Mathematical Finance

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Module 3: Modern Portfolio Theory
Lecture 6: Performance Analysis

Hello viewers, welcome to this course on Mathematical Finance. Todays lecture is going to be the twelfth lecture and it is going to be the last lecture of module 3, where we will look at certain implications of the capital market line and the capital asset pricing model. With a particular emphasis on carrying out performance analysis particularly in case of assets and this will show us the practical applicability of CML and CAPM.

And also we will look at the formulation resulting from the offshoot of the CAPM to justify the usage of the nomenclature that this is going to be an asset pricing model.

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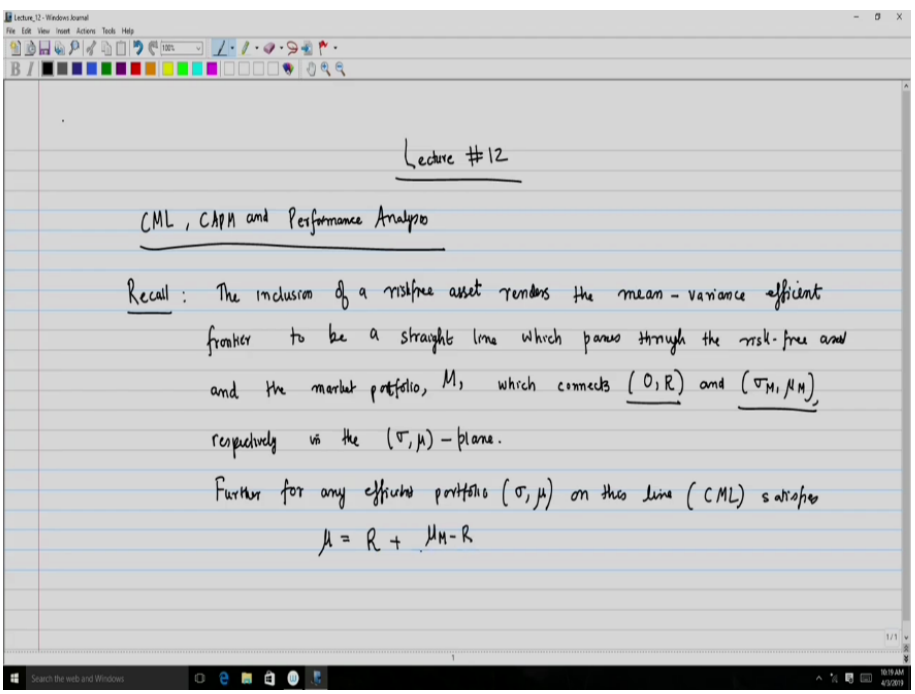


So, accordingly we start this lecture, so as I said this is going to be on CML, CAPM and performance analysis. So we begin with a recap of what you have done in the last couple of lectures. So recall that the inclusion of a risk-free asset renders the mean variance efficient Frontier that we had seen when you consider only a portfolio of risky assets. So this efficient Frontier that you had seen earlier and then had included the risk-free asset is leads to a straight line which will pass through two portfolios namely the one which is exclusively of the risk-free assets and the other which was the market portfolio M.

And this connected the point $(0, R)$ where $R$ was the risk-free rate and where $\sigma_M$ and $\mu_M$ were the risk and return of the market portfolio respectively in the sigma mu plane. Now further we had seen that for any efficient portfolio sigma mu on this line that is the line connecting $(0, R)$ and $(\sigma_M, \mu_M)$ which you called as the CML satisfies $\mu = \frac{R+\mu_M-R}{\sigma_M}\sigma$.

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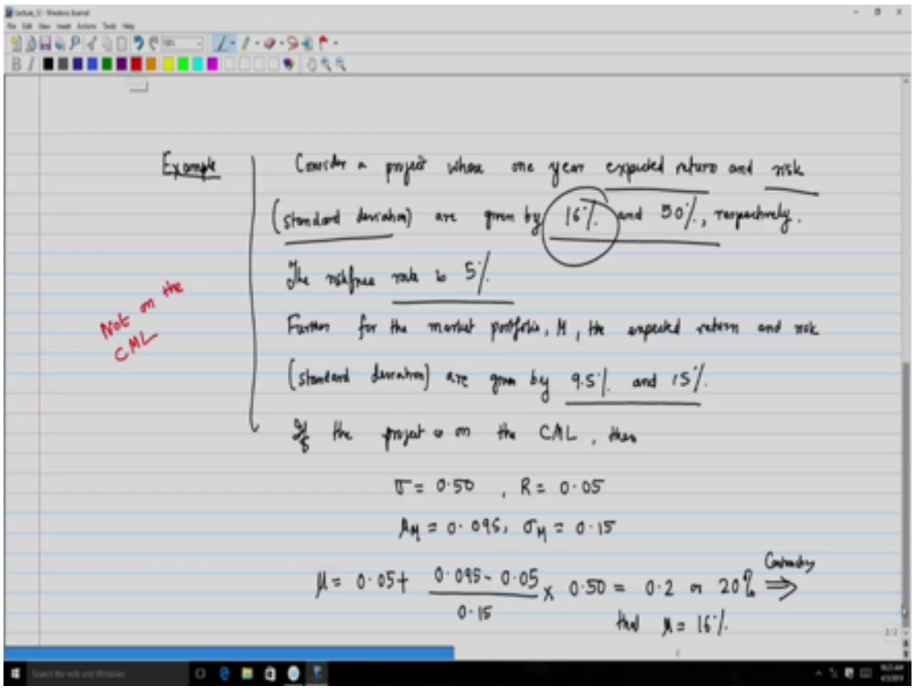


So to illustrate this, we consider a little example as follows. We consider a project, this is the project or one-year horizon, so accordingly whose one-year expected return and risk and by risk I mean the standard deviation of returns are given by $16\%$ and $50\%$ respectively. The risk-free rate is $5\%$ further for the market portfolio $M$ the expected return that means its characteristic on the $(\mu, \sigma)$ plane. And risk again given by the standard deviation are given by $9.5\%$ and $15\%$.

If the project is on the CML then what will you get? So here, let us identify what the parameters are? So, we look at the expected return and risk for the project to be $16\%$ and $50\%$. So here sigma is going to be equal to $0.50$. The risk-free rate is 5 percent, so that means R is going to be $0.05$ and the expected return and risk in case of the market portfolio that is given by, so that means $\mu_M$, which is $9.5\%$ will be $0.095$ and $\sigma_M$ which is $15\% = 0.15$.

Now, if this was on the CML then I would have $\mu = 0.05 + \frac{0.095-0.05}{0.15} \times 0.50 = 0.2 = 20\%$, which essentially contradicts that $\mu = 16\%$. So, hence this particular project that we have this is not on the capital market line.

Okay, so this is just slide background recap of what we had done in case of the CML.

Now, let us go back to our CAPM framework, so recall that in CAPM we stated the following, that is that, in equilibrium the expected return $\mu_i$ for any asset $i$ will satisfy $\mu_i = R_i$ which is the risk-free rate into $\beta_i$ that is $\beta$ of the asset multiplied by $\mu_M - R$ where you would recall that $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$. So here I just want to make an observation that this particular term $\mu_i - R$ this is basically the expected excess rate of return and by expected excess I basically mean that this is an excess over the risk-free rate which is $R$.

So, let us now use this to get some interpretation of the CAPM. So interpretation of the CAPM is that the expected rate of return of an asset, where the expected excess rate of return that means this term often asset is equal to the product of its beta with the expected excess rate of return of the market portfolio. So to put it explicitly I will make use of this relation and what it says is that, this $\mu_i - R$ which is the expected excess rate of return of an asset this is going to be equal to the product of the $\beta$ into the expected excess rate

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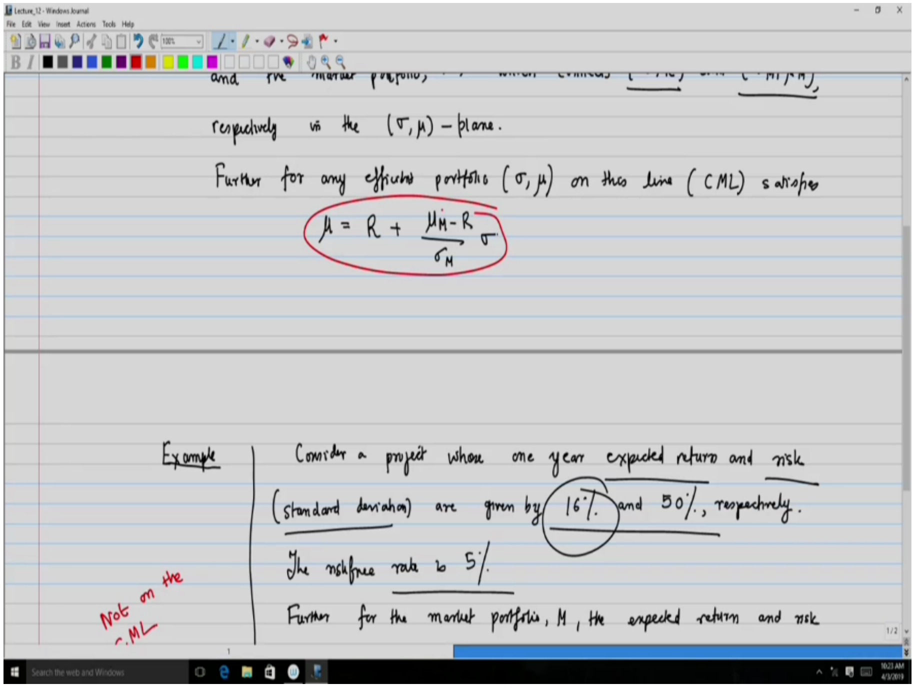


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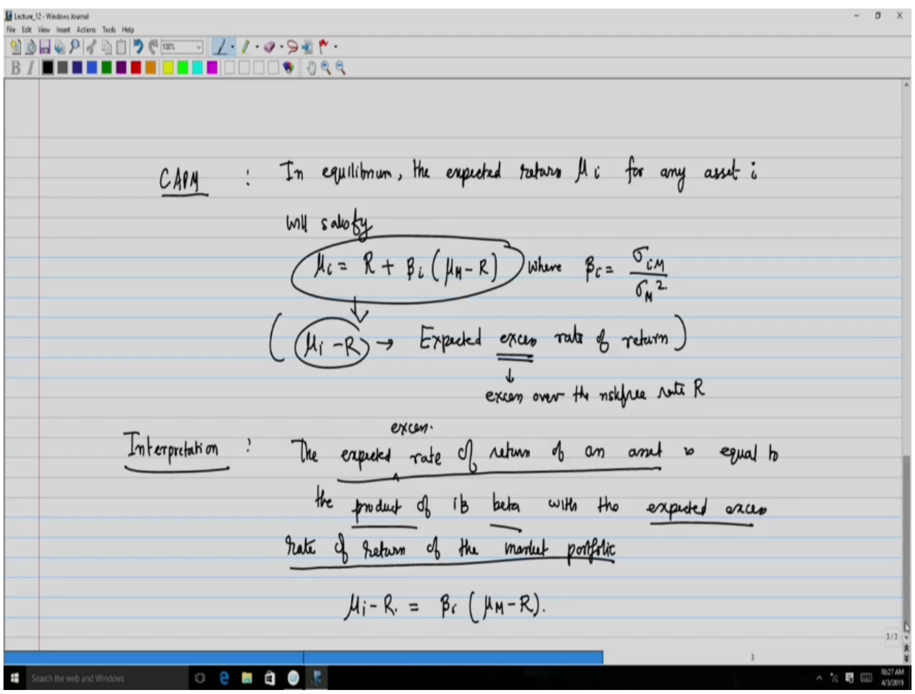


of return of the market portfolio which is $\mu_M - R$.

Now, we recall that for a single period model, say we consider the two-time period $0$ and $1$ and suppose the stock price at time $t = 0$ is $S(0)$ which is known and stock price at time $t = 1$ is $S(1)$ which is a random quantity then and suppose we input the subscript $i$ for the $i$-th asset then the return for the $i$-th asset is simply going to be $S_i(1) - S_i(0)$ that is the net profit or loss that you make as a percentage of the original price. And then this is the return of the asset $i$.

Now, we defined two quantities motivated by the excess return of the asset and the excess return of the market portfolio over the risk-free rate and we define a new variable $R_i$ which I will define as $R_i - R$ which

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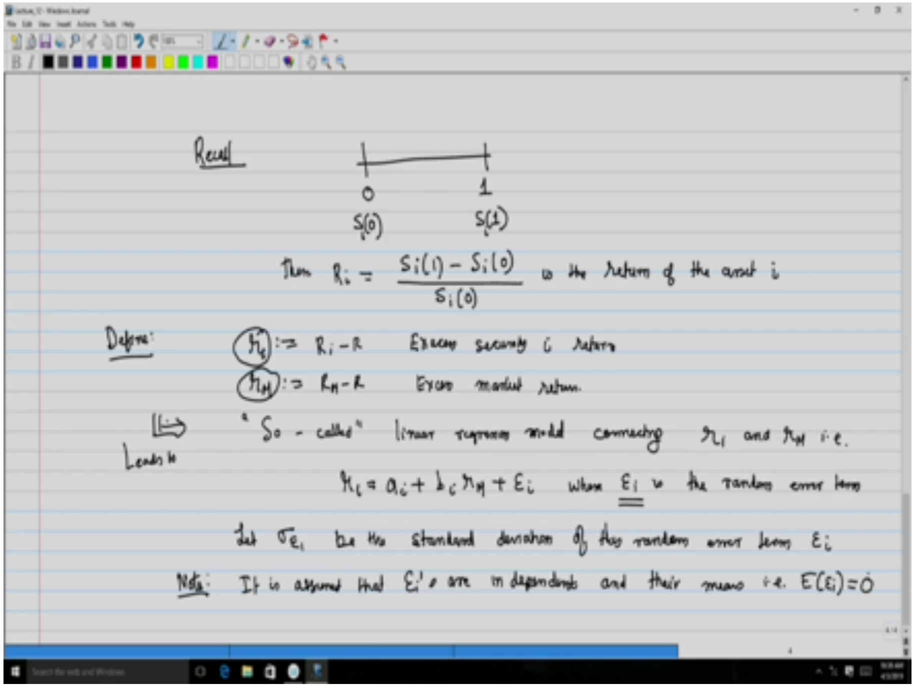


is the excess security $i$ return and small $r_m$ to be $R_M - R$ which is the excess market return.

Now, this leads us to the so-called quote unquote linear regression model connecting $r_i$ and $r_M$ as follows. We will write this as $r_i$ is equal to some $a_i$, so I am basically trying to find a linear relation that will connect my $r_i$ with $r_M$, so this will be $r_i = a_i + b_i r_M + \epsilon_i$, where $a_i$ and $b_i$ had to be determine and where $\epsilon_i$ is basically the random error term.

So, this is obviously a random quantity, so accordingly we define $\sigma_{\epsilon_i}$ to be the standard deviation of this random error term $\epsilon_i$. Now, note that it is assumed that $\epsilon_i$ are independent and their means that is expected value of $\epsilon_i$ these are going to be equal to $0$.

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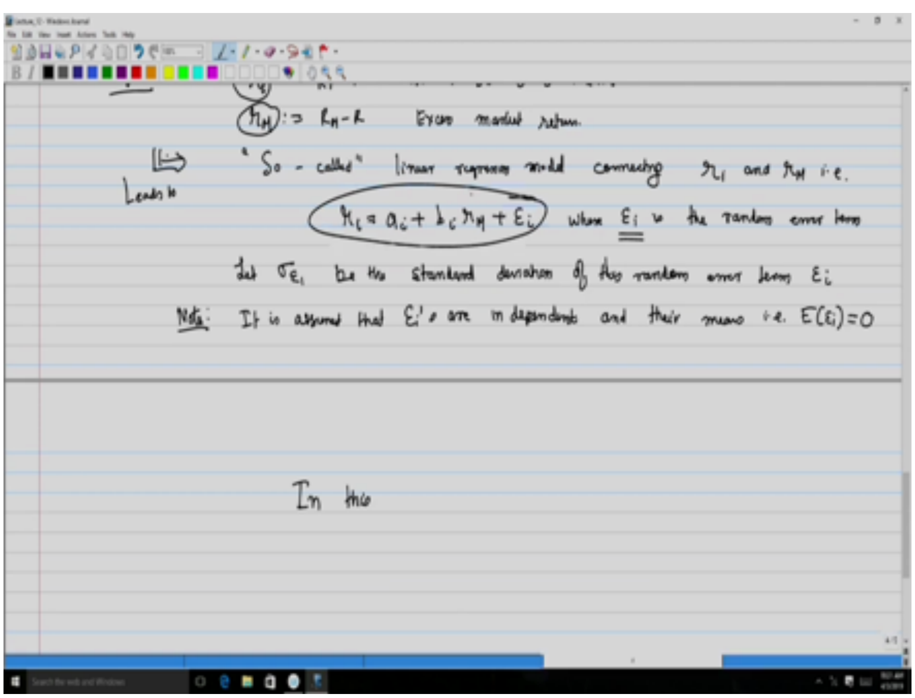

So in this case do can make use of this particular relation that we have done here. Take expectation on both the sides.

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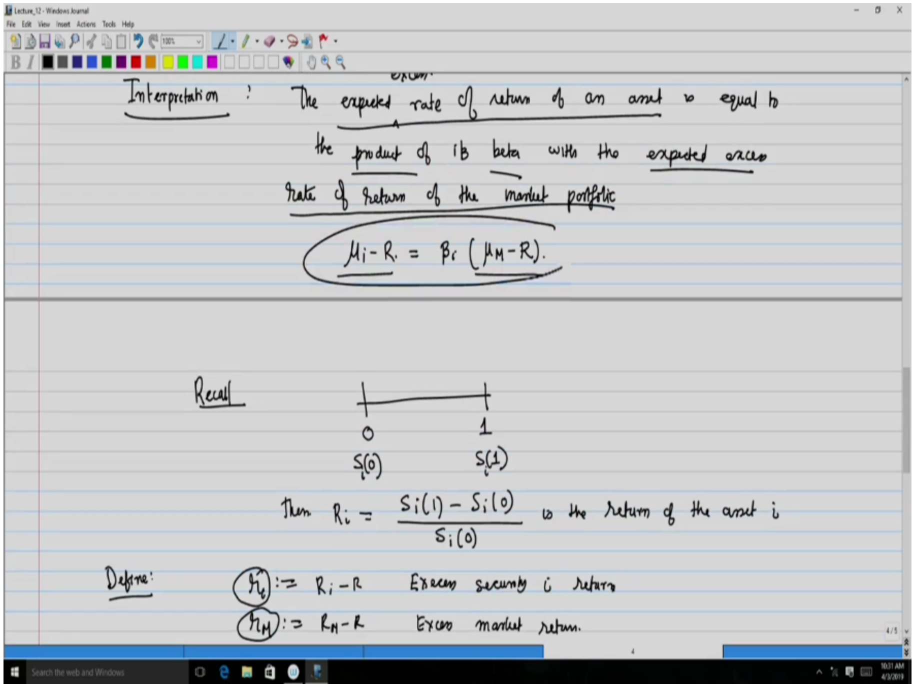


And make a comparison with the CAPM and what you will be able to achieve is the following.

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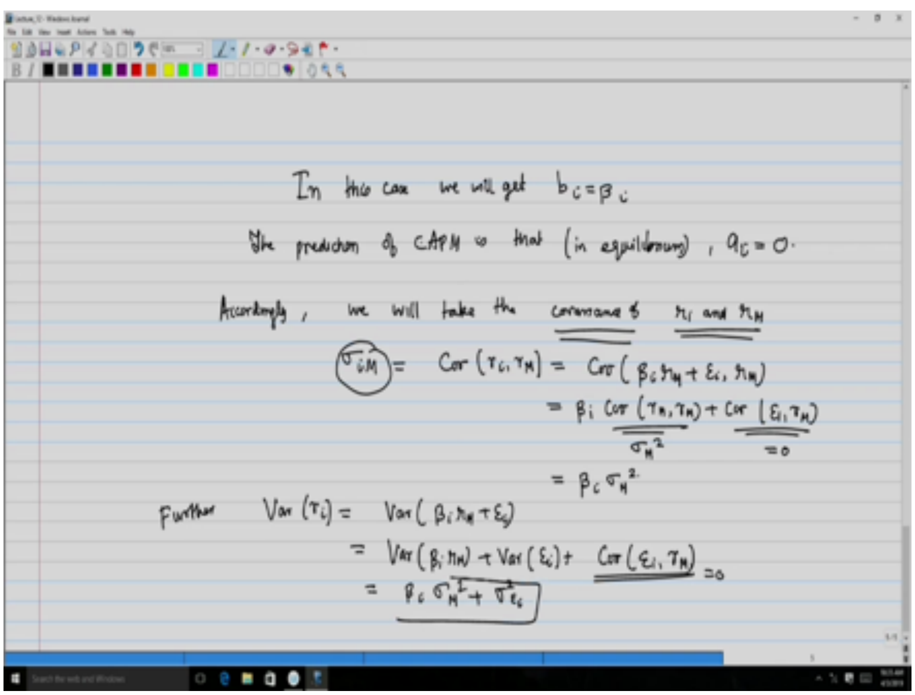


That in this case we will get that $b_i = \beta_i$ and the prediction of CAPM is that in equilibrium $a_i = 0$. So, what I will do is, since $r_i$ is suggestive of the returns, it gives basically the excess return for each individual asset in terms of the excess return we have for the market portfolio. So we need to look at what is going to be the expected risk in this case, the expected return. And more importantly we need to see basically the volatility that is expected by $r_i$ and also see how it actually fairs with the market excess return of $r_M$.

And so accordingly what we will do is that, we will take the covariance of $r_i$ and $r_M$ and this is necessary because you want to figure out what is going to be the beta. So accordingly $\sigma_{iM} = Cov(r_i, r_M)$, this is simply going to be covariance, so this is $Cov(r_i, r_M)$, this can be written as what is $r_i$, so covariance of $r_i$.

Remember here we had got $a_i = 0$ and $b_i = \beta_i$, so accordingly this is going to be, $r_i$ will be replaced by $\beta_i r_M + \epsilon_i$, this covariance with $r_M$. So, if you use the properties of covariance this is going to be $\beta_i Cov(r_m, r_m) + Cov(\epsilon_i, r_M)$ and what is this going to be? By assumption this term, 2nd term is going to be equal to 0 and 1st term is simply going to be $\sigma_M^2$, the variance of the excess market return and this is simply going to be $\beta_i \sigma_M^2$.

Okay, accordingly, so further in addition to looking at $Cov(r_i, r_M)$. We further get variance of $r_i = Var(\beta_i, r_M) + \epsilon_i$ and this is going to be using the properties of variance, this is going to be $Var(\beta_i r_M) + Var(\epsilon_i) + Cov(\epsilon_i, r_M)$.

So, this is going to be $\beta_i Var(r_M)$, which is $\sigma_M^2 Var(\epsilon_i)$. We have already defined this to be $\sigma^2 \epsilon_i$ and $Cov(\epsilon_i, r_M)$. We have already seen this to be equal to 0, so what you end up getting is, basically a decomposition of variance of $r_i$ into two terms.

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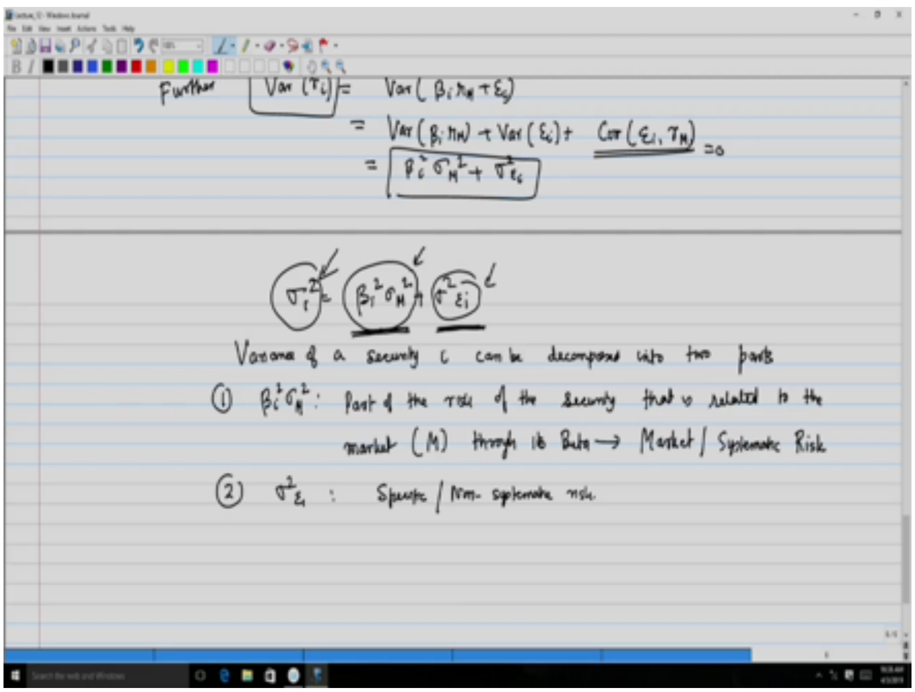

So, let us examine this in a little more detail that I have got $\sigma_i^2 = \beta_i^2$, this is going to be $\beta_i^2 \sigma_M^2 + \sigma^2 \epsilon_i$, so this can be said that, this can be interpreted as variance of a security i that means this term can be decomposed into two parts, right? Namely this part $\beta_i^2 \sigma_M^2$ and $\sigma^2 \epsilon_i$.

So, the first part that is $\beta_i^2 \sigma_M^2$, this is the part of the risk that means the part of $\sigma_i^2$ risk of the security that is related to the market $M$ through its $\beta$ and sometimes this is what is known as or called as Market or systematic that means something that is related to the system as a whole, some kind of a global factor, so this is what is known as the market or the systematic risk. So this is the market or the systematic risk.

And secondly we have this term $\sigma^2 \epsilon_i$ and this is what is known as the specific or non-systematic risk. So, this means that the overall risk of the asset is basically a composition of the risk that is arising from its interaction with the market as seen through this term containing $\beta_1$, in addition to something that is endemic or idiosyncratic are just locally specific to this individual asset.

And this is you would recall what you call this as, this is what is known as the diversifiable risk, a part of the risk and this is what is known as the non-diversifiable part of the risk. Okay now let us dig a little deeper into this and illustrate this whatever we have discussed in terms of CAPM as well as the systematic and the nonsystematic risk through an example.

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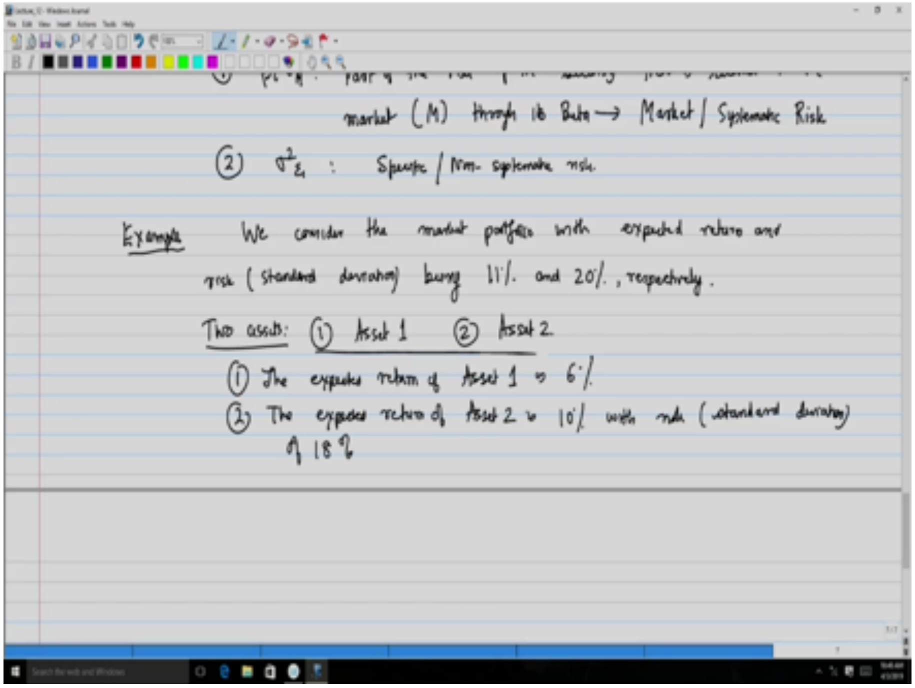


So, the example is the following, we consider the market portfolio with the two parameters that means with the expected return and risk that means your standard deviation being, so you take the expected return to be (say) $11\%$ and the risk to be $20\%$, respectively.

Now, we consider two assets call them asset $1$ and asset $2$, so for asset $1$, the expected return of asset $1$ is $6\%$ and the expected return of asset $2$ is $10\%$ with risk as given by standard deviation of $18\%$.

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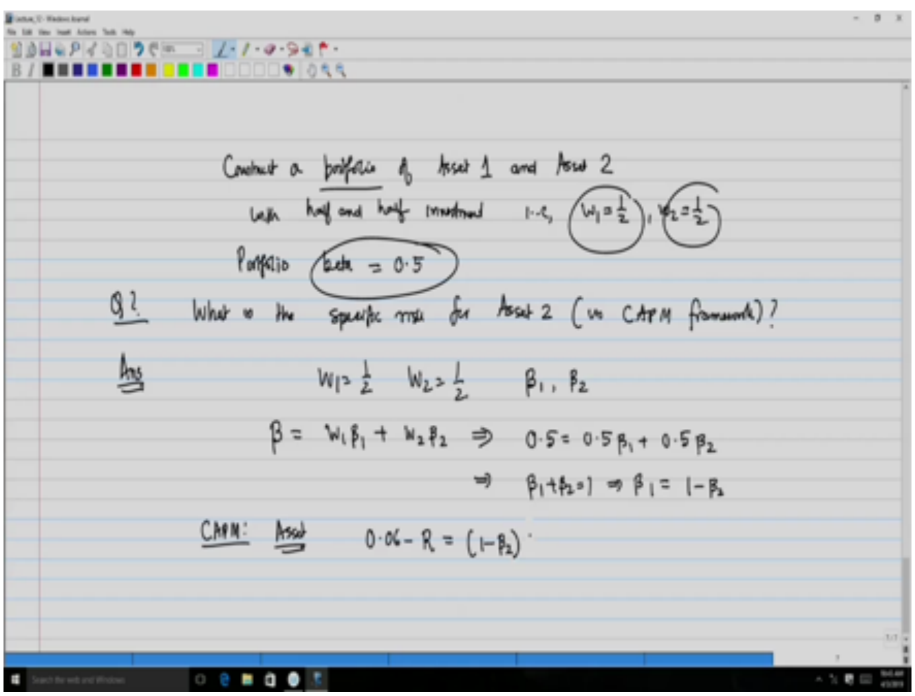


Alright, so now we take these two assets and construct the portfolio out of it, so we construct a portfolio of Asset $1$ and Asset $2$ with half and half investment that is $w_1$ is equal to half the way the and $w_2$ is going to be equal to half.

Further we have the portfolio, $\beta = 0.5$ and the question is, what is the specific or non-systematic risk for Asset 2 in CAPM framework? So lets try to answer this questions in the CAPM framework. So, here we have $w_1$ equal to half and $w_2$ also equal to half then we have $\beta_1$ and $\beta_2$ be the $\beta$'s of the two Assets, so then the beta for the portfolio is going to be $w_1\beta_1$ and it is left as an exercise for you to prove that this is true.

You just make the use of the definition of covariance and that the properties of covariance. So this will give me that the $\beta$ of the portfolio is 0.5, so this will be 0.5, that is half $\beta_1 + 0.5\beta_2$, which leads to the relationship that $\beta_1 + \beta_2 = 1$, that is I can write $\beta_1 = 1 - \beta_2$.

So, now I will invoke CAPM for both asset 1 and Asset 2. So for Asset 1 I will have, for Asset 1 what was the return? Return was 6%.

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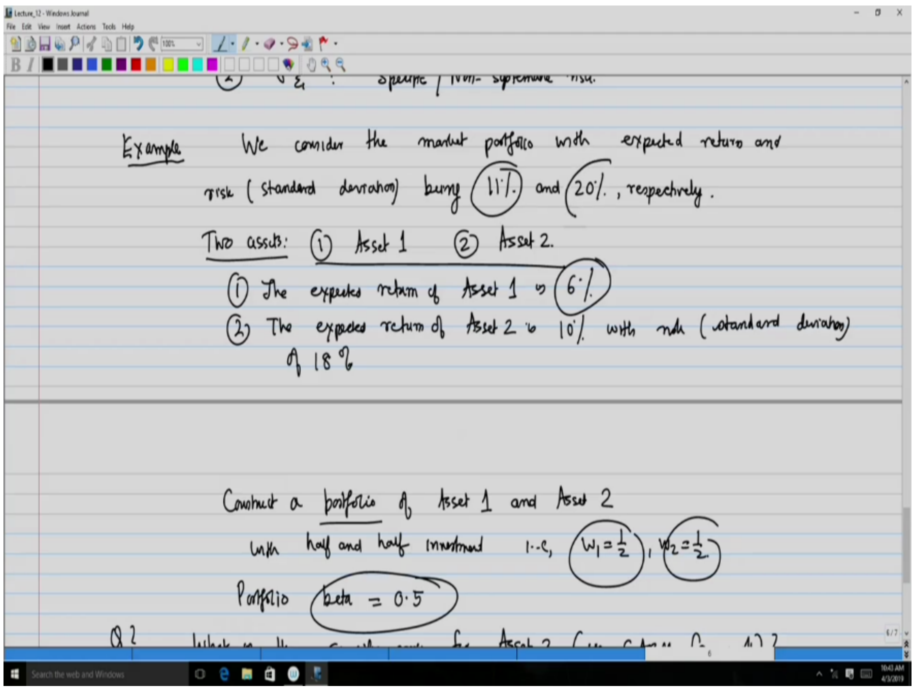


So accordingly for Asset 1, I will have $0.06 - R = (1 - \beta_2)(0.11 - R)$.

And for Asset 2, you would see, go back and see that the expected return was 10 percent. So this is going to be $0.10 - R = \beta_2(0.11 - R)$. So you solve this you can add them up to obtain $R$ and then consequently $\beta_2$, we solve this to get $R = 0.05 = 5\%$ and $\beta_2 = \frac{5}{6}$.

So then what was my question? My question is what is going to be the specific risk for Asset 2, so this means that I am trying to figure out what is going to be my $\sigma\epsilon_2^2$, right? That is the thing I want to figure out. So then, we will make use of the formula for $\sigma_i^2$, so $\sigma_2^2 = \beta_2^2\sigma_M^2 + \sigma^2\epsilon$.

So I am just basically making use of this relation for $i = 2$. So this means that what is $\sigma_2$?

$\sigma 2$ is basically the risk for the second asset, go back, what is the risk for the second asset? It was 18%. So accordingly what I can do is, so this will give me $\sigma^2\epsilon_2 = \sigma_2^2 - \beta_2^2\sigma_M^2 = 0.18^2 - (\frac{5}{6})^2(0.2)^2 = 0.0046$.

sigma M was 20 percent here.

So, this is what is going to be the specific risk or the nonsystematic risk for Asset 2.

Alright, so let us now move onto an application of CML and CAPM from the point of view of performance analysis that means suppose we have two Assets and or two portfolios and how can we make use of the CAPM framework any particular the risk of each of the assets and the betas of the assets or equivalently for the portfolios. So in order to make a comparative analysis of preferring one particular investment in an asset over other.

So, this is what is known as the performance evaluation. So, what the performance evaluation does the purpose itself is that this model and remember that we will work in the CAPM framework. So, this model

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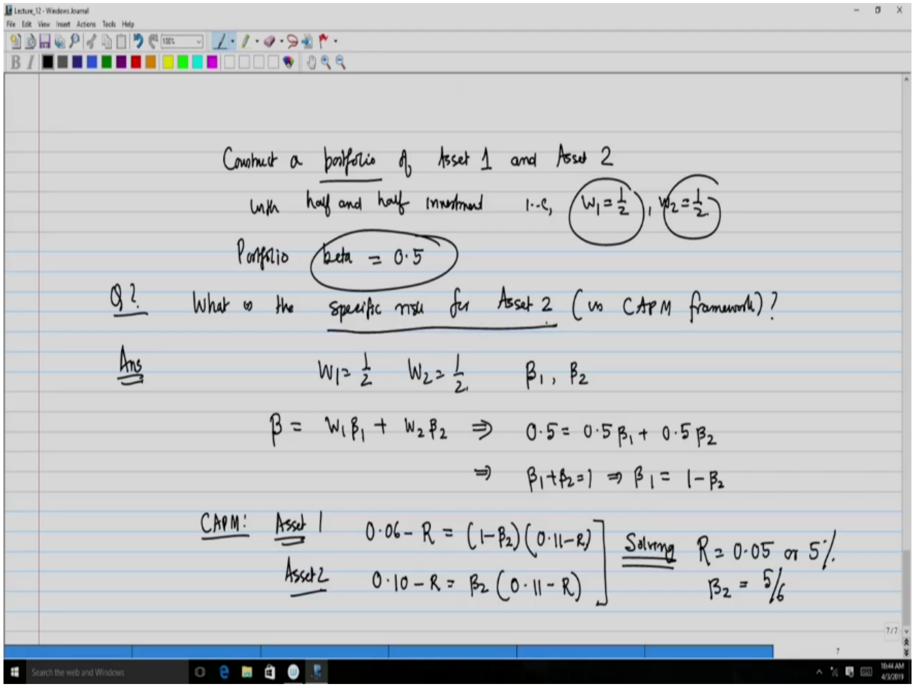


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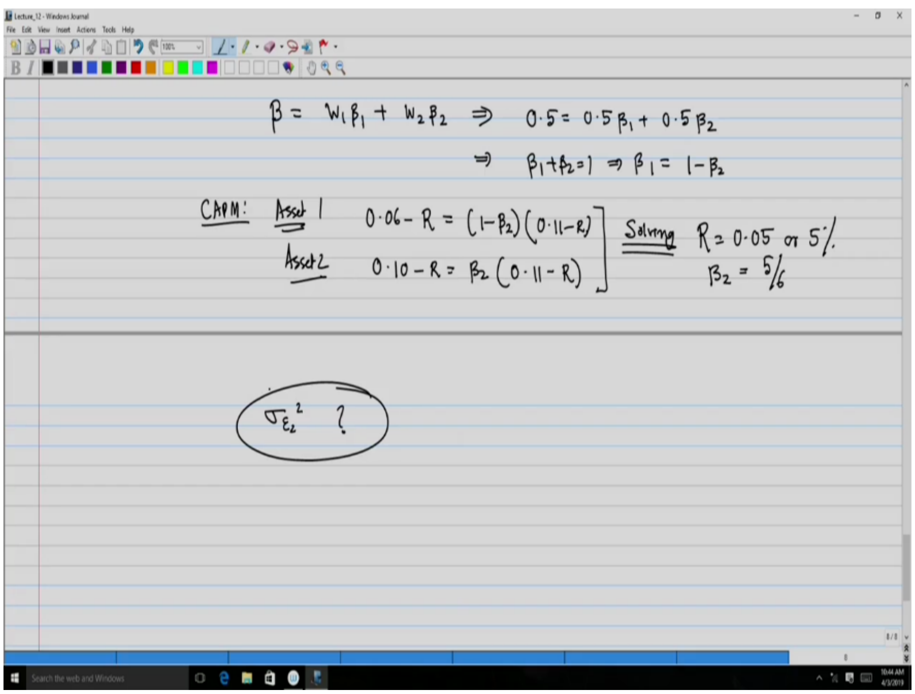


gives a benchmark of some sort of criteria or a test to evaluate securities and decide whether the price of the security or the asset is correct or whether they are incorrect in terms of being overvalued or undervalued.

So this will sort of give you the justification for the usage of the term capital asset pricing model. So, to this end we will look at basically 3 index or measures for performance. So one is Jensens index, the second is the Treynor index and the third is Sharpe index or Sharpe ratio. So, we will first start off with the Jensens index, so consider a regression type model. Say, $\hat{r}_i = \hat{\alpha}_i + \beta_i \hat{r}_M$ with a hat where.

So, I put the hat because these are values that are going to be estimated based on empirical data, so these are going to be based on empirical data that means past historical data.

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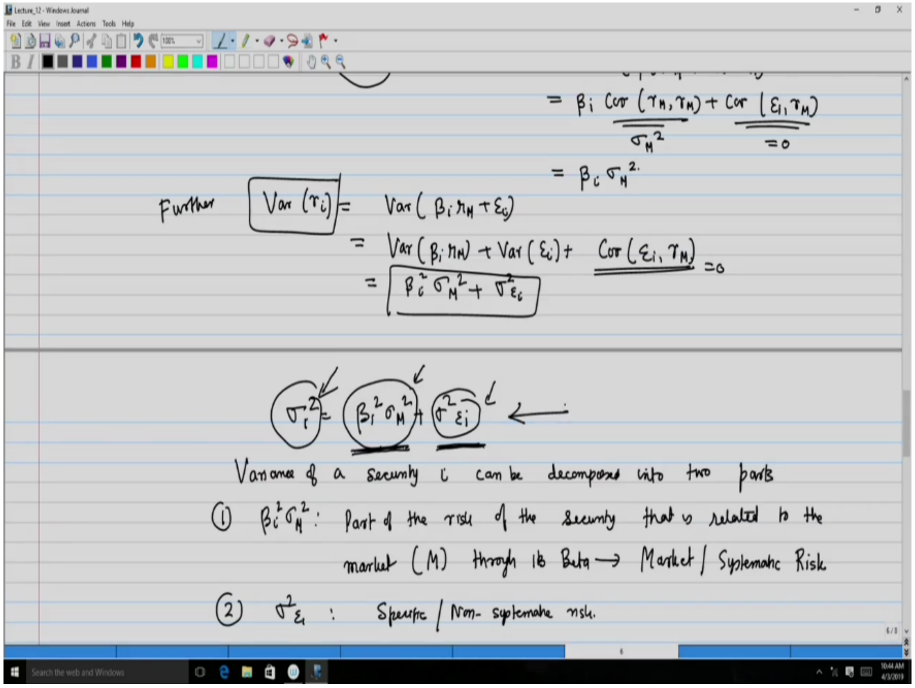


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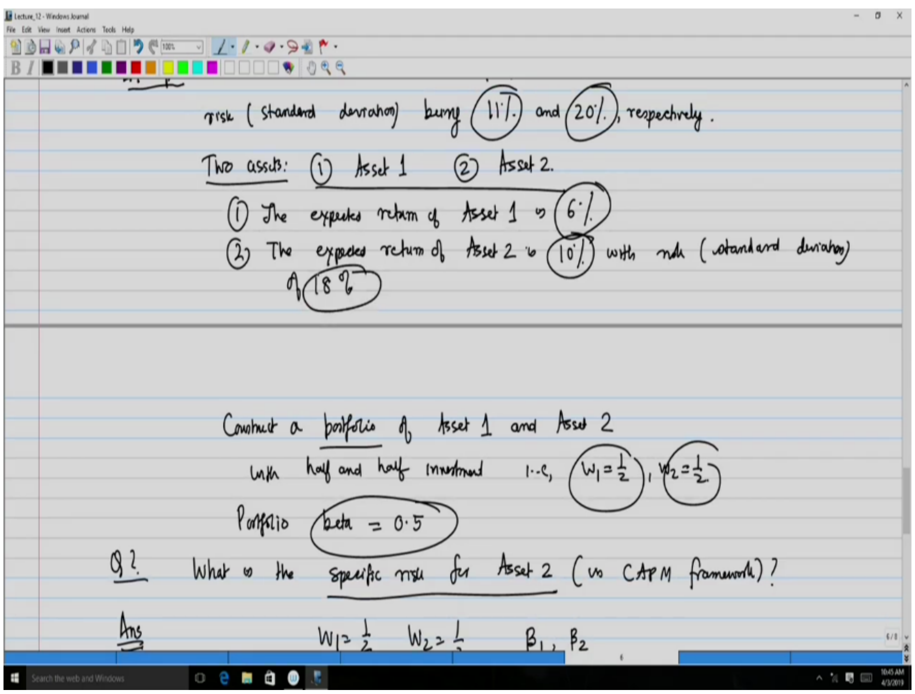


So, here $\hat{\beta}_i$ is the estimate of $\beta$ for security $i$ or the Asset $i$, $\hat{r}_i$ is the mean return of security $i$ and $\hat{r}_M$ is going to be the market average estimated data, so this is the estimation of the average market return typically would be looking at some sort of a market index to make this estimation.

Alright, so here we have seen that, okay fine, we have define what is $\hat{r}_i$, $\hat{\beta}_i$ and $\hat{r}_M$ and these are estimated, so then from this relation what you get is that $\hat{\alpha}_i$ is going to be $\hat{r}_i - \hat{\beta}_i\hat{r}_M$, so this relation $\hat{r}_i - \hat{\beta}_i\hat{r}_M$ is going to be the estimate of the true $\alpha$ which is a measure of overpricing. So sometimes Jensens index is also known as Jensen is $\alpha$ because of this term because this is the one which will actually do a performance analysis. So which is a measure of overpricing or underpricing of the security. So, in case of

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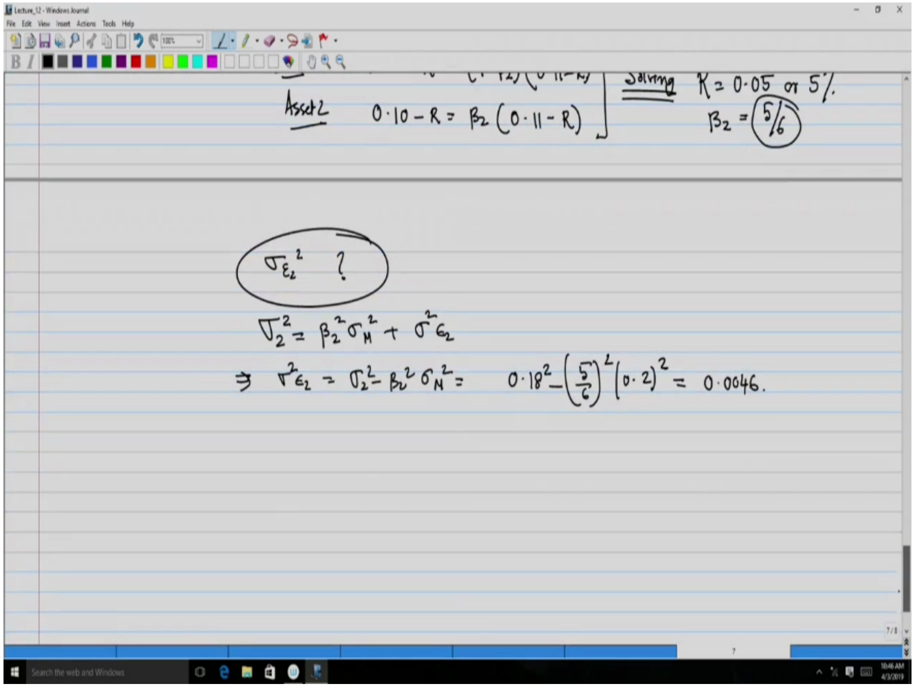


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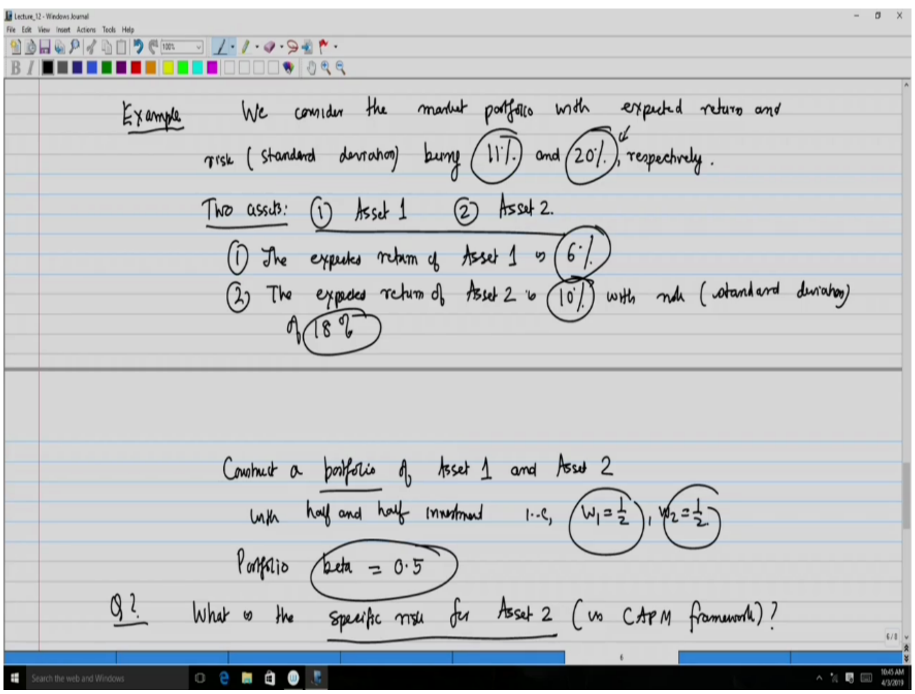


CAPM your $\alpha_i$ obviously is going to be equal to $0$.

Now, however we need to explore the possibilities of alpha i being positive or negative, so accordingly we will look at each of those subcases individually. So if $\hat{\alpha}_i$, the estimate for $\alpha > 0$, this means that the security seems to be paying, right? On an average a return that exceeds the return it should pay in the CAPM equilibrium.

By this I mean the following that if you go back and look at this here, now ideally if $\alpha = 0$ but if you have $\alpha > 0$, that means that, in an equilibrium state your return will be $\beta_i r_M$, however the return you are getting has an additional term $\alpha_i$, so which means that if this is positive that means your return actually

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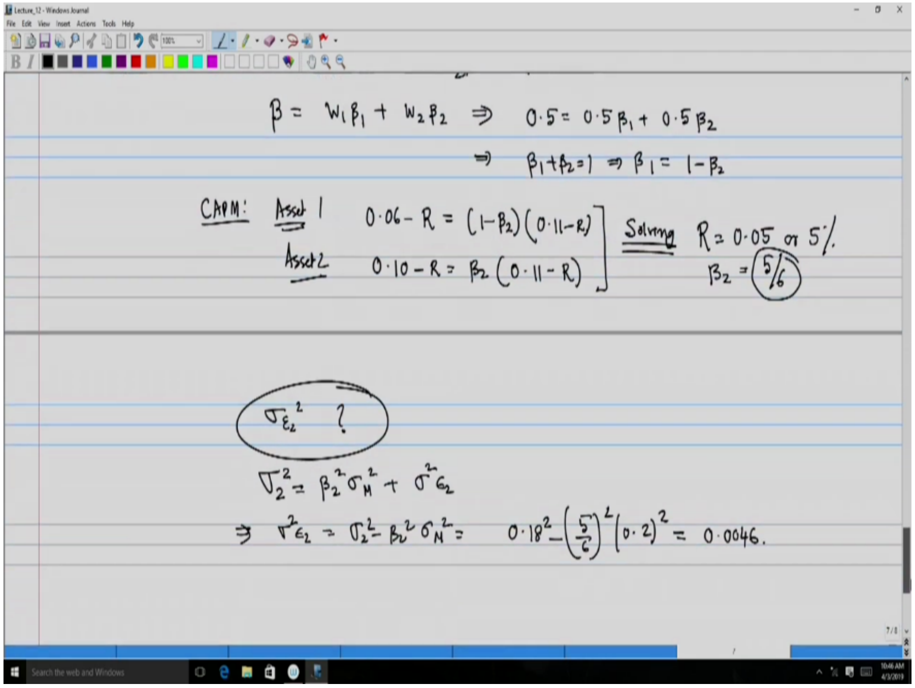


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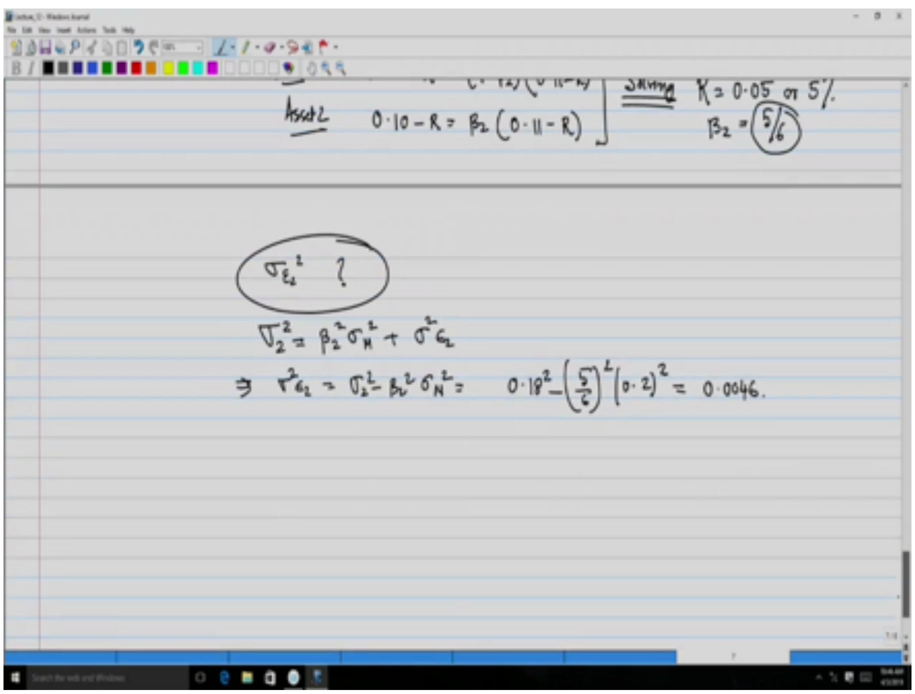


exceeds the return that you expect from the CAPM framework. So that means it is actually performing better than you expect in case of a CAPM equilibrium.

And in case your $\hat{\alpha}_i$ is less than the security seems to be underperforming. So what is the, what is the fallout of these particular two observations that we have made here. The fallout here is and this is given in terms of this parameter $\hat{\alpha}$.

So, the parameter alpha or its estimate given by a hat is called Jensens index or Jensen's $\alpha$ and it says the following. That if you have $\alpha > 0$, then the security is performing better than what CAPM would predict and accordingly this means that the security is actually underpriced.

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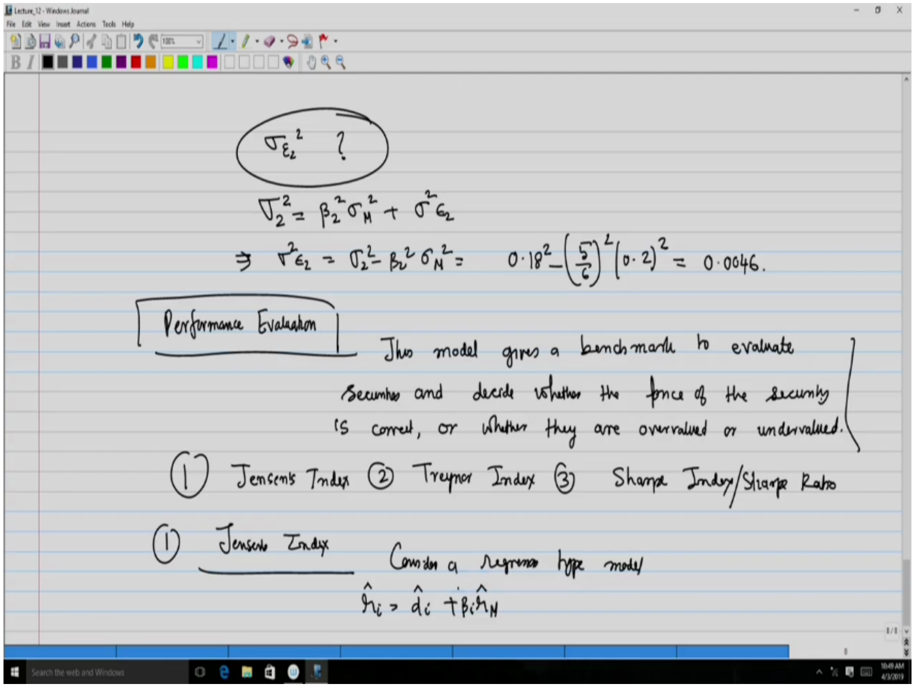

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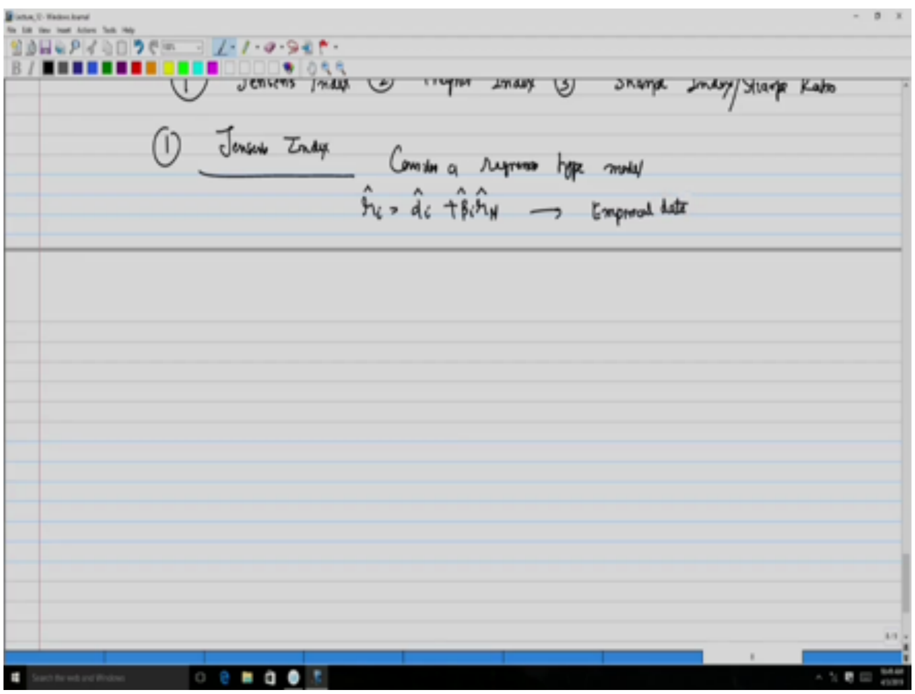

And if $\alpha < 0$, that means the security is actually performing worse than it should as you would expect it to happen in case of CAPM and consequently, we say that the security is going to be overpriced. Now we have stated what is the Jensen's index or the Jensen's $\alpha$? Now let us just point out a particular drawback regarding the usage of the Jensen's index.

So a drawback of this index is the following that the Jensen's index does not give an indication of the actual risk level of the security. So for example just to illustrate this point or drive home the point that suppose $\alpha > 0$ and they are same for two securities, so it should not suggest you that they are equally attractive in terms of investments because you also need to look at the possibility that $\beta_i$'s are different. And

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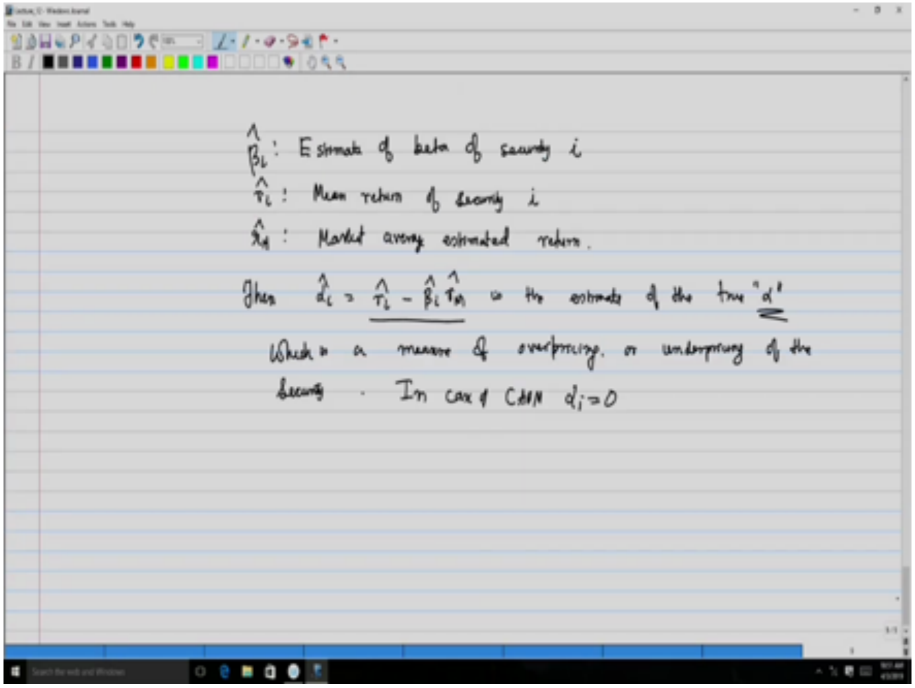

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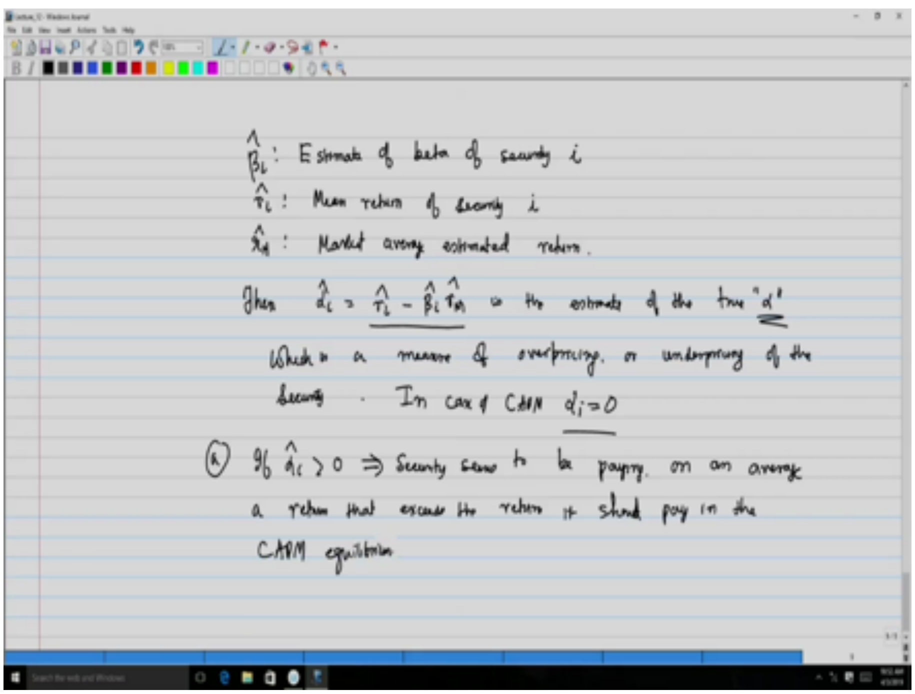

obviously in this case the one over $\beta_i$ is more attractive from the point of view of the investment.

Okay, so we have taken care of the Jensens index, lets now move onto Treynor index.

So, let us start off with this particular index and the Treynor index which is denoted by $T_i$ for the $i$-th asset is defined as $\frac{\mu_i - R}{\beta_i}$, so this is like an excess return over risk kind of a thing except that your $\beta_i$ is just the risk of the suggestive of the risk associated with the market where $\mu_i - R$ is the as you recall is the expected excess return of security $i$.

Now, if the Treynor index of a given security is greater than the Treynor index $\mu_M - R$ remember that in case of a market portfolio what would be the Treynor index? The Treynor index is going to be $\frac{\mu_M - R}{\beta_M}$ and

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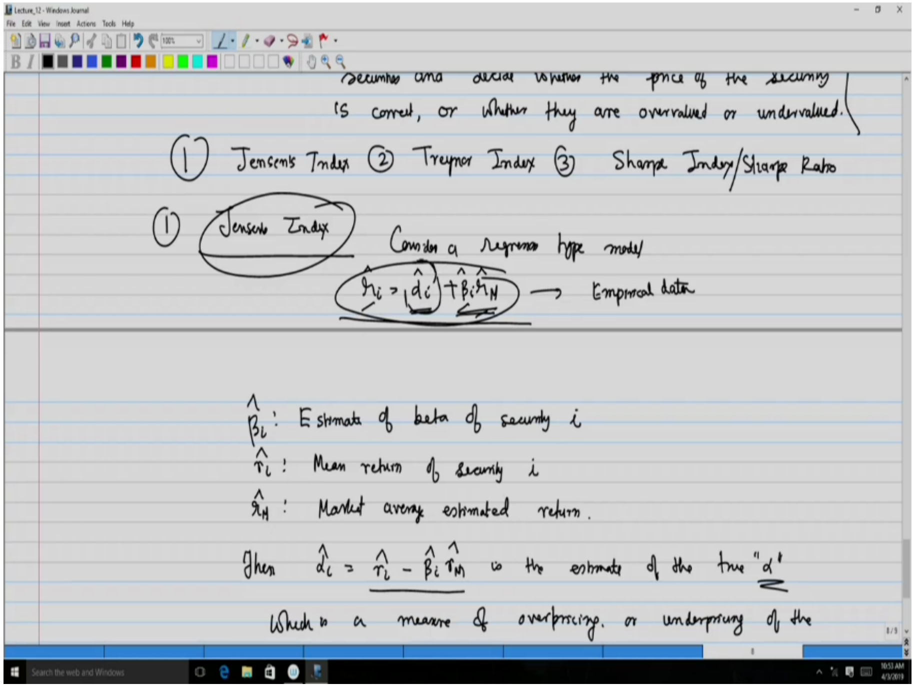

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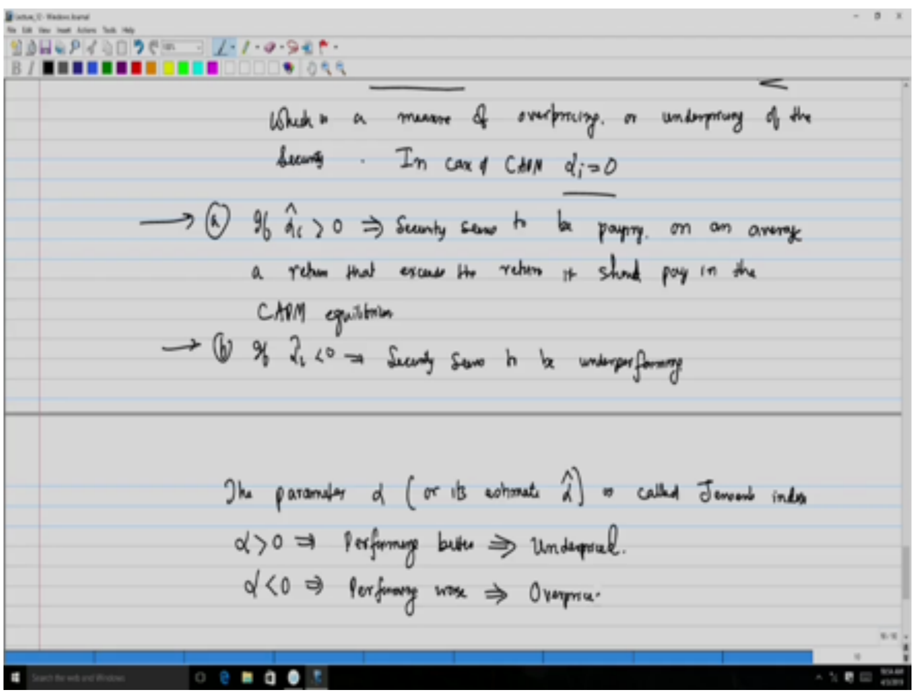

$\beta_M = 1$.

So, in this case the Treynor index, so in case the Treynor index $T_i$ of the asset is greater than the Treynor index $\mu_M - R$ of the market portfolio then what you get is that, you can say that then the security is actually performing better than it should according to CAPM, right?

So, according to CAPM you are actually expecting that the Treynor index should be $\mu_M - R\mu_M - R$, so essentially this means that if your Treynor index for the asset turns out to be bigger than this, so it is obviously suggestive that the performance of the individual asset as measured by the Treynor index is actually better than the performance as would be suggested by the CAPM framework.

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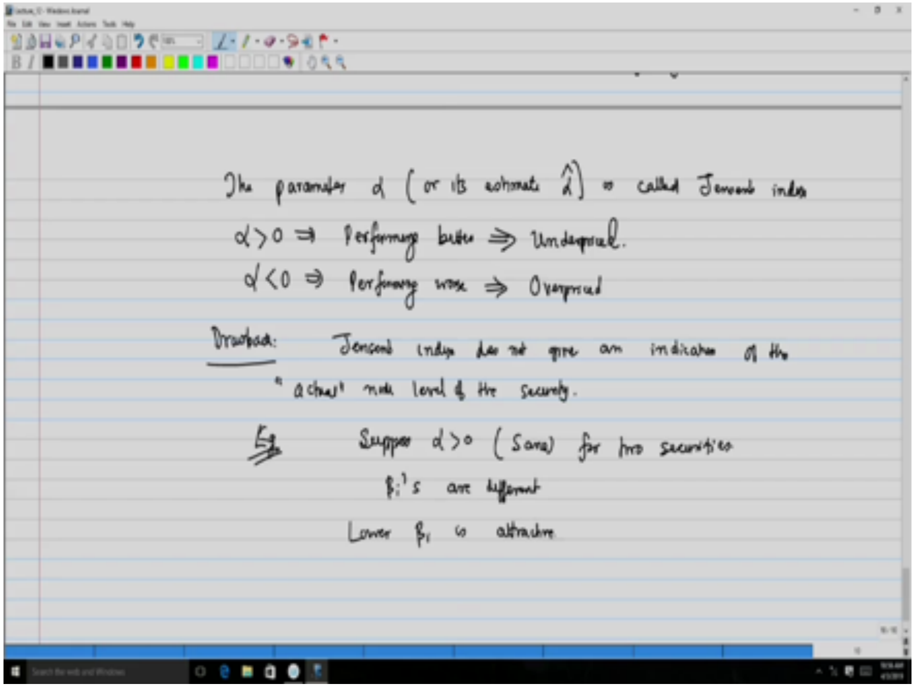


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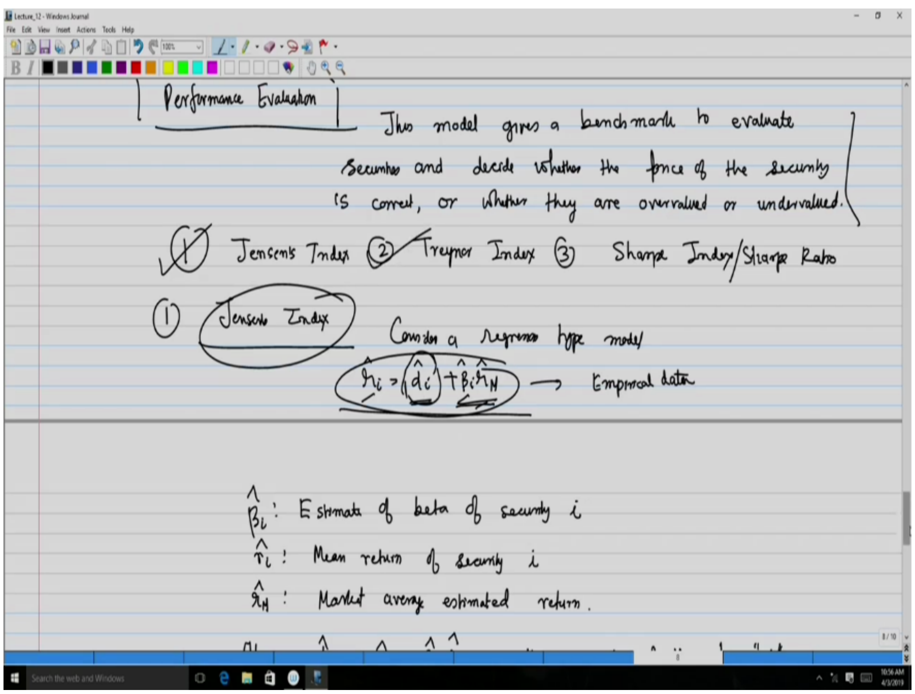


Okay, now just an observation here again. Treynor index is something that is actually not applicable in a blanket fashion but has some sort of a rider that comes along with it or the condition that it has to be diversified portfolio and we accordingly make the observation regarding this diversification necessary in case of a Treynor index.

So this observation is that diversification in case of Treynor ratio is necessary since a security with good performance vis--vis CAPM as a benchmark, as given in case of a Treynor index might actually have high specific risk which will not go away if the portfolio is not well diversified.

So, it is with this shortcoming that we actually go and define our third performance measure namely the

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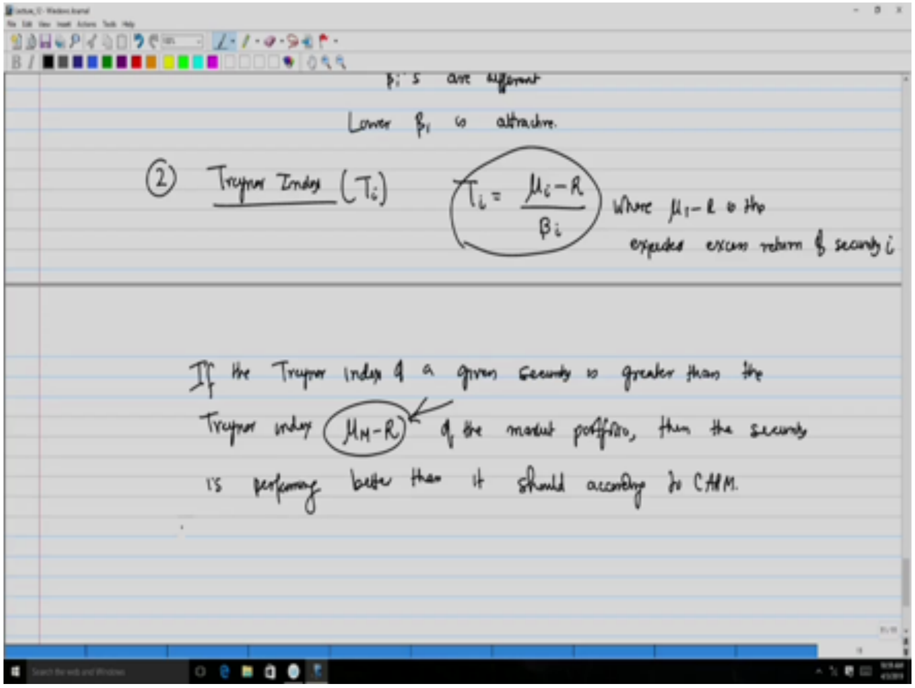

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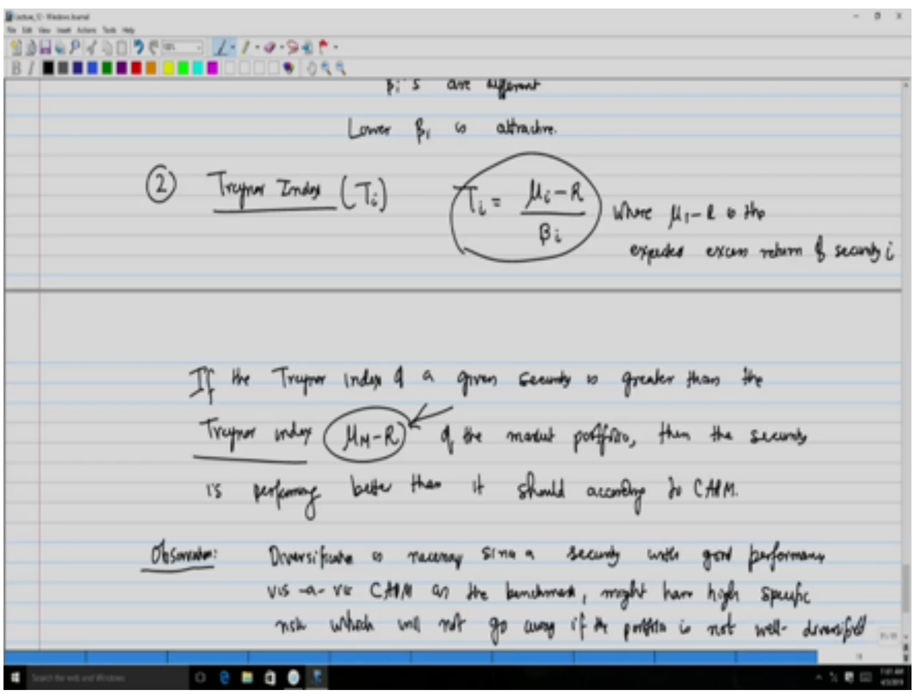

Sharpe index of the Sharpe ratio.

So the third one is the Sharpe index or Sharpe ratio which will denote by $S_i$ for the $i$-th asset and is defined as $S_i = \frac{\mu_i - R}{\sigma_i}$. So the interpretation is that the higher the Sharpe index the better the security is in the mean variances.

And we can view this by just recalling that $\frac{\mu_i - R}{\sigma} = \frac{\mu_M - R}{\sigma}$. So that means the higher you have this term ideally more than $\frac{\mu_M - R}{\sigma}$ the better it is that the particular asset is actually performing well.

Okay, so we have looked at these three different ratios, Jensen's, Treynor and Sharpe ratios. We just summarize whatever we have seen so far. So, if we have a good Jensen index typically this will be on both

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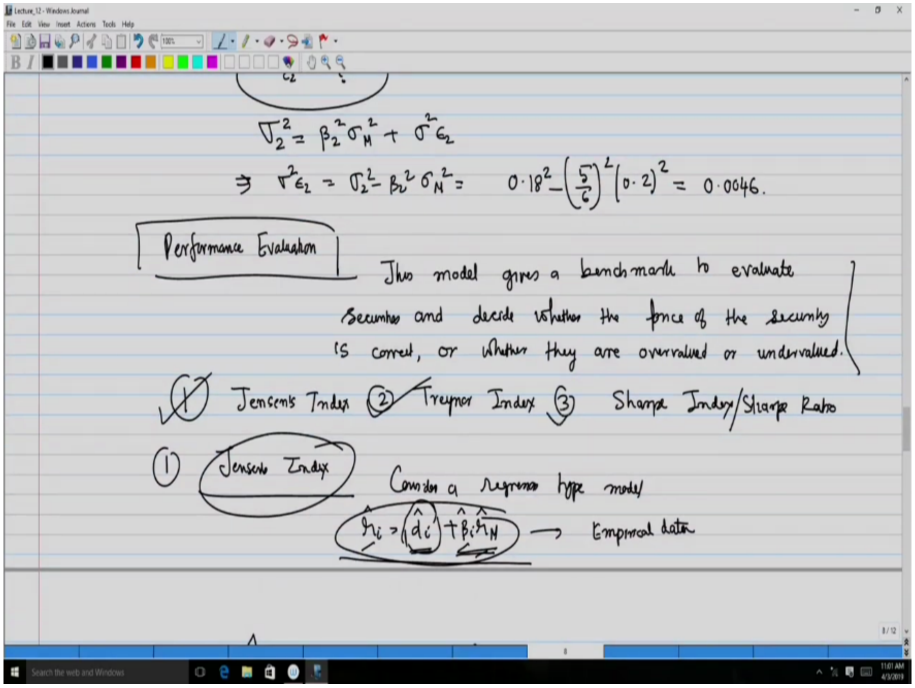


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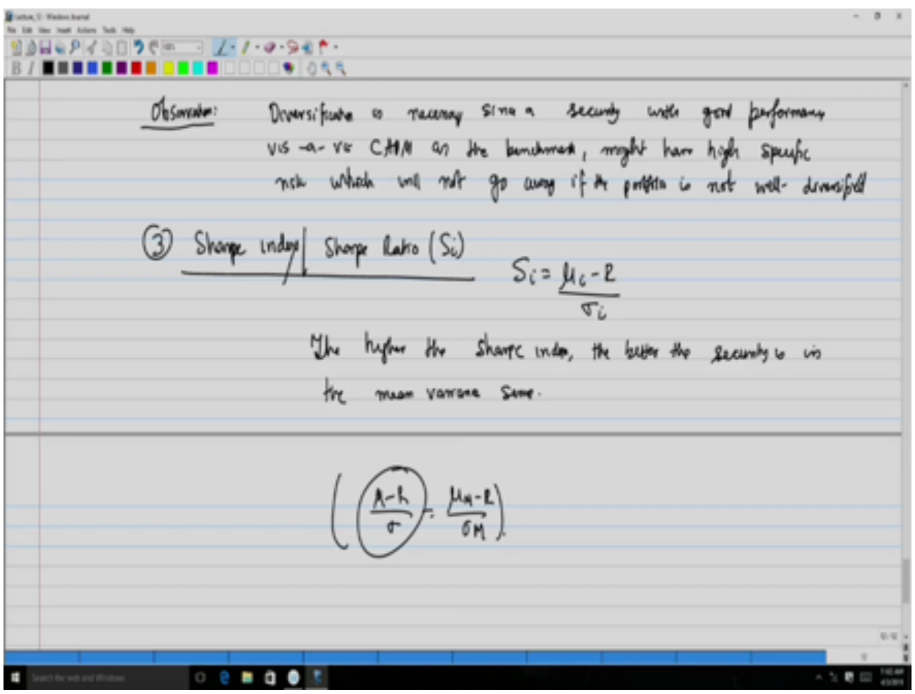


sides that you will have a good Treynor ratio or a good Treynor ratio both ways. Secondly so obviously fallout of the statement, first statement is that a bad Jensen index will mean bad Treynor index ratio and vice versa.

The third observation is that good Sharpe ratio implies good Treynor but good Treynor can happen with a bad Sharpe and likewise bad Treynor implies bad Sharpe but a bad Sharpe can occur with good Treynor. Anyway I mean these two statements sort of we can reconcile these two statements and we can of course reconcile these two particular statements.

So, now we actually come to the last topic that is the viewing the CAPM and the CML in the context of

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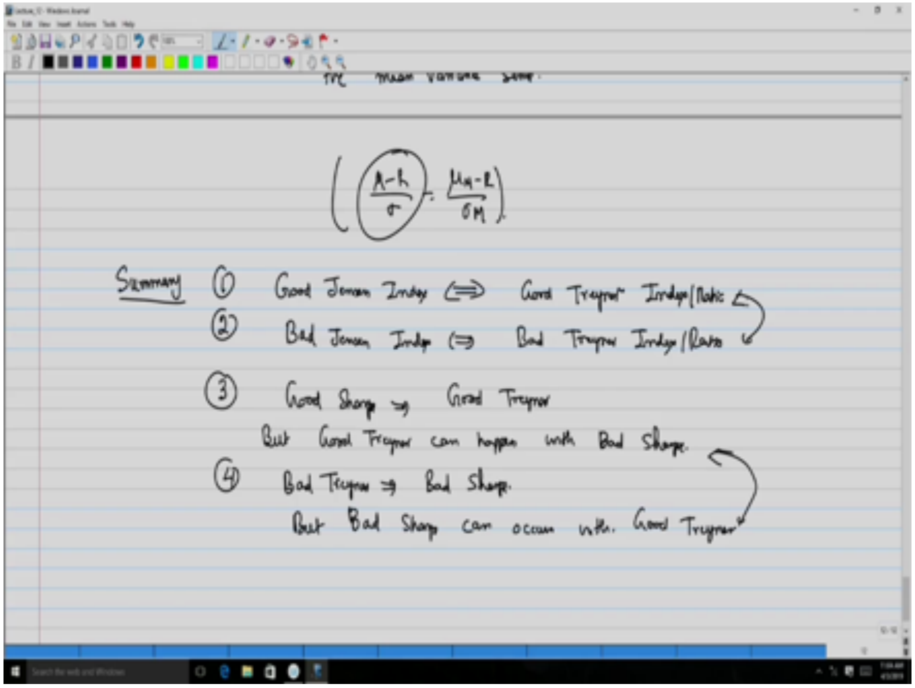

an asset pricing model. So we will give a couple of formulas that are actually obviously derivable from the relation for CAPM.

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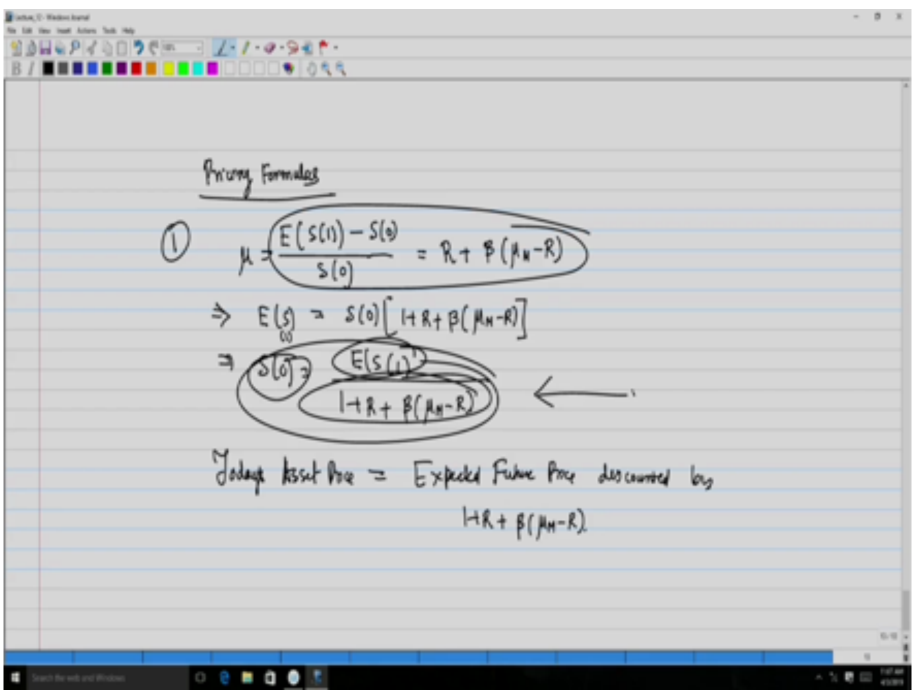

So, just look at the pricing formulas. First one, now you recall that again we look at the single period model. So then the return is going to be or the expected return is going to be $\frac{E(S_1)-S(0)}{S(0)}$, which we will denote by $\mu$ and this according to CAPM this is going to be $R + \beta(\mu_M - R)$ which means that $E(S_1) = S(0)\,[1 + R + \beta(\mu_M - R)]$.

And what this gives you is that, $S(0) = \frac{E(S_1)}{1+R+\beta(\mu_M-R)}$. So the interpretation of this is the following that today's asset price is going to be equal to that means $S(0)$ is going to be equal to the expected future price that means $E(S_1)$ discounted by this particular factor $1 + R + \beta(\mu_M - R)$. So you see now out of the formula CAPM formula we are now able to actually get a pricing formula.

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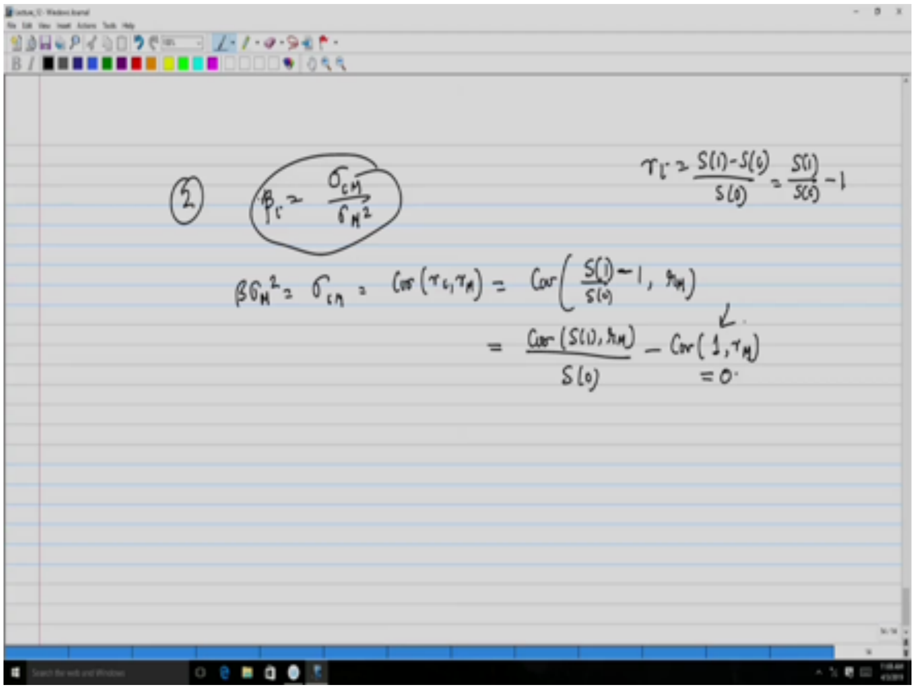

Our second pricing formula would be like $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$ you would recall the definition of $\beta_i$. So, if I combine $\beta_i$ to your $\beta$, so this becomes $\beta\sigma_M^2 = \sigma_{iM}$ and this is going to be the $Cov(r_i, r_M)$, by definition. Now, what is $r_i$? $r_i$ in a single period model is just going to be $\frac{S_i}{S(0)} - 1$.

$r_i = \frac{S_(1) - S(0)}{S(0)}$ and this is $\frac{S_i}{S(0)} - 1$ and a covariance of this random variable with $r_M$, this can be written as $\frac{Cov(S(1), r_M)}{S(0)} - Cov(1, r_M) = 0$.

So, therefore if you replace this value for $\beta$ in the previous that we have here.

Then we obtain that $S(0) = \frac{1}{1+R}\left[E(S1) - \frac{1}{\sigma_M^2} Cov(S(1), r_M)(\mu_M - R)\right]$. So you get another pricing formula in a different form.

So, this brings us to the end of the discussion on module 3 which was on portfolio theory. Just to do the recap of the module we started of the defining what a portfolio is. We then moved onto the Macrovit framework where we considered our portfolio comprising of completely risky assets and we looked at what is the efficient frontier corresponding to all those portfolios which are either the minimum variance portfolio or the minimum variance portfolio for given amount of return, PDS signed returns.

The portfolio which gives the maximum return for a predetermined value of risk and then we added a risk free portfolio to this portfolio of risky assets to obtain the efficient frontier and henceforth give the CML and the capital framework. And then we saw how this can be actually used to measure the performance of assets and portfolios in terms of several well established ratios and we finally concluded by talking about a couple of pricing formulation that arises out of the CAPM framework. Thank you for watching.

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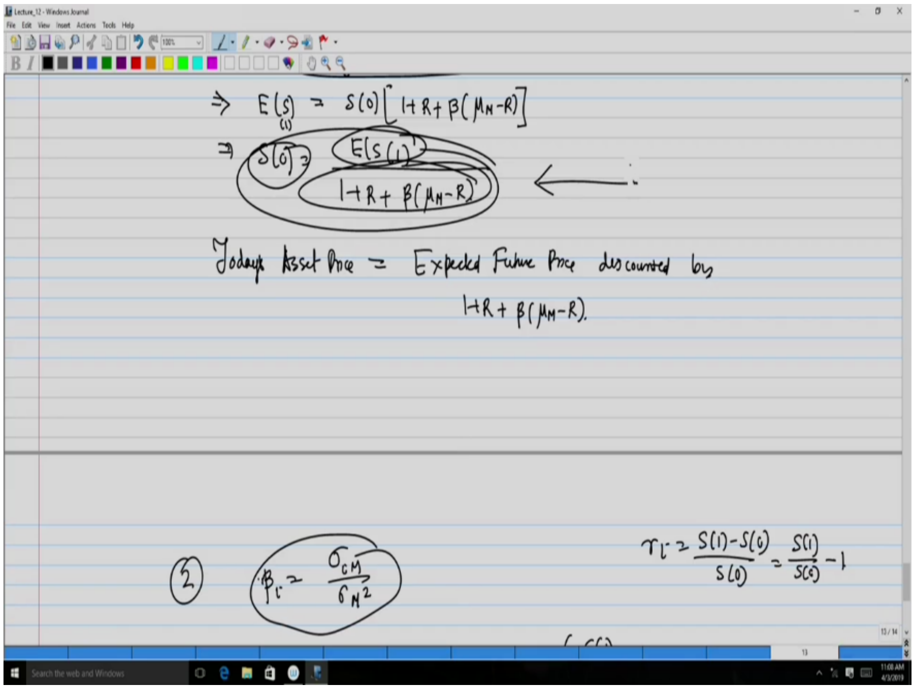


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