Mathematical Finance

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Module 4: Fundamental of Derivatives
Lecture 2: Futures, Options and Put-Call-Parity

Hello viewers! Welcome to this class on Mathematical Finance. Today we will continue our lecture on properties of derivatives. Remember that in the last lecture we had talked about forwards and we had derived using the 'no-arbitrage' principle as to what the appropriate forward price should be.

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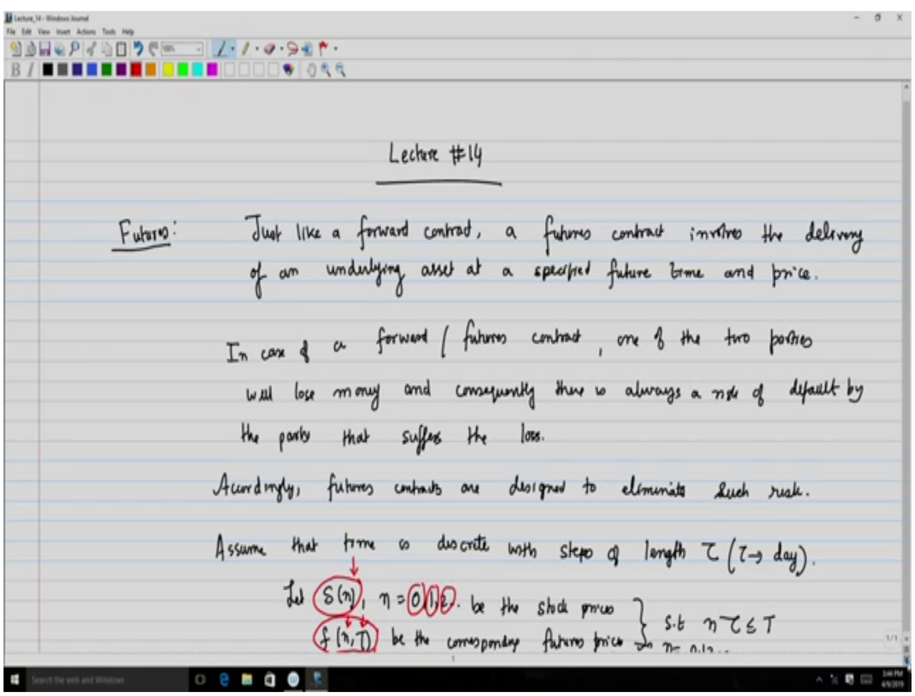

In today's class, we will look at the analogous derivative of forwards, namely, futures and then we will also discuss a little bit about options and one important property of option, namely, the put-call-parity. All right! So, we start off with futures. Now, remember that just like a forward contract, futures contract involves the delivery of an underlying asset at a specified future time and price. That means both the time and the price have to be specified.

But how are they different from a futures contract? So, in case of a forward or futures contract, one of the two parties will lose money and consequently, there is always a risk of default by the party that suffers the loss. And it is for this reason that we accordingly futures contracts are designed to eliminate such risk. So, let me illustrate this little more in a descriptive way.

Suppose that there is a party A and party B, and party A and B both agree on forward slash futures contract, that they are going to buy or sell the underlying asset for a fixed price at a future time.

Now, what might happen is that once the price is agreed upon and it is observed that after some time the price of the asset is slowly going down, so in that case, there is always the possibility that the party which has agreed to buy the underlying asset in the agreement is likely to default because he or she instead of honoring this contract and paying the forward price or the futures price may instead decide to buy it at a lower price in the market if the downward trend of the price continues.

In an analogous manner, suppose that the price of the underlying asset is slowly going up, then the party with the short position in the forward-futures contract, that is, the party which had agreed to sell the underlying asset, they might view this as an opportunity that they could default because if they default then they can sell the underlying asset in the market for a higher price which is what is expected if the trend of the higher prices continue, thereby defaulting.

So, that means that depending on the direction of movement of the prices of the underlying asset one party or the other in the forward or the futures agreement is likely to default. Now this risk of default certainly is more pronounced in case of a forward contract since those are over-the-counter contracts. However, the futures contract which are traded on exchanges have a mechanism to prevent such defaults taking place or more specifically they have a mechanism by which the party, if a party defaults then the other party they are compensated from the losses that they would incur by as a result of the default by one of the other parties.

And this is why forwards and futures if you compare them, the advantages of a forward is that you can set your own contract but the risk is that there might always be a default. On the other hand, futures have much more restricted terms and conditions but at the same time it offers the advantage that the presence of the clearing house or the exchange ensures that in the event of a default the other party is actually compensated for the losses that they might incur.

Okay, now we assume that time is discrete with steps of length tau and typically this tau is going to be a day. Now, let Sn where $n = 0$, 1, 2, so on, be the stock prices and f n T be the corresponding futures prices such that n of tau is less than or equal to capital T. And here of course, $n = 0, 1$, to et cetera. So by this I mean here, the following data, if we have this time $n = 0$, $n = 1$, $n = 2$ and so on, then Sn is going to be the price of the stock and consequently $f(n, T)$ will be used to denote the futures price for a maturity of T, but decided at the time point n when the price of the underlying asset or the stock is $S(n)$.

All right! So now accordingly it is obvious that only $f(0, T)$ is known initially which means that all the other futures prices $f(n, T)$ for $n = 1, 2$, so on, are random variables. That means setting time equal to 0, you know what is going to be the futures price for expiration T, but you cannot predict exactly what the futures price are going to be at all subsequent time points, or what exactly is going to be the evolution of the agreed upon futures price after the initial time point $T = 0$, and which is why $f(n, T)$ is considered to be a random variable.

Now, at time $t = 0$, there is no cash flow but there are cash flows during the reminder of lifetime of the futures contract. Now contrast this. So in contrast a forward contract, remember forward contract was over-the-counter, a forward contract where you contrast this with the forward contract where there is a single cash flow at expiration. So now this is one of the fundamental differences between forward and futures contract, and we have said that there are cash flows during the reminder of lifetime of the futures contract.

So this means, now this cash flows are essentially designed in a way so that they exactly compensate for any losses by a party in case of the default by other party. So accordingly the futures contract involves these random cash flows that are known as, and the term is "marking to market" or sometimes it is denoted as "m-2-m".

So normally, now typically how does this "marking to market" happens? So normally at each $n = 1, 2$, so on, that means on a daily basis the holder of the long position, that means the party which has agreed to purchase the underlying asset, will receive or pay an amount $f(n, T) - f(n-1, T)$, if $f(n, T) - f(n-1, T)$ is positive or negative respectively. And this is since the long position, so accordingly the short position the

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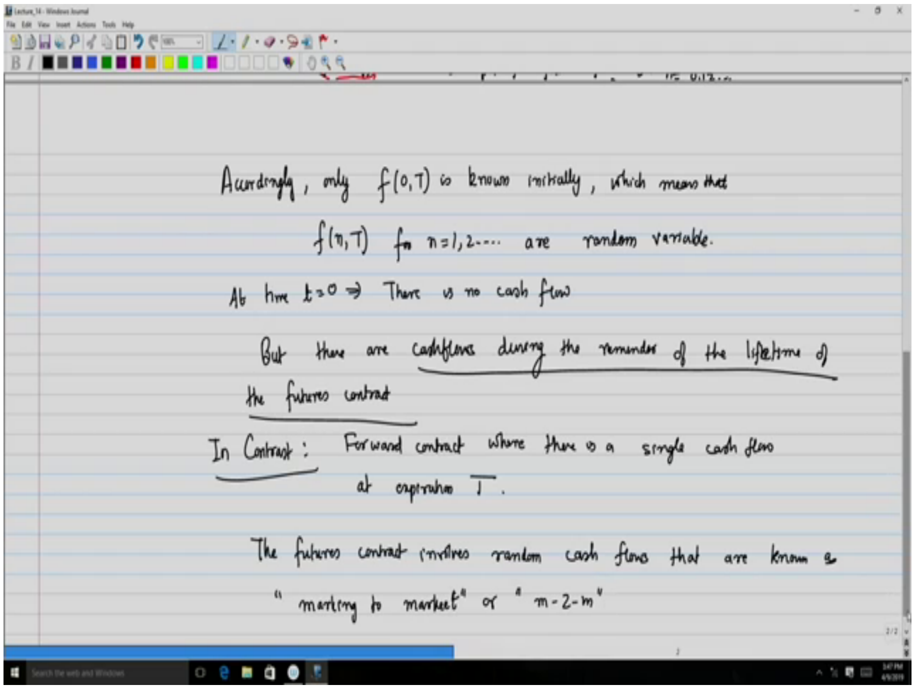

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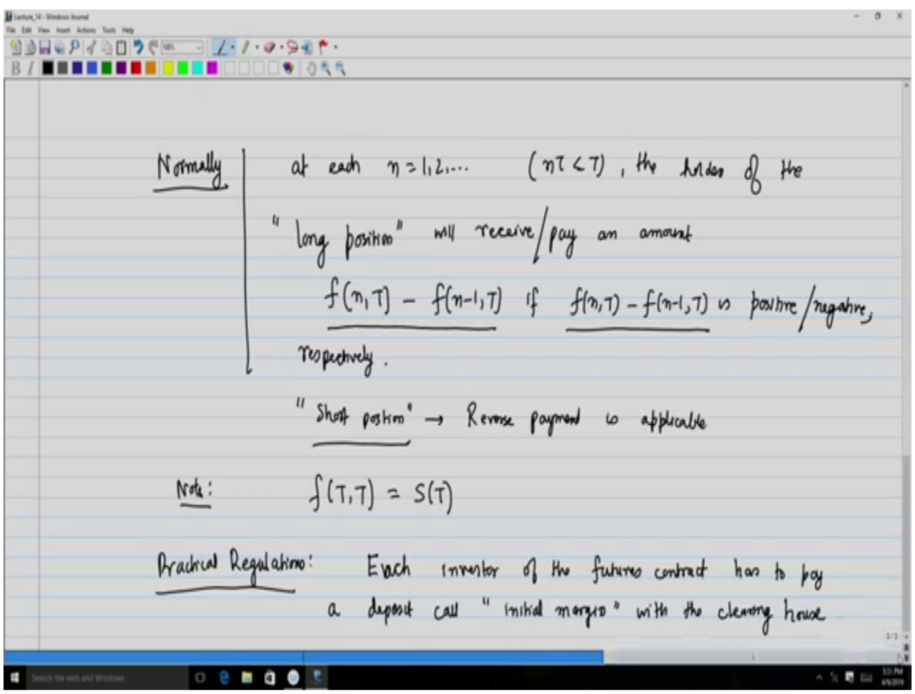

reverse payment is applicable.

So, this means that if this quantity is positive, $f(n,T) - f(n-1,T)$, then the party with long position will receive $f(n,T) - f(n-1,T)$ and they will receive this from the party with the short position and vice-versa. So one little note is that the futures price for immediate execution is just nothing but the price of the underlying asset or the spot price of the underlying asset at that particular time T, because it is, so this is basically in accordance with the 'no-arbitrage' principle that if you are agreeing to undertake the transaction to a futures contract and if it is happening immediately then obviously in order to avoid a price mismatch you need them both to be identical.

Okay, so now let us look at the practical regulations, I mean how do futures actually work in real terms. So, in a from practical point of view when a futures contract is actually executed over in an exchange or through a clearing house then each investor of the futures contract, that means any investor who gets into a futures contract, they have to pay a deposit called 'initial margin' with the exchange or the clearing house.

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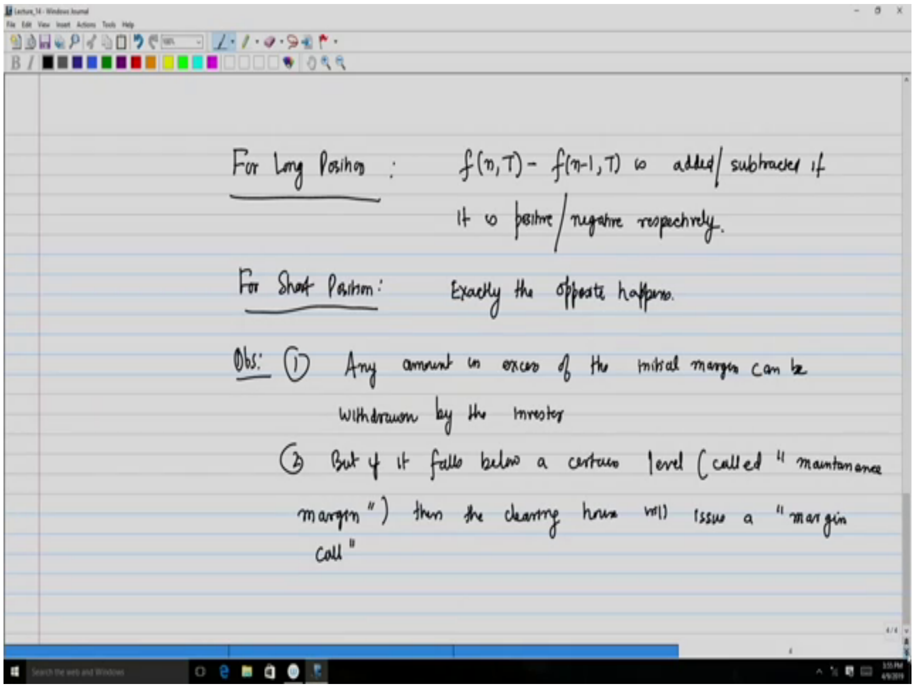

Now for long position, as already noted, an amount of $f(n, T) - f(n-1, T)$ is added or subtracted if it is positive or negative respectively. So for short position exactly the opposite happens. Now, here I want to make a few observations. The first observation is any amount in excess of the initial margin, so the initial margin is like initial security deposit, can be withdrawn by the investor from their margin account.

And the second observation is that, but if it falls below a certain level, and we will call this since this is maintenance account or a margin account, this is what is called the maintenance margin. That means it is the minimum amount of security deposit that has to be put up with the clearing house, then the clearing house will issue a request for more top-up deposit and this process is known as a margin call that means it is a call to actually replenish your margin account.

So, let us illustrate this through a little example as follows: Suppose that the initial margin or the initial deposit in the margin account is set at 10% and maintenance margin is set at 5%, and these percentages are of the futures price prevailing at that time. Now we will look at a table to illustrate certain scenarios of futures prices. So we will have the small column. So this column is n, that means I will take $n = 0$, that means the 0-th day, 1, 2, 3, 4, and so on.

Now suppose the futures prices on those days are $140, 138, 130, 140, 150$. That means these are the futures prices that are actually agreed upon on that particular day. That means on the initial day the agreed upon price to be paid in futures is 140, the next day it is 138, then the day after is 130, then on the third day it is 140 and the last case it is 150. So, let us look at what are the things that you need to take into consideration.

So, we will have the cash flows, margin 1, we will have the payment and margin 2. So margin 1 basically is the deposit at the beginning of the day and margin 2 is basically end of day. And payment you know, this is the amount received or paid. So, let us start off with $n = 0$ when the price of the futures is 140. So the cash flow is that this is the opening and the initial margin is 0. Now, we open the margin account. Remember that you have to put up 10% of the futures price for the margin account. So 10% of 140 is 14.

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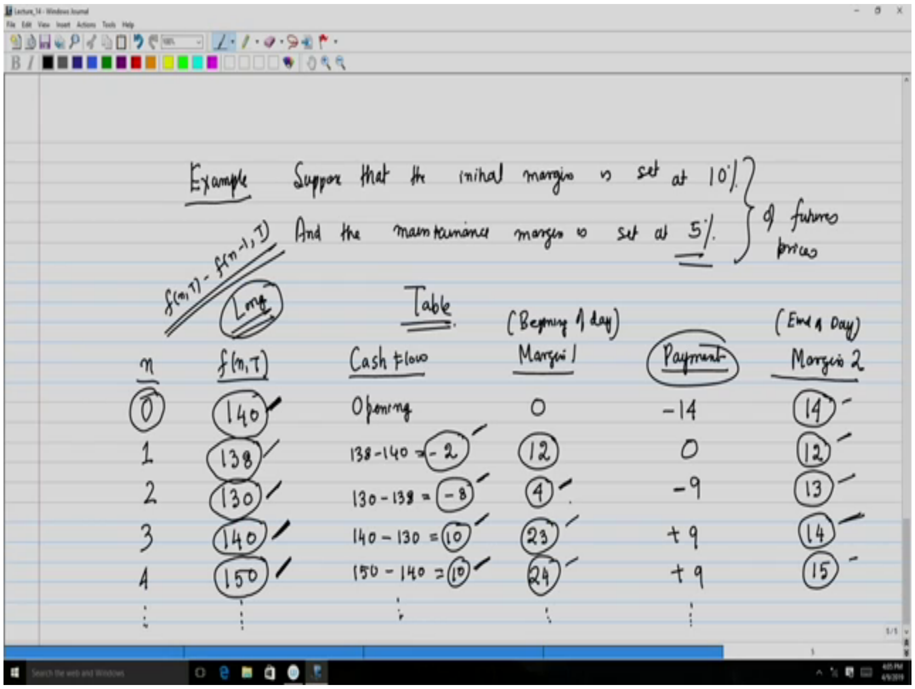

Now, since we make a payment of 14, so this is going to be -14, which you have paid to the clearing house and that means that your balance in the margin account at the end of the day is going to be $+14$. Now, suppose next day the futures price for the same expiration and for the same underlying asset, it falls to 138. Then, what is to be the cash flow? Remember the cash flow was nothing but $f(n,T) - f(n-1,T)$. So in this case this is going to be $138 - 140 = -2$.

Now, the previous day the margin had an amount of 14 and remember that we are talking this from the point of view of the party with the long position. So since the previous the margin was 14 and the cash flow has to be -2, that means this is the amount of money that will go to the party with the short position to their margin account. That means your remaining balance is going to be $14 - 2 = 12$.

Now, since the 10% of $138 = 13.8$, but remember that we had given the condition that the maintenance margin is set at 5 percent, so since an amount of 12 that is there in the margin account, is still more than 5% of 138, so no payment is actually required to be made to the clearing house and so your balance at the end of this day 1 is going to be 12.

Now, suppose that the day after that the price of the futures goes to 130. That means the payment that the party with the long position has to be made to the short position, that means the amount that will be transferred from the margin account of the party with the long position to that of the short position, this is going to be $130 - 138 = -8$. Now if this -8 is taken out of the remaining balance of 12, that means the margin account will then just have an amount of 4. Now this is going to be a problem because this amount of 4 is less than 5% of 130 which is 6.5.

So, that means that in view of this margin being followed or the amount in the margin account falling below 5%, the party will be getting a call from the clearing house asking them to bring the balance up to 10% of the prevailing price. So accordingly this means that 4 is there in the margin account but 10% of 130 is 13, so in order to increase 4 to 13 an additional payment of 9 has to be made, so an amount of -9 goes out of the party with the long position, resulting in the balance in the margin account held the clearing house being equal to 13.

Now, suppose that the day after the futures price goes to 140, so the cash flows, so for the first time the party with the long position will gain an amount of $140 - 130 = 10$. Now the previous day the balance was 13 and they have received an amount of 10 in their margin account from the margin account of the party with the short position, so that means the new balance becomes $13 + 10 = 23$.

Now, the party with the long position is only required to maintain the balance at 10% of the futures price which in this case is 14, but they actually have a higher amount 23, so the excess amount of $23 - 14$, they can take out, so they will receive an amount of $23 - 14 = 9$. And as a result the balance comes down to 14.

Now, finally suppose that the price of the, the value of the futures or the agreed upon price goes to 150, then the cash flow for the long party is going to be $150 - 140 = 10$. Now the previous day's balance was 14 and with the current balance of 10, the balance in the margin account becomes 24. That means you had 14 the previous day and they will receive 10 from the margin account of the other party and so you have a balance of 24, but you are only required to maintain a balance of 10% of 150 which is 15.

So that means out of 24 you can take out an amount of 9 and the remaining balance is going to be equal to 15. And this goes on until the final execution of the contract. So, let us see it from the point of the party with the long position. So they started off, they agreed upon price of 140, the next day the price goes to 138, which means that from the point of view the party which is going to buy the asset, they see that well, it is showing a downward trend. That means that there is a likelihood that they might default and so that is the result they have to pay an amount of -2.

Then when it goes down further to 130, there is now even a greater incentive for the party with the long position to actually default. That means they have to make another additional contribution because they are greater in likelihood of actually defaulting. So accordingly they pay an amount of 8. However, subsequently the futures price goes up to 140. So the trend which was earlier downward is upwards and so this means that the party which will sell the underlying asset, they are likely to default if this upward trend continues and that is why they have to pay this party with the long position an amount of 10.

And again when it further increases, this means the party with the long position is seeing that there is an upward trend and that means if the contract is executed and the trend continues they will be able to actually get it for a lower price of 140 that was agreed at time $T = 0$ despite the upward movement of the price. This means that the party with the short position even has now higher chance defaulting and that is the reason why from their margin account an amount of 10 is taken out and put into the margin account of the party with the long position. And this trend continues until the final execution of the contract. And if there is, at any point both the parties they can actually move out of the particular position if they want with the respective remaining balance in the margin account.

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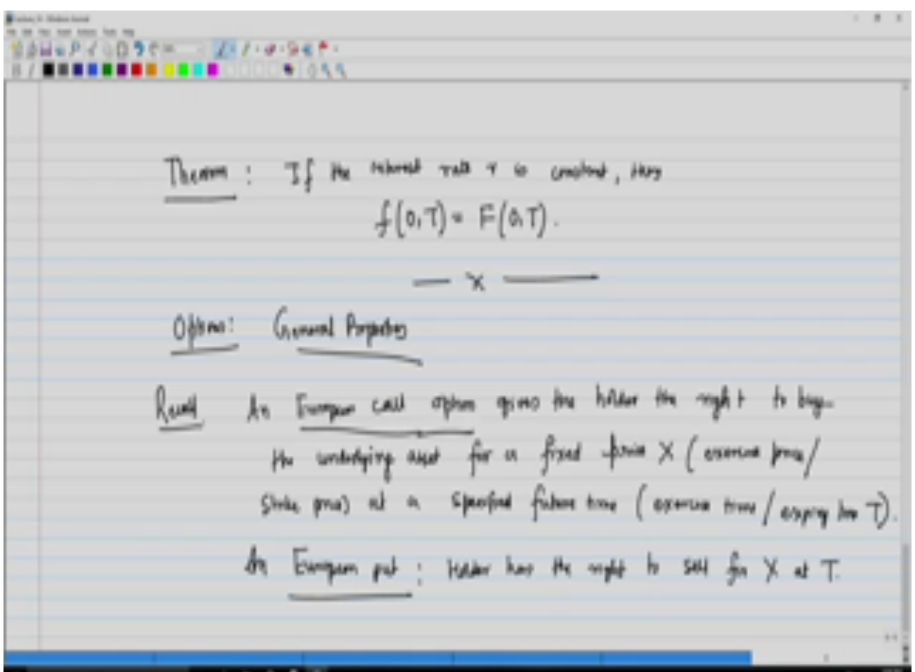

All right! So we conclude our discussion on forwards and futures with just a simple theorem. And the theorem says that if interest rate r is constant then you can show that the futures price $f(0,T)$ is going to be the same as the forward price, $F(0,T)$. So we now come to the second topic, that is some general properties of options. So for this let us just begin by recalling the following:

You would recall that an European call option gives the holder the right to buy the underlying asset for a fixed price X which is known as the exercise price or sometimes the strike price at a specified future time called the exercise time or expiry time T. In analogous way we would recall that we had the definition for European put option. And in case of European put option the holder has the right to sell for the exercise price X at expiration T.

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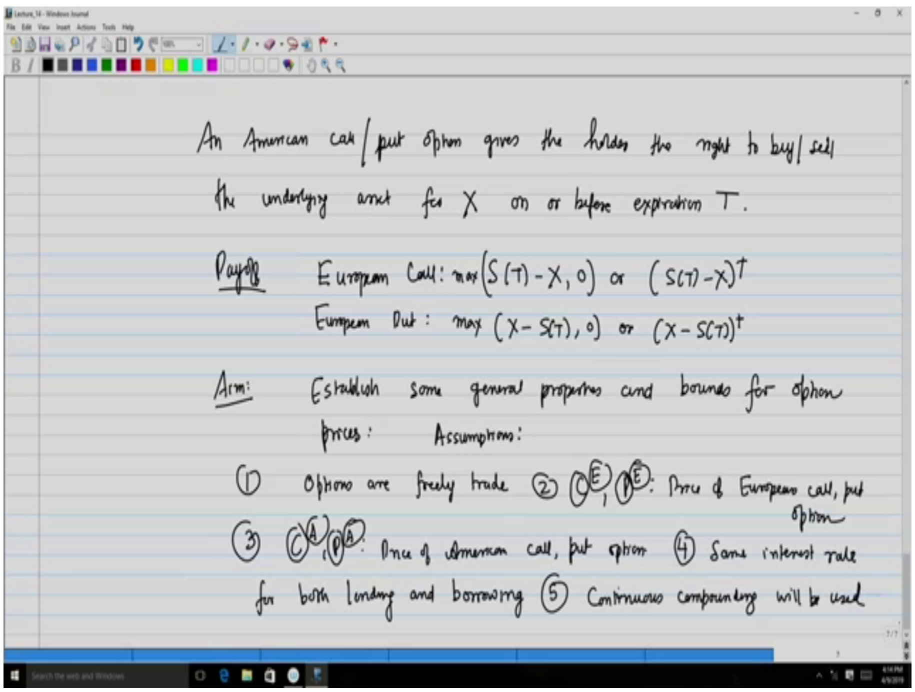

Now, likewise an American call or a put option will give the holder the right to buy or sell the underlying asset for strike X but on or before expiration T. So now let us recall what were the payoffs. So the payoff for a European call was maximum of $S(T-X,0)$ or sometimes this is denoted as $(S(T)-X)^+$. And the payoff in case of a European put option is maximum of $(X-S(T),0)$ or $(X-S(T))^+$.

Now, what we are going to do is we will try to establish some properties, general properties pertaining to options, a lot of which will be dealt with in terms of the price of the option. Just to recap what we had talked about earlier about the price of the option is that unlike forwards and futures contracts options are legally binding on only one party. So this means that it is a situation where only the holder of the option has a position or leverage or advantage, putting the writer of the option at disadvantage. In order to accept this certainly weakened position.

The writer of the option will expect a certain upfront payment at time equal to 0 from the holder of the option or the buyer of the option. And this particular initial payment or compensation is what is known as the price of the option. So accordingly our aim here is to establish some general properties and bounds for option prices, for which there are certain assumptions that are made. So accordingly we will make the following assumptions:

First assumption is that options are freely traded, that means they are completely liquid. Secondly, we will use C^E and P^E as price of European call or put option. Thirdly, we will use the notation C^A, P^A to denote price of American call or put option. So C will be used to denote call options, P will be used to denote put options and E will be used for European option and A for American option. And remember these are the prices, that means the premium that the buyer of the option pays to the holder of the option.

Number four is that we will assume the same interest rate for both lending and borrowing. So typically there is a difference between these two but for the purpose of this discussion we will assume that you can borrow or lend at the same interest rate. And for the purpose of calculating interest we will make use of continuous compounding.

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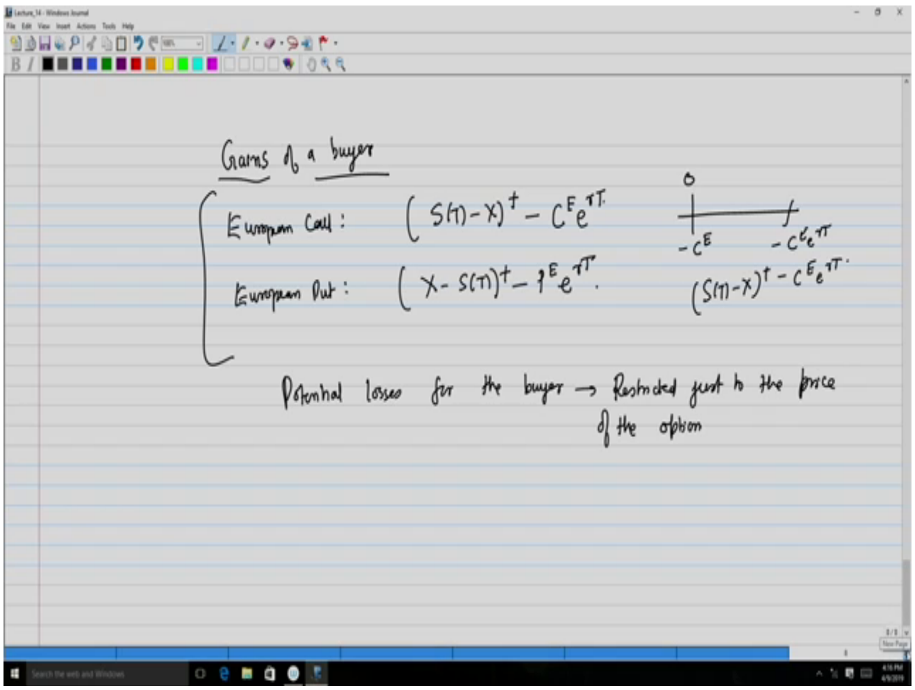

So, once we have these notations set up what we can say is that what is going to be the gain, so I will just make a remark about the gains of a buyer. So in case of a European call the gain will be whatever is the payoff, $(S(T) - X)^+$, minus, the premium they have paid at the initial time $T = 0$, whose future value will be given by $C^E e^{rT}$. So this means that at the initial time $T = 0$, they pay an amount of C^E which had they not paid but have invested in the money market account or a bank account, would have grown to an amount $C^E e^{rT}$.

So this is the amount that actually that goes out of their pocket. But then they stand to gain an amount of whatever is the payoff $(S(T) - X)^+$, minus this quantity $C^E e^{rT}$. Similarly, in case of a European put, as you can guess analogously, this is going to be $(X - S(T))^+$, minus the price of the European put into e^{rT}. So that means that the potential losses for the buyer is restricted just to the price C^E or P^E of the option.

So then, we now prove a very important result which is known as the put-call-parity. And this will be revisited later in the course also. So as the name suggest, put-call-parity is nothing but a relationship between the put and the call option or rather the price of them. And put-call-parity states the following:

For a stock that pays no dividends the following relation holds between the prices of European call and European put options, with the same, that means both have the same exercise or strike price X and exercise time T. And the relation is $C^E - P^E = S(0) - Xe^{-rT}$. So the proof goes as follows: Again, we will make use of 'no-arbitrage' principle.

So, suppose we assume that this equality does not hold, so we explore the two possibilities. There is case I: Suppose that $C^E - P^E > S(0) - Xe^{-rT}$.

Then what strategy should we adopt? Then at $T = 0$, what we will do is that we will buy or share for $S(0)$, we will buy a put option for what price? For P^E. Thirdly, we will sell a call option for price C^E and what is the remaining balance? So the remaining balance is going to be, see you have bought a share for $S(0)$, that means you have spent an amount of $S(0)$, so you have $-S(0)$.

You have bought a put option for P, so you have spent an amount of P^E, so you have $-P^E$. And you have sold a call option for C^E, so this is going to be $+C^E$. And this amount, we will invest this at rate r or

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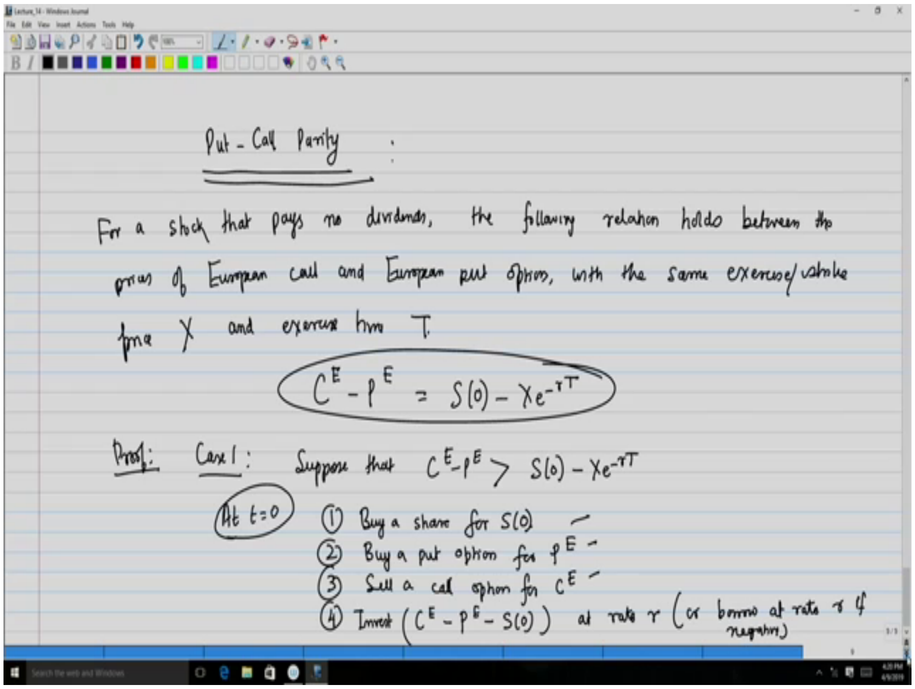

borrow at rate r if this quantity is negative. See, that is the reason why we had to assume that the lending and borrowing, or investment and borrowing both actually can take place at the rate r.

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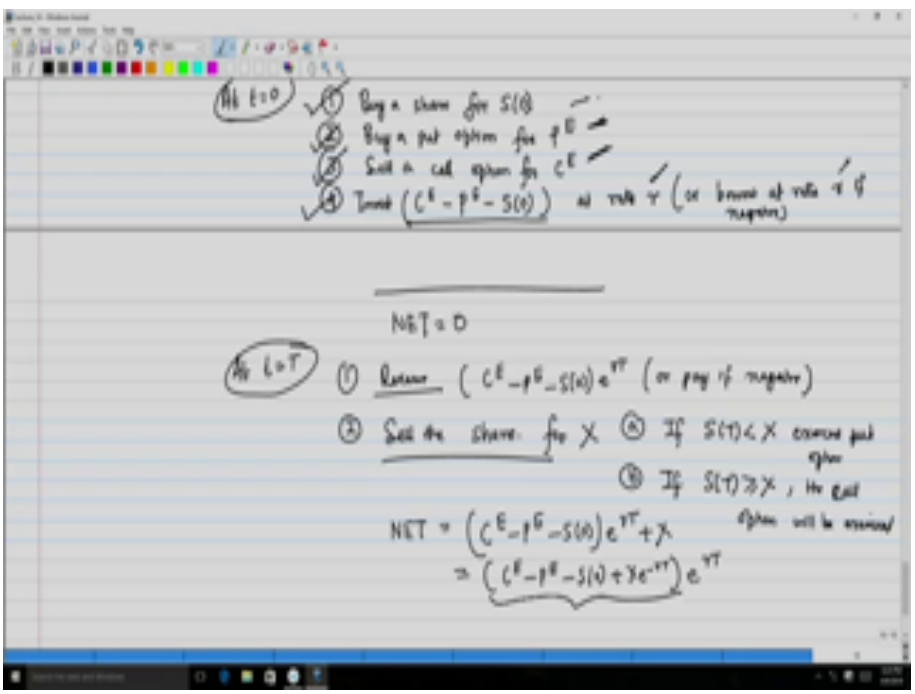

So what is your net here? So accordingly your net, whatever money was there you have invested the balance or borrowed, so the net is going to be equal to 0. Now let us see what happens to this position at time $t = T$. At $t = T$, let us close the positions one by one. First of all, we had made this investment at the rate r, so that means we will receive $[C^E - P^E - S(0)]e^{rT}$. Or, pay if the original amount was negative. So this position is closed. Now remember we had bought a share for $S(0)$, so we will sell the share.

Now for what price do we sell the share? We sell the share for X. Now, how are you going to guarantee that we are going to receive this amount of X? So we are going to sell the share for X in two ways: Firstly, either in case if $S(T) < X$, then what is going to happen? Remember that you had bought a put option for P, that means you have the right to sell, which means that if the prevailing stock price $S(T) < X$ at time T, then you can exercise your right under the put option and receive an amount of X.

So that means if $S(T) < X$, then you exercise put option. However if $S(T) \geq X$, this means that the party with the call option, remember you have sold the call option, that means you are under obligation to sell. So if $S(T) \geq X$, that means definitely the other party is going to exercise the option and that means you have to sell it, rather you are obliged to sell it for an amount of X.

So this means that if $S(T) \geq X$, rather the call option will be exercised. So in either case you will at least get the amount of X. So this means that this option has been called closed and this position has been closed. So, this means the following: So once we have done all these transactions, what will you get? So the net amount is going to be equal to what? You have sold the share for X and you have received this amount, so the net amount is that you have received $[C^E - P^E - S0]e^{rT}$, plus, you have received an amount of X by selling the share.

So, then, this can be rewritten as $C^E - P^E - S(0) + Xe^{-rT}$, whole thing, into e^{rT}. So we can factor out the e^{rT} and remember what is this expression.

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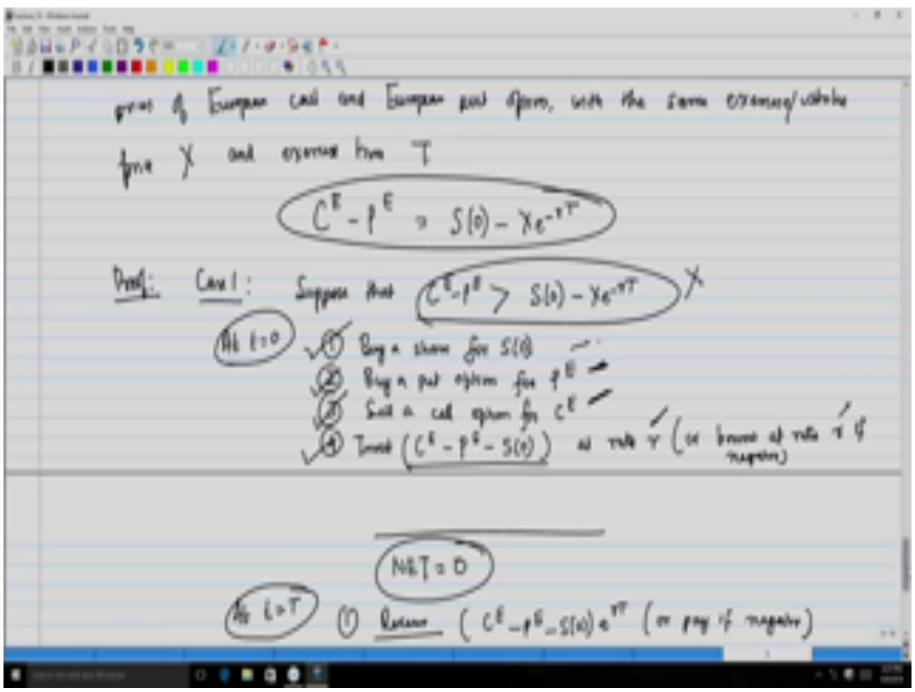
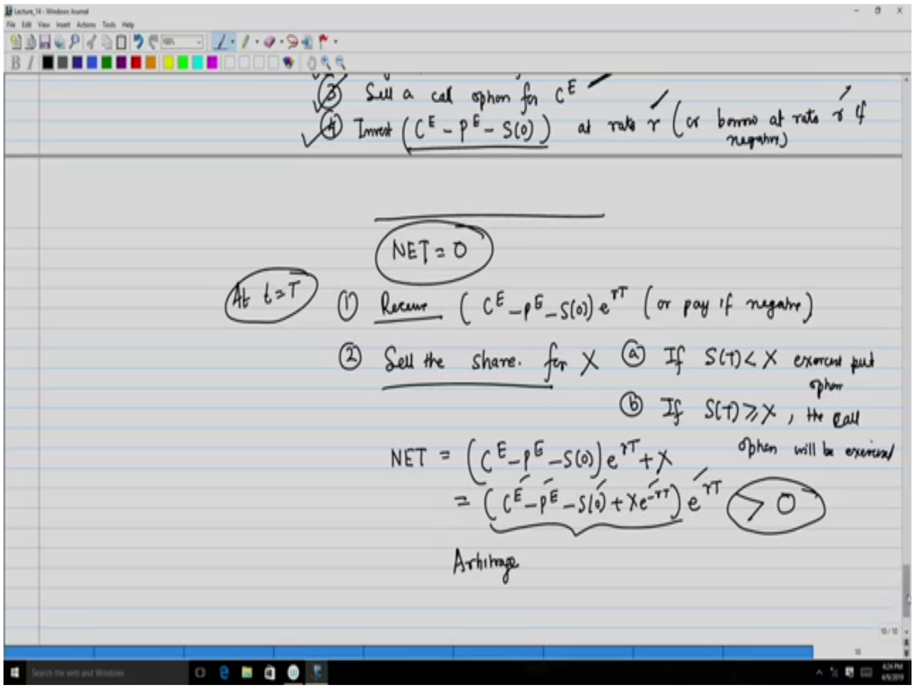


This expression is exactly this minus this: $C^E - P^E - S(0) + Xe^{rT}$. This is the positive quantity. This obviously is a positive quantity, so this is going to be greater than 0. So this means that an initial investment of 0 results in a net positive profit. And so this means that there is an arbitrage opportunity, which shows that this possibility has to be ruled out.

Now, let us look at the second case: Now we suppose that $C^E - P^E < S(0) - Xe^{-rT}$. Then what we are going to do at time $T = 0$? At $t = 0$, we will do the following: We will short sell one share and receive an amount of the current spot price which is $S(0)$. Write a put option for P, that means sell a put option. Then buy a call option for C^E.

So when you short sell one share, you have received $S(0)$. When you sell the put option, you have received P^E. And when you buy the call option, you have spent an amount of C^E. So then, with this balance you invest this entire amount at rate r. Or in case this term is negative, this expression is negative, then you need to borrow. Now, let us see what happens at time $t = T$. At $t = T$, you will basically receive



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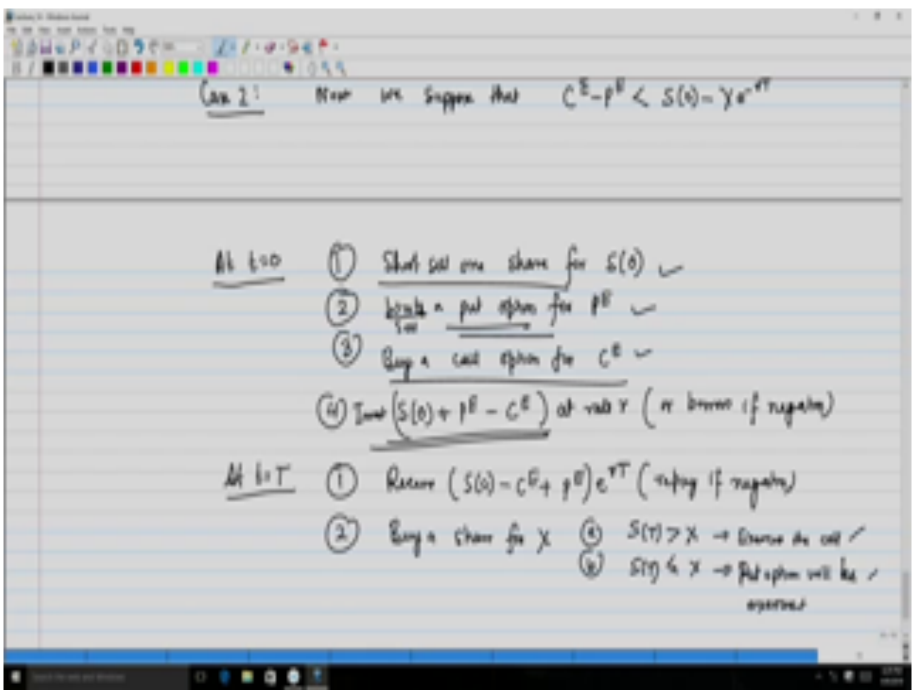

your investment amount along with the interest. So you will receive $[S(0) - C^E + P^E]e^{rT}$, or basically repay if it was negative.

Now, you buy a share for X. Why do you need to buy a share for X? Because remember you have short sold one share. So you will need to procure a share from the market in order to return this particular share to the owner of the share. And so you have to buy a share. Now you can always buy a share for an amount $f(X)$. And how can you do that? You can do that in the following way:

That, if your $S(T) > X$, then you are the owner of a call option. So that means you can exercise the call. And if $S(T) \leq X$, then you know that since you have written a put option, so that means the party which had bought the put option for you will obviously going to exercise that option because they can sell it to you for a higher price of X as compared to the prevailing market spot price of $S(T)$. So, in this case the other party, they will, the put option will be exercised.

So, irrespective of whether $S(T)$ is greater than, equal to or less than X, you can actually sell the share for an amount of X only in one case. It will happen because you are exercising a call option and in the other case the party which is the owner of the put option that you are sold, that party will exercise the put option.

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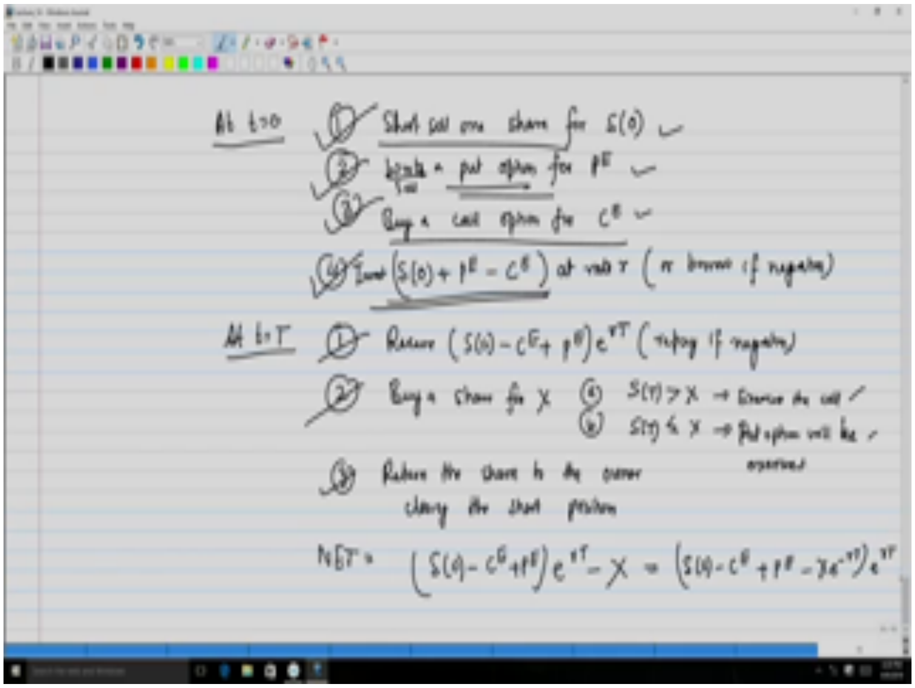

So, that means they have bought, you have bought this, so that means that this position is closed, this is closed, this is closed. And once you have the share, you then return the share to the owner closing the short position. So, then what is going to be your net? Your net is, you have received an amount of $[S(0) - C^E + P^E]e^{rT}$. And you have bought a share for X, that means you have spent an amount of X. And how can we write this? This can be written as $[S(0) - C^E + P^E - Xe^{-rT}]e^{rT}$.

And remember from here you will get $S(0) - C^E + P^E - Xe^{-rT} > 0$.

So that means that this quantity is going to be positive, this is anyway positive. So the entire thing is positive. So that means the net investment of 0 at time equal to 0 results in a positive profit, which means that there is an arbitrage.

So the other possibility here also does not hold.

So once we have excluded both the inequalities, this means that we have the only other possibility is that they are at equal. That is, $C^E - P^E = S(0) - Xe^{-rT}$. Okay, so we conclude todays class with couple of observations or rather extension of the put-call-parity. So the first observation is if div 0, please be careful with the notation div 0, is the present value of a dividend paid by the stock between times 0 and T, then $C^E - P^E = S(0) - div\ 0 - Xe^{-rT}$.

So please be careful, if a dividend div is paid, then div 0 is going to be the present value of the dividend at time $T = 0$. Then for this particular underlying asset the put-call-parity gets modified to this relation to include the dividend. Likewise, the second observation is that if dividends are paid continuously at a rate r div, then $C^E - P^E = S(0)e^{-rdiv\ T} - Xe^{-rT}$.

This brings us to the conclusion of the put-call-parity. In the next class what we will do is that we will look at what is put-call-parity estimate. And this will hold in case of American options.

And this will be given in terms of a relation like $S(0) - Xe^{-rT} \geq C^A - P^A \geq S(0) - X$. Since the American options do not have a pre-specified time point at which they can be exercised, by which I mean that even though the T is specified, the expiration T is specified but the actual exercise of the option can

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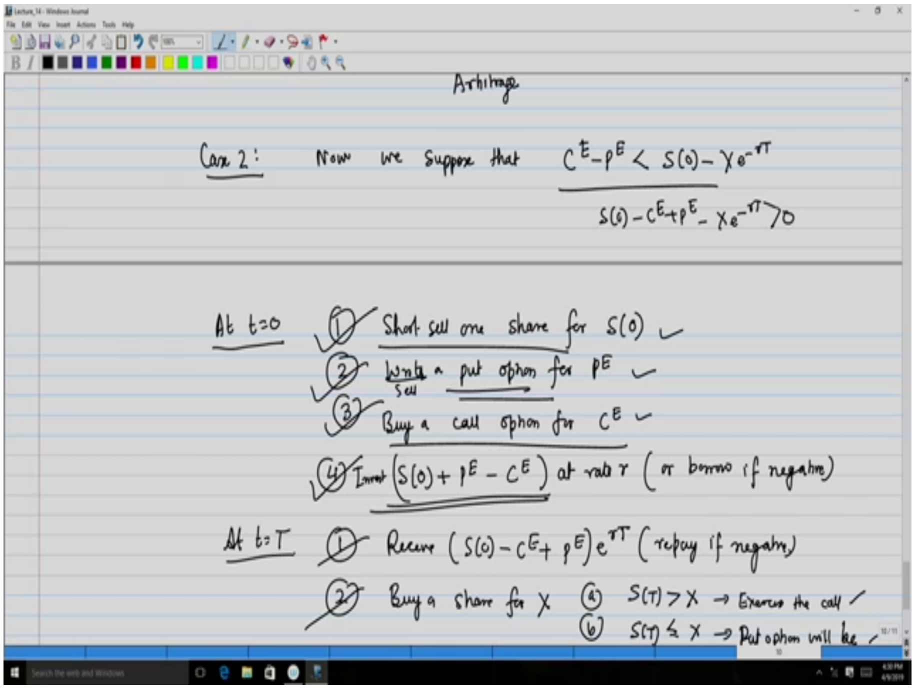

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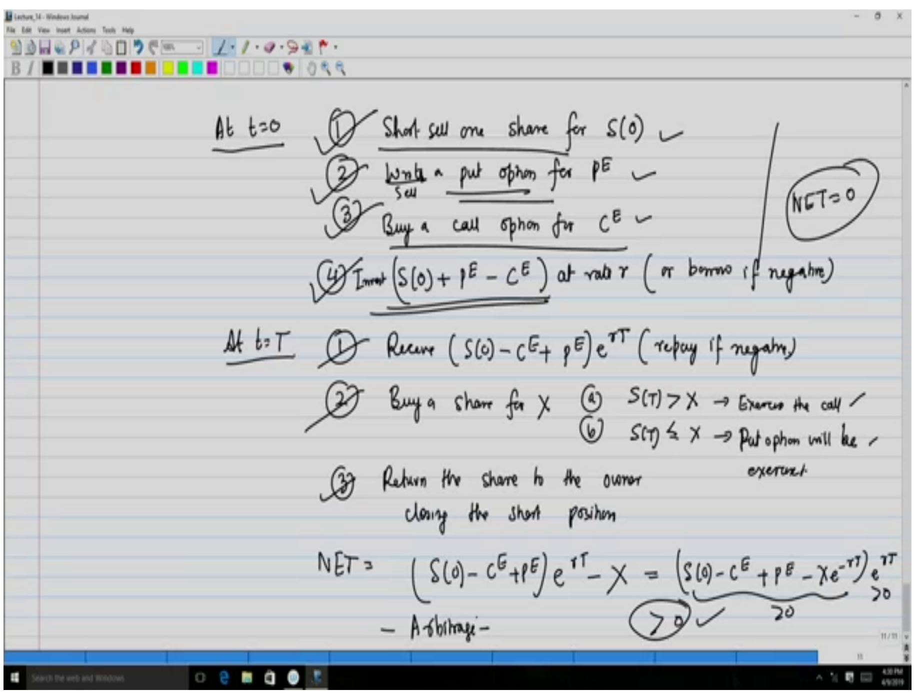

be done by the owner of the option at any time on or before T. That means that the chances of equality holding like the put-call-parity does not exist and instead we will basically have estimates or bounds for the put-call-parity.

So, just to summarize what we did today, we have essentially looked at two things. One is that we extended the forwards contracts in case of a futures contract through an illustrative example of what is known as the 'marking to market' and how a margin accounts actually function in a real-life setting. And we also made the observation that under a specific circumstance how the futures price that is agreed upon is equal to the price of a forward contract.

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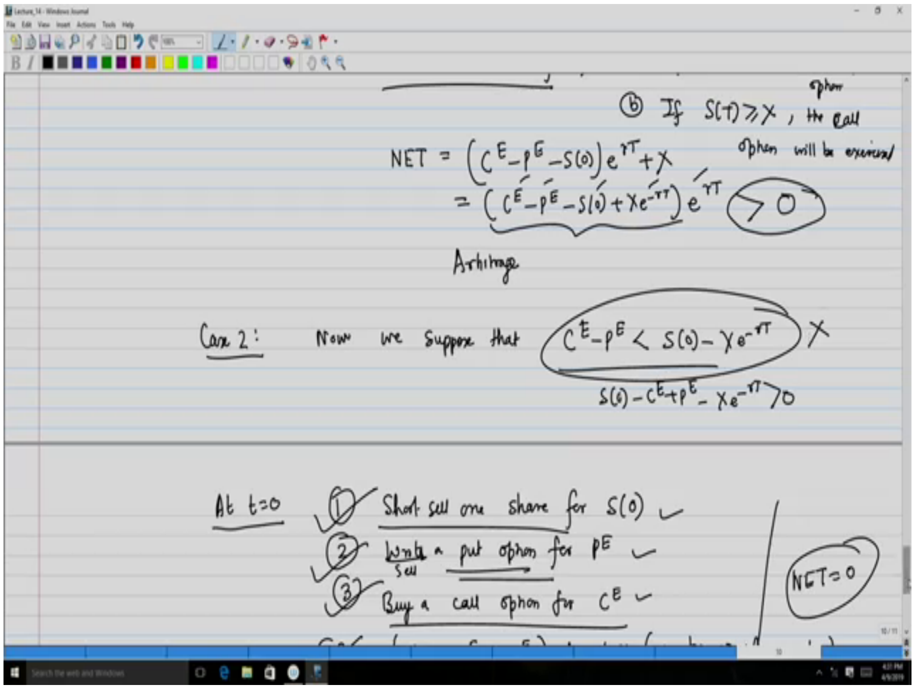

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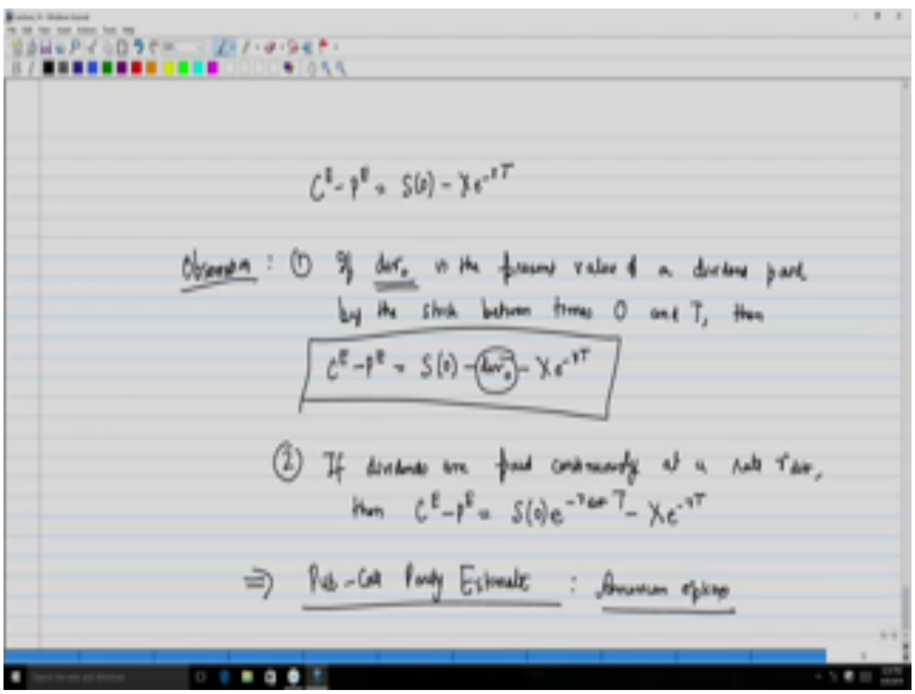

We also talked a little bit about the price of the option and then how it can be used to actually specify the gains from the point of view of the buyer of the option, wherein the gain is going to be the payoff that can be expected minus whatever premium or price of the option that is paid by the holder of the option. And finally, we looked at an important result connecting the price of the European call and the European put option, namely, the put-call-parity. Thank you for watching.

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