Mathematical Finance

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Module 5: Derivative Pricing by Replication in Binominal Model
Lecture 2: Derivative Pricing in a Multiperiod Binomial Model

Hello viewers, welcome back to this course on Mathematical Finance. This is the second lecture of this module. Remember that in the previous class where you begin this topic, we talked about the very important question of pricing of options and in particular we looked at a model for the underlying asset evolution or the asset price evolution, namely the binomial model and we looked at a single step model where is we basically looked at an initial time $t = 0$.

And then we looked at two possible asset price movements from $t = 0$ to time $t = 1$. And had obtained a pricing formulation for this particular option. And we had looked at a little example illustrating how this process ensures that the price should exactly match the value that is required to replicate the eventual pay offs in case of a European a call option.

So, in today's lecture what you are going to do is that we are going to extend the same concept from a single period model that is form $t = 0$ to $t = 1$ into a scenario where there are two periods and also three periods and then extend this, and give the option price formulation in case of capital N number of periods.

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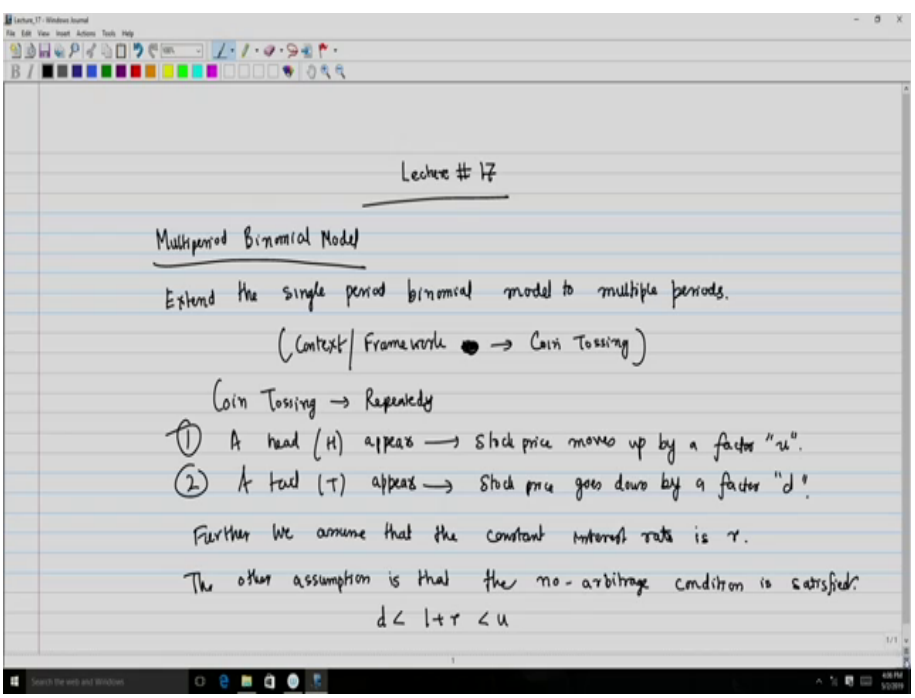

So, we begin our lecture on this topic of Multiperiod Binomial Model. So, in this case what we do is that we extend the single period binomial model to multiple periods. And so we do this in the context or the framework of our previous formulation namely the coin tossing problem.

Remember that the coin tossing problem is ideally suited for this binomial framework and so here what we do is that we do the coin tossing repeatedly and it results in two possible outcomes namely a head appears appears in which case the assumptions is that the stock price moves up by a factor, remember we have the up factor of u And secondly a tail appears and in this case the stock price goes down by a factor of d.

So, now, what I want to point out here is that, what I am saying is that earlier we started off with the initial stock price at S_0 and we said that what the prediction regarding the stock price at time $t = 1$ as a random variable will be driven by a coin tossing problem where if there are just two possible outcomes namely a head and a tail.

And we had assume in the event that there is a head then the stock price will go up by a factor of u that means it will go from S_0 to uS_0 and in the event of a tail it is going to go down from S_0 to dS_0. Now, when we are going to extend this in case of a multiperiod model, the assumption is that we will repeatedly do the coin tossing problem so if you have three number of period that you are likely, we are going to consider in that case we will have to toss the coin consecutively three times and depending on each of those coin tossing whether a head or a tail comes we model it by a upward or a downward movement accordingly.

So, this in general will be extended to a n period model where you essentially have to do n number of tosses and for each toss a head will indicate that there is a upward movement from the present time period to the subsequent time period and in case of a tail there is going to be a downward movement of a stock price from the present stock price to the stock price as a random variable in the next time point.

Okay, so also another assumption here we make is that the up factor and the down factor it will be taken as a constant across the entire all the periods, that means as you move from $t = 0$ to 1, $t = 1$ to 2 and so on, in each of the those time periods the assumption is that the up factor u and d are going to be constant every time.

Further we assume that the constant interest rate for a single period is r as before. So, the other assumption is the one due to no-arbitrage, so it is that the, is the, no, is that the no-arbitrage condition is satisfied. So, recall that what are the no-arbitrage condition, it is that d is strictly less than u and $1 + r$ is basically a sandwich between these two values.

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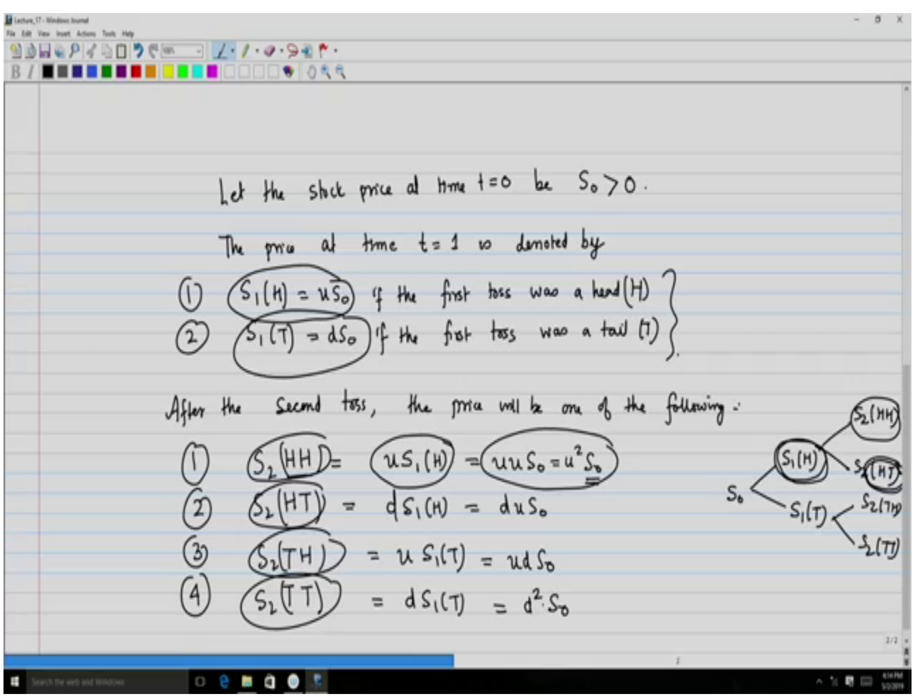

Okay, now, what we are going to do now is actually look at the model setup, so accordingly let the stock price at the initially time point $t = 0$ be a positive number S naught as before. The price at the end of the first period of the binomial model namely that time $t = 1$ is denoted and this is a random variable.

So, as before we will denote this by $S_1(H) = uS_0$ if the first toss was a head and secondly $S_1(T) = dS_0$ if the first toss was a tail. So, this is basically exactly the same setup that we had in case of the single period model. Now, after the second toss, what is going to happen? What are going to be the possible prices? So, the price will be one of the following.

Firstly, so, let us look at, so there are four cases that will actually come out of it. First of all we will have the case, suppose that the first toss was a head, in this case then it will either followed by another head or tail, and if the first toss was a tail it will also be either followed by a head or a tail.

So, this is the complete exhaustive list, so the price S_2, so there are 4 possible values of S_2 depending on the sequence of outcomes of two coin tosses. Now, S_2HH is what, S_2HH is nothing but the fact that you had an upward movement of a factor of a from S_1. So, you start off with S_0 and this is $S_1(H)$ and this is $S_1(T)$.

So, $S_2(HH)$, so $S_2(HH)$ that you are considering here is a result of an upward movement from $S_1(H)$. So, I have written this as $uS_1(H)$ and you know that $S_1(H) = uS_0$, this simply becomes uuS_0 which is equal to u^2S_0. Also this sort of follows obviously because it is two heads so two heads means two upward movements from the initial price of S_0, so you can straight you can see that this is u^2S_0.

Likewise, $S_2(HT)$ here was, is basically a movement, of downward movements which is followed from $S_1(H)$, so you had a $S_1(H)$ and you made a downward movement to arrive at $S_2(HT)$, so this is going to be equal to $dS_1(H) = duS_0$. So, likewise this is going to be nothing but $uS_1(T)$ and this is going to be $dS_1(T)$. Remember $S_1(T) = dS_0$, so this will become udS_0 and this will simply become d^2S_0.

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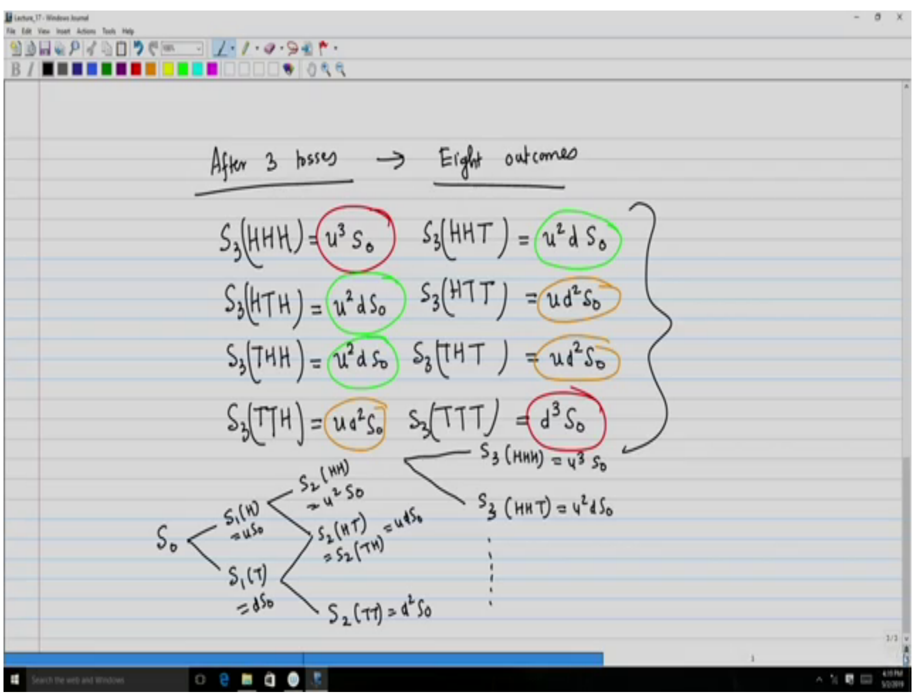

Now, let us look at the case of the third period, namely what happens between time $t = 0$ and time $t = 3$. So, after three tosses, how many outcomes you will get. You will basically get eight outcomes of coin tosses which means that you will get 8 possible values of the stock price not necessarily all distinct, so what we have, if you have now at the end of the second toss you had 4 possibilities, head head, head tail, tail head, and tail tail.

Now, each of this possibilities results in two possibilities, new possibilities, so you will have a head head followed by a head, a head head followed by a tail, a head tail followed by a head, a head tail followed by a tail, a tail head followed by an head, a tail head followed by a tail, tail tail followed by a head and tail tail followed by tail.

So, this is the complete and exhaustive list of all possible sequence of coin tosses when you have tossed the coin three time, so depending on each of the circumstance, what is going to be the stock price S_3 under each of those circumstances. So, let us articulate each of this cases one by one, so when there are three heads that means there has been three upward movements so this is going to be $uuuS_0 = u^3S_0$.

In this case there are two heads and one tail so it is going to be u^2dS_0. In this case also there are two heads and 1 tails so it is u^2dS_0. In this case HTT, there are going to be two tails or one head so it is ud^2S_0. In case of THH, again there are two heads so it is going to be u^2dS_0. For THT, it is one head so I get u and two tails so I get d^2S_0.

For two tails and one head I will again ud^2S_0 and the case of all tails I will basically get d^3S_0. So, you see that u^3S_0 is one value, d^3S_0 is another value, and u^2dS_0, there are three cases, one here, one here and one here and similarly there are three cases of ud^2S_0, so one I have here, one here and one value is here.

So, we can put this in a form of a tree. So, we start off with S naught, we have $S_1(H)$, which is equal to uS_0, we have $S_1(T) = dS_0$. From here we get $S_2(HH)$ which is u^2S_0. From here we get $S_2(HT)$ which is going to be the same S, $S_2(TH)$, namely, udS_0 and from here we arrive at $S_2(TT)$, which is d^2S_0.

So, likewise from here we will get $S_3(HHH) = u^3S_0$ and likewise we will get S_2, S_3, HHT, which is u^2dS_0 and likewise you can generate that entire tree based on whatever observations we have made here.

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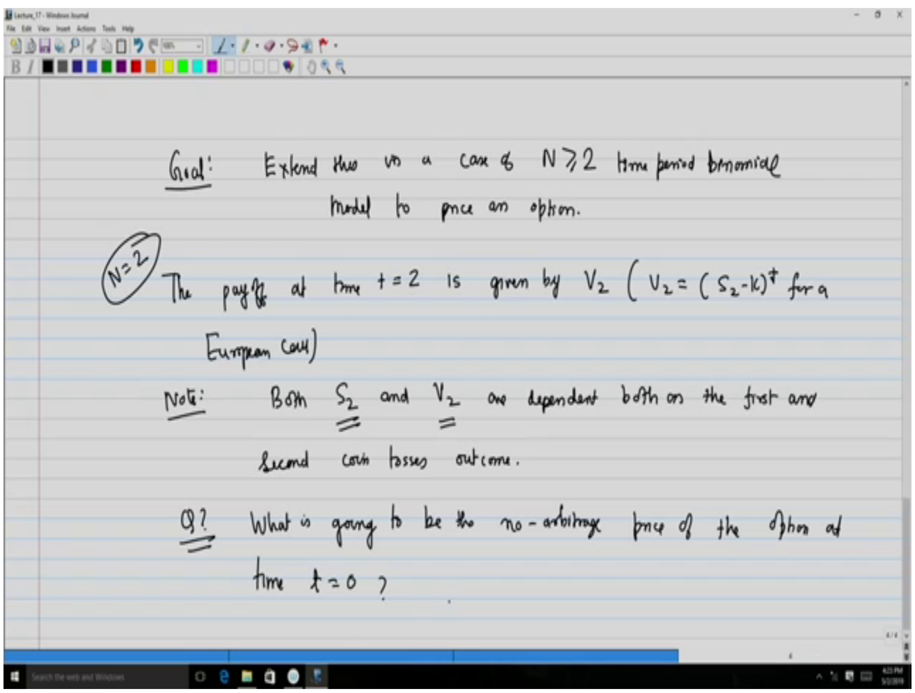

Alright, so now let us see how we are actually going to extend this and how does it actually work. So, our goal, eventual goal is to extend this in case of a N greater than or equal to 2 time period binomial model to price an option. Remember, now that we have explained the, how the trees actually generated and how naturally it can be extended to N number of time periods.

We can now start expanding the fundamental principle of the risk neutral pricing that you have done in case of one period model by making use of replication and the next goal is extend this replication strategy in case of two period model and then obviously naturally to a N period model.

So, let us look at the case of $N = 2$, remember we had done the option pricing for $N = 1$, so accordingly we will look at the pay off at time $t = 2$ is given by say, V_2. So, $V_2 = (S_2 - k)^+$ for a European call. Now, note that here both S_2 and V_2, remember S_2 had two components, S_2 as you remember, S_2 was something that is dependent on both the first two coin tosses and sequence in which they appear and more important is the number of heads and tails that come.

So, obviously S_2 is dependent on what is the outcome of the first two coin tosses and likewise, and obviously consequently because if you are just considering the value of a derivative in a two period model which will be dependent on S_2, so obviously if S_2 is dependent on the first two coin tosses, naturally V_2 is also going to be dependent on the first two, the outcomes of the first two coin tosses. So, both S_2 and V_2 are dependent both on the first and second coin tosses outcome.

So, the immediate that question that arises is, what is going to be the no-arbitrage price of the option at time $t = 0$. Please, bear in mind that whenever you are talking about the price of an option it is customary to refer to it has the price that the buyer of the option pays at time $t = 0$ and subsequently when you talk about the evolution of the price of the option over a period of time, then it will be specified that it is the price of the option at that particular intermediate time period but unless otherwise stated the price of an option will always mean that what is the price of the option at time $t = 0$ when the option was initiated.

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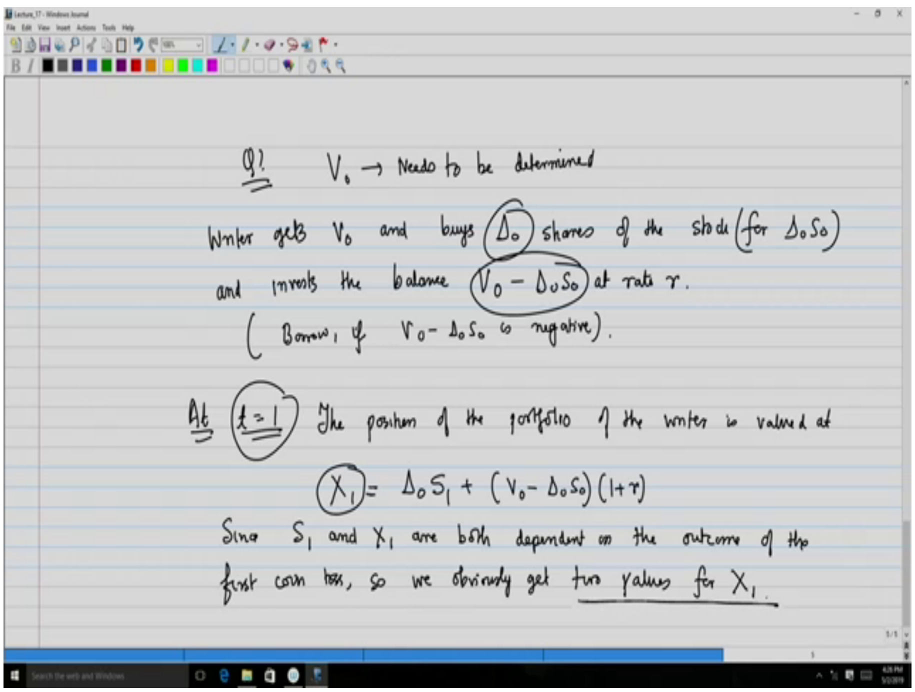

So, in the mathematical formulation this basically means that we need V_0 needs to be determined, so the idea is the following is that the writer gets V_0 and buys Δ_0, just like we had done in case of the single period model, buys Δ_0 shares, Δ_0, yet to be determined of the stock for a total amount of $\Delta_0 S_0$ and invests the balance, what is going to be the balance, the balance is going to be $V_0 - \Delta_0 S_0$ at rate r.

And of course if this turns out to be negative so it will mean that you have to borrow if $V_0 - \Delta_0 S_0$ is negative, just as we had seen in the numerical example that we saw in the last class. Now, what happens at time $t = 1$? So, the argument of what happens at time $t = 1$ will follow more or less on the lines of what you have already done in the single period model.

So, time $t = 1$, the position of the portfolio, what was the portfolio, Δ_0 shares and the remaining amount in a risk free account, so the position of the portfolio of the writer is valued at, so you had Δ_0 shares which now has a value of S_1, so the total value is $\Delta_0 S_1$, plus the remaining amount of $V_0 - \Delta_0 S_0$ has now grown to by a factor of $1 + r$.

And this is a random variable because X_1 is a random variable and this is going to be equal to X_1. Now, since S_1 and obviously consequently X_1 are both dependent on the outcome of the first coin toss, so we obviously get two values for X_1. Exactly, the same way we had done in the case of a single period mode where X_1 is basically a random variable and what I am saying by, what I mean by two values for X_1, it means that it is a random variable which takes two values of X_1.

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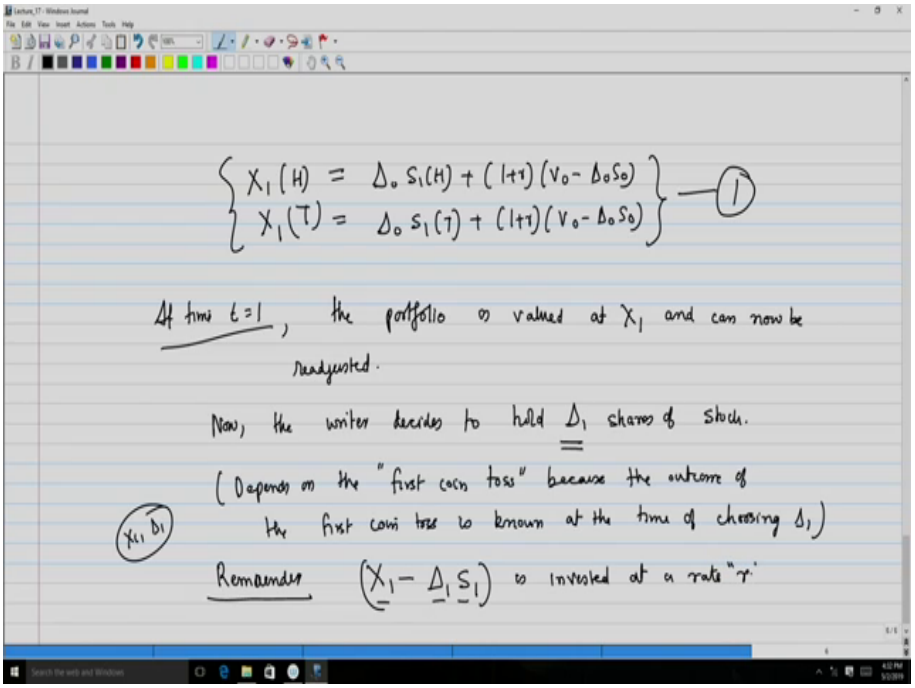

So, what are these two values, this is contingent on the outcome of the first coin which could either be head or it could be tail. So, according to the values of X_1 under these two possible outcomes of coin toss is going to be $\Delta_0 S_1(H)$ in the first place plus $1+r$, $V_0 - \Delta_0 S_0$ and $\Delta_0 S_1(T) + (1+r)V_0 - \Delta_0 S_0$.

So, I call this set of two equations as equations 1. Now, we have the complete setup of what has happened at time $t = 1$. Now, at time $t = 1$, the portfolio, what was the portfolio, delta naught share and remaining in a risk free account, so this portfolio is valued at X_1 and can now be readjusted. This means that I can actually reshuffle the number of units of the stock and what proportion of money I put in the risk free account.

So, accordingly, so now, the writer what he or she will do, is the following that the write decides to hold Δ_1 shares of stock, so remember that at time $t = 0$, the number of shares that was purchased was Δ_0 and one you arrive at time $t = 1$, then the portfolio can be reshuffled and the number of shareholdings can now be changed from Δ_0 to Δ_1.

So, the writer now decides that he or she wants to have Δ_1 shares of stock and so this is allowed, and this is what something that depends on the first coin toss, right? So, the decision of Δ_1 which is yet to be ascertained will depend on what has happened in the first coin toss so there will be Δ_1 value it will be asserted if the first coin toss was a head and in case the first coin toss was a tail, the strategy in terms of number of shares that you own that is Δ_1, is going to be something different or is likely to be something different.

So, since this depends on the first coin toss, because, so another reason we talk about that this dependency is because the outcome of the first coin toss is known at the time of choosing Δ_1. So, this is strongly related to a concept called filtration that you will see later in the course where basically a decision is being made at time point, intermediate time point which is one of the drivers of the decision is that all the information as far as the movement of the asset price is concerned up to the particular point is known when you are making the decision.

So, in this case what has happened is that when you are sitting at time $t = 1$ by that time you know whether a head (and it) or a tail has taken place. So, suppose that a head has taken place, so you obviously know that at time $t = 3$ you cannot have a outcome where there are three tails, so that is ruled out.

So, this is what is known as basically you know the information is passing or in the parlance stochastic calculus this is what is known as a filtration. Alright, so what happens is that once we have sorted out this X_1 and Δ_1, what you do is that, in that case once you have invested Δ_1 so what is the remainder?

So, the remainder is going to be, you had an amount of X_1, which is of course of dependent on head or tail, and you have invested $\Delta_1 S_1$ in the shares and the remaining amount, this remaining amount that you have which is $X_1 - \Delta_1 S_1$, where each of this is basically going to be dependent on the first coin toss is invested at a rate small r.

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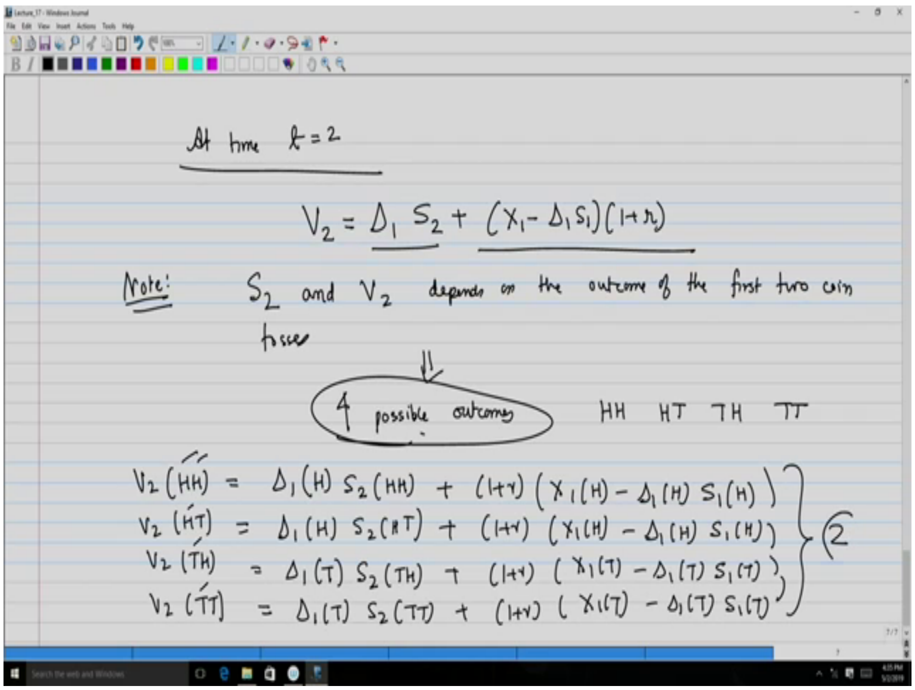

So, at time $t = 2$, what will we have? At time $t = 2$, you had Δ_1 shares, you bought a time $t = 1$, so its value at time $t = 2$, is going to be $\Delta_1 S_2$ plus, you had invested this amount at rate r so this will now become $X_1 - \Delta_1 S(1 + r)$ and these must match the value V_2 at time $t = 2$.

Now, obviously we can make an observation here that S_2 which is the price of the stock after two coin tosses is obviously dependent on the outcome of the first two coin tosses and so obviously V_2 will also depend on the to come of the first two coin tosses. Okay, so this results in what, this results in four possible outcomes.

So, this is good, this is good news, so what are the four possible outcomes, remember the four possible outcomes was head head, head tail, tail head and tail tail. So, accordingly, I will have four values of V_2, $V_2 HH$, $V_2 HT$, $V_2 TH$, and $V_2 TT$. And this is going to be Δ_1, now remember in the first case what is going to be the delta function of, see in this case and this case the head appears first in the first toss.

So, Δ_1 decision was made with the information available that a head has happened in the first coin toss, in the third and the fourth case there was a tail in the first coin toss so accordingly for these two scenarios the Δ_1 will depend on tail which was the first outcome and this will be multiplied by S_2, obviously of HH, same as HH here, this will be second case, it will be $S_2(HT)$, in the third case it will be $S_2(TH)$ and in the fourth case it will be $S_2(TT)$.

And now we look at, this term is done, and we will look at this particular term. So, this particular term we will get X_1, so we will have a factor of $1 + r$ that is common for all the cases, and also here, so in this case we will have X_1, remember X_1 depends on the first coin toss which was head so this is going to be $X_1(H)$ and $X_1(H)$ and here, for the third and the fourth case X_1 will depend on H and T.

Minus of course $\Delta_1(H)$ and $\Delta_1(H)$ for the first two cases, and $\Delta_1(T)$ for the third and the fourth cases.

Multiplied by S_1, so again S_1 is something that is dependent only on the first coin toss, it is going to be $S_1(H), S_1(H), S_1(T)$ and $S_1(T)$. So, I call this set of equation to be equation (2).

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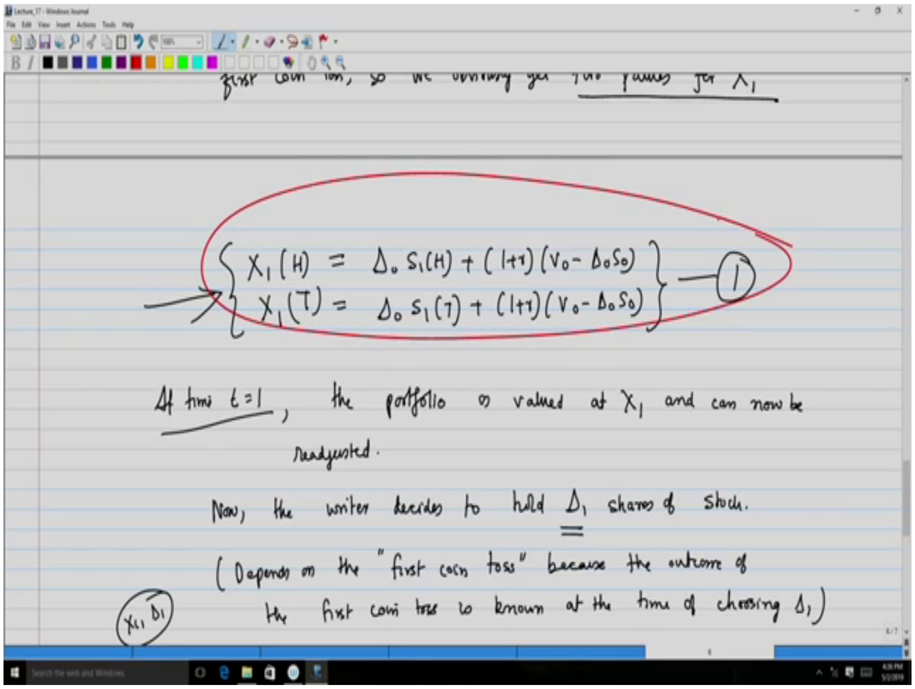

So, remember what was equation (1), equation (1) was basically the portfolio valuation at the end of time $t = 1$, so this is 1st set of equations.

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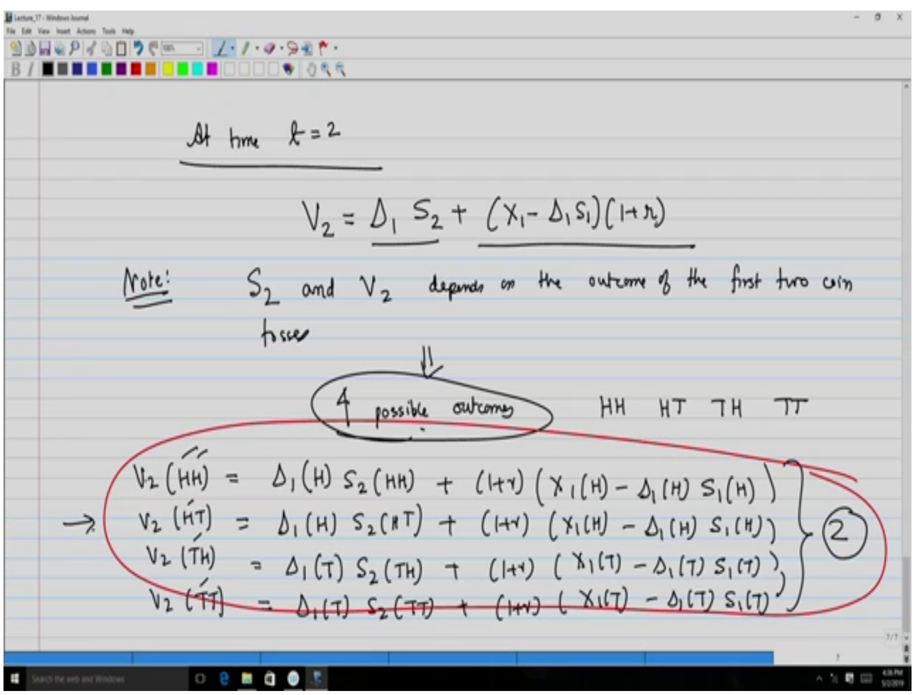

And this is the second set of equations, equation (2) is the portfolio of values which must match V_2 in all cases at time $t = 2$.

So, in equation (1), we had two equations and in equation (2) we had four equations, so these two sets give total of six equations, so in (1) and (2) we thus have six equations. And what are the unknowns, so

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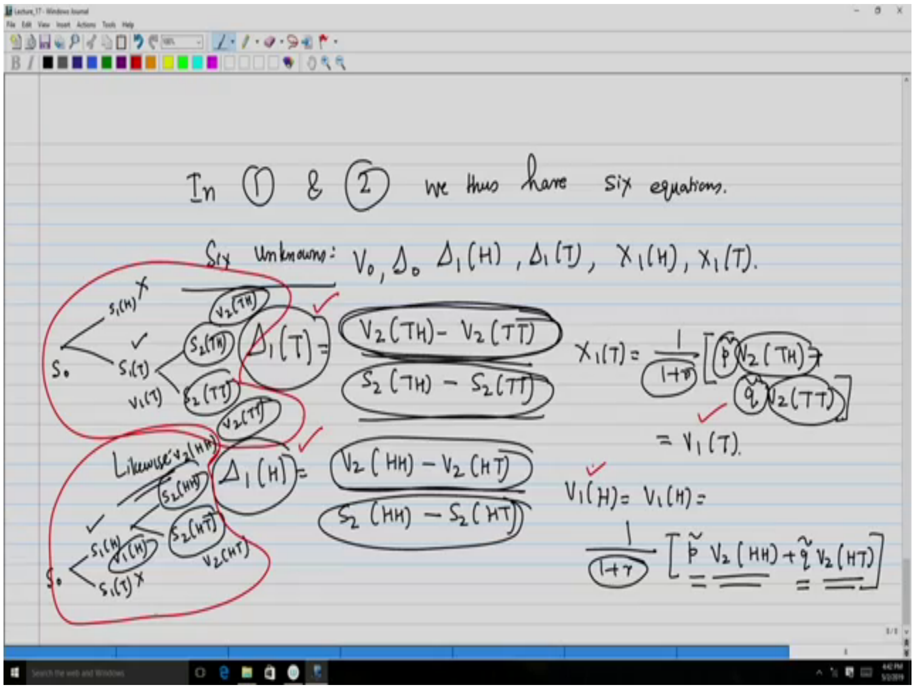

there are six unknowns that are accompanying it. What are these unknowns? These unknowns are basically $\Delta_1(H)$, $\Delta_1(T)$ and Δ_0 to be ascertained.

You need to ascertain what your V_0 is and you want to know what is $X_1(H)$ and $X_1(T)$. Alright, so now solving this set of equations we can actually obtain $\Delta_1(T) = \frac{V_2(TH)-V_2(TT)}{S_2(TH)-S_2(TT)}$. And this will imply that $X_1(T) = \frac{1}{1+r}(\tilde{p}V_2(HT) + \tilde{q}V_2(TT))$.

And this is nothing but, this is going to be same as $V_1(T)$. Likewise we can obtain, so likewise we can obtain $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(TT)}$ and $X_1(H) = V_1(H) = \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)]$.

So, you observe very carefully how we actually got it. So, here $\Delta_1(T)$ is nothing but change in the value of V_2 divided by change in the value of S_2. So this means that the outcome from, so the first you had started off with S_0 and then you actually go up to $S_1(H)$ or down to $S_1(T)$. Now, when I am saying that I want to ascertain my $\Delta_1(T)$ that means that this event has not happened but this has happened.

So, in this case this leads us to two values of S, namely of $S_2(TH)$ and $S_2(TT)$. So, the denominator is going to be the difference of $S_2(TH) - S_2(TT)$ and the numerator is going to be the corresponding difference of the values for the same outcome in that order. Now further in this case the value is going to be $V_2(TH)$ if there is an upward movement from time 1 to 2 and this is going to be your $V_2(TT)$, so, the value of $V_1(T)$ is going to be the expected discounted value of $V_2(TH)$ and $V_2(TT)$.

So, accordingly we have $V_2(TH)$ and $V_2(TT)$ multiplied by the corresponding this neutral probabilities $\tilde{p}$ and $\tilde{q}$, and then discounted $1 + r$ and this is going to be $V_1(T)$ at time $t = 1$. Alright, in a similar fashion you can have exactly the same argument in the second case where you again start off with S_0.

You go to $S_1(H)$ and $S_1(T)$, now since you are considering the case of $\Delta_1(H)$ that means this event is sold out and if it consider $S_1(H)$. Now, from $S_1(H)$ your value of the stock can go up to $S_2(HH)$ or it can come down to $S_2(HT)$. Okay? So, this actually should be $S_2(HT)$.

So, as before you see the denominator is nothing but the difference of $S_2(HH) - S_2(HT)$ and the numerator is $V_2(HH) - V_2(HT)$, so it is a corresponding difference that you see in the denominator and what is going to be the value $V_1(H)$, the value $V_1(H)$ will be the discounted value of the expected value of $V_2(HH)$ and $V_2(HT)$. So, what do I get, I get $V_2(HH)$, $V_2(HT)$, I multiply it by the corresponding restricted probabilities $\tilde{p}$ and $\tilde{q}$ and I discount it by a factor of $1 + r$.

So, by looking at this, the binomial tree, you are actually able to ascertain what is Δ_1 for both the circumstances the two possible Δ_1 values at times $t = 1$ are resulting in the corresponding V_1 values at time $t = 1$. Now, we are going to use the same tree kind of an argument that we have used in these two cases to find out what is value of the option V_0 at time $t = 0$ which was the first question that we sort to answer.

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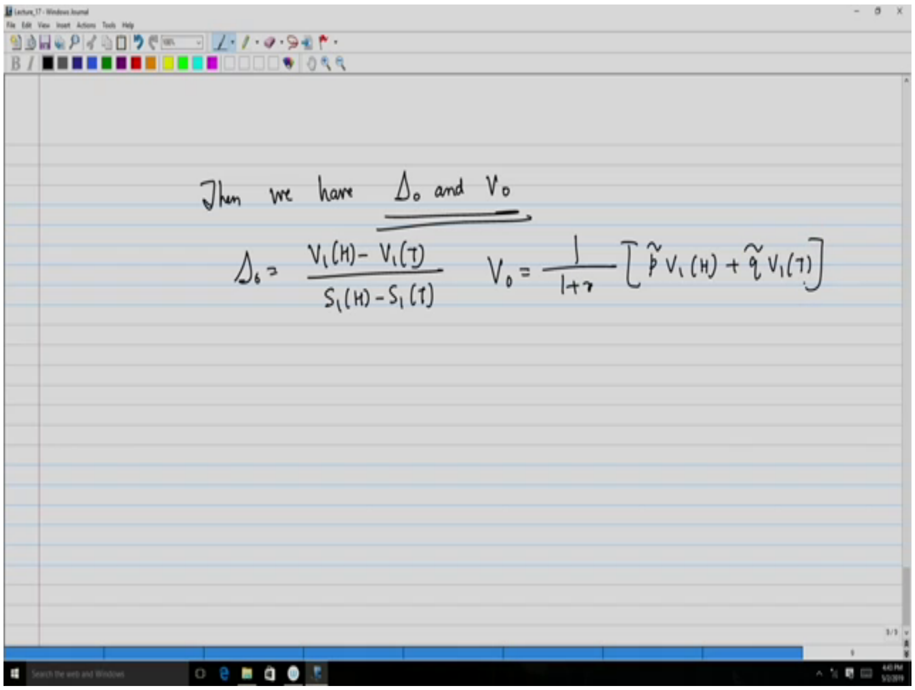

What you can have now, so then we have Δ_0 and V_0 which you can actually obtain by solving the equations which you have already done in the previous class. So, remember the delta naught is $\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$ and V_0 is going to be $\frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$.

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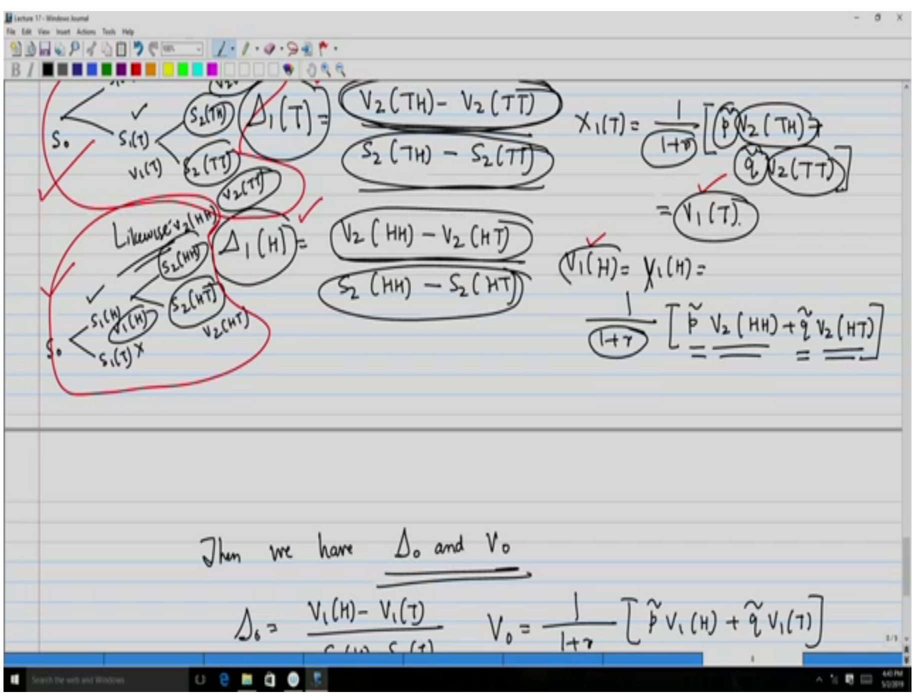

So, remember in the previous step we had actually figured out is going to be $V_1(H)$, so this should be $X_1(H)$, so $V_1(T)$ and $V_1(H)$, so this information is available.

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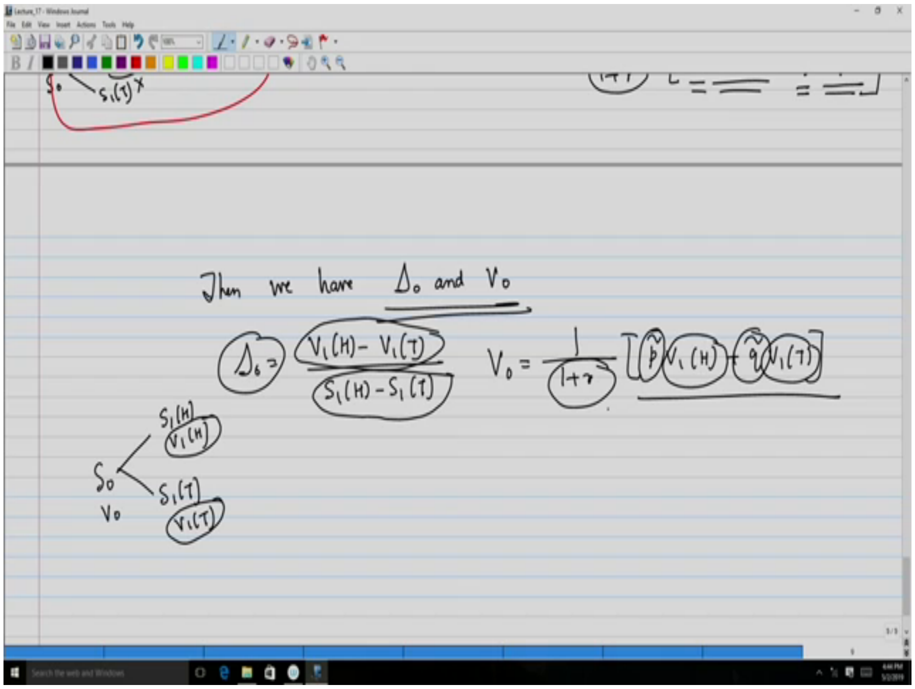

So, now again view this binomial tree. So, for S_0, you can go up to $S_1(H)$ or you can come down to $S_1(T)$, so the denominator is nothing but, the denominator in case of Δ_0 is nothing but the difference of the S values and numerator is the difference of the corresponding V values, namely, $V_1(H)$ and $V_1(T)$.

So, what you do is that, now, you want to find out what is V_0, so V_0 is going to be the discounted value of the expected values, so one value that you can attain at time $t = 1$ is $V_1(H)$ with probability $\tilde{p}$ and the other value that you can obtain is $V_1(T)$ or with probability $\tilde{q}$, so this will give the expected value of V_1 discount by $1 + r$.

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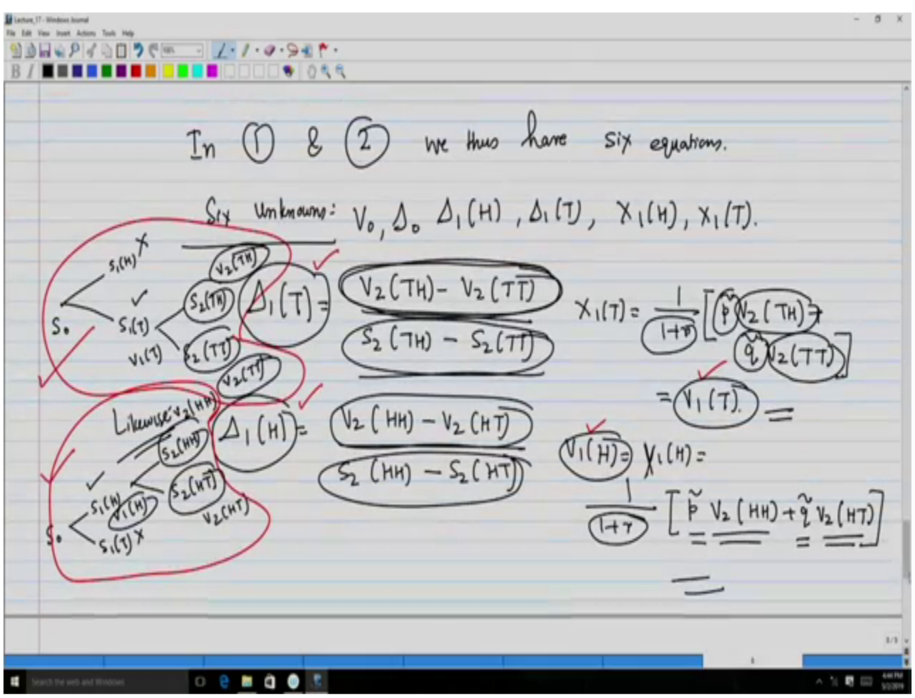

It is exactly the same manner that we had done in the case of figuring out what is $V_1(T)$ and $V_1(H)$.

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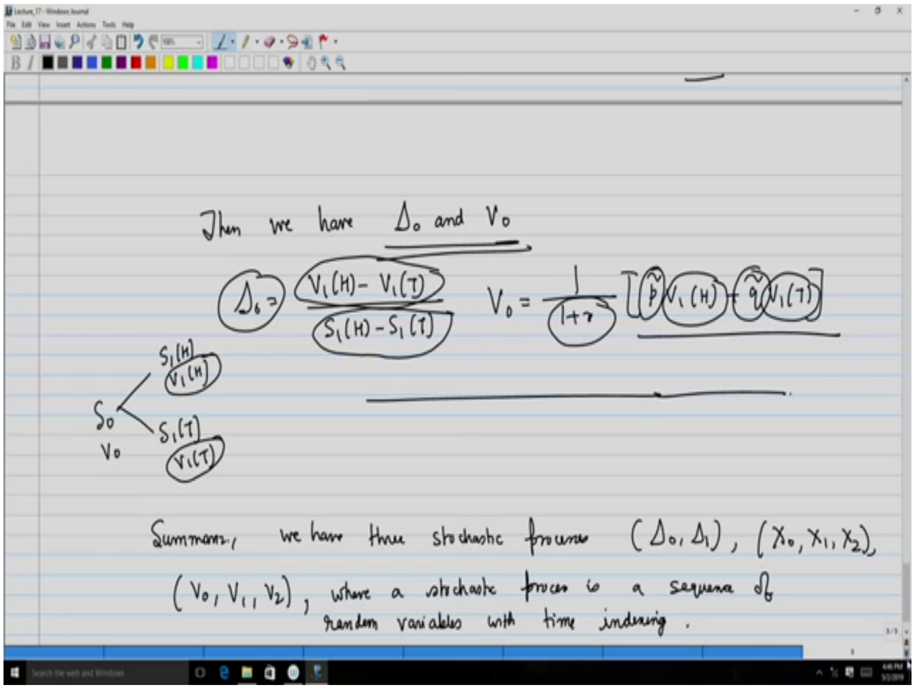

And this is again the same formula that you had obtained in case of the single period binomial model. Now, summarizing we have three stochastic process, namely delta naught and delta 1 so the evolution of deltas then the evolution of $X_0,\ X_1,\ X_2$ that means the evolution of the replicating portfolio and correspondingly the evolution of $V_0,\ V_1$ and V_2. And note that so where a stochastic process is a sequence of random variables with time indexing.

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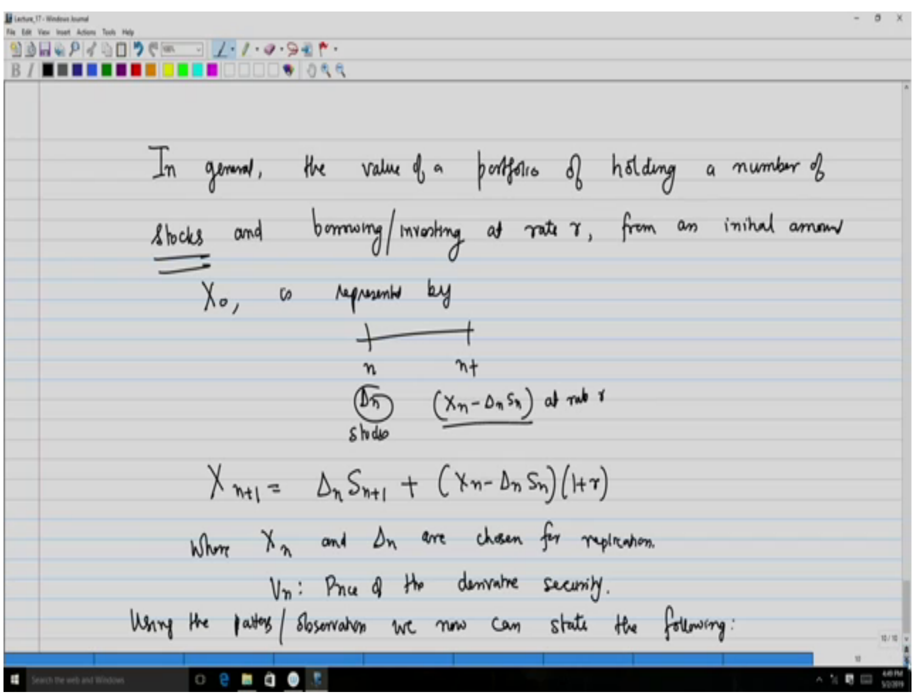

Okay, so now let us see how we are actually going to extend this in a generalized case. So, in general the value of a portfolio, which is the portfolio, the portfolio as before, it is a portfolio of holding a number of stocks namely delta naught, Δ_1 and so on that you have seen previously, so number of stocks and then borrowing or investing depending on whether the remaining amount that you have after the investment in stocks whether it is positive or negative, so borrowing and investing at (say) rate r, from an initial amount X_0, is represented by...

So, for this you actually consider this time point n and $n+1$, so you can write this as at n you buy Δ_n number of shares, right? And then the remaining amount, so at time n you have an amount of X_n out of which we had spent an amount of $\Delta_n S_n$ in buying the stock and so you invest this at rate r and you buy Δ_n stocks.

So, its valuation at time $n+1$ will be the value of the stocks, price of each stocks S_{n+1} multiplied by the delta number of stocks you had purchased plus the amount of $X_n - \Delta_n S_n$ that was invested at rate r which has now gone to by a factor of $1+r$ where X_n and Δ_n are chosen for replication.

And correspondingly the V_n that you have, this is going to be the price of the derivative security which in our case for most part will be treated as option. Now what you can do is that, using the patterns or observations we now can state the following, and I will state this particular theorem.

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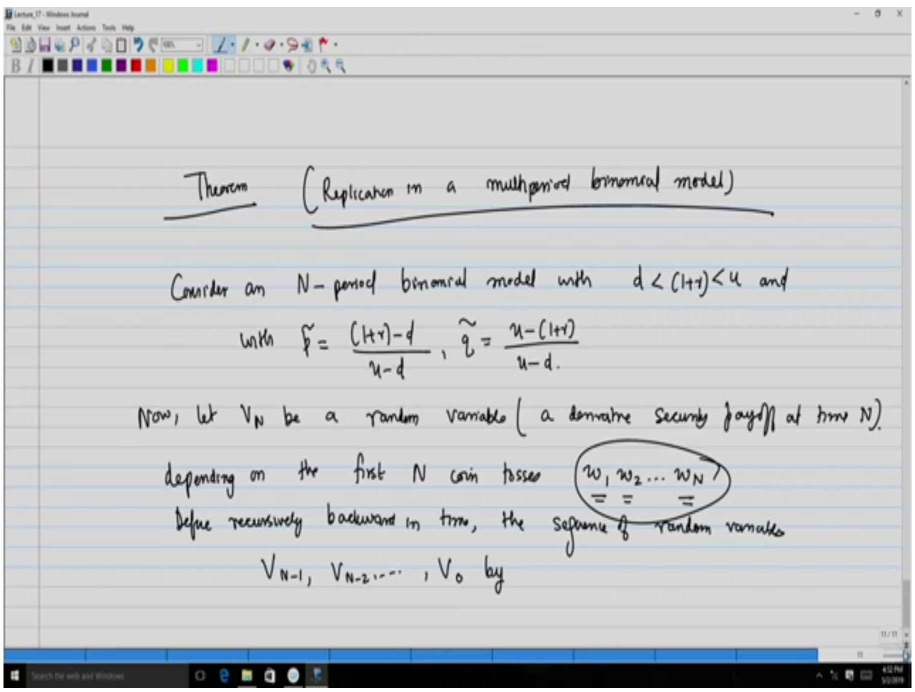
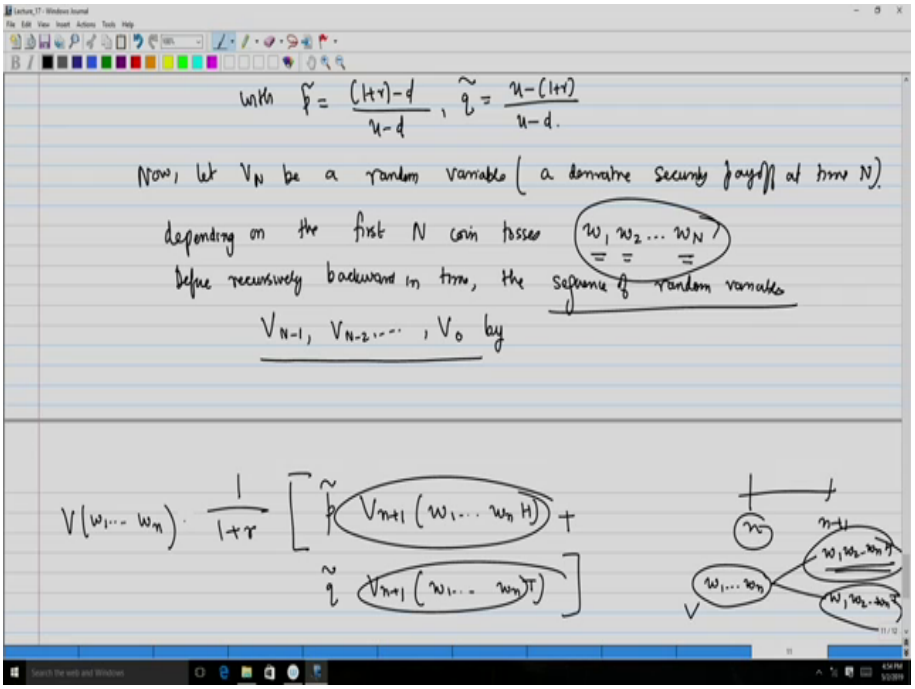


So, here goes the theorem. This theorem is about replication in a multiperiod binomial model so what you are going to do is that the way we do the pricing and we have seen the pricing being done in case of one time period in the last class and for the two time period in the present class, so we are basically going to extend this idea in a generalized case where there are n number of periods where this n, rather $N > 2$.

So, then we consider an N period, so the way this is stated now of course it is also applicable for $N = 1$ and 2 which you have looked at separately so we consider the N period binomial model with parameters $d,\ u$ and r satisfying this no-arbitrage condition and with $\tilde{p}$ as already defined being $\frac{1+r-d}{u-d}$ and $\tilde{q}$ you would recall is defined as $\frac{u-1+r}{u-t}$.

So, now let V_N be a random variable so this is a derivative security pay off at time N and this V_N depends on the first N coin tosses and we will denote this N coin tosses as $\omega_1,\ \omega_2,\ldots,\omega_n$, where each of those omegas can either be head or a tail. So, basically this particular notation indicates capital than number of consecutive outcomes of coin tosses.

So, it is basically a sequence of capital of length N which can either be, where each of those characters can actually have be either head or a tail, so in case $N = 1$ you just have ω_1 which is H or T in case of $N = 2$, you will have $\omega_1,\ \omega_2$ where there are four cases, namely, $HH,\ HT,\ TH$, and TT.



So, then we will define recursively backward in time, the sequence of random variables $V_{n-1}, V_{n-2}, \ldots, V_0$, so we will define it in the usual fashion, so what we do is the following, we will look at two time points, so I will just explain this on the side, we look at two time points n and $n+1$, so this means that at time n, you already have a sequence ω_1 to ω_n and at time $n+1$, the sequence will either be ω_1 to ω_n head.

Remember this is the filtration that is available and this is either we are going to go up, because of a head or in the second case when we go down because of the tail it is going to be $\omega_1, \omega_2, \ldots, \omega_n T$. So, this means that I will look at the value at $n+1$ for $\omega_1, \ldots, \omega_n H$, this is one value, here. And V_{n+1} in this case is going to be $V_{n+1}(\omega_1, \ldots, \omega_n T)$.

And what is going to be the expected value of this, this happening has a probability $\tilde{p}$ and this happening has a probability of $\tilde{q}$, so the expected value will be sum of this and I will discount this by a factor of $1+r$, and what is this going to give me, this is going to give me the value at V at which was a consequence of n number of coin tosses. So, I define the sequence of random variables $V_{n-1}, V_{n-2}, \ldots, V_0$, in this particular fashion.

So, because you remember that V_n you already know it is simply going to be the payoff of the derivative and you use V_n to calculate $Vn-1$ from this particular formula and then once you have obtained V_{n-1} by replacing n to be equal to $n-1$, you can then use this in case of $n-2$ to find V_{n-2} and so on until you arrive at time $V=0$.

So, only V_n can be calculated by looking at the final possible values of the asset price and the consequent pay off and also subsequent intermediate values all the way up to the initial value of the derivative or in particular if you want to view it as an option can be figured out by making use of the recursive relation.

So, this particular recursion we have already seen, specific cases of it earlier when you were calculating V_1 and V_0 from V_2. So, the continue with the statement of the theorem, so we define recursively this sequence of random variables, so once I have defined this $V_n, V_{n-1}, \ldots, V_0$, so that each V_n depends on the first n coin tosses, namely, this $\omega_1, \ldots, \omega_n$, right?

Where of course your $n = N-1, N-2, \ldots, 0$, so next we define $\Delta_n(\omega_1, \omega_2, \ldots, \omega_n)$, so Δ_1 is the strategy that you actually adopt here and which you hold onto this final time point, so remember what is this? This is going to be the difference of the price of the asset for this minus this.

So, this can be written as $S_{n+1}(\omega_1, \ldots, \omega_n H)i - S_{n+1}(\omega+1, \ldots, \omega_n T)$ and the numerator will going to be exactly the same thing, that is ω_1 through ω_n head and tail but the valuation for V_{n+1}. So, you have now defined what is V_n, and also you have defined what is going to be your δ_n.

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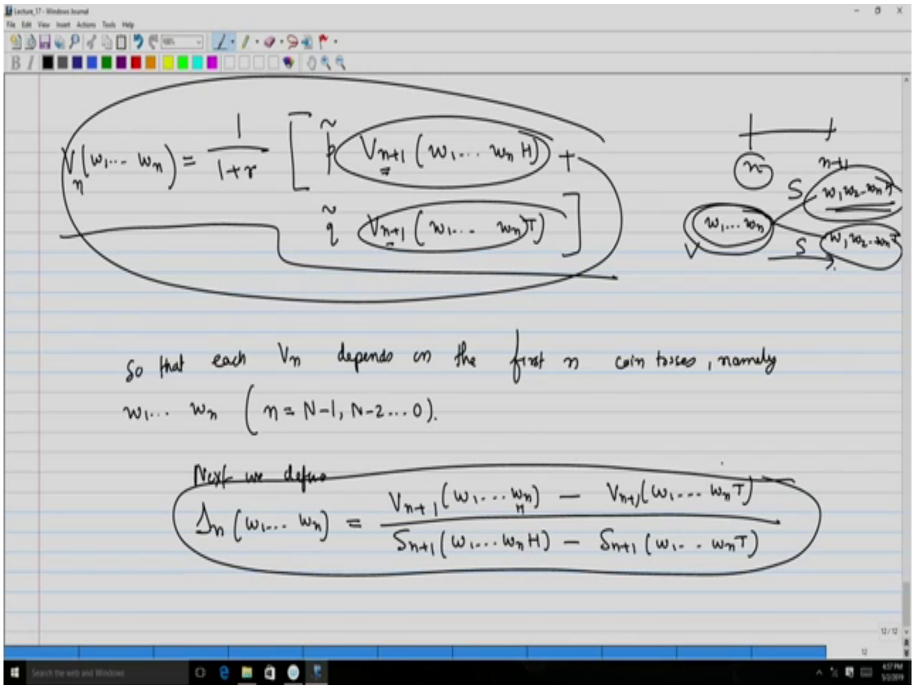

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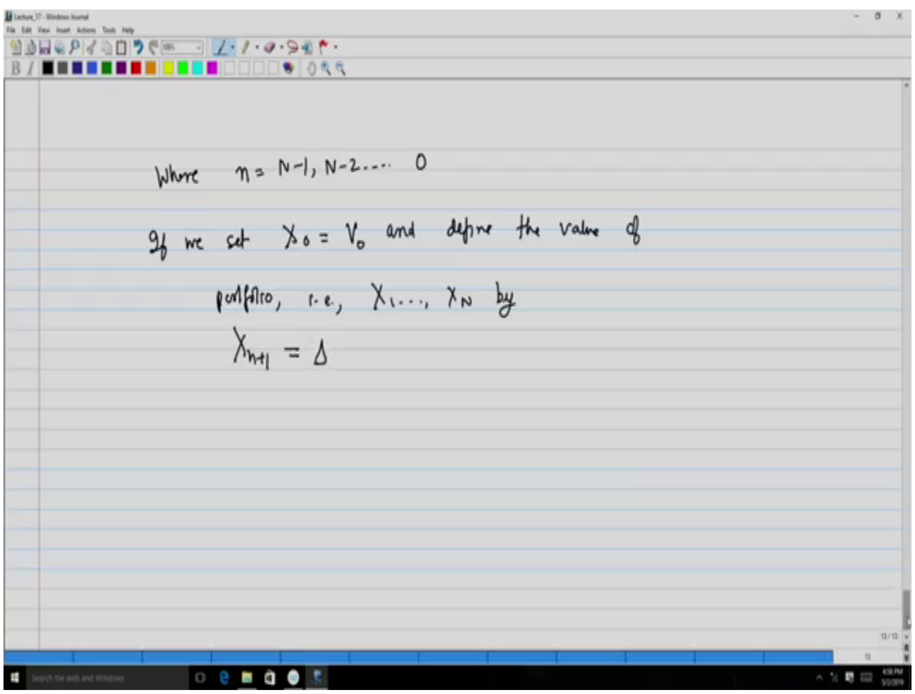

So, where of course your $n = N-1, N-2, \dots, 0$. Now, if we set $X_0 = V_0$ and define the value of portfolio that is $X_1, \dots, X_N$ by the relation X_{n+1} which you have already seen X_{N+1}.

Remember this is just the same relation as we had seen here, exactly the same relation, so I am just reproducing it.

So, $X_{n+1} = \Delta_n S_n + (1+r)(X_n - \Delta_n S_n)$. Then we will have $X_n(\omega_1, \dots, \omega_n) = V_n(\omega_1, \dots, \omega_n)$, right?

So, what it means is the following, so just to see the things in context, consider a binomial, n period binomial model which satisfied this no-arbitrage condition with the following define risk neutral probabilities

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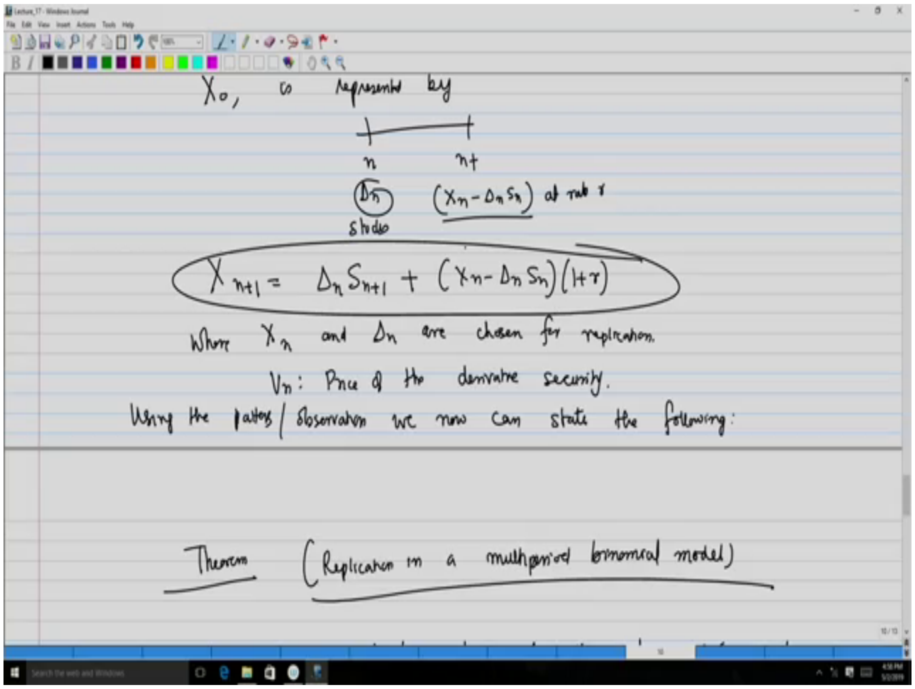

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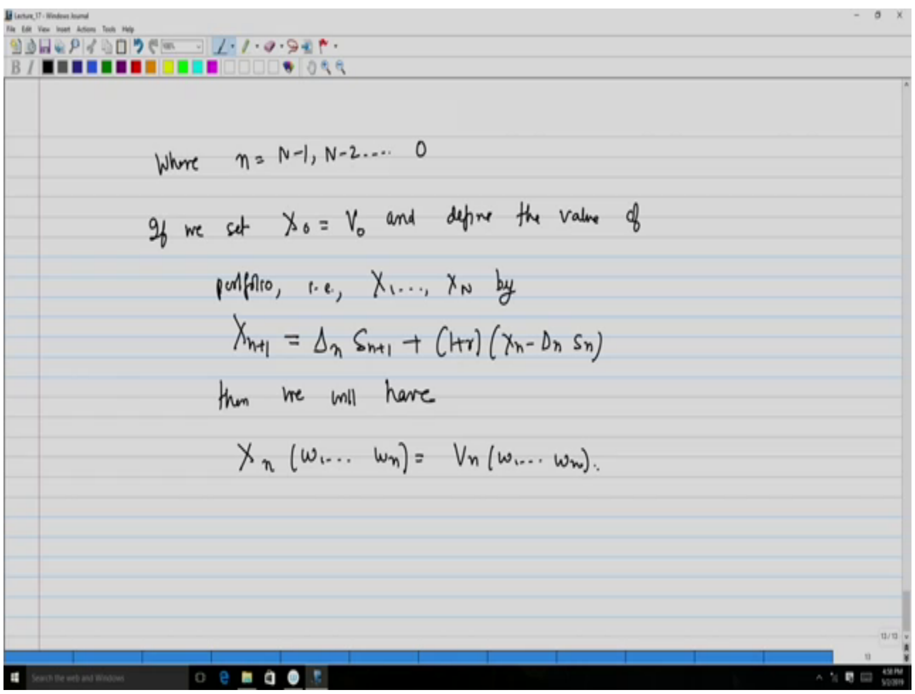

and if I define the values of V_n in this manner, starting off with V_n as the pay off the derivative and if we define the Δ_n to be this.

Then a process where I start off $X_0 = V_0$ as already defined and then all the subsequent X_n's are defined by making use of the Δ_n's which I have defined here.

So, remember I first figured out what is going to be my V_n, I then use my V_n to get my Δ_n, right I need these V_n values and once I get this Δ_n I can use this to define my X_{n+1} so this X_n's that I have here will match all the values of V_n. So, basically this is the theorem which talks about the replication that means you are replication stock value of the replicating portfolio X_n will always be matching V_n if you define V_n

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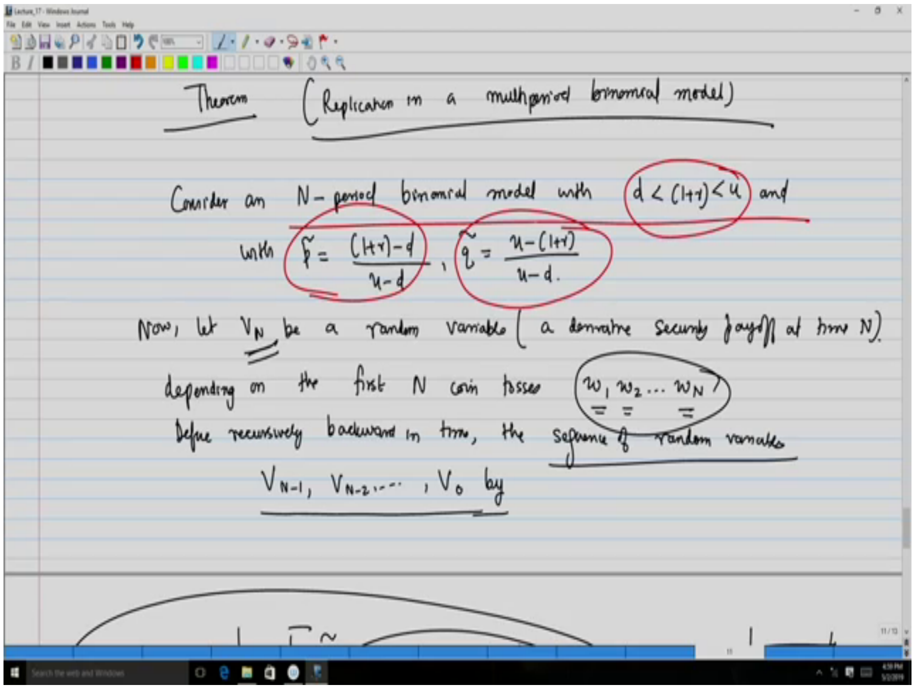
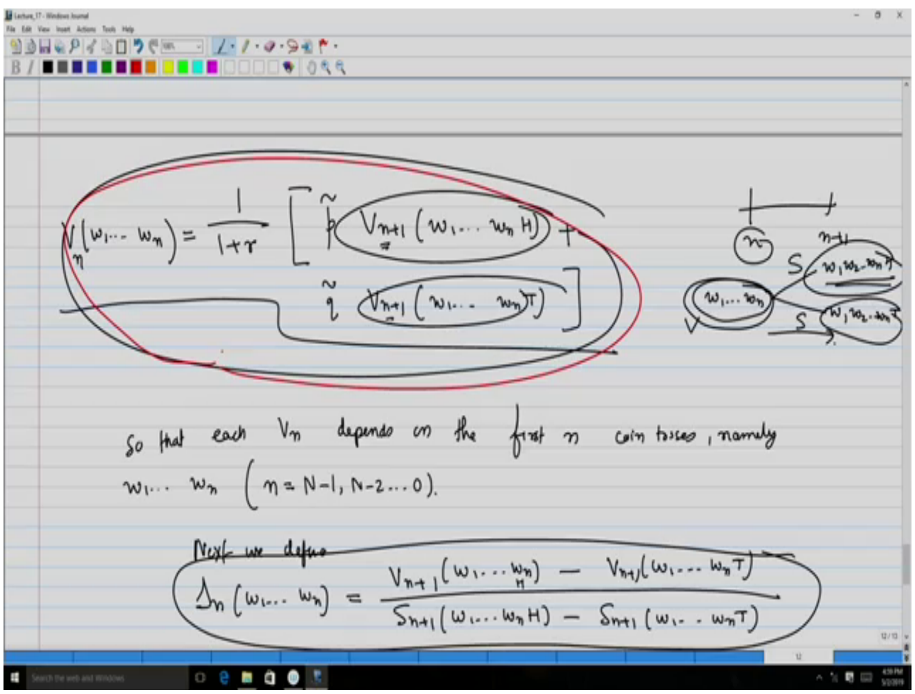
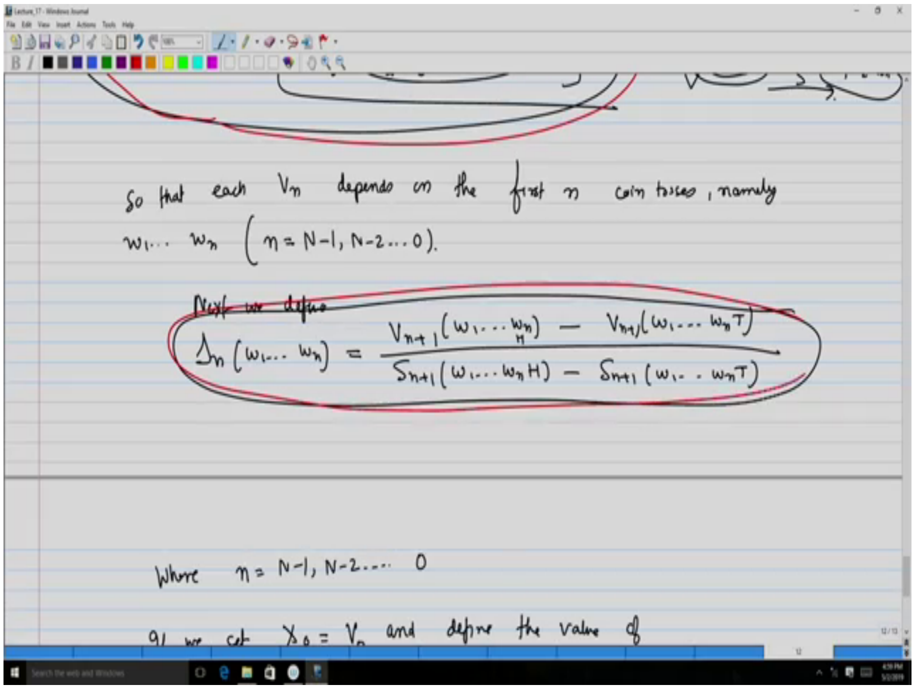




and Δ_n as specified earlier in the theorem. So, this does two things, what it does is that it gives you the replication strategy and that it actually works and second thing also it gives you a formula for valuation of the option price. So, just to sum up whatever we have discussed today, we mainly looked at two things, we looked at two period binomial model and a three period binomial. And we sort of extended the one pricing of an derivative, of a derivative or an option in case of the one period model to the two period model and observing the pattern we went ahead and looked at this last theorem which gives us the valuation at any given time from 0 to $n-1$ and resulting in the consequent values of delta which is used to design a replicating portfolio such that the value of the replicating portfolio at every time point without any exception must be matching the value of the derivatives at all those time points. So, in the next class we will continue a little more on this binomial model and option pricing and that will conclude this first part of this discussion



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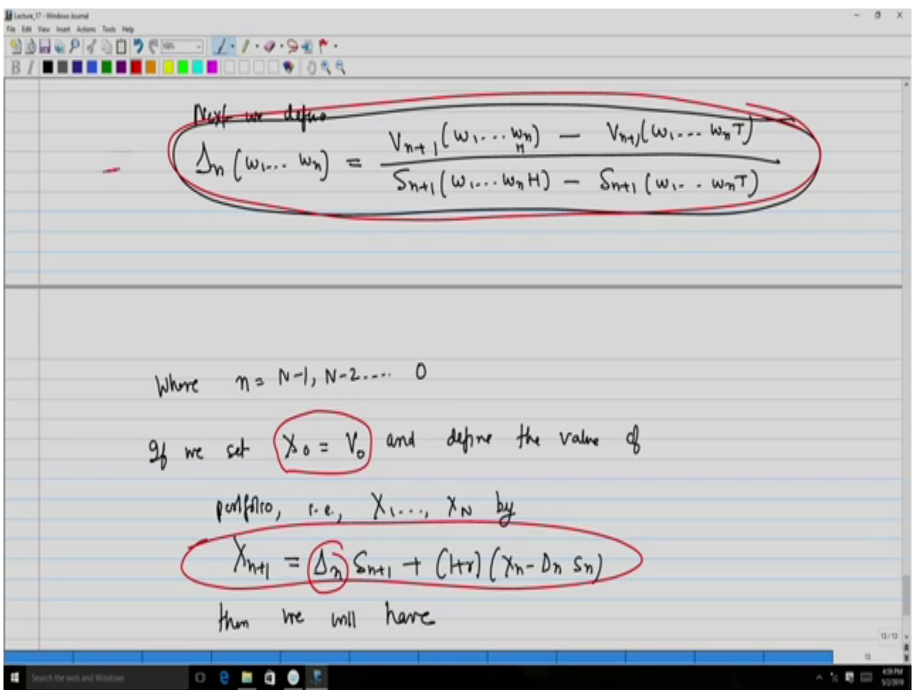

on option pricing in the binomial setup. Thank you for watching.

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