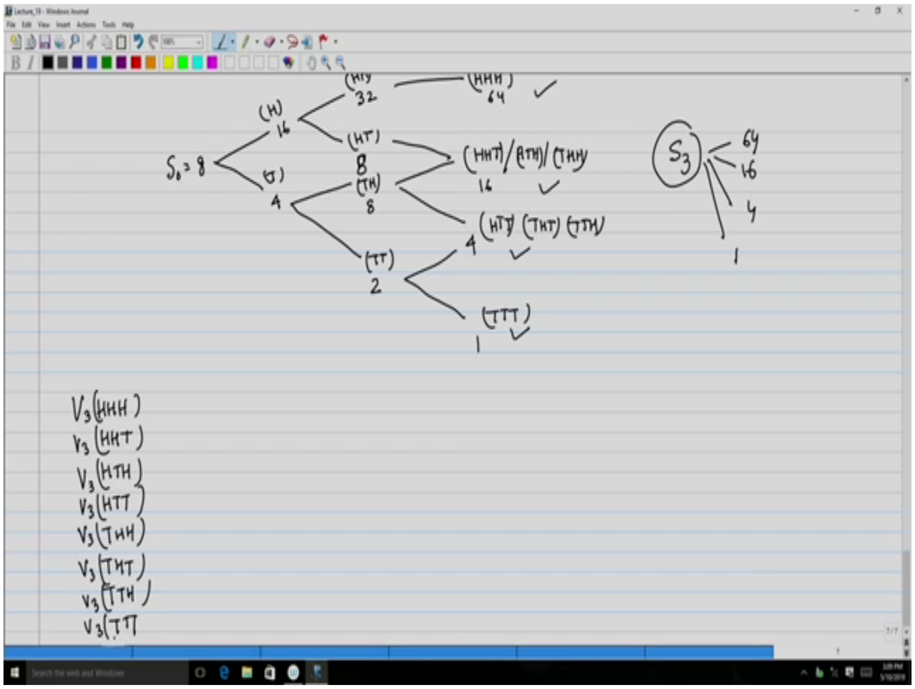

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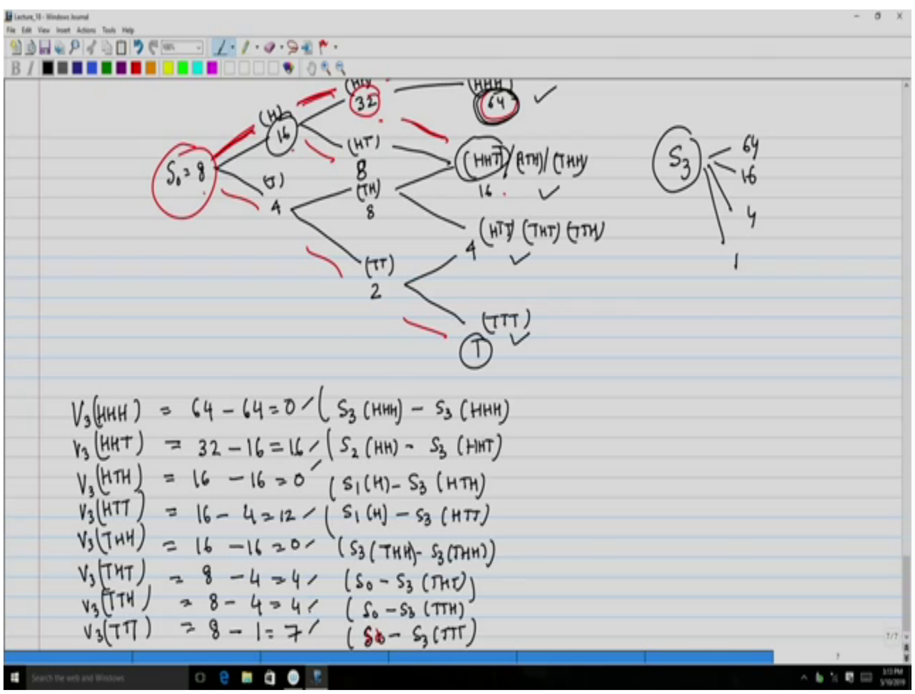

So, what is going to be the V_3 in each of those cases? So, let us look at the first case of head head head, so in this case what is going to be the payoff. So, a head head head happens if you start off here, go here and come here. So, you see that along this path the maximum value of S is going to be equal to 64, so here you will get this to be equal to 64 minus what is the S_3 in event of all head.

In the event of all head the S_3 is also going to be equal to 64. So, this is $64 - 64 = 0$. Next, so this is nothing but S_3 head head head, right? Because the maximum will happen here, this is the maximum of stock price along this particular path of head head head minus S_3 of head head head.

Okay, now let us see what happens in case of head head tail. So, head head tail means it will be head

head and then it will be tail along this path. Now, along this path we look carefully and observe that the largest value of the stock along this path is going to be 32, because it follows $8, 16, 32, 60$. So, in this case it is nothing but S_3, so the largest value is going to be these 32 value which is two heads so this is nothing but S_2 of head head $-S_3$ of head head tail, so this is 32 minus, what is head head tail? Head head tail has the value of 16, so this is $32 - 16 = 16$.

Alright! So, next look at head tail head, so what is this head tail head, head tail it is, we have a head, then we have a tail, and then we will have a head, it is followed by another head. So, in this case what we will have is the following, that this is going to be equal to $16 - 16 = 0$ and this is going to be nothing but, so the highest will happen at this 16, so this is $S_1(H) - S_3(HTH)$.

So, I will just enumerate the remaining cases, so $HTT = S_1(H) - S_3(HTT)$ and this is going to be $16 - 4 = 12$. Then THH, this case it will be S_3, the maximum will happen at $S_3THH - S3THH = 16 - 16 = 0$, then in case of THT this is going to be $S_0 - S3THT$, so this is going to be $8 - 4 = 4$.

For $TTH = S_0 - S3TTH$, so this is again $8 - 4 = 4$. And finally, we have this particular path, all tail so in this case the smallest, the largest value is going to be your S_0 and $S3TTT$, so the largest value $S_0 = 8 - 1 = S3TTT$, so this is going to be $8 - 1 = 7$. So, the values are $0, 16, 0, 12, 0, 4, 4, 7$.

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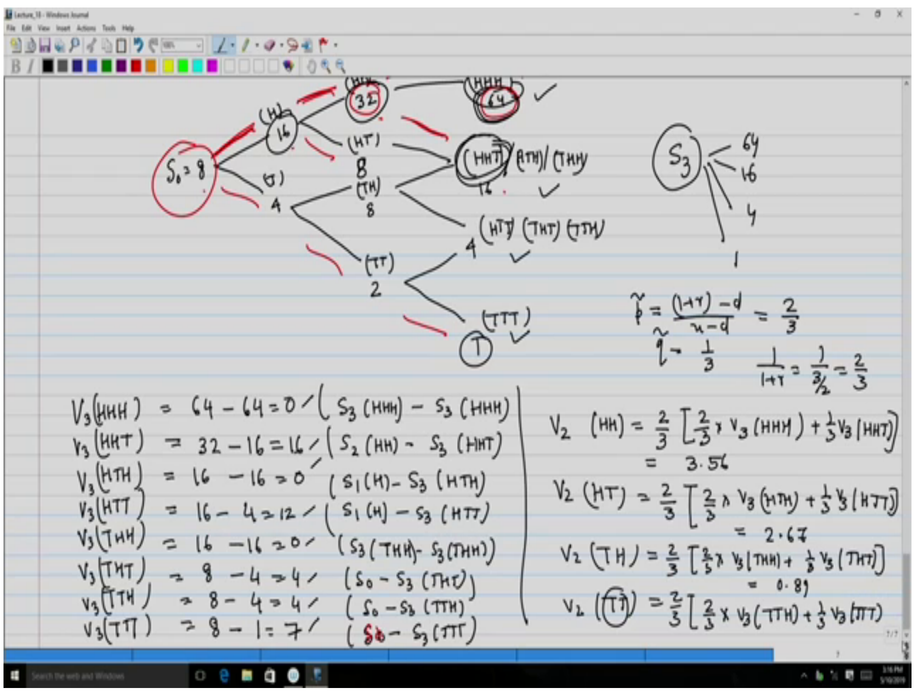
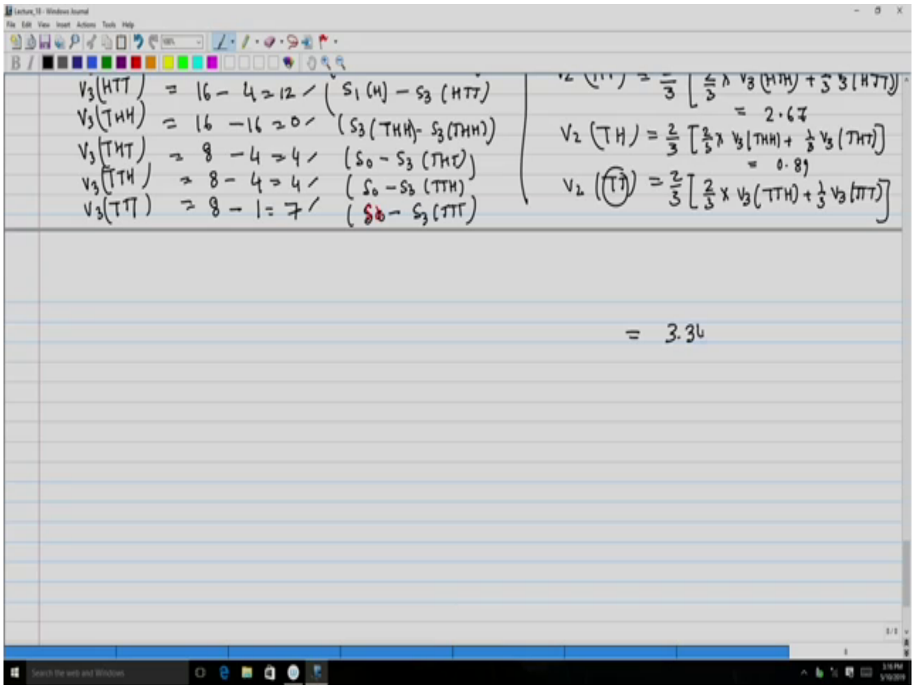


Alright, now recall that your $\tilde{p} = \frac{1+r-d}{u-d}$, this we have already calculated to be $\frac{2}{3}$. And $\tilde{q}$ was obviously then $\frac{1}{3}$. So, then what is going to be my V_2 values, now V_2, there are 4 possible values at time $t = 2$ or $n = 2$, we have 4 values or 4 outcomes namely head head, head tail, tail head, and tail tail. So, what are going to be these values, these values as per the theorem, this is going to be a $\frac{2}{3}$, remember $\frac{1}{1+r} = \frac{2}{3}$, so I applied the discount factor of $\frac{2}{3}$ everywhere.

And this is going to be equal to $\tilde{p}V3HHH$, remember $V_2(HH)$ is going to be given by the risk neutral valuation of $V_2(HHH)$ and $V_2(HHT)$, so it is going to be $\tilde{p}V_3(HHH) + \tilde{q}V_3(HHT)$. Alright? So, this is going to be equal to, this turns out to be equal to 3.56. For the next one this is going to be $\frac{2}{3}V_3(HTH) + \frac{1}{3}V_3(HTT)$ and this turns out to be equal to 2.67.

The third case this is $\frac{2}{3}V_3(THH) + \frac{1}{3}V_3(THT)$. And this is going to be equal to 0.89. And finally we have $V_2(TT) = \frac{2}{3}VT$, so this is basically the value where you have TT followed by $H + \frac{1}{3}$, while we have V_3TT followed by T. So, in this case this turns out to be equal to 3.34.

Okay, so now we are all set to figure out what is going to be V_1. Remember that we have two possible of



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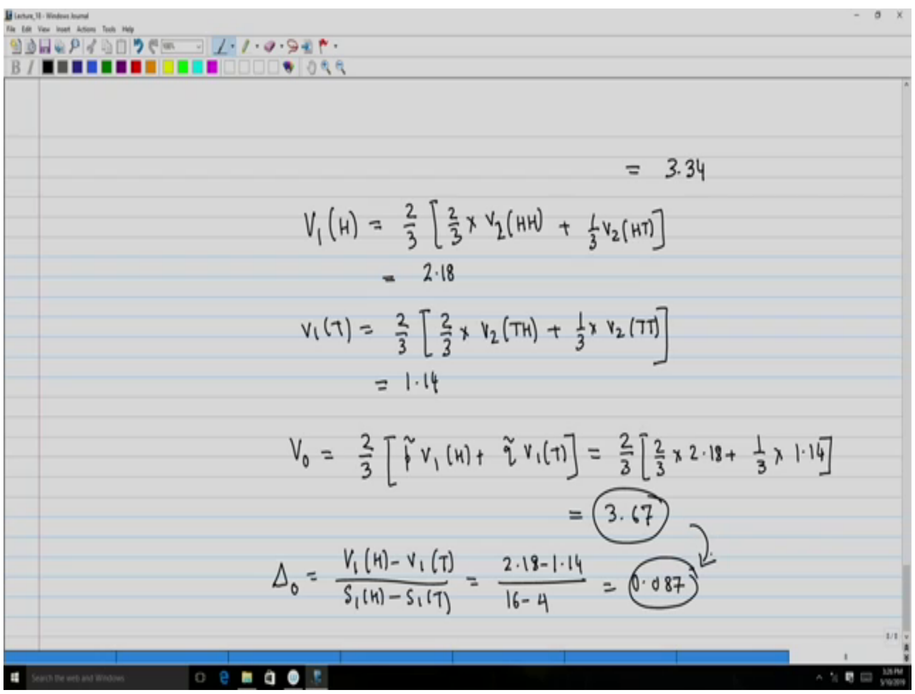

values of V_1 and that is $V_1(H)$ and $V_1(T)$, so this is going to be equal to two thirds of $\frac{2}{3}(V_1(H)$, followed by H, actually this is $V_2 + \frac{1}{3} = \tilde{q}V_2(H)$ followed by T. So, likewise $V_1(T) = \tilde{p} = \frac{2}{3}V_2(T)$ followed by $H + \tilde{q}$, which is $\frac{1}{3}V_2(T)$ followed by T.

So, once you substitute the values of V_2 in this case, so this turns out to be equal to 2.18 and the second case, this is 1.14. So, now we have the last step in our reverse recursion, so $V_0 = \frac{2}{3}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = \frac{2}{3}[\frac{2}{3} \times 2.18 + \frac{1}{3} \times 1.14] = 3.67$.

So, that means the price of the option in this case has to be 3.67 and the replication strategy in terms of number of options or underline stock that you need to buy is going to be $\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$, so this is $\frac{2.18-1.14}{16-4} = 0.087$. So, this means that you start off with an amount of 3.67 and then you use the part of that amount to purchase 0.087 number of shares.

So, this concludes our discussion on binomial model and replication in case of option pricing. Just to sum up whatever we did in three lectures on this topic, we started off with by defining what is a binomial model, as specifying what is going to be my down and up factor, and we looked at this binomial setup and the consequent option pricing for a single period binomial model.

And then this was extended in case of multiperiod binomial model followed by a result on how to price the option in the multiperiod binomial model framework by making use of new probability measure namely risk neutral measure which we denoted by $\tilde{p}$ and $\tilde{q}$ and we looked at two examples of how one can make use of this binomial asset pricing model for the underlying asset in order to ascertain what is going to be the price of the option in case of the European option and in case of a path dependent option following what is known as the replication strategy. Thank you for watching.