Mathematical Finance

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Module 4: Fundamental of Derivatives
Lecture 3: Bounds on Options

Hello viewers! Welcome to this course on Mathematical Finance. You will recall that in the last class we talked about options and we looked at some of the basic properties of option. In particular, we looked at how we can make use of the no-arbitrage principle in order to prove a very important result, namely the put-call-parity and then we stated that particular result in the case of dividends.

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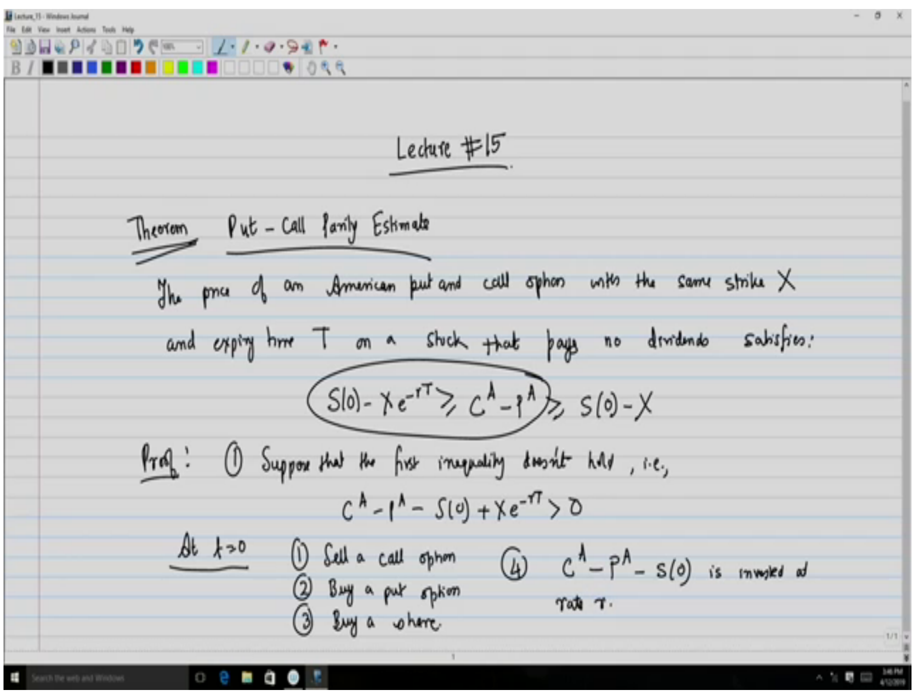

So in today's class we will discuss a few more results on several properties of options and we will begin with a particular property in case of American options. So accordingly, now as I mentioned towards the end of the previous class that in case of American option given the nature of the American option, that is you have a right to actually exercise any time on or before the expiration date. So you no longer have a put-call-parity in terms of equality but rather you actually have inequality which we call as the put-call-parity estimate.

So we can state it as follows: The price of an American put and call option with the same strike, as I had assumed in the previous case of European option, so we assume that they have the same strike and expiration on a stock, as before we assumed the stock pays no dividends, so on a stock that pays no dividends satisfies the following estimate:

That is, $S(0) - Xe^{-rT} \geq C^A - P^A$. Remember, C^A and P^A are the price of the American call and put respectively. And this is going to be greater than or equal to $S(0) - X$. Okay. So the proof for this, again we will make use of the 'no-arbitrage' principle. So first of all, we assume, suppose that the first inequality does not hold. So by this I mean that I assume that this inequality does not hold.

So this means that I am assuming that $C^A - P^A - S(0)$, that means $C^A - P^A > S(0) - Xe^{rT}$. So this is $C^A - P^A - S(0) + Xe^{-rT} > 0$. Then what is the strategy I am going to adopt? That means that at some time t is equal to 0, first of all we will sell a call option, then secondly we will buy a put option. Both of them are American call and put. Then, thirdly we will buy a share.

And what we do is that with the remaining amount, that is when I sell a call I receive an amount of C^A. When I buy a put, I spend an amount of P^A and when I buy a share, I spend an amount of $S(0)$. So this amount is invested at rate r.

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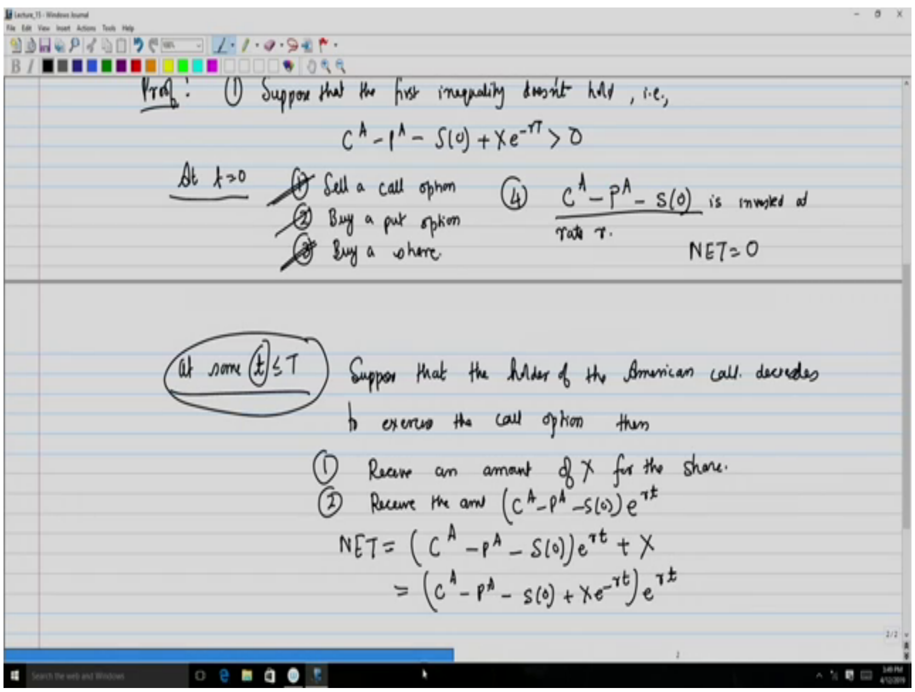

Now, at some small intervening time $t = T$, suppose that the holder of the American call, remember we had sold an American call option, so that can be exercised any time on or before T. So we now consider the possibility that yes, indeed, the holder of the American call decides to exercise the call option, then what will happen? First of all, when the call is exercised, so that means that you will receive an amount of X for the share.

So this means that your position under the call option is now covered and the share you had bought you have managed to sell it off, and the put option that you had bought that sort of goes without exercise. So you end up receiving an amount of X for the share. So now again, at that point if you decide that you want to actually withdraw, so here the net investment is equal to 0, so here the net amount is that first of all, you receive an amount of X for the share and secondly, you had invested $C^A - P^A - S(0)$ at rate r. So this you will receive the amount $C^A - P^A - S(0)$ with accumulated interest.

Remember, this is time t, so accumulated interest will be given by a factor of e^{rT}. Hence, your net gain is going to be $[C^A - P^A - S(0)]e^{rT}$. Now, this can be written as $[C^A - P^A - S(0) + Xe^{-rt}]e^{rT}$.

Now we just observe that since $t \leq T$, so therefore $e^{-rt} \geq e^{-rT}$. So this expression means that this will be greater than or equal to $[C^A - P^A - S(0) + Xe^{-rT}]e^{rT}$. And remember that this quantity by assumption

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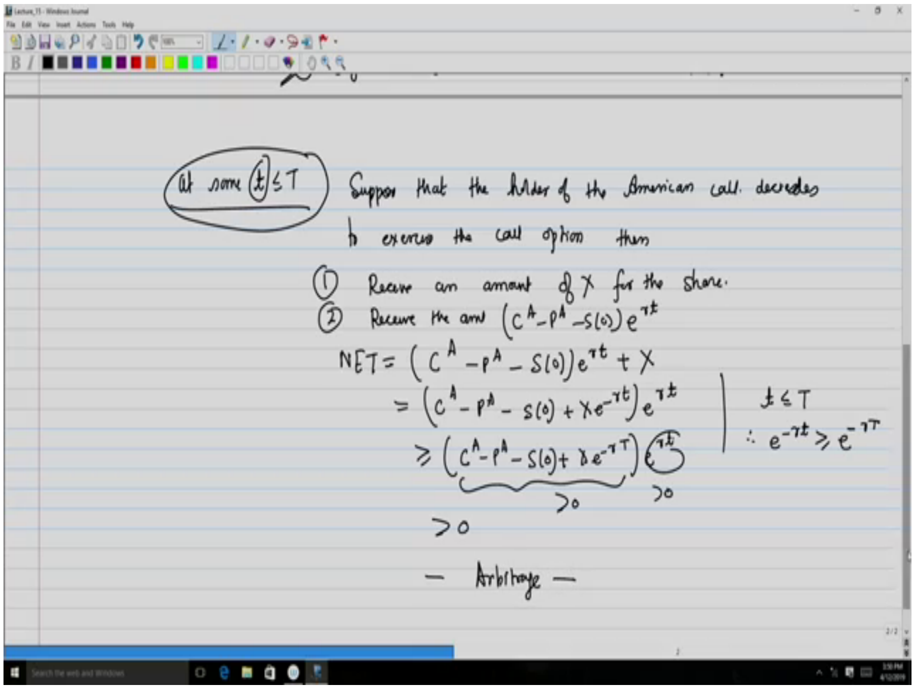

this was a positive quantity. So this means that this entire exercise, this value, this is anyway is positive, so the whole thing is going to be positive. So, this means that we have an arbitrage.

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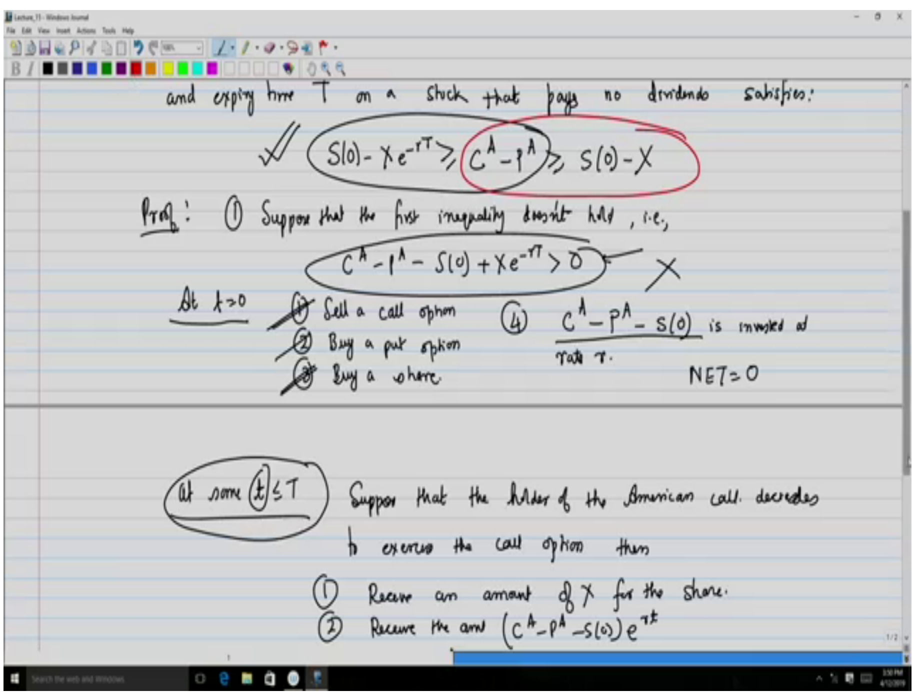

And so our assumption here is incorrect, so that means that the first inequality is indeed actually true. Okay, now I suppose that the second inequality actually, namely this one does not hold true. So in this case what will you assume? We will assume that.

So for the second case we suppose, so what is the assumption? I will assume that $C^A - P^A - S(0) + X < 0$. So I assume here $C^A - P^A - S(0) + X < 0$. So then what is going to be by strategy which could lead to arbitrage? So at time $t = 0$, what we do is we will first of all sell a put. Secondly, we will buy a call;

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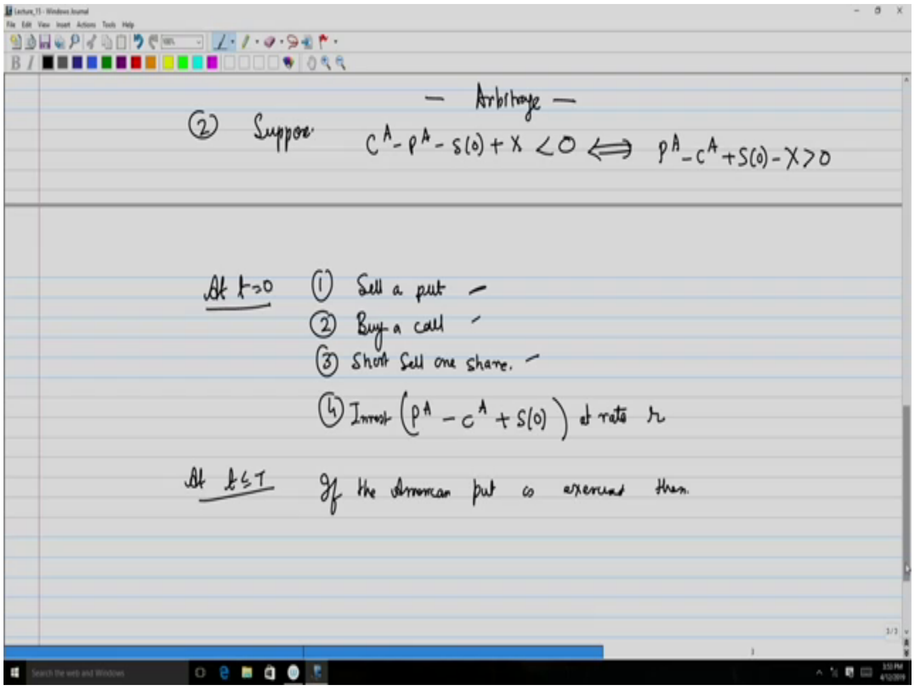

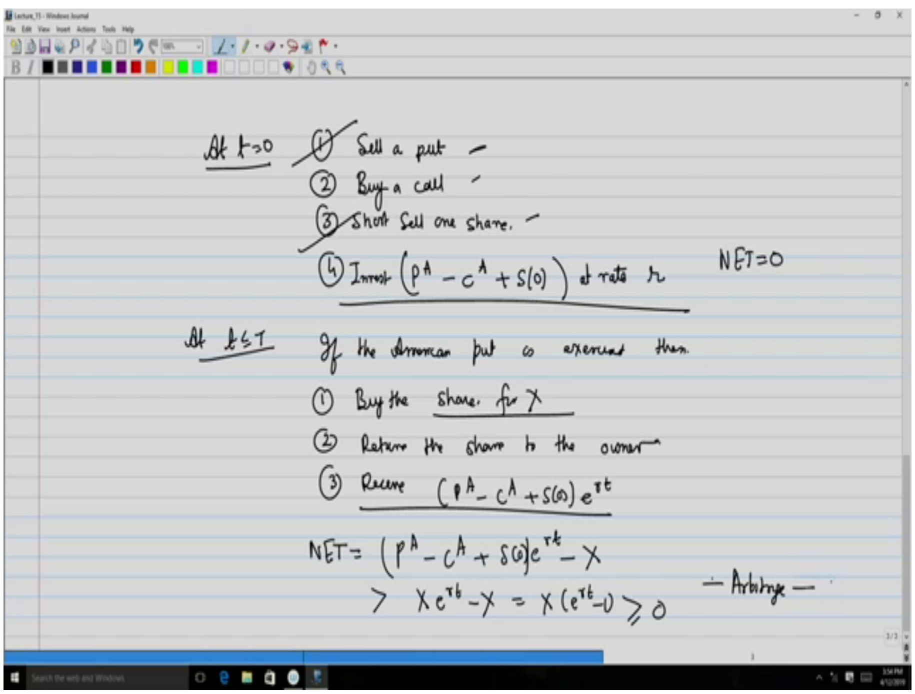

thirdly, we will short sell one share.

So this actually I can view, this is being equivalent to $P^A - C^A + S(0) - X > 0$. So, when I sell a put, that means I will receive an amount of P^A. When I buy a call I will spend an amount of C^A. And when I short sell one share, that means I will receive an amount of $S(0)$. And that means that I now can invest this amount at rate r, or borrow in case, this is a negative quantity.

Now, suppose, now here you have sold an American put which could be exercised at some time. So, at some time t less than or equal to T, if the American call is, actually American put is exercised, then what is going to happen? In this case that means that you will actually have to buy the share.

First of all, you will buy the share. So that means your position on the put is closed. And you have short sold the share, so that means you return the share to the owner. And then you receive your return on this investment, so that is going to be $[P^A - C^A + S(0)]e^{rT}$. So that means what is going to beSo the first case your net is equal to 0 and in the second case your net profit is going to beSo you have received this amount, so it is going to be $[P^A - C^A + S(0)]e^{rT}$.

And you had bought the share for X, so you spend an amount of X. And now the assumption was that $[P^A - C^A + S(0)] > X$. So this is going to be greater than $Xe^{rT} - X$, which is $Xe^{rT} - 1$, and this is strictly greater than or equal to 0. So that means that you again have an arbitrage opportunity.

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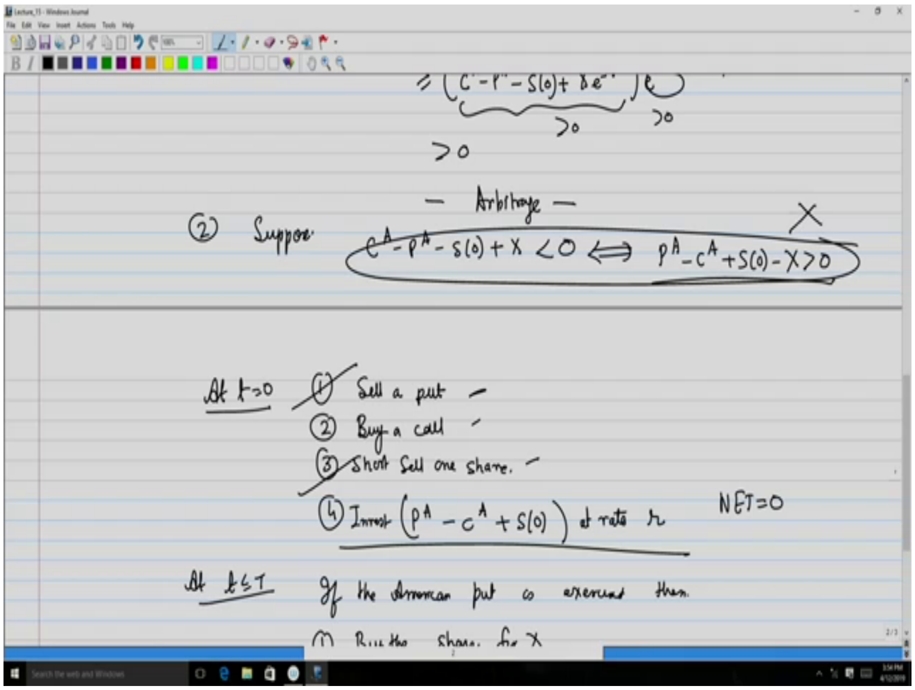

And so your basic assumption here was incorrect.

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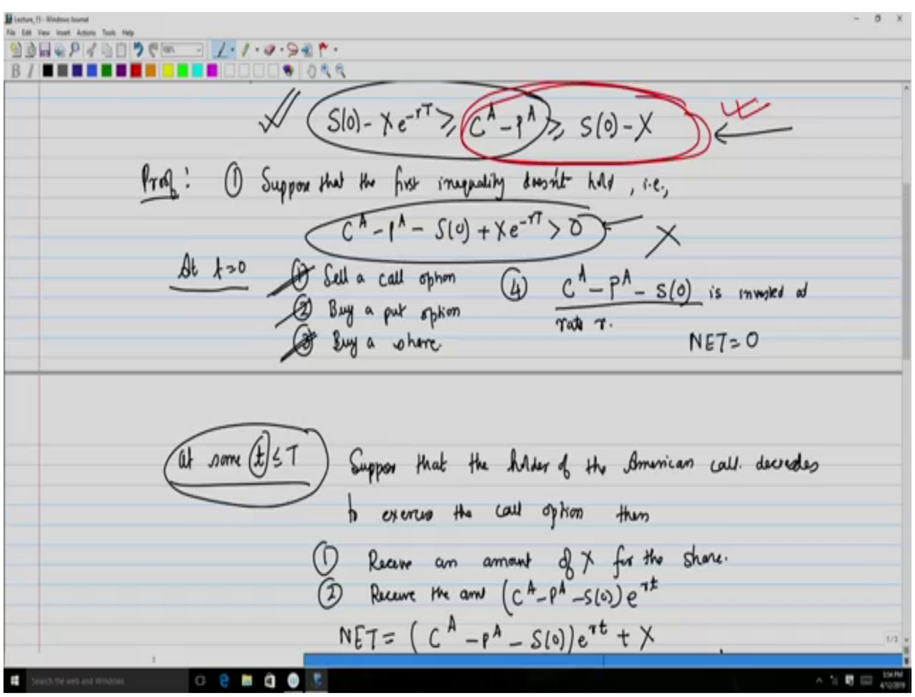

And consequently this relation actually holds. Now just one more observation before I finish off.

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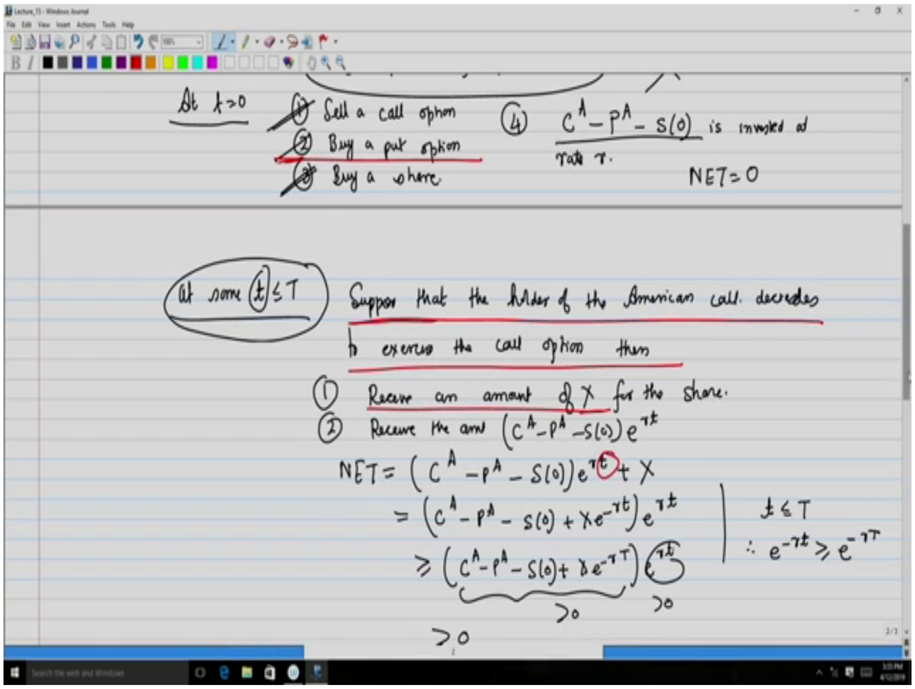

Now, here I had used the argument that suppose that the holder of the American option decides to exercise the call option for which you receive an amount of X. In case it is not exercised, then you still have the possibility because you are the owner of the option where you have the leverage and so you can actually receive an amount of X by selling the put option because remember you had bought the put option.

So even if the call option is not exercised, as I had assumed here, you can still receive an amount of X by exercising the put option. So that means your argument here does not change except that you will have to wait up till time $t = T$.

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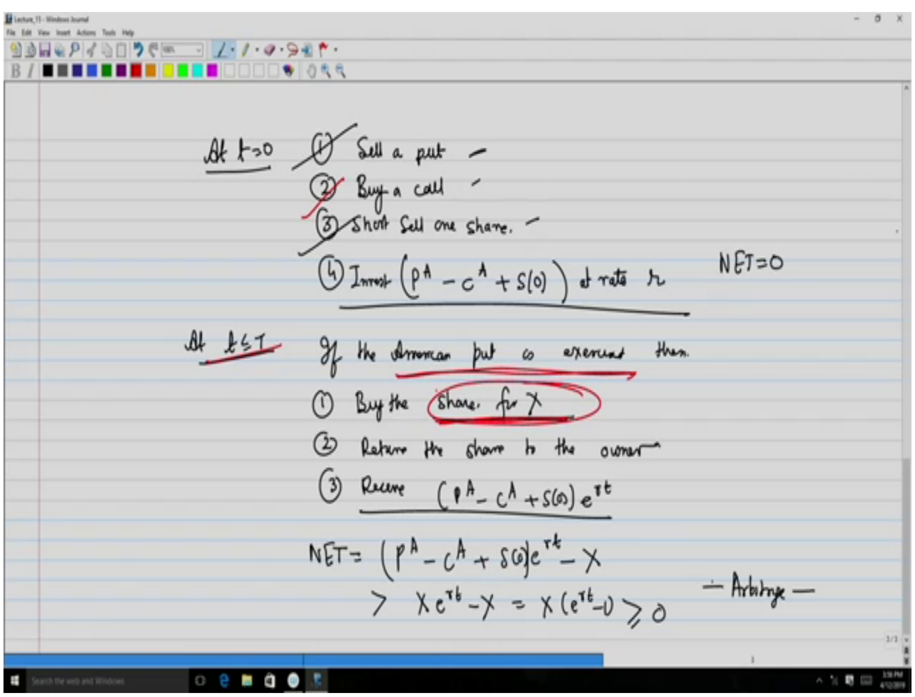

Likewise, the same thing here: I have given the argument here by assuming that American put is exercised and then you buy the share for X. In case the American put is not exercised, then at time t equal to T, you can make use of your call option and buy the share for an amount of X. So, you can obviously still get the share for the amount of X whether the other party exercise the put option or not. If they exercise the put option, well and good, then you get the share. Or in case they do not exercise, then at time t equal to T, you can actually decide to buy the share for X.

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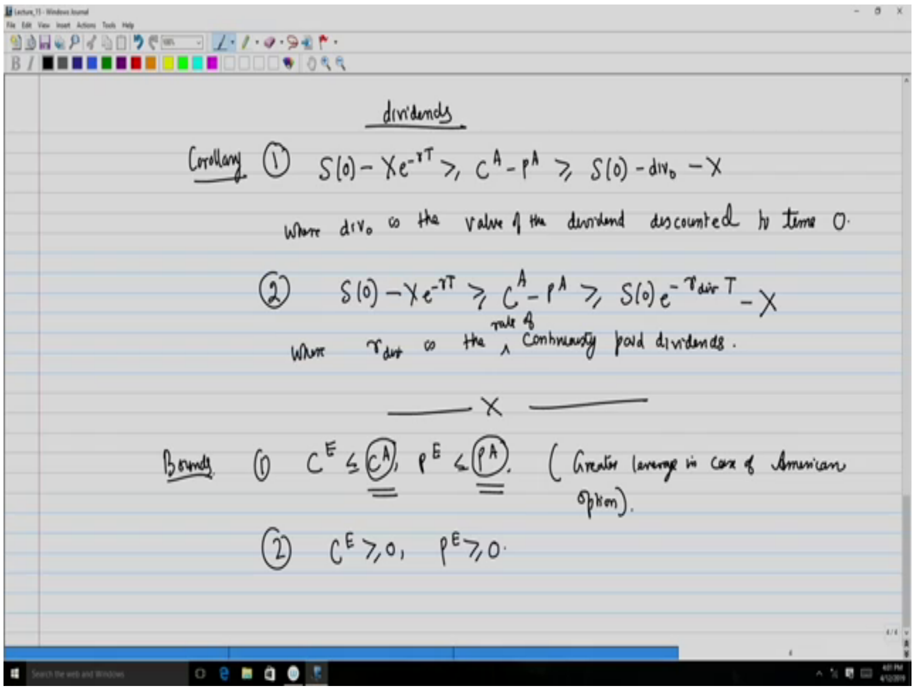

So, next what you look at is that, you basically look at a couple of corollaries out of this result. The first corollary is as follows: So they will be the corollary which take care of dividends. So remember we had obtained similar results in case of the European option. So we will actually state the analogous result in this case in case of American option. So first of all, let us look at the first corollary.

So this first corollary, following that we can show that, $S(0) - Xe^{-rT} \geq C^A - P^A$ and this is greater than or equal to $S(0) - X - div_0$. And recollect that where div_0 is the value of the dividend discounted, so the dividend could be paid at any intermediate time point but div_0 is going to be the value of the dividend discounted to time 0.

And secondly, another relation would be $S(0)Xe^{-rT} \geq C^A - P^A$ and this is going to be greater than or equal to $S(0)e^{-r\ div\ T} - X$, where $r\ div$ is the continuously is the rate of continuously paid dividends.

So we have obtained the results in case of European option, namely the put-call-parity and the put-call-inequality or bounds in case of American option, and we have looked at stating the analogous results for dividend that is actually paid once and dividend that is actually being paid continuously, both in case of European and American option.

Now let us look at couple of more sort of obvious properties, not just relating to the price of a call and put option but just comparing the price of an American call and European call and likewise an American put and a European put.

So, then here we will have the bounds. So the obvious inequalities are $C^E \leq C^A$ and $P^E \leq P^A$. And obviously you can state this because the price of the American option both in case of call and put option is going to be larger than or greater than or equal to than their corresponding European counterparts. And the simple reason is that because there is a greater leverage in case of American option. And because as a buyer of an American option you have a greater leverage, so obviously we have to pay a larger premium.

And one can of course also use some 'no-arbitrage' argument to prove this. Next, we have another obvious bound, that is $C^E \geq 0$ and $P^E \geq 0$. So obviously this has to be both non-negative because obviously nobody is going to actually pay you for taking up disadvantageous position. The obvious way of arguing this is that if your $C^E < 0$, that means you actually pay the buyer of the option an amount when you are actually having a disadvantageous position; likewise, in case of a put option.

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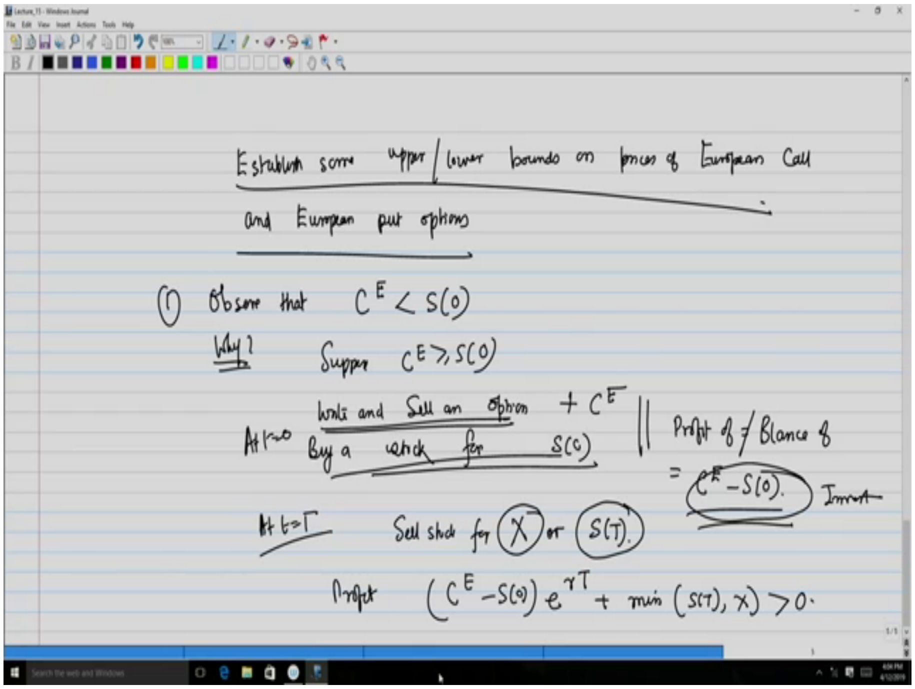

The next is, we establish some upper and lower bounds on prices of European call and European put options. So first of all, we observed that $C^E < S(0)$. Why is this so? It is so because, suppose $C^E \geq 0$, then what we would do is that we will just write and sell an option. So you will receive an amount of C^E. And then we buy a stock for $S(0)$.

That means that here in this case I made a profit of or rather I should say balance of $C^E - S(0)$. Now that I have actually made a profit here, but I have a position of obligation because I have written an option but it does not matter because I have already purchased the stock for $S(0)$. And so this means that at time $t = T$, I can simply sell the stock for either the strike price X or the prevailing stock price $S(T)$.

So this means that at time t equal to 0, I am making this profit and at time $t = T$, I am making initial amount of this X or $S(T)$ depending on whether the call option is exercised or not respectively. So this means that my profit is going to be $C^E - S(0)$.

Remember, I have made this profit at time $t = 0$, which I will of course invest. So at time$t = T$, this will go to an amount of $(C^E - S(0))e^{rT} + \min\{S(T), X\} \geq 0$.

So, what we have seen here is, we have this one particular bound which is the upper bound.

Now, let us look at a lower bound and that is going to be $S(0) - Xe^{-rT} \leq C^E$.

Remember this is the upper bound because it puts an upper cap on the price of the option C^E, namely $S(0)$.

And likewise, I will have a lower bound on $C(E)$. So again, using 'no-arbitrage' principle we try to address why this is so. So we observe the following: So recall that $C^E - P^E = S(0) - Xe^{-rT}$, this was the put-call-parity. Now this can be rewritten as $P^E = C^E - S(0) + Xe^{-rT}$. But we know that the price of any option must be greater than or equal to 0, so this is going to be greater than or equal to 0.

So from here I can use this 'to be greater than or equal to zero' to show that $S(0) - Xe^{-rT} \leq C^E$. So both these results, now that you have obtained the lower and the upper bound, so I can sum this up as a single relation as follows:

So, $C^E < S(0)$. And this is from here going to be greater than or equal to $S(0) - Xe^{-rT}$. So this is a key result that we have obtained. Now also remember that there was another result, namely that $C^E \geq 0$.

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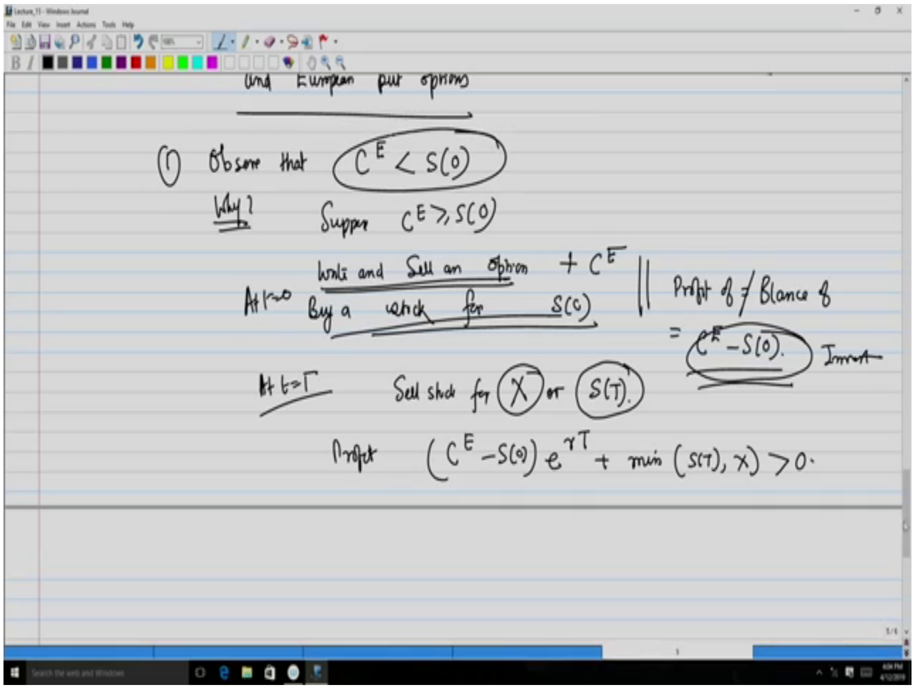

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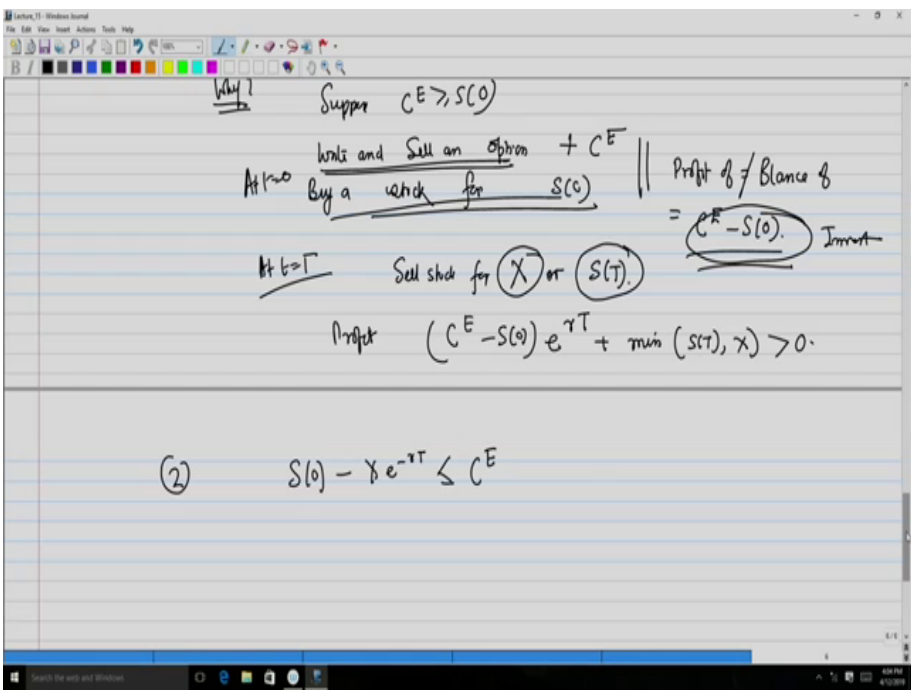

So C^E actually has two lower bounds, 0 and this respectively. And so accordingly we have the following observations:

First of all, $C^E < S(0)$. But C^E, now taking into account both these lower bounds can be rewritten as $\max\{0, S(0)\} - Xe^{-rT}$. And this is going to be less than or equal to C^E. Likewise in case of the put option this is going to be $\max\{0, -S(0) + Xe^{-rT}\} \leq P^E < Xe^{-rT}$.

Now in case of dividends being paid this relation gets modified to $\max\{0, S(0)\}$. So all the $S(0)$ get replaced with $S - div_0$, so this becomes $S(0) - div_0 - Xe^{rT} \leq C^E < S(0)$. So this is $S(0) - div_0$. And this relation becomes $\max\{0, -S(0) + div_0 + Xe^{-rT}\} \leq P^E < Xe^{-rT}$.

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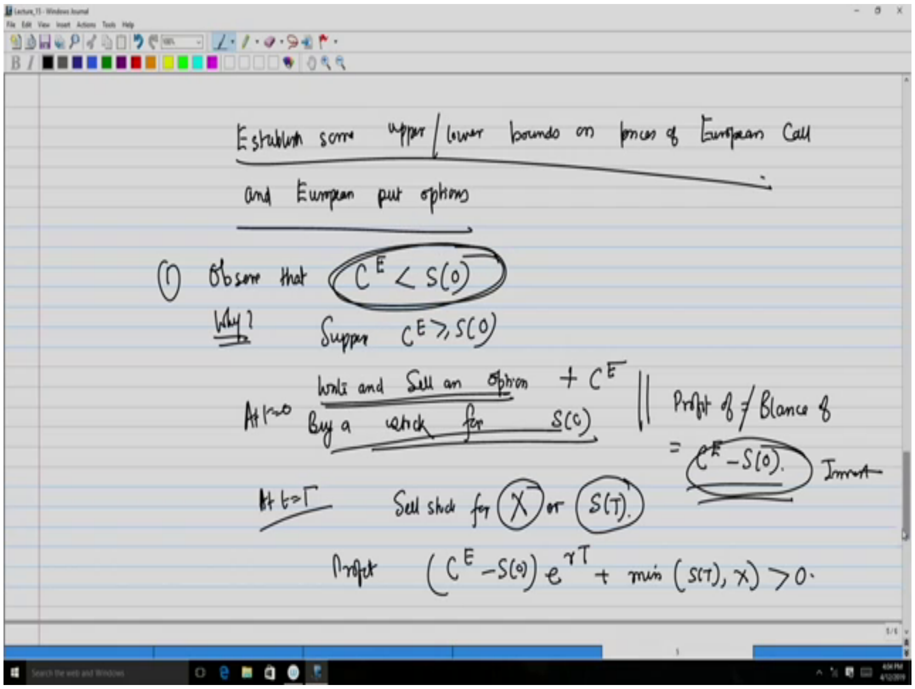

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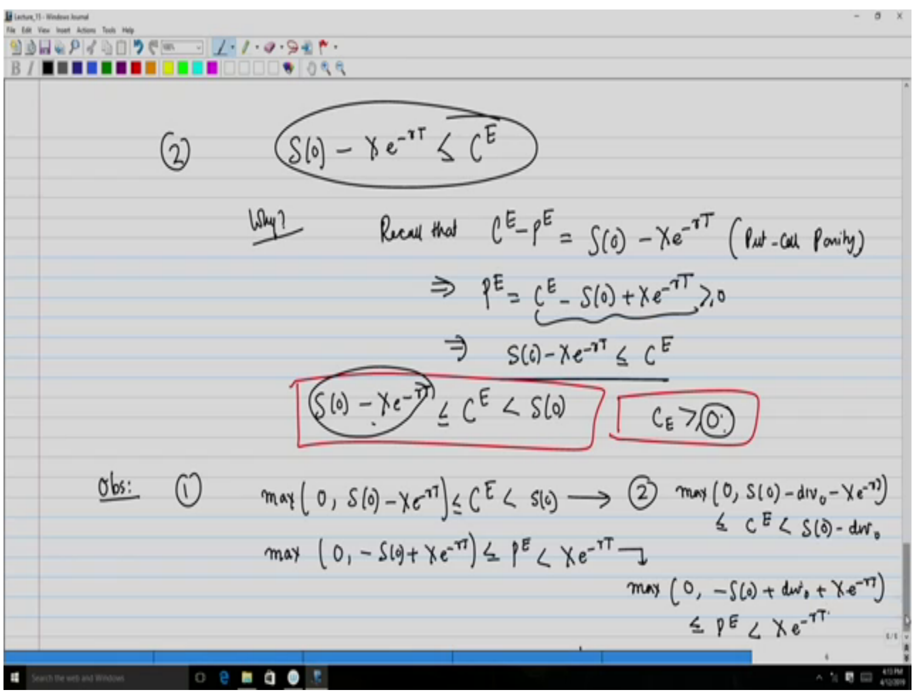

Now, we state the analogous results on the bounds in case of American option in a similar way. So analogously we can show that $\max\{0, S(0) - Xe^{-rT}\} \leq C^A < S(0)$ and $\max\{0, -S(0) + X\} \leq P^A < X$. And similarly, in case of, so here this is the case when no dividends are paid and in case of dividends we have similar results.

That is, $\max\{0, S(0) - div_0 - Xe^{-rT}, S(0) - X\} \leq C^A < S(0)$. And likewise I have $\max\{0, -S(0) + div_0 + Xe^{-rT}, -S(0) + X\} \leq P^A < X$.

So, now what I am going to do is that I am going to state a few results or propositions. And I will actually just do one or two cases and leave the rest as exercises. So some propositions, and these propositions are in

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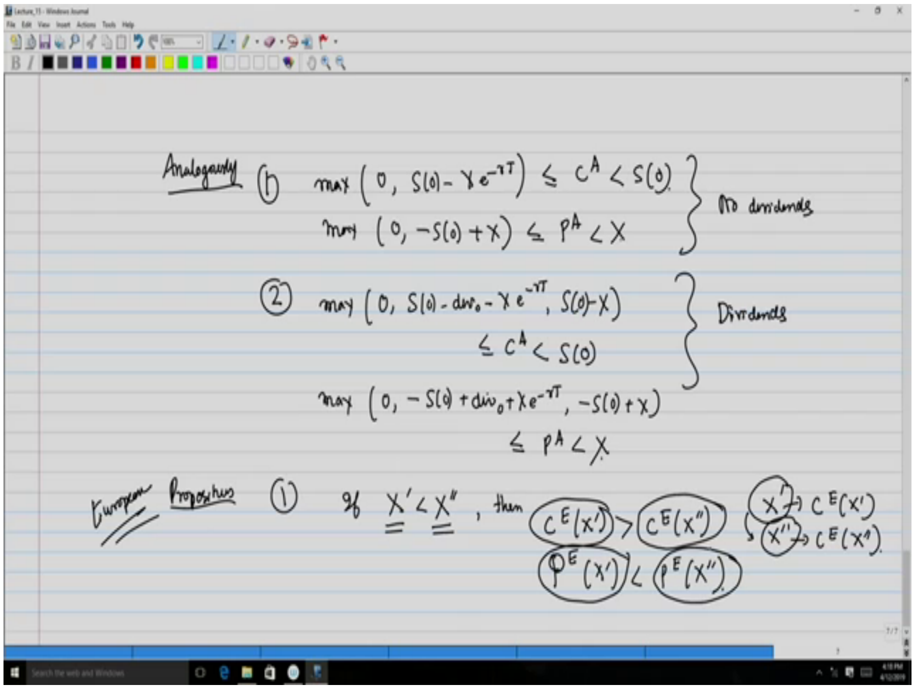

case of European option. So the first proposition we state is the following:

That, if $X' < X''$, that means there are two call options, there are two options whose strike prices are different. Then, $C^E(X') > C^E(X'')$ and $P^E(X') < P^E(X'')$. Now, the results are very sort of obvious, why this is true? See, for example, here you understand that you have X', which is a strike price and which is less than X''.

That means the price of an option with a strike price X', the price for this would be the, for the call option is $C^E(X')$. And for X'', this is going to be $C^E(X'')$. Now, what happens is that when $X' < X''$, it means that it gives the holder of the European call option to purchase the underlying asset for a price X'.

Now since in this case $X' < X''$, this means that in the second case the purchaser of the option if they exercise will have to exercise and buy the underlying asset for a higher price. And which means that, in that case they should expect to actually pay a lower premium which is why the price of the option in case of the higher strike price is actually going to be lower than when the strike price is lower. And similar argument can actually be made for this particular inequality.

Now, we look at another proposition. If $X' < X''$ as before, then

$$C^E(X') - C^E(X'') < e^{-rT}(X'' - X')$$

and

$$P^E(X'') - P^E(X') < e^{-rT}(X'' - X').$$

Thirdly, let us look at another proposition. Again, we select $X' < X''$, and let $\alpha \in (0, 1)$, then

$$C^E[\alpha X' + (1-\alpha)X''] \leq \alpha C^E(X') + (1-\alpha)C^E(X'')$$

and

$$P^E[\alpha X' + (1-\alpha)X''] \leq \alpha P^E(X') + (1-\alpha)P^E(X'').$$

So, this is to say that $C^E(X)$ and $P^E(X)$ are convex functions of X. Next, we state this proposition four. Now here we assume that $S' < S''$, then

$$C^E(S') < C^E(S'')$$

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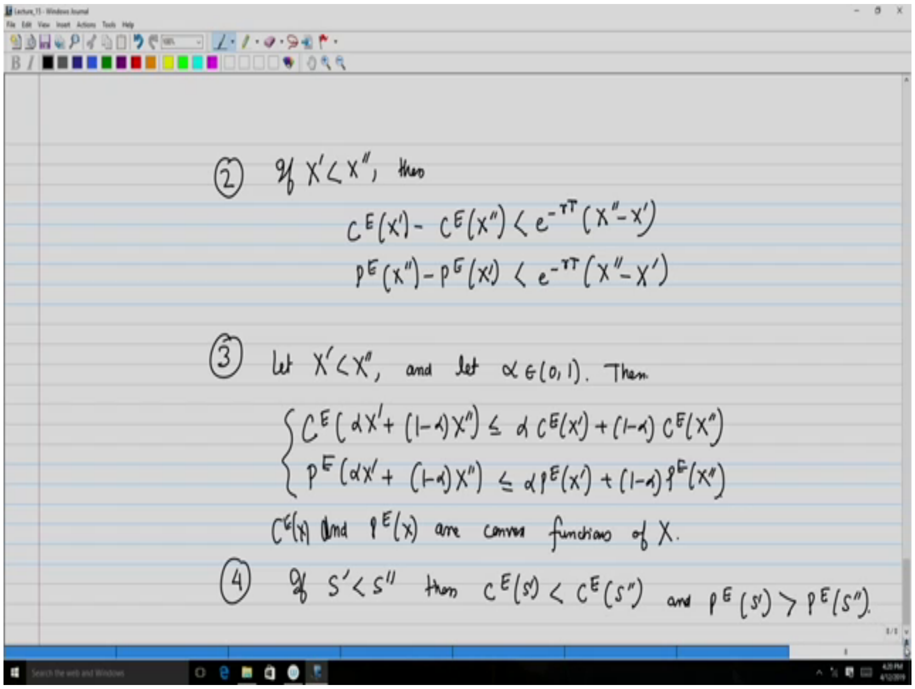

and

$$P^E(S') > P^E(S'').$$

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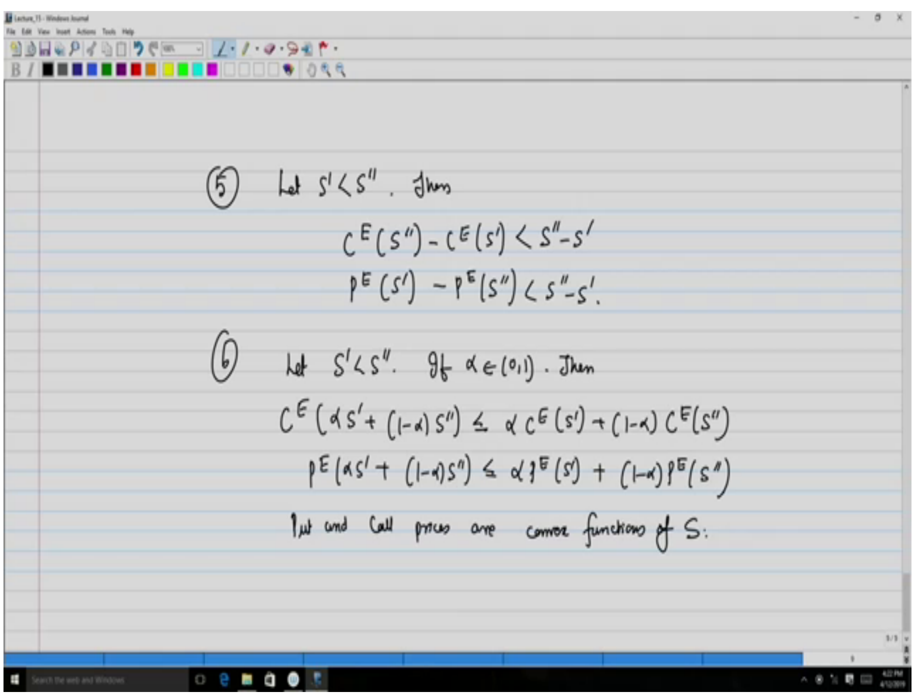

Now, another relation. Again, let $S' < S''$. Then, just as in case with the strike price we obtain a similar relation of

$$C^E(S'') - C^E(S') < S'' - S'$$

and

$$P^E(S') - P^E(S'') < S'' - S'.$$

Finally, one last proposition:

Again, let $S' < S''$. If we have an $\alpha \in (0,1)$, then

$$C^E[\alpha S' + (1-\alpha)S''] \leq \alpha C^E(S') + (1-\alpha)C^E(S'')$$

and

$$P^E[\alpha S' + (1-\alpha)S''] \leq \alpha P^E(S') + (1-\alpha)P^E(S'').$$

So that means the put and call prices, in case of the European option are convex functions of S.

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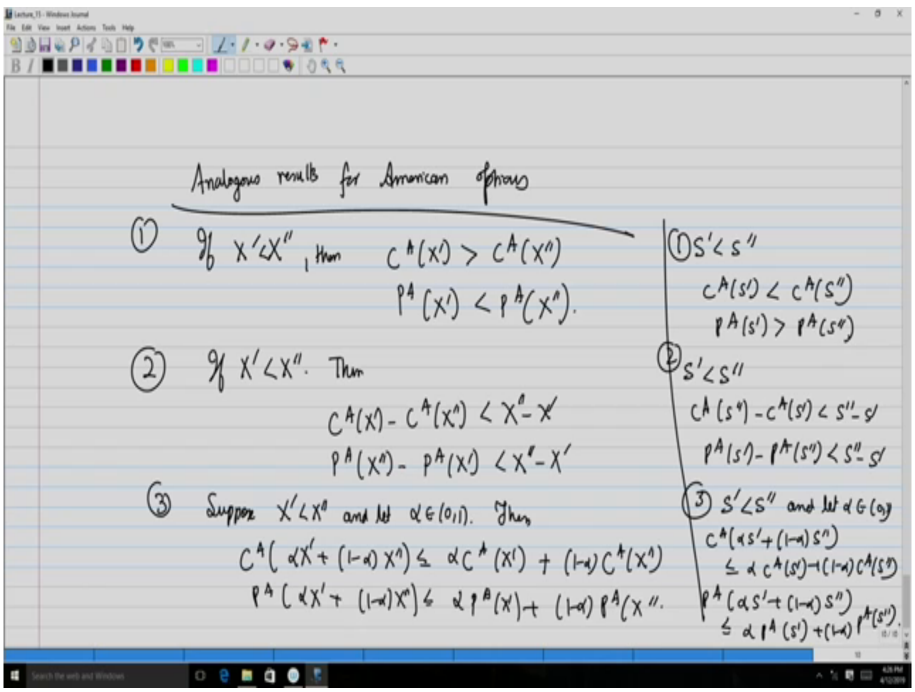

Now, you move next onto the analogous results for American options. One, if $X' < X''$, then we have $C^A(X') > C^A(X'')$. The argument is very similar to that in case of put option. Intuitively you can see that, in case of European option, you can intuitively see that $P^A(X') < P^A(X'')$. The second proposition is, if $X' < X''$, then $C^A(X') - C^A(X'') < X'' - X'$.

And, $P^A(X') - P^A(X'') < X'' - X'$. Now, for the convexity proof, suppose $X' < X''$ and we let $\alpha \in (0,1)$, then

$$C^A[\alpha X' + (1-\alpha)X''] \leq \alpha C^A(X' + (1-\alpha)C^A(X'').$$

And likewise, in case of the American put option, we have an analogous result. Similarly, in case we have $S' < S''$, then $C^A(S') < C^A(S'')$ and $P^A(S') > P^A(S'')$.

Again, when I have $S' < S''$, then

$$C^A(S'') - C^A(S') < S'' - S'$$

and

$$P^A(S') - P^A(S'') < S'' - S'.$$

And finally, for the last case when $S' < S''$ and let $\alpha \in (0,1)$, then we have

$$C^A[\alpha S' + (1-\alpha)S''] \leq \alpha C^A(S' + (1-\alpha)C^A(S'')$$

and

$$P^A[\alpha S' + (1-\alpha)S''] \leq \alpha P^A(S' + (1-\alpha)P^A(S'').$$

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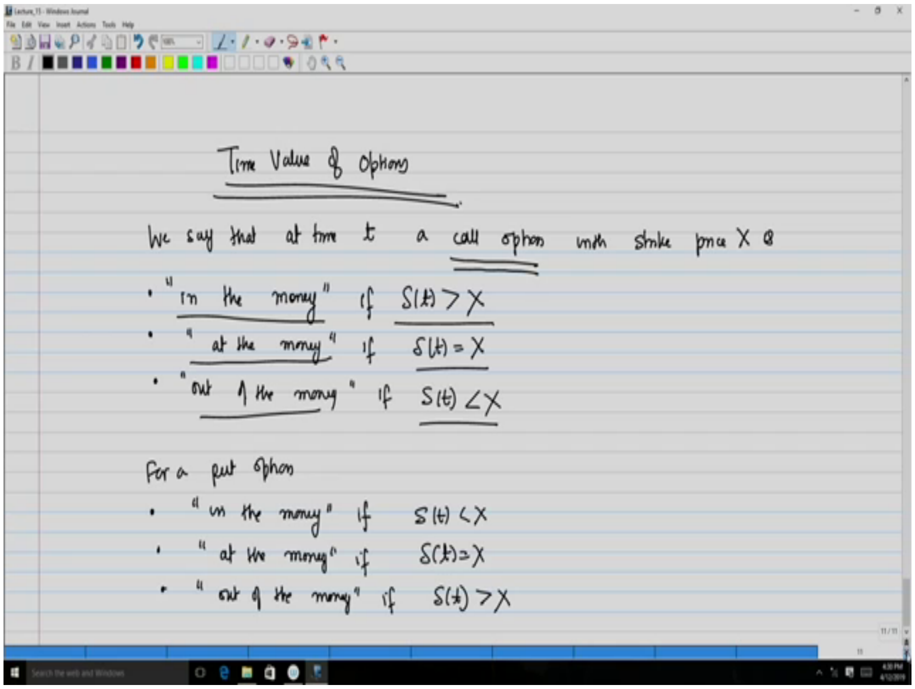

So finally, we come to what is known as 'The time value of options'. So we say that at time t a call option with strike price X is 1 in the money if $S(t) > X$, at the money if $S(t) = X$ and out of the money if $S(t) < X$. So if you observe carefully here, this is the statement that I am making in general for a call option. So both in case of European call and American call option. So whilst it is in the money because $S(t) > X$, here signifies a situation where if the holder of the option exercises at that particular time point, they will make a profit of $S(t) - X$.

Likewise, at the money means that it is essentially some sort of an equilibrium but there is neither loss nor gain if the exercise actually takes place at time t. And out of money means that there is no incentive for the holder of the option to actually exercise the option because your prevailing price of $S(t)$ in the market is less than the price that is actually agreed upon.

In a similar way, for a put option we say that it is in the money and it is in the money when the owner of the put option starts to gain, which is when $S(t) < X$. It is at the money when both of them are identical, $S(t) = S$. And it is out of the money if the holder of the option has nothing to gain, that means it is when $S(t) > X$.

So, this brings us to a definition: At time some $t \leq T$, that means on or before expiration, the intrinsic value of a call option with strike price of $X = (S(t) - X)^+$. That means the call option will be in the money if this holds. And at the intrinsic value, so at time $t \leq T$, the intrinsic value, so it is essentially the payoff at any given point of time t, not necessarily only at the final time.

So, it is the intrinsic value of a put option with strike $X = (X - S(t))^+$. So this means that when you are out of money or when you are at the money, then in both cases the intrinsic value is equal to 0.

So now we talk about the 'Time value of an option'. So time value of an option is the difference between the price of the option and its intrinsic value. So for European call, this is going to be $C^E(t) - (S(t) - X)^+$. For European put, this is going to be $P^E(t) - (X - S(t))^+$. In case of American call, this is going to be $C^A(t) - (S(t) - X)^+$. And in case of American put, this is going to be $P^A(t) - (X - S(t))^+$.

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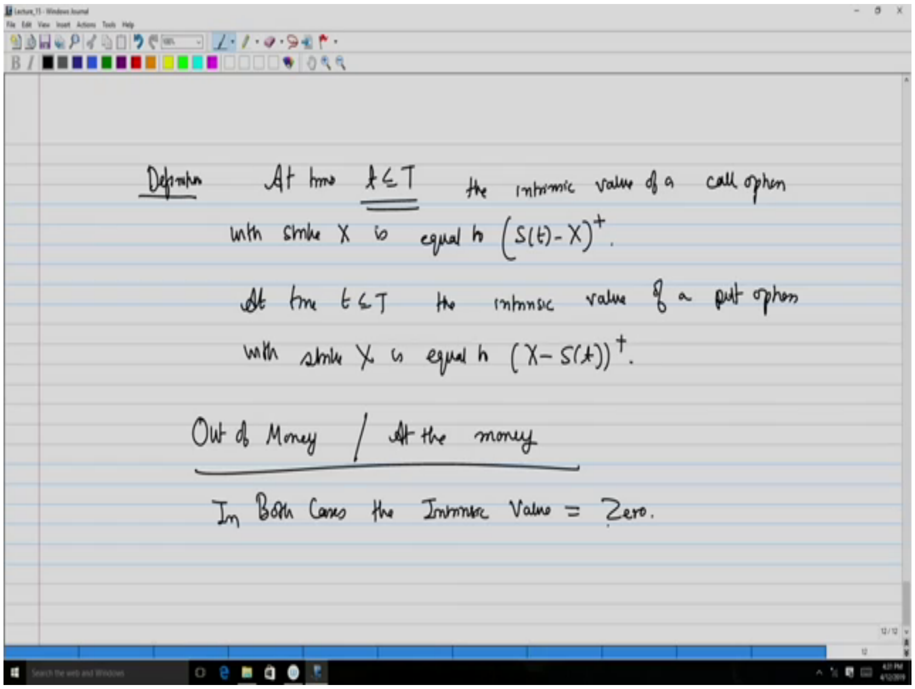

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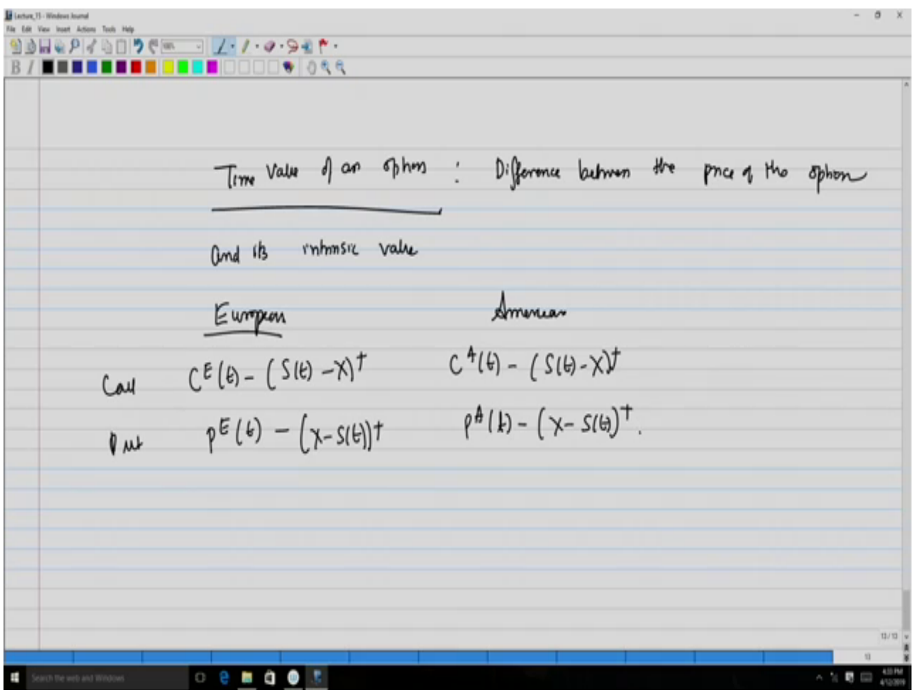

So this concludes our discussion about the various properties of options. Recall that in the previous class we had talked about the put-call-parity which was equality that connected the price of the European call and put option and the strike price and included the current value of the stock and what is going to be the time T for expiration. In today's class we have extended this to inequality which is analogous to the put-call-parity, we looked at bounds on the prices of the European call and American call as well as European put and American put options, namely, the lower and the upper bounds.

And we stated a large class of properties, especially bound properties in case of both call and put options for both European and American type. And finally, we talked about the intrinsic value of the option which leads us to the definition of what is the time value of an option. So this concludes our discussion on properties of derivatives, namely, forwards, futures and options. Thank you for watching.