Mathematical Finance

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Module 5: Derivative Pricing by Replication in Binominal Model
Lecture 1: Derivative Pricing in a Single Period Binominal Model

Hello viewers, welcome to this next lecture in this course on Mathematica Finance. We will begin a new module today and this will be the starting point of the major component of the course namely pricing of options. Recall that the price of an option is essentially the compensation or the upfront premium that is paid by the buyer of the option to the owner of the option or rather the seller of the option because the seller of the option takes up an obligatory position with basically no rights at all. So, the key question that opens up is that what exactly is the appropriate or the correct compensation that actually needs to be paid by the buyer of the option and we will begin the answer to this particular question from today's class. In particular, what we will look at is that we will first look at a binomial model in a single time period for the asset price and then in particular we will look at the strategy of the replication in order to ascertain what this important question, what is the answer to this particular important question, namely the pricing of the option.

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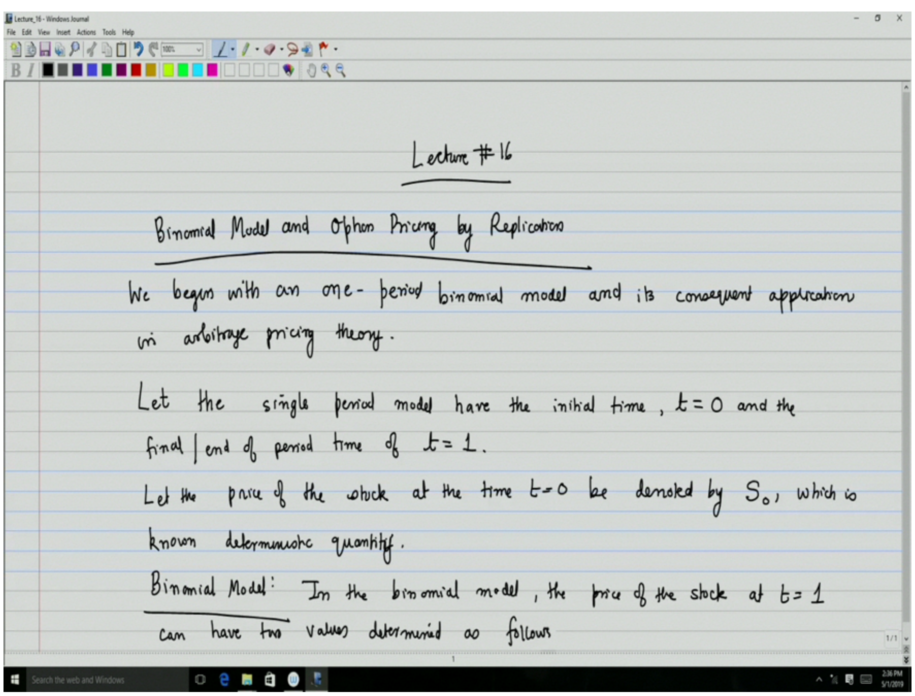

So, we will start this lecture and the topic is the Binomial Model and Option Pricing by Replication. So, we begin with a one-period binomial model and its consequent application in arbitrage pricing theory. So, we first begin with a discussion the binomial model and we will look at the simplest case of the binomial model, then, namely the one for a single period, so let the single period model have the initial time, namely $t = 0$, and the final or end of period time of $t = 1$.

Further, let the price of the stock at the time $t = 0$ be denoted by S_0, which of course is known, it is a deterministic quantity, so which is known deterministic quantity, that means we know this when you were actually starting the entire exercise. Now, what is the binomial model? See, I am talking about a single binomial model, so I will first start out with what is the binomial model. So, in the binomial model as the name suggest, the price of the stock, or the underlying asset at time $t = 1$ can have two values determined as follows.

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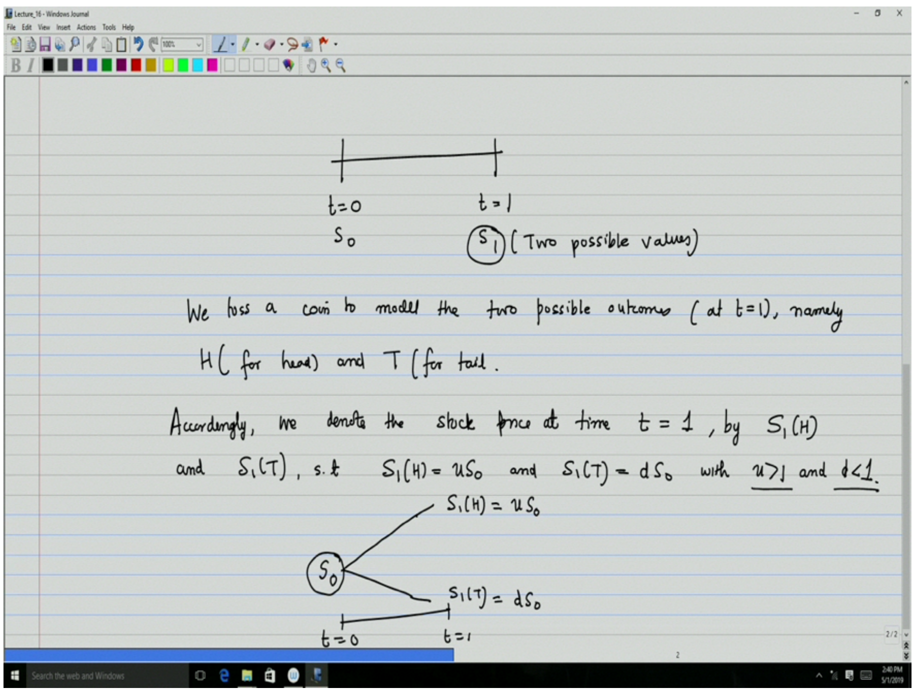

So, we do it in the following ways, so essentially, there we start off with this is single period binomial model, so $t = 0$, you know what is the stock price and at $t = 1$, basically the stock price, which will denote by S_1, this will essentially have two possible values and these two values are basically, the two values of the random variable S_1 takes at time $t = 1$. Now, typically when you talk about a binomial model or binomial distribution driven model, you are always looking at scenario where there are essentially two possible outcomes because it is a binomial model and so the easiest way to sort of generate the entire process is to look at the simplest case of a binomial distribution or a binomial setup namely a coin tossing problem.

So, what we will do is that we will take this two possible values of stock at time $t = 1$, namely, S_1 is something that is actually being determined in a random fashion by the outcome of the toss of a coin, so accordingly, we toss a coin to model the two possible outcomes, and remember that you are just talking in the context of the binomial model, so this two possible outcomes at time $t = 1$, namely head denoted by H and tail denoted by T.

So, accordingly, we denote the stock price at time $t = 1$ by $S_1(H)$ and $S_1(T)$ such that $S_1(H) = uS_0$ and $S_1(T) = dS_0$ with $u > 1$ and $d < 1$. So, this means that I mean, if you want to visualise this, I start off with the value of the stock at time $t = 0$ as S_0, which is the deterministic quantity and the binomial model then assumed that the price of the stock at time $t = 1$ is either going to be $S_1(H)$ or $S_1(T)$ or most specifically in the context of S_0 this will either go up to uS_0 because I have taken $u > 1$ or it will go down dS_0, because I have taken $d < 1$.

So, this means that starting off with S_0, there are only two possible price movements at time $t = 1$ either it will go up or it will go down, whenever it goes up will denote the new price to be uS_0 while $u > 1$ and whenever it goes down we will denote it by dS_0, where $d < 1$.

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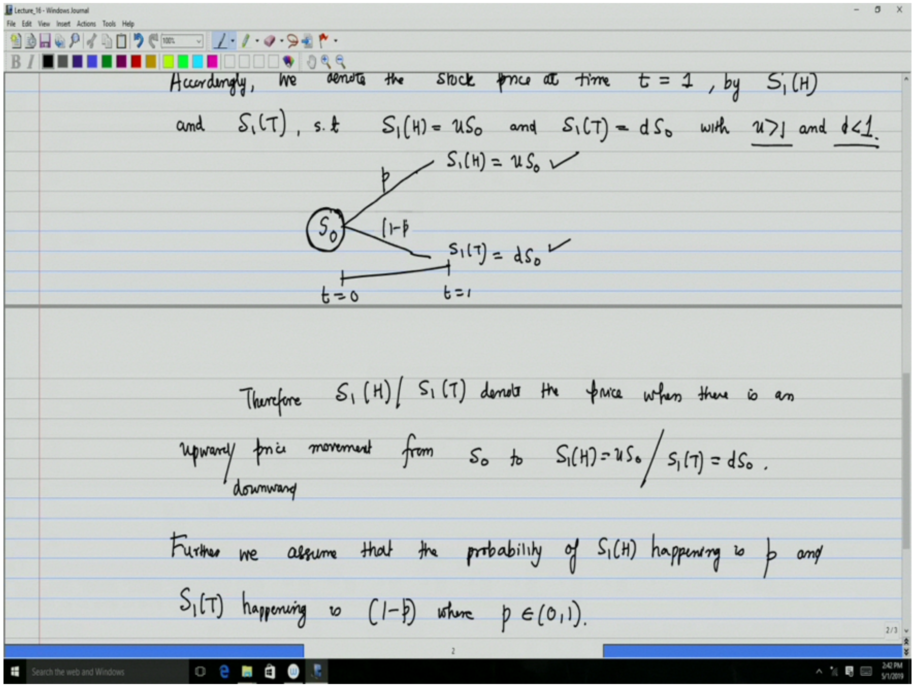

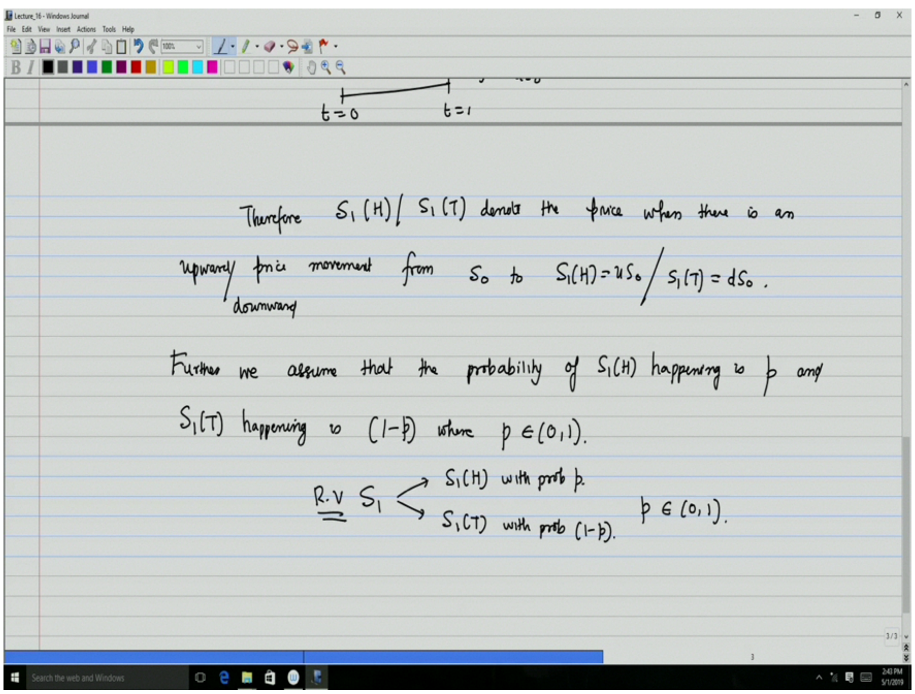

Now, therefore we can make the statement that $S_1(H)$ or $S_1(T)$, it they denote the price when there is an upward price movement from or respectively a downward price movement from S_0 to $S_1(H) = uS_0$ or respectively $S_1(T) = dS_0$. So, anyway you have seen this already graphically. Now, further we assume that the probability of $S_1(H)$ happening is p and $S_1(T)$ happening is $1 - p$ where $p \in (0, 1)$. So, this essentially means if you go back to the diagram, this means that the probability of the movement from S_0 to $S_1(H)$, this probability is p and moving downwards to $S_1(T)$, which is dS_0, this is going to be $1 - p$.

So, this means that the random variable S_1, this random variable S_1 essentially takes two variables of $S_1(H)$ with probability p or it takes the value $S_1(T)$ with probability $1 - p$. And here it is essential because we have taken p between 0 and 1, because if your p becomes either 0 or equal to 1, then basically the value of S_1 does not remain a random variable or does not remain uncertain anymore.

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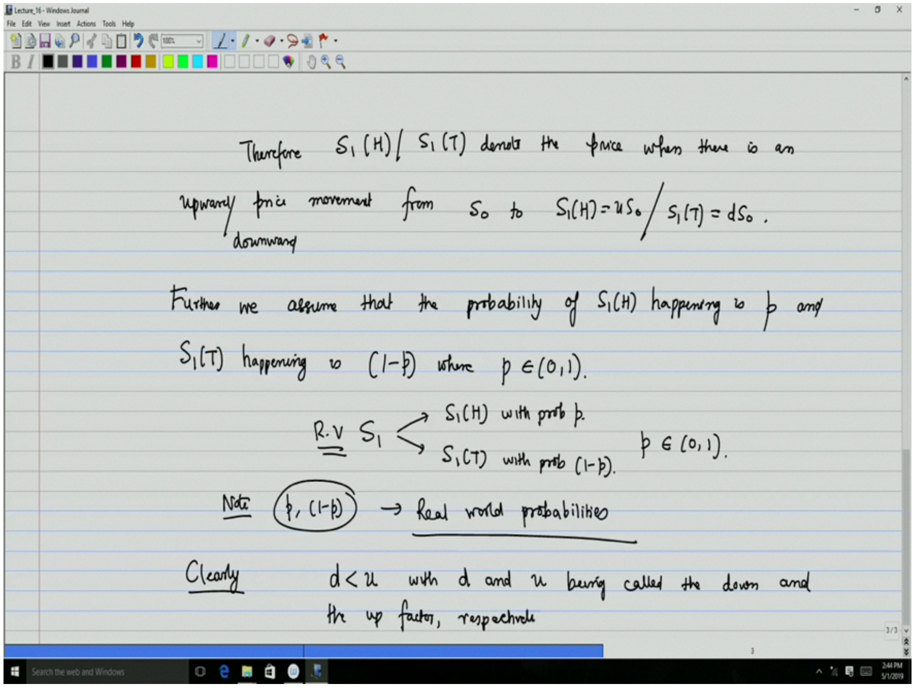

Okay, now we have this setup, now here we note that this p and $1-p$, these are what are known as real world probabilities and I just mention this here because at a later stage will encounter a new probability measure which will be call the risk neutral probability measures. Now, clearly since $d < 1$ and $u > 1$, so obviously $d < u$ with d and u just to introduce the nomenclature with d and u being called the down and the up factor respectively.

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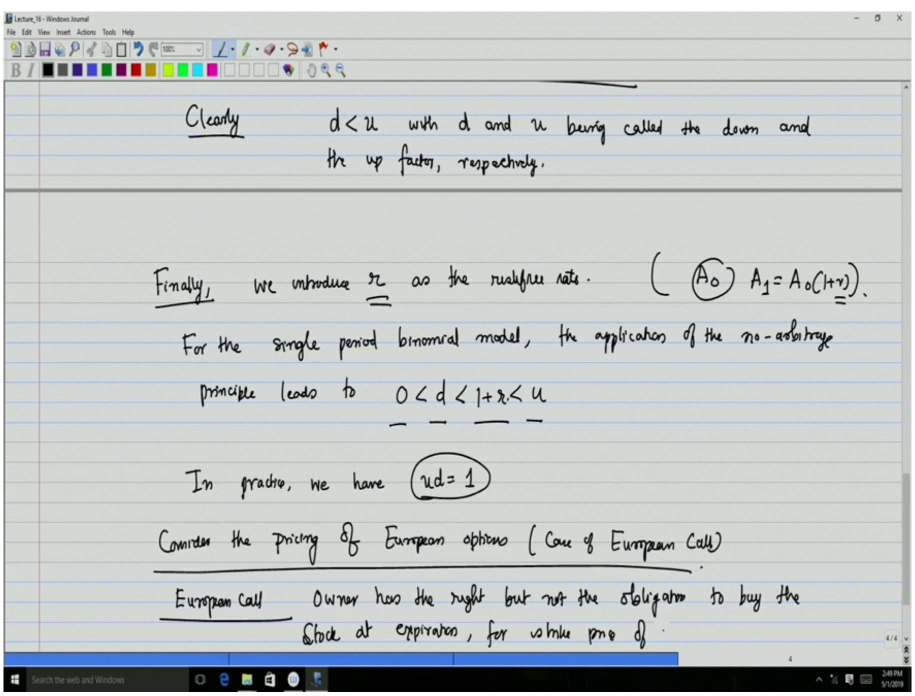

Okay, now finally there is one notation that we have to introduce in this context, so finally we introduce R as the notation for risk free rate. So, what you can say is that for the single period binomial model, the application of the no-arbitrage principle leads to $0 < d < u$ and with $1 + r$ lying in turn. Remember are is the risk-free rate, so they basically mean that if you invest an amount of A_0, then at some time your amount will grow to $A(0)(1+r)$, so this is what I mean by the risk-free rate in this context. So a simple no-arbitrage argument will lead you to be able to prove that not only is $d < u$, but actually $0 < d < 1 + r < u$ in order to satisfy the no-arbitrage condition.

So, it has left as an exercise for you to actually give an argument, it is a very simple argument which can lead you to this proof and obviously $d > 0$ because dS not is the stock, one of the is possible stock prices at time $t = 1$ and you cannot have negative stock prices, so that make sure that you always need here $d > 0$, alright, so now while there is no real restriction on u and d, except the fact that $d < u$ and then $d < 1 + r < u$.

However, in practice, we have $ud = 1$, and the reason behind this is something which is basically known as recombining tree and this rational behind this assumption of $ud = 1$, we will become more obvious when we move from the single period binomial model to the multi period binomial model just to briefly mention is that in case of a two period binomial model, you might have a upward and a downward movement, so in which case, if you sort of with S not then the price after two time periods will be S_0ud which is S_0 and likewise a downward movement followed by an upward movement would again give you $S_0du = S_0$.

So, you see that there are two possible different paths which actually leads to the same possible value of the asset at time $t = 2$, because you had assume that $ud = 1$, so the more details of this will actually be encountered when you are actually going and moving into the multi period binomial model.

So, continuing a discussion with this single period binomial model, we now consider the pricing of European options and will also look at an example of European call option, so recall that in case of European call the owner has the right but not the obligation to buy the stock at expiration, I remember say whatever $t = 1$ in this case for the strike price of K, so we denote strike price to be equal to K.

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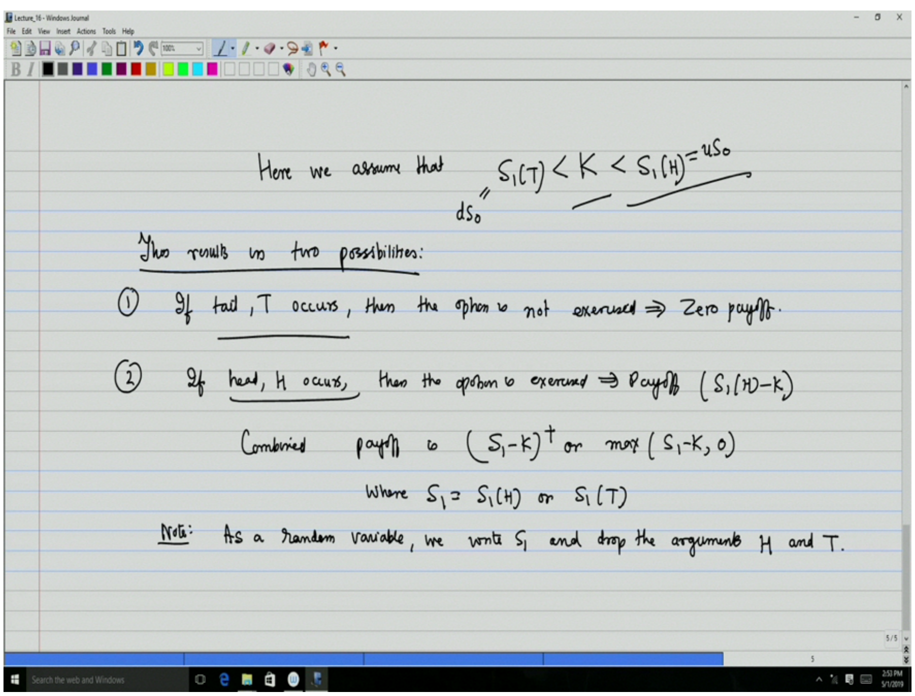

Alright, so here we assume that the strike price that is decided upon this $K < S - 1(H)$, which is the upper value of the prize at time $t = 1$ and greater than $S_1(T)$, remember $S_1(H)$ was uS_0 and $S_1(T)$ was dS_0. Now what does this do? So, this results in two possibilities, so let us examine this possibilities one by one. Okay, so if tail T occurs, then the option is not exercised which means that the payoff is 0, right, because if tail T occurs, then essentially the price of the stock that is available for purchase in the market is going to be $dS_0 < K$.

So, since the owner of the option has the right to buy the stock, but not the obligation, so obviously they are going to choose to buy it if they want to at the price of the dS_0, because it is lower than the price K, that is the predetermined price they have to pay in case they decide to buy the stock by executing the call option.

Now, secondly if head H occurs, then the option is exercised and this gives the payoff, of what? It will give you $S_1(H) - K$, the difference between this price at which the owner of the option can actually sell minus the price K they have paid for the option. So combined as you already seen as a combination the payoff can be written is as $(S_1 - K)^+$ or this means this is $\max\{S_1 - K, 0\}$ where this random variables S_1 can either be $S_1(H)$ or $S_1(T)$, so just to note that as a random variable, we write S_1 and drop the arguments H and T for the head and tail respectively.

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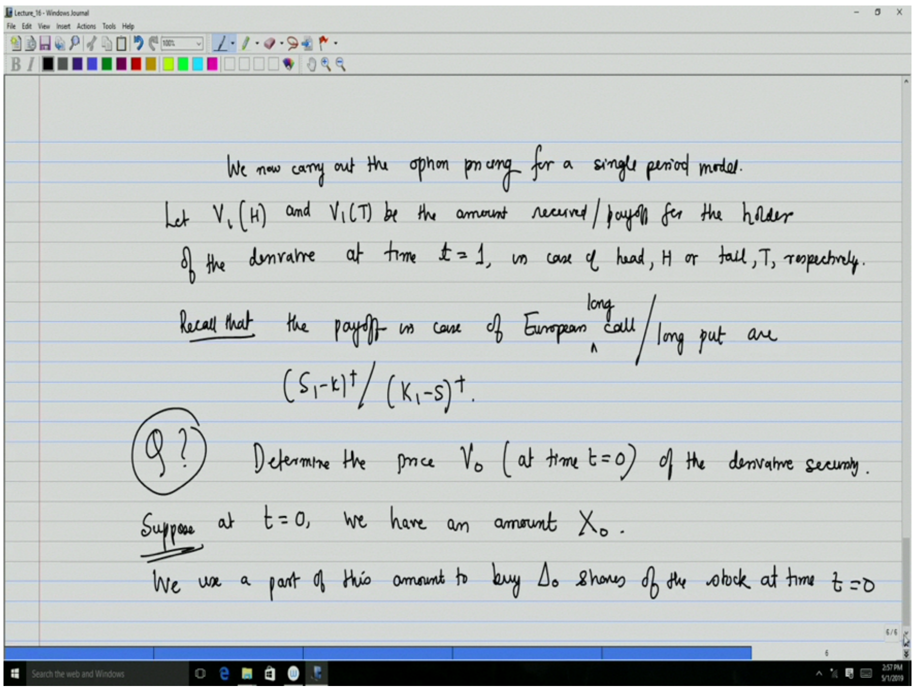

Okay, now we come to the main question of the pricing of the option, so we now carry out the option pricing for a single period model, so now let, so obviously we have to introduce the whole bunch of notations here, so we will say that let $V_1(H)$ and $V_1(T)$ be the amount received or basically the payoff for the holder of the derivative at time $t = 1$ in case of head H or tail T respectively.

So, just recalled that the payoff in case of European call, a long position of European call or a long put are $(S_1 - K)^+$ or $(K_1 - S)^+$, alright, now that we have the complete setup, we now actually set the question that you want to address, the question is the following, "Determine the price V_0 obviously at time $t = 0$?", remember that the price of the option or the derivative is a premium or a compensation that has to be paid at the initial time $t = 0$, when both the parties actually get into this particular option arrangement.

So, we determine the price V_0 at time $t = 0$ of the derivative security and this is the question that you are going to answer. Okay, so how we will go about this, suppose at $t = 0$, we have an amount X_0, what is X_0 will remain unspecified for the time being, it will become clear as we move along the discussion. So at $t = 0$ we have an amount X_0, now how we are going to use this X_0, so we are going to use this X_0 in the following way, so we use a part of this amount to buy Δ_0 shares of the stock at time $t = 0$. (Refer Slide Time: 24:56)

So, at time $t = 0$, so essentially what you is have is that you have an amount of X_0 and you are first of all what you will do is that you will use it to by Δ_0 stock, so you will spent an amount of $\Delta_0 S_0$, so what is going to be the balance? So the balance is going to be equal to $X_0 - \Delta_0 S_0$ and what you do is the following that the balance, so what do we do with this balance, so this balance $X_0 - \Delta_0 S_0$ is invested at rate r, in case

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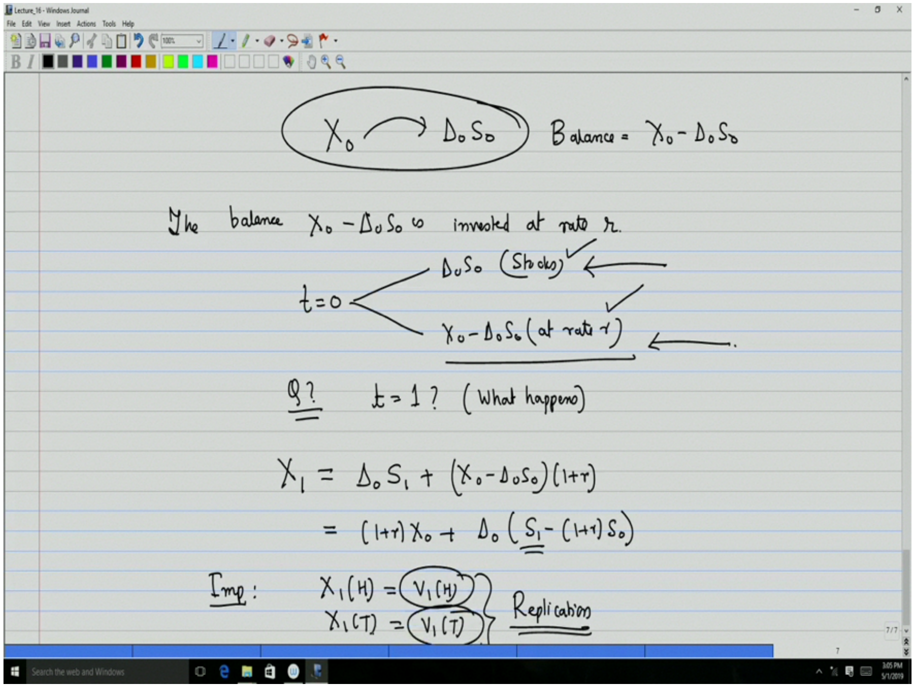

it turns out to be negative that means the quantity $X_0 - \Delta_0 S_0$ turns out to be negative as you will see in an example later on, in that case, instead of investment obviously you have to borrow this amount of money at interest rate of r.

Okay, so now once you have made this investment, so this means that time $t = 0$, you have put $\Delta_0 S_0$ in stocks, alright and $X_0 - \Delta_0 S_0$ at rate r, so the question is what is going to happen to our investment at time $t = 1$? What happens? So, now at time $t = 1$, what is going to the value of this portfolio? What does this portfolio comprise of? The portfolio comprises of Δ_0 number of stocks and this investment at rate r.

So, at time $t = 1$, your value of the portfolio is going to be X_1, now you have purchased Delta not shares of the stock at a value of S not at time $t = 0$. Now the value of each stock has moved from S_0 to S_1 which is a random variable, now since you have purchased $Delta_0$ number of stocks, so that means the value of Δ_0 number of stocks that you own at time $t = 1$ is simply going to be Δ_0 into the price of the stock at time $t = 1$, namely S_1.

Please keep in mind that this S_1 that you have here can take two possible values, namely $u(S_0)$ and $d(S_0)$, okay, so this means that, this is going to be the current valuation of this initial investment and what is going to be the valuation of your investment $X_0 - \Delta_0 S_0$, so your investment of $X_0 - \Delta_0 S_0$, this will now grow by a factor of $1 + r$ at the end of time $t = 1$, so that means the total amount or the value of your portfolio at time $t = 1$ is going to be the sum of this two.

Okay, now this can be rewritten as follows, I take this first term $(1 + r)X_0$ separated out this particular term, plus I will take the factor of Δ_0 as common multiplied by $S_1 - (1 + r)S_0$. Now there is a very critical statement that I am actually going to make here, so this important statement is the following, is that the natural question is that if you start off with an amount of X_0 what should be the value of Δ_0, right, I mean you have open choice of Δ_0 and what will be the appropriate choice of Δ_0 and then what should be your appropriate choice of X_0.

Now, what is this driven by the choice of Δ_0 and X_0, it is driven by in such a manner, so now let me try to actually give you the motivation of what exactly is happening here. The idea behind is that you are basically the writer of an option X_0 is going to be the compensation that you actually receive for the price of the option that you receive from the buyer of the option. Now, what you do is that you take that amount of X_0 and you decide to invest in Δ_0 number of shares and the remaining money of $X_0 - \Delta_0 S_0$ is going to be invested at a risk-free rate of r.

Now, you have to choose your X_0 and Δ_0 in such a manner that this particular investment strategy that you have adopted namely, Δ_0 in shares and remaining as risk-free rate should be such that at time $t = 1$, whatever is the value of the portfolio resulting from your investment that should give you enough money to give the payoff to the holder of the option, so this means that whatever is the payoff that you actually have to give to the owner of the option, your amount of X_1 must match exactly that, so that you do not suffer any particular loss.

So, this is the first time you are intuitively looking at the idea that being a writer of an option does not always necessary mean a loss because the prices designed in such a way that with the price an investment can be made to exactly match the all the possible different values that you actually have to give to the owner of the option, namely all the possible values of payoffs, so in this case there are just two possible values of payoffs because the payoff is dependent on the stock price at time $t = 1$.

So, accordingly, the value of X_1 should be such at time $t = 1$ that it matches the value of the payoff, so this is what is known as basically replication that you are essentially replicating in such a manner that in the event of a head or in upward movement of the stock price the payoff will match exactly the value of the portfolio X_1 and likewise if there is a tail on a downward movement on the stock whatever be the payoff of the option at that time again you will notice that the value of X one of the portfolio exactly matches there.

So, in both the scenarios your portfolio is designed in such a way that you are able to match and meet your obligations under the payoff. Remember in case of, if the option actually exercising in case of a call option you can either give the stock to the owner or equivalently you can just pass on the profit that the owner of the option will get by executing that particular option.

So, in mathematical terms this means that since X_1 is dependent on S_1 and S_1 has two values, so it should be such that X_1 in the event, S_1 becomes $S1(H)$, X_1 must match the $V_1(H)$ and $X_1(T)$ must match $V_1(T)$ and this is what is known as replication, that you are actually able to replicate $V_1(H)$ or $V_1(T)$, which is an obligation to the buyer of the option by making an appropriate choice of $Delta_0$ and X_0 resulting in precise X_1H and X_1T and in other words X_1 will replicate V_1 and $V_1(H)$ and $X_1(T)$ will replicate $V_1(T)$.

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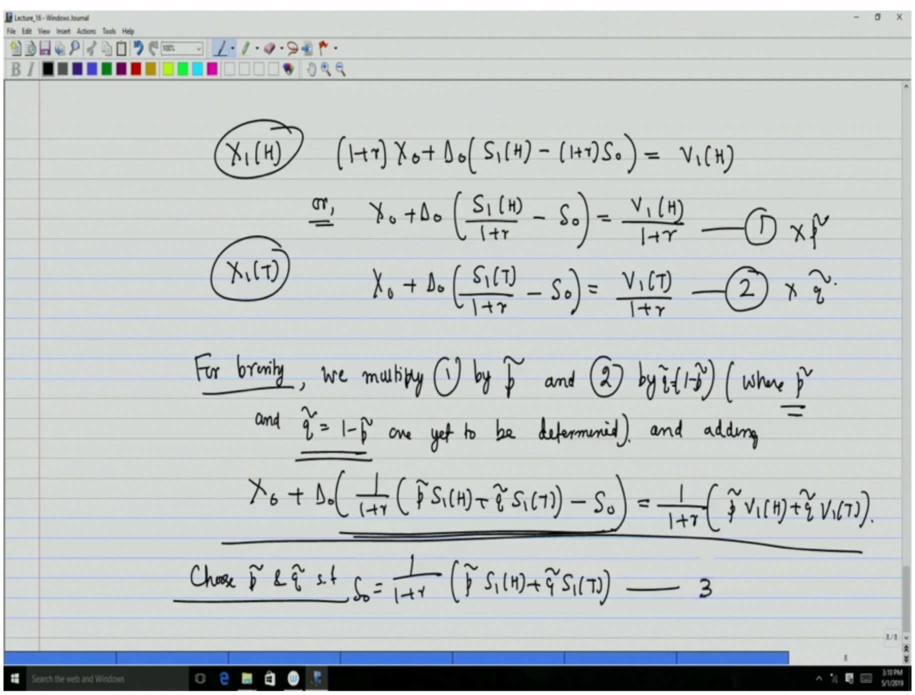

Now let us look at the two scenarios, what is $X_1(H)$? $X_1(H)$ is a scenario when a head has happened, so this is going to be $(1 + r)X_0 + \Delta_0(S_1(H) - (1 + r)S_0$ and from the first condition here this is going to be equal to $V_1(H)$ or equivalently we can write this as $X_0 + \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right) = \frac{V_1(H)}{1+r}$, I am just dividing both sides by $1 + r$.

In case of $X_1(T)$ you can in a similar way you can obtain $X_0 + \Delta_0 \left(\frac{S_1(T)}{1+r} - S_0 \right) = \frac{V_1(T)}{1+r}$. So, let me call this equation (1) and call this equation (2). Okay, so for brevity or simplicity we multiply $\tilde{p}$, we multiply actually question (1) by $\tilde{p}$ and we do not specify $\tilde{p}$ right now but we will come back to it and equation (2) by $1 - \tilde{p}$, where, so I will call this to be $\tilde{q}$, so where $\tilde{p}$ and $\tilde{q}$ which is 1 minus $\tilde{q}$ are yet to be determined.

Now, once you have done this, you multiply this and adding, so we multiply this by $\tilde{p}$ and this by $\tilde{q}$ and added so what do we get? Will get X not into $\tilde{p}$ plus $\tilde{q}$ which is 1 which is $X_0 + \Delta_0(1 + r)$ and within bracket I will get $\tilde{p}(S_1(H) + \tilde{q}S_1(T))$ and of course $-\tilde{p}S_0 + \tilde{q}S_0$, this is going to be simply $-S_0$ and this is going to be $\frac{1}{1+r}(\tilde{p}V_1(H) + \tilde{q}V_1(T))$.

Now, what we can write is that, I can write this as and this, so what you can do now is, now that you have this particular expression we can actually address the question of what is going to be my $\tilde{p}$ and $\tilde{q}$, so what we can do is that we can choose $\tilde{p}$ and $\tilde{q}$ such that this quantity here, this quantity essentially becomes equal to 0, so then this becomes S_0 is equal to, so if I choose $\frac{1}{1+r}(\tilde{p}S_1(H) + \tilde{q}S_1(T)$, which choose this to be equal to S_0, now this is very convenient, so let me called this actually equation (3).

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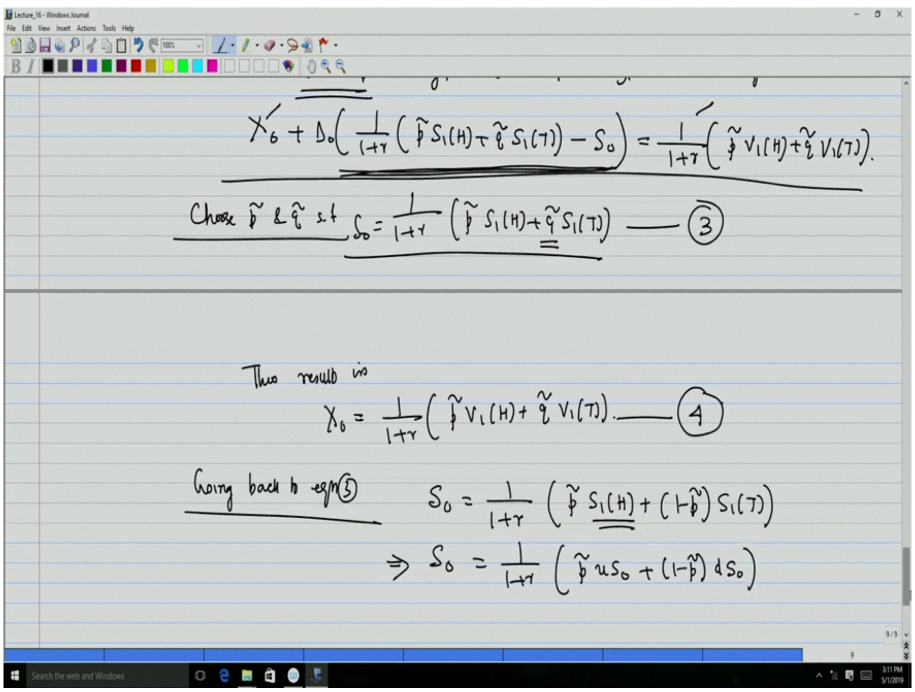
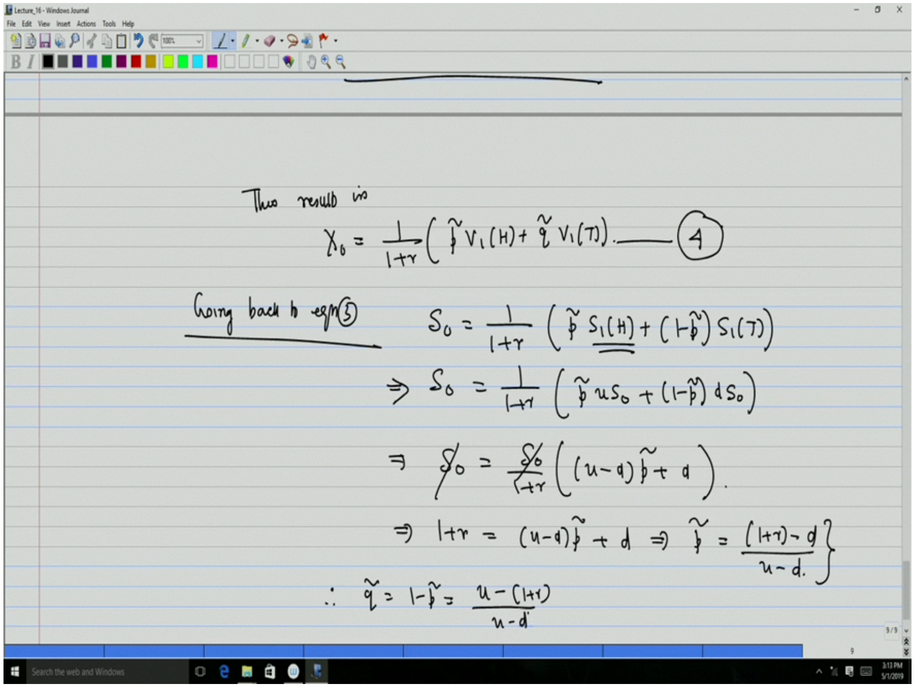


So, this results in what? So if you go back to previous relation, once I have this expression my choice of $\tilde{p}$ and $\tilde{q}$ this expression here becomes equal to 0, so only you have this and this expression which remain, so this becomes $X_0 = (1+r)\tilde{p}V_1(H) + \tilde{q}V_1(T)$, so what do we get? We basically get that, so I will call this equation (4).

Now, going back to equation (3), what do we do? In case of equation (3) we have S_0, now I will replace $\tilde{q}$ by $1 - \tilde{p}$, so S_0 becomes $\frac{1}{1+r}(\tilde{p}S_1(H) + (1 - \tilde{p})S_1(T)$ and this gives me $S_0 = \frac{1}{1+r}$, remember what is $S_1(H)$? $S_1(H)$ is nothing but uS_0 and $S_1(T)$, so $(1 - \tilde{p})S_1(T)dS_0$. And so what do we get here is the following, so this gives me $S_0 = \frac{S_0}{1+r}$, I will take the common factor of S_0 out and within bracket I have $(u - d)\tilde{p} + d$.

So, now this S_0 will cancel out, so what you end up getting is, so this will give you $1 + r = (u - d)\tilde{p} + d$ and this gives you $\tilde{p} = \frac{1+r-d}{u-d}$. So the $\tilde{p}$ and $\tilde{q}$ they have, you have introduced earlier, now without actually specifying can now has been determine and this is turns out to be $\frac{1+r-d}{u-d}$. Remember that u and d are just the model parameters because you have specified what your binomial model is, so you already know what u and d are and also you know what the risk-free rate r is.



So, you can actually uniquely determine what your $\tilde{p}$ is dependent on the model parameters. So, accordingly, you will get your $\tilde{q}$ which is $1-\tilde{p}$ and this is going to be $\frac{u-(1+r)}{u-d}$.

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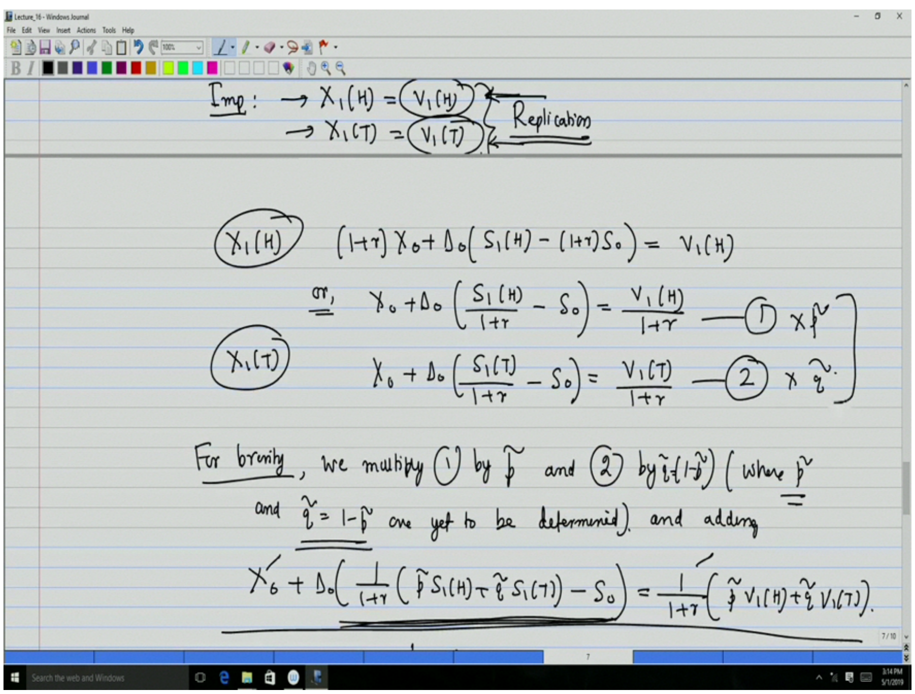
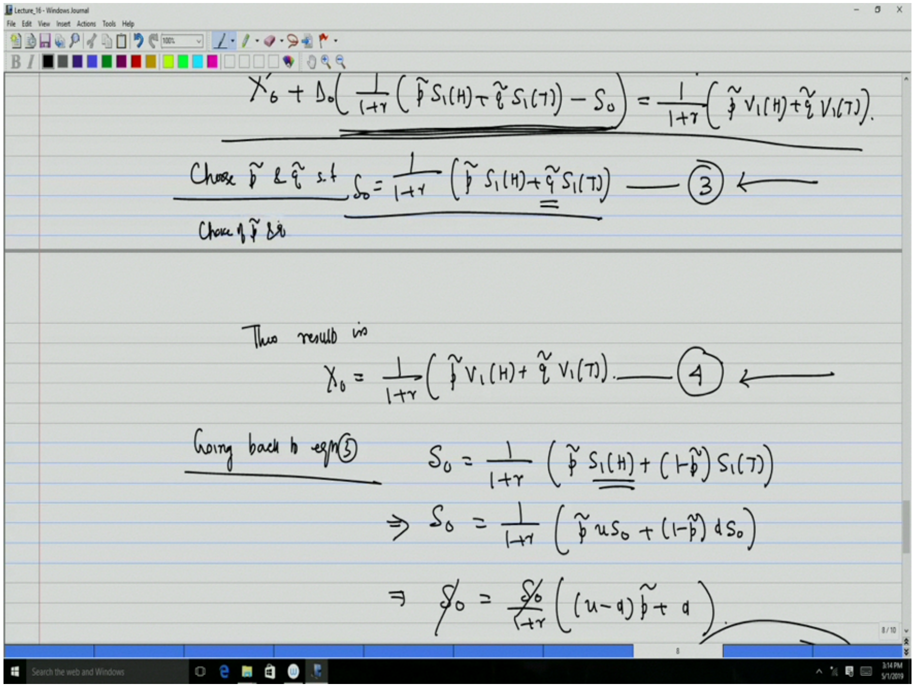
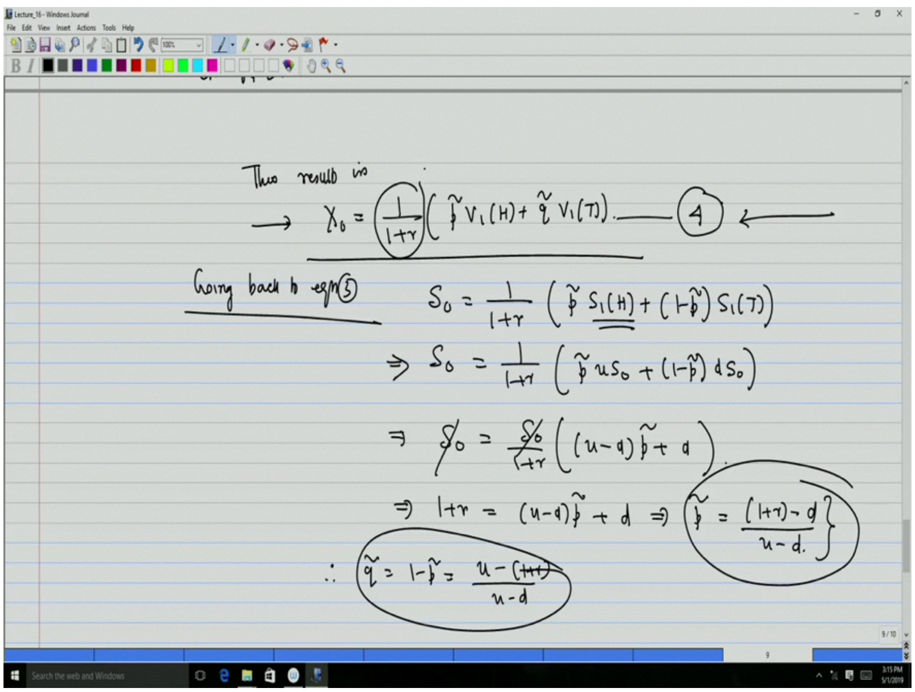


Okay, so this resolve the issue of figuring out what is $\tilde{p}$ and $\tilde{q}$? Then what is going to be equal to, what is going to be your Δ_0, in this case, you can so that your Δ_0 is nothing but $\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$, so this is what is known as the delta and will see a lot of it in the later heart of the course and this basically this is what is known as delta hedging or a delta replication.

So, this means what? This means that if an investor, so will now what we are going to do is that we have recall that we have made this replication strategy here and this replication strategy resulted in my equation number (3) and equation number (4), equation number (3) gave you the choice of $\tilde{p}$ and $\tilde{q}$ and using that

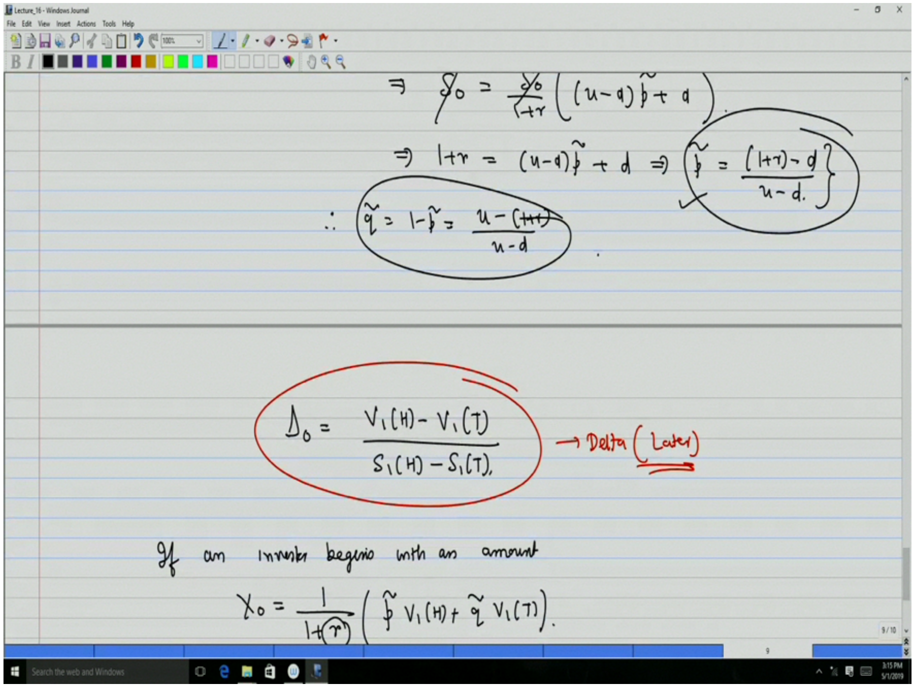


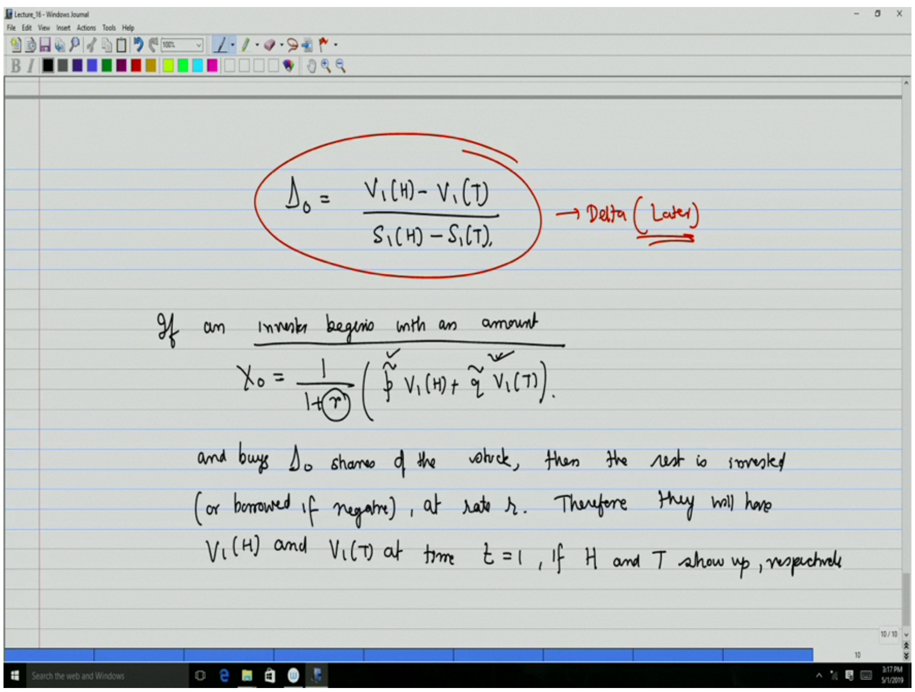


we have asserted what is my $\tilde{p}$ and $\tilde{q}$ and consequently I am able to revaluate what is going to be my Δ_0, so this means that, so I am now going to basically make use of this relation, now that I know what is going to be my $\tilde{p}$ and $\tilde{q}$ that is known.

So, therefore, if, so now my X_0 has been determine, so if an investor begins with an amount X_0 equal to, so I will just make use of relation (4) here, what was X_0? $X_0 = \frac{1}{1+r}(\tilde{p}V_1(H) + \tilde{q}V_1(T))$, so if I make a choice of X_0 according to equation for that is if I begin with $X_0 = (1+r)\tilde{p}V_1(H) + \tilde{q}V_1(T)$, now the interesting part here is that your r is a known quantity, your $\tilde{p}$ and $\tilde{p}$ you can figure out from the formulas here.

Right and $V_1(H)$ and $V_1(T)$ can easily be ascertained by depending on a, by ascertaining what is going to be the payoff, so remember here the $V_1(H)$ and $V_1(T)$ can, is applicable irrespective of whether you are choosing put or a call option and so is actually a generalise setup not specific just to European call option and once you actually have, so that means that every quantity here on the right inside of this particular





expression, every quantity here actually is, can actually be determined and so you know exactly what is going to be your value of X_0.

So, if this says that the investor begins with X_0 and buys delta not shares of the stock, then the rest is invested or borrowed as I have already mentioned if negative at rate r and therefore because of the way I have asserted $\tilde{p}$ and $\tilde{q}$, therefore, they will have $V_1(H)$ and $V_1(T)$ at time $t = 1$, if H and T show up respectively.

So, then we can say that the investor which will see that in this case is actually the writer of the option, the investor then has hedged a short position in the derivative security, the derivative that pays V_1 at time $t = 1$ should accordingly priced as $V_0 = \frac{1}{(1+r)}(\tilde{p}V_1(H) + \tilde{q}V_1(T))$.

Alright, so now what you can do is that, you know, let us move from the is abstract setup and try to see this through an example and this is sometimes called hedging or sometimes this is what is known as replication and, so this pricing should be done at as V_0 at time $t = 0$. So, we illustrate this through an

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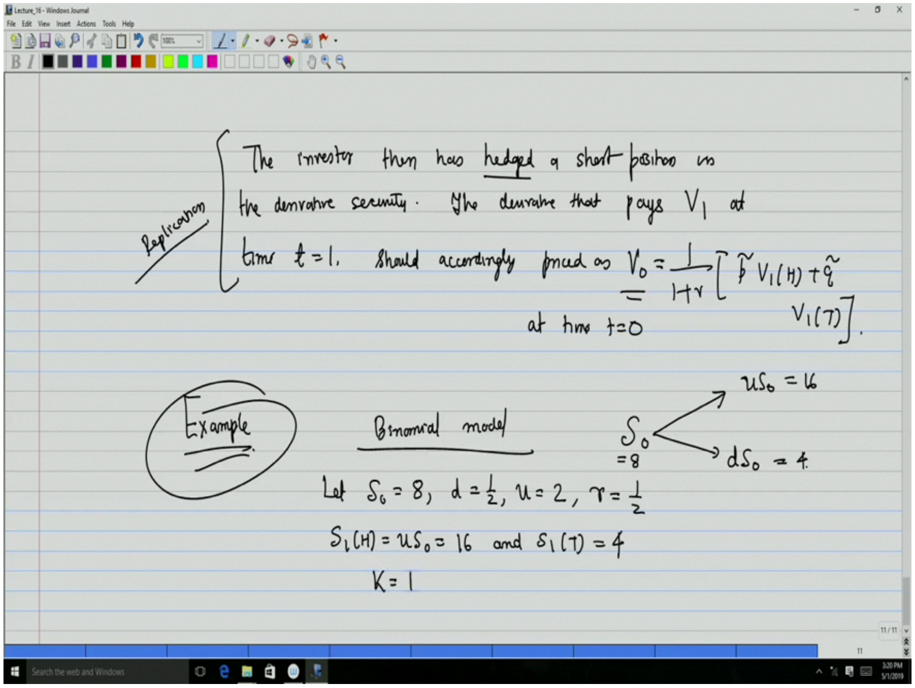

example, now, so suppose we consider a binomial model, what do we need in case of binomial model, right? In case of binomial model recall that you have initial price it is S_0 and you can go up to uS_0 or come down to dS_0.

So, we need the values of S_0u and S_0d, so let $S_0 = 8$, $d = \frac{1}{2}$ and $u = 2$ and also we did the risk-free rate, so let $r = \frac{1}{2}$, so what does this mean? This means that your stock price S_0 can either go up, so $S_0 = 8$ and it can either goes up to $uS_0 = 2 \times 8 = 16$ or it can come down to $dS_0 = 8 \times \frac{1}{2} = 4$, so basically this means that $S_1(H) = uS_0 = 16$ and $S_1(T) = 4$, so now we check two things, so further we also did the strike price, so let us choose $K = 10$.

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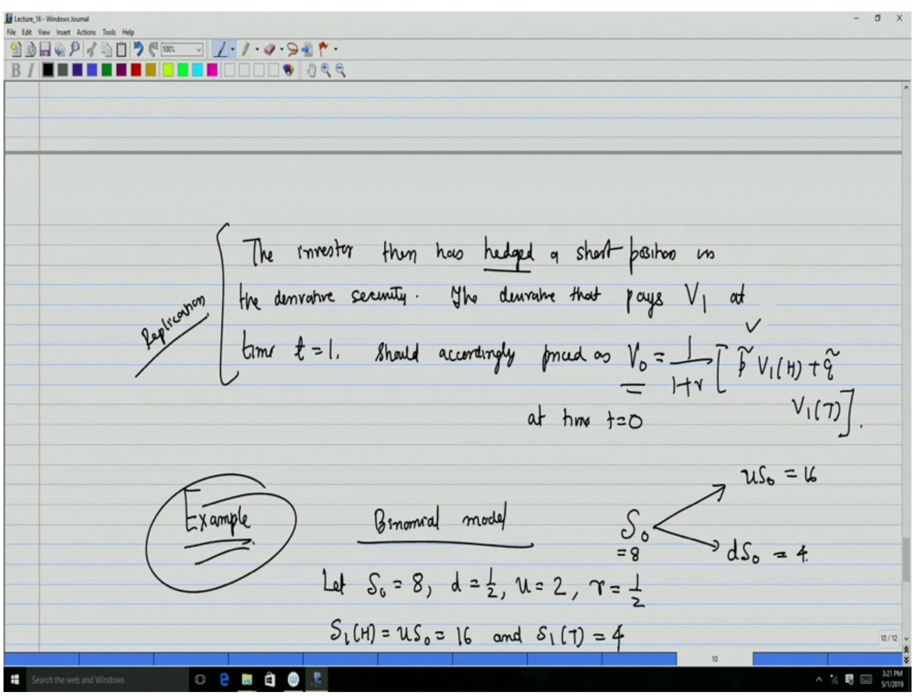

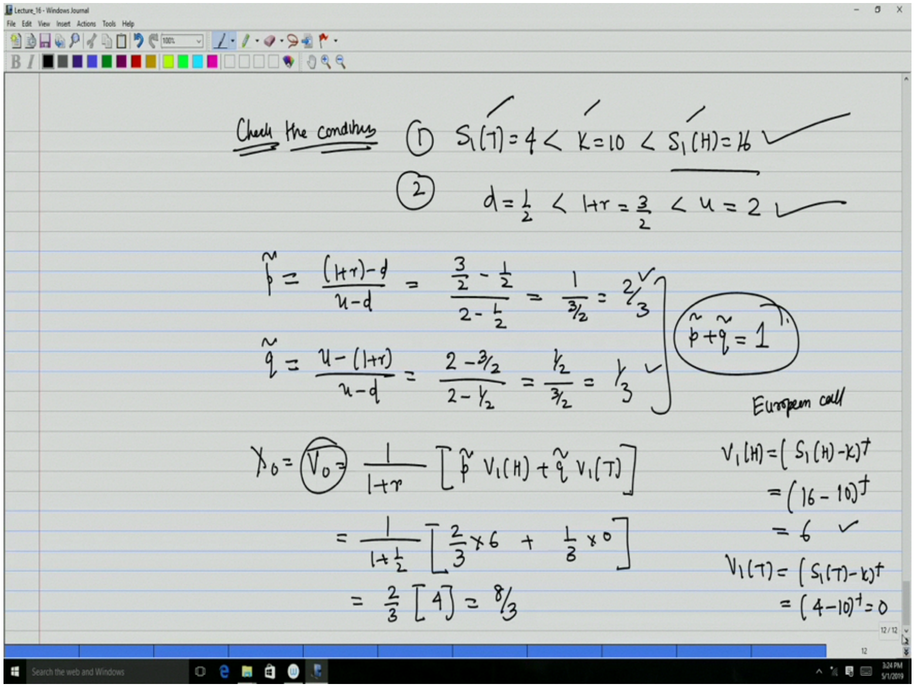

So, let us check whether all the condition is satisfied, so let us checks that the conditions, what are the conditions? The first condition was that your $S_1(T) = 4 < K = 10 < S_1(H) = 16$, so $S_1(T) < K < S_1(H)$, so this is satisfied.

The second condition you need to check that no-arbitrage condition, so that for that you need $d < 1 + r < u$, so $d = \frac{1}{2} + r = \frac{3}{2}$ and $u = 2$, so this condition is satisfied, so this is basically a legitimate binomial asset pricing model setup. So, now let us calculate what is going to be my $\tilde{p}$ because I need $\tilde{p}$ and $\tilde{q}$ to figure out what is going to be my V_0. So $\tilde{p} = \frac{1+r-d}{u-d} = \frac{2}{3}$.

Likewise $\tilde{p} = \frac{u-(1+r)}{u-d} = \frac{1}{3}$.

Okay, so then my $X_0 = V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = \frac{1}{1+\frac{1}{2}}[\frac{2}{3} \times 6 + \frac{1}{3} \times 0] = \frac{8}{3}$, so $V_1(H) = (S_1(H) - K)^+ = (16 - 10)^+ = 6$ and $V_1(T) = (S_1(T) - K)^+ = (4 - 10)^+ = 0$.

So we substitute all the values here now, and this becomes $\frac{1}{1+\frac{1}{2}}$, remember $\tilde{p} = \frac{2}{3}$, $\tilde{q} = \frac{1}{3}$, $V_1(H) = 6$, $V_1(T) = 0$. So, what does this becomes? This becomes $\frac{2}{3} \times 4 = \frac{8}{3}$, so this should be V not and this must exactly be the same as X_0, so V_0, here basically means the value of the payoffs or the expected value discounted of the payoffs from the European call option, discounted back to the time $t = 0$ and this must exactly match the money that, the money V not must exactly match X_0, which is the money that the writer of the option needs in order to replicate or hedge against any potential losses.

So, this means the following, now since $V_0 = \frac{8}{3}$, so $X_0 = \frac{8}{3}$, so this means that the writer of the option receives $X_0 = \frac{8}{3}$. Now this writer of the option now has to invest in Δ_0 stocks and rest at rate r. So, how do I determine? I have determine what is my X_0 this is going to be as same as V_0, but now I want to determine what is going to be my Δ_0.

Now, recollect after solving we had mentioned that this Δ_0 is nothing but $\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$, so we will use this formula, so in this case than Delta not is going to be $\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)} = \frac{6-0}{16-4} = \frac{1}{2}$.

So, this means the strategy that the writer of the option takes is that first of all you buy $\Delta_0 = \frac{1}{2}$ shares or stocks by investing, now since the price of the stock is S_0, so Δ_0 share will cost $\Delta_0 S_0$, which is $\frac{1}{2} \times 8 = 4$ and so what is going to be balance? So this balance will be you started off with $\frac{8}{3}$ and you have invested an amount of 4 in the stock, so this becomes $-\frac{4}{3}$, so that means you have to borrow this.

You had $\frac{8}{3}$, which is less than the 4 that you need, so you had to borrow an amount of $\frac{4}{3}$, now so this is something that is happening at time $t = 0$. Now at time $t = 1$ what happens? There are two possibilities, the first possibility is a head and second possibility is a tail, so then what is going to be this particular portfolio

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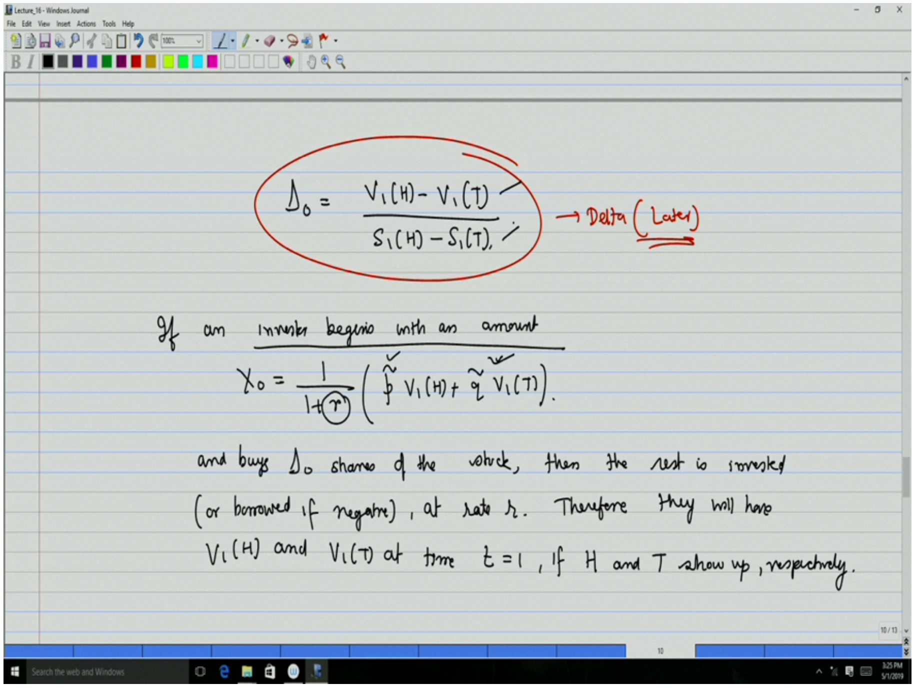
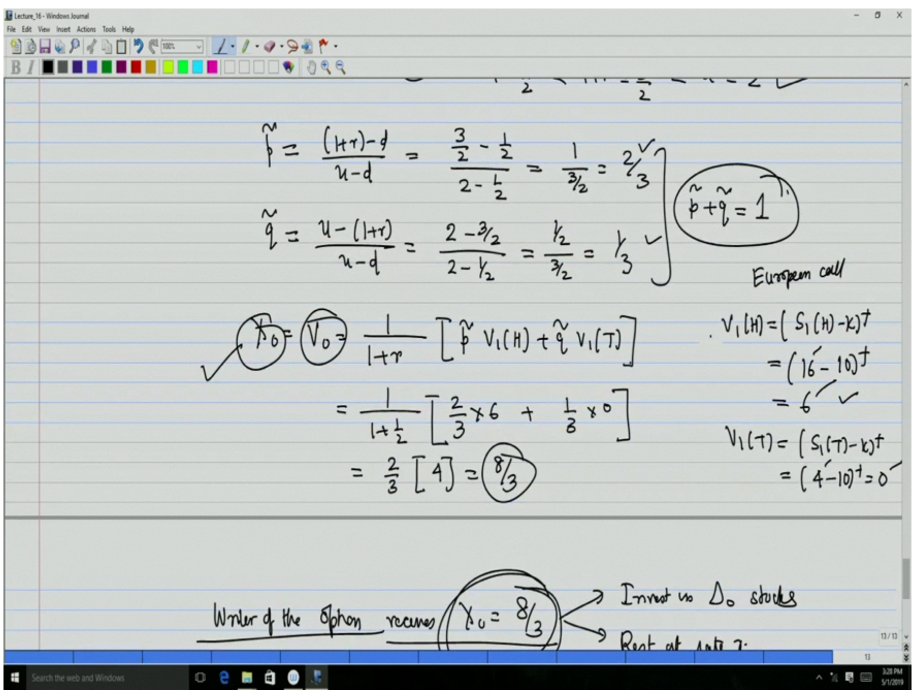
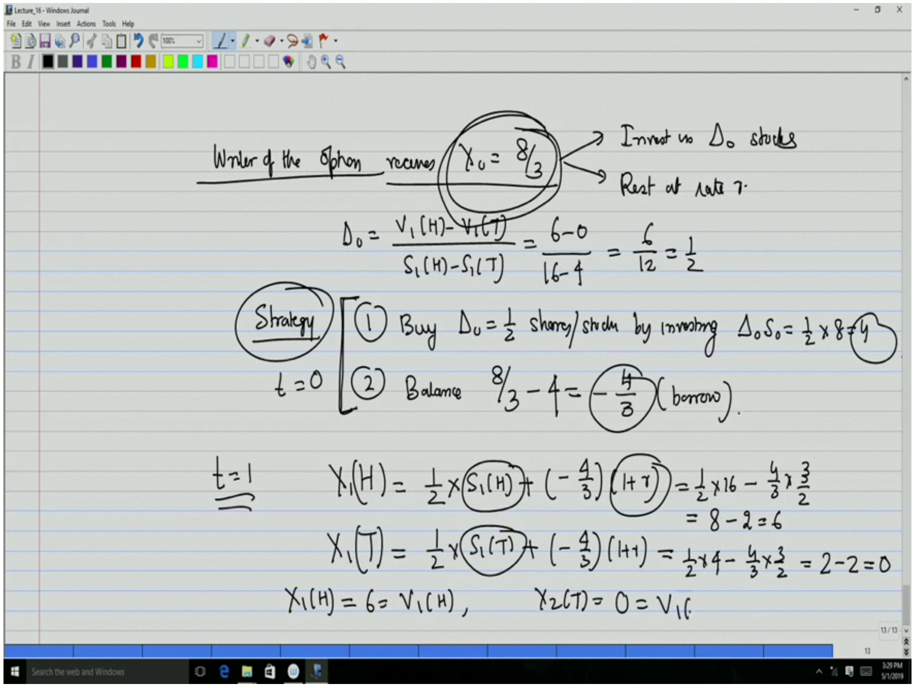




at the point, so $X_1(H)$ is going to be, remember you had half shares, so it is going to be half into $S_1(H)$ plus this balance amount minus $\frac{4}{3}$ into $1+r$.

Remember $S_1(H) = 16$, so this becomes $\frac{1}{2} \times 16$ and remember $1 + r = \frac{3}{2}$, so this is $-\frac{4}{3} \times \frac{3}{2}$, so this is nothing but $8 - 2 = 6$ and remember that means the value of the portfolio in case of an upper stock movement is going to be equal to 6 and in case of $X_1(T)$, that means downward stock movement this is going to be half into $S_1(T)$ plus this minus $\frac{4}{3}$ into $1 + r$, now in case of $S_1(T)$, $S_1(T) = 4$, so this is $\frac{1}{2} \times 4 - \frac{4}{3} \times \frac{3}{2} = 2 - 2 = 0$. So, you see that $X_1(H) = 6$ which is the same as $V_1(H)$, so you have replicated $V_1(H)$ and similarly $X_2(T) = 0$, which is the same as $V_1(T)$, which is 0, so this is $V_1(T)$.

So what does this mean? This means the following, it says the following that, so what is, so let us look at the interpretation of this, the interpretation of this example is as follows that you start off with a stock S_0



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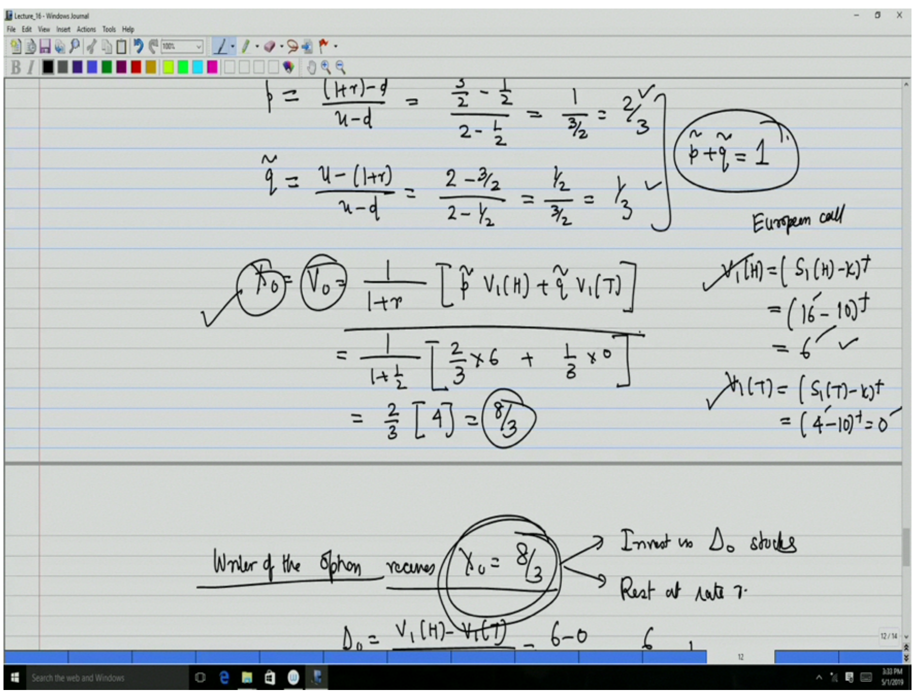
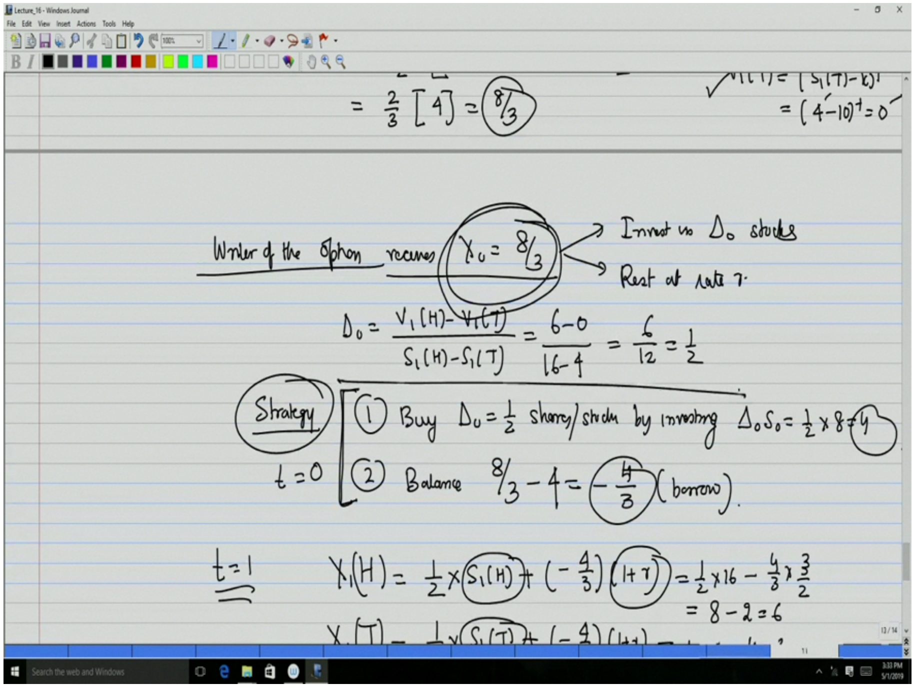
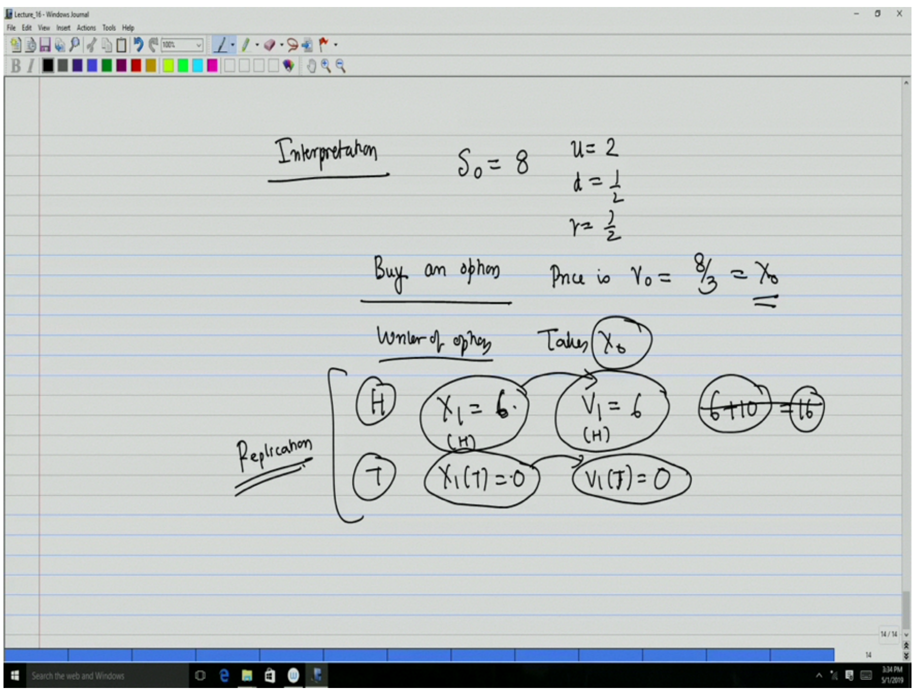


at the current prevailing price of the stock is $S_0 = 8$, the binomial model is designed with $u = 2$, $d = \frac{1}{2}$ and $r = \frac{1}{2}$. So, suppose somebody wants to buy an option than the price of the option is $V_0 = \frac{8}{3}$ and this is same as X_0.

So, this means that the following that the writer of the option takes X_0 and makes an investment, so what happens is the following that in case there is a head this investment with this amount of $X_0 = 6$ and the payoff is also going to be equal to 6, so this means that taking the price of the option X_0 and investing by the appropriate choice of Δ_0, which is equal to half, the owner, the writer of the option ends up with the amount of 6 and this is exactly enough to match the payoff that they have to pay to the owner of the option.

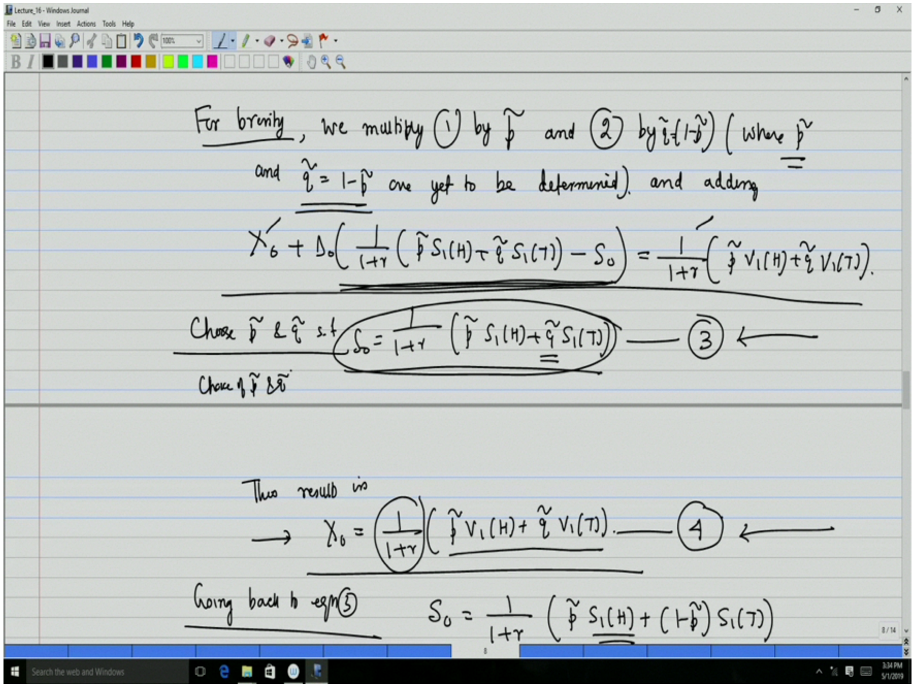
The owner of the option what they are going to do? So what are going to do is that they are going to get this 6, alright, they are going to receive an amount of 10 from the owner of the option, they have this entire





amount of 16, alright and then they use this 16 to buy the stock at the prevailing price and give this stock to the owner or equivalent what they do is that, they tell, they have an amount of 6, they tell the owner of the stock, rather the owner of the option saying that, "Look you normally pay me an amount of 10 for the stock, which was the agreed strike price and now I know that you will immediately sell it in the market for 16 and you are interested in only the payment of 6."

"So, let us not go through this entire exercise, since you are only interested in this 6, I am just going to hand over this amount of 6 which I have to you." So, that means the amount of 6 that they are obligated to pay is exactly matched because they made a correct choice of X_0 which was $\frac{8}{3}$ and a correct choice of Δ_0, which was equal to half. Now on the other hand suppose the tail happens, in case of a tail what is going to happen? $X_1(T) = 0$, but in case of a tail the strike price is 10 and the current market price is 4, so the option will not be exercised and that means $V_1(T) = 0$, I mean they will find that they have portfolio 0, but they did not worry about it because clearly the other party who are the owner of the option, they obviously



are not going to be exercise and so there is no liability.

So, this means that given that the right choice of X_0 as given by this particular formula, this particular formula right choice of X_0 and the right choice of Δ_0, that we had as given by this particular formula has ended up ensuring that in both the scenarios of the stock price movement, namely the up and downward which is denote by head and tail, whatever amount they have 6 and 0 exactly matches the amount that they own or they have to pay to the buyer of the option and this is what is basically a typical example of the replication.

So, just to sum it up or whatever we discusses is that we started off, we essentially look today at a single period binomial model for asset prices, we started off with a known stock price and S_0 and the binomial model said that this value of S_0 the future value of the stock at time $t = 1$ is going to be equal to uS_0 when there is an upward movement, obviously, $u > 1$ and if there is a downward movement, it is going to be duS_0.

And you essentially then can write an option on it and you can use our replication strategy to determine what is going to be the Fair Price of the option and just to make one last point that I had missed out on, that point is that remember we had made a choice of $\tilde{p}$ and $\tilde{q}$ so that I choose my $\tilde{p}$ and $\tilde{q}$, so that is not is basically $\frac{1}{1+r}(\tilde{p}S_1(H) + \tilde{q}S_1(T))$, so what you can do is that you can revisit this.

So, just coming back to our final observation that I want to make is $S_0 = \frac{1}{1+r}(\tilde{p}S_1(H) + \tilde{q}S_1(T)$, so basically this means that here it is like $S_1(H)$ is a random variable, $S_1(T)$ is a random variable, if I take my $\tilde{p},\ \tilde{q}$ to be probabilities which I can do at least you know intuitively because $\tilde{p} + \tilde{q} = 1$, you can view this as the expected value of S_1 with probability $\tilde{p},\ \tilde{q}$ instead of $p,\ q$, and this is nothing but $\frac{E(S_1)}{1+r}$.

So what it is saying is basically that $E(S_1) = S_0(1 + r)$, so what it is saying that, if you start off with an investment of S_0 at time $t = 0$, at time $t = 1$, it will grow to a factor of $S_0(1 + r)$, if this investment was risk-free but what you are seeing now is basically that $S_0(1 + r)$ is becoming the same as the expected price of the stock, so this might give you an impression that an investment of a stock is basically the same thing as a investment in a risk-free asset, but please remember that this is equal to this expected value of the stock price at time $t = 1$ under the probabilities $\tilde{p}$ and $\tilde{q}$ and not allowed the original probabilities of p and q.

So, normally $E(S_1)$ or if I call this $\tilde{E}$, $E(S_1) = pS_1(H)o + qS_1(T)$, this would typically be greater than S_0 and this is what are known as the real world probabilities and this, since this choice of probabilities results in the expected values of S_1 being the same as the value that you would get as a risk-free investment that is the reason why $\tilde{p}$ and $\tilde{q}$ are known as risk neutral probabilities, where is in the real world since you

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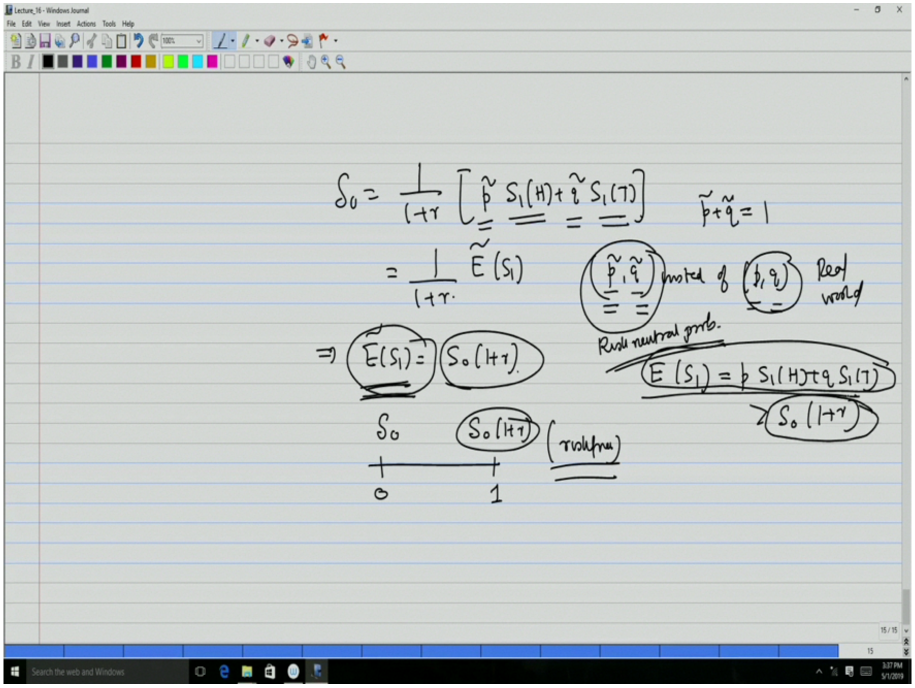

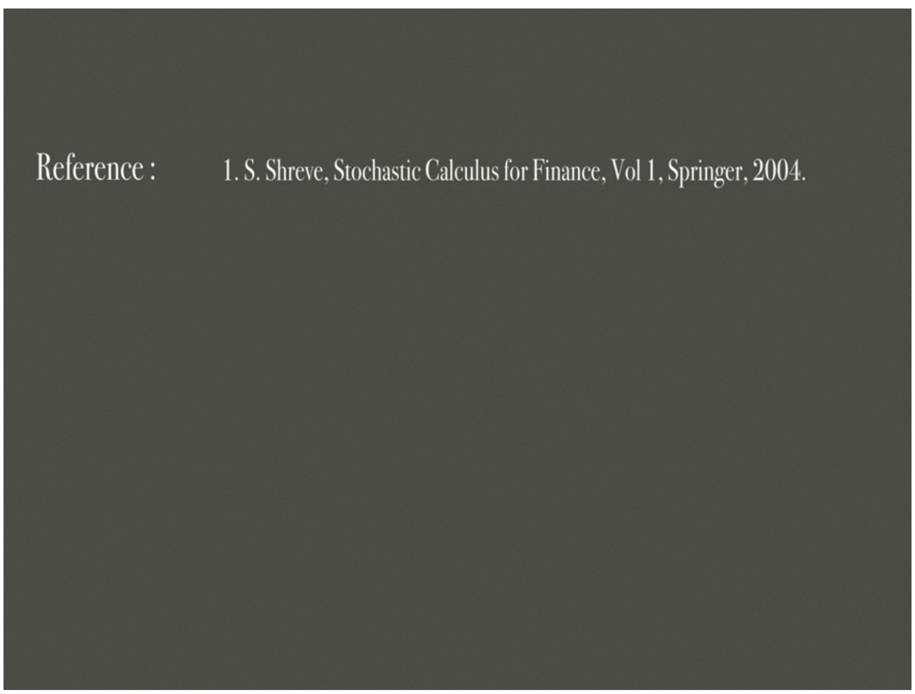

are taking a risk, this is $S_0(1+r)$, you expect that your expected return on the investment in the stock should be more than what you would get in case of an risk-free investment.

So, more details of this will be taught as you move along this discussion on Option Pricing. So this brings us to the end of today's class, in the next lecture we will continue more on the replication strategy to determine the price of an option, this time in case where you go beyond the single period framework. Thank you for watching.