Mathematical Finance

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Module 3: Modern Portfolio Theory
Lecture 5: Capital Market Line and Capital Asset Pricing Model

Hello viewers, welcome to this lecture on the course on Mathematical Finance. This is going to be eleventh lecture of the course where we will continue our discussion on portfolio theory and remember that we have already talked about the minimum variance line and we had just given some of the introductory basic framework for the capital asset pricing model.

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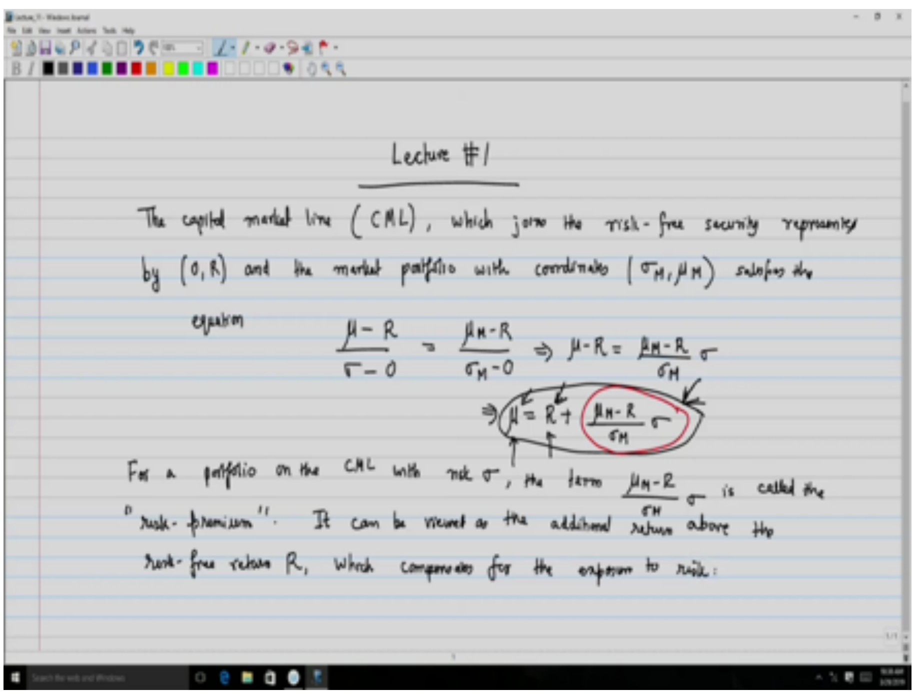

So accordingly, so first talk about the capital market line. The capital market line which we will abbreviate or acronym as CML and this is the line you would recall which joins the portfolio of a purely risk-free security and no risky security represented by the point $(0, R)$ in the (μ, σ) plane and the market portfolio with coordinates (σ_M, μ_M), the weights of which we derived in the last class satisfies the equation and this equation is nothing but the equation of a line joining two points.

So, it is going to be $\frac{\mu-R}{\sigma-0} = \frac{\mu_m-R}{\sigma_M-0}$, which implies that $\mu - R = \frac{\mu_M-R}{\sigma_M}\sigma$ and this can be written in the form $\mu = R + \frac{\mu_M-R}{\sigma_M}\sigma$. So, you can make the following observation given this particular relation for the CML.

Remember here R is the risk-free rate and mu is essentially the return that you actually get for your risky investment. So for a portfolio which lies on the CML and has the risk, sigma of course with the corresponding return μ, the term $\frac{\mu_M - R}{\sigma_M}\sigma$ that means this particular term here, this term is called the risk premium. Why do I call this as a risk premium?

It is called the risk premium because given a choice between a completely risk-free investment and an investment which incorporates to a certain extent the risk as given by any portfolio on the CML, then the return of the portfolio will be the return that you get from the risky asset, the risk-free asset rather and the risk premium. So in some sense it means that if you are not willing to take any risk at all then the return that you are assured is capital R.

However, in the event of taking a risk than you can expect that you will basically get an additional return given by this term here over and above the risk-free return that you are assured if you do not want to invest in any of the risky asset. So this is the additional component or incentive which is referred to as a premium because you have decided that you are going to take position in risky asset also in addition to the risk-free asset. So now, I can state this as, this can be viewed as the additional return above the risk free return R which is essentially a compensation for the exposure to risk.

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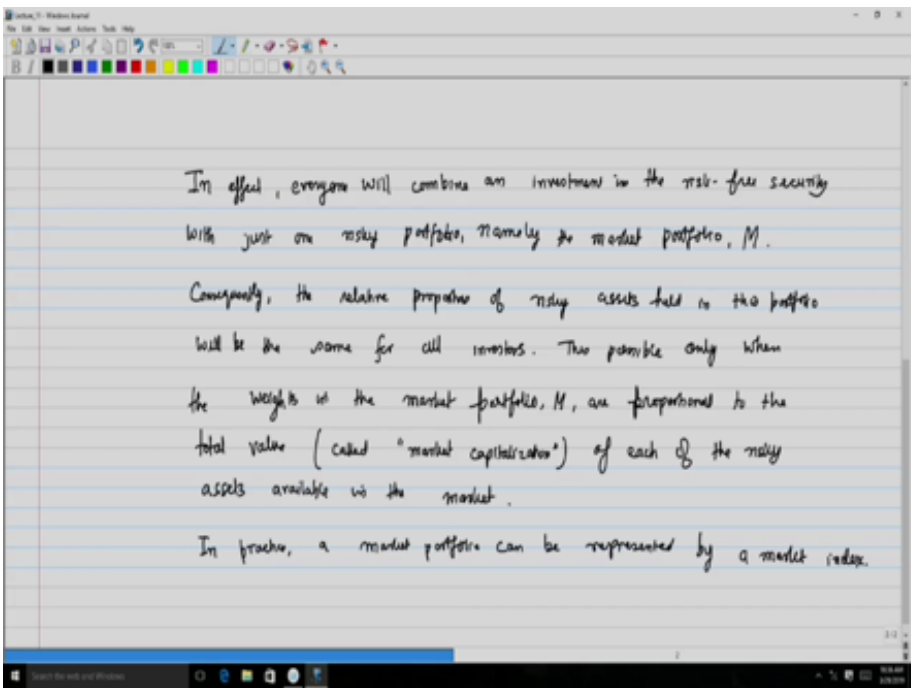

So, to elaborate a little more on this, in effect we can state the following that everyone will combine an investment in the risk-free security with just one risky portfolio, namely, the market portfolio. So, instead of going and investing in individual securities and working out the weights one can basically take up a risky position simply by making an investment in the market portfolio.

Then consequently the relative proportions of risky assets held in this portfolio will be the same for all investors. So this is possible only when the weights in the market portfolio M are proportional to the total value called market capitalization of each of the risky assets available in the market. So what I mean to say here is the following, is that for the investment in risky assets all investors take up an investment in the market portfolio and if all investors are investing in the market portfolio that means the proportion of the money invested in each of the assets of the market portfolio or the words of the market portfolio are going to be identical across all investors.

And this can only happen if this particular portfolio reflects a portfolio where you include all the assets that are available in the market with the weights being the same as the market capitalization of each of the asset divided by the total market capitalization of all the assets that are available in the market, which is a justification of or why this particular portfolio is referred to as the market portfolio.

So since you cannot actually invest directly in all the assets in the market, so the next best thing that you can do in terms of in a practical view of investing is the following: That in practice a market portfolio can be represented by a market index. So what this means is the following, is that, if you decide to follow the capital market line and you want to make an investment in the risk-free and the risky asset then all you need to do is that for the risky component of the asset you invest in the market portfolio which in practice would be equivalent to investing in just a market index like say Sensex or Nifty. We now start off on the discussion on the capital asset pricing model which is another key component of portfolio theory.

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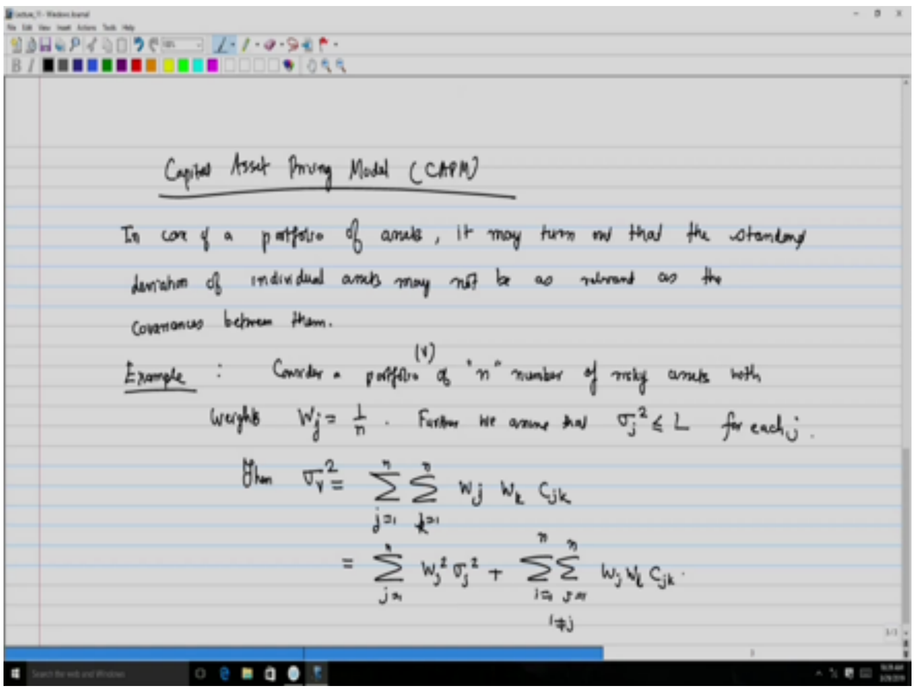

So capital asset pricing model is typically abbreviated as CAPM or usually called as CAP M. So in case of a portfolio of assets, it may turn out that the standard deviation of individual assets may not be as relevant as the core variances between them. So let us first sort of elaborate on this through a little example, what I mean by that the variances of the risk of the individual assets might have a lesser impact on the overall risk of the portfolio as compared to the collective behavior as given by the co-variances between the returns of the assets and this is what we are going to illustrate through a very a simplified example of a portfolio comprising of N number of assets.

So in this example we consider a portfolio of n number of risky assets with weights $W_j = \frac{1}{n}$. Further, we assume that the individual variances for each asset σ_j^2 is bounded by some L for each j, then

$$\sigma_v^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} W_j W_k C_{jk} = \sum_{j=1}^{n} W_j^2 \sigma_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} W_j W_i C_{jk},\ i \neq j.$$

$$\sigma_v^2 \le \frac{1}{n} L + \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_{jk}.$$

Assume further that all off diagonal elements of C are equal.

$$C_{jk} = c > 0,\ \forall j \neq k.$$

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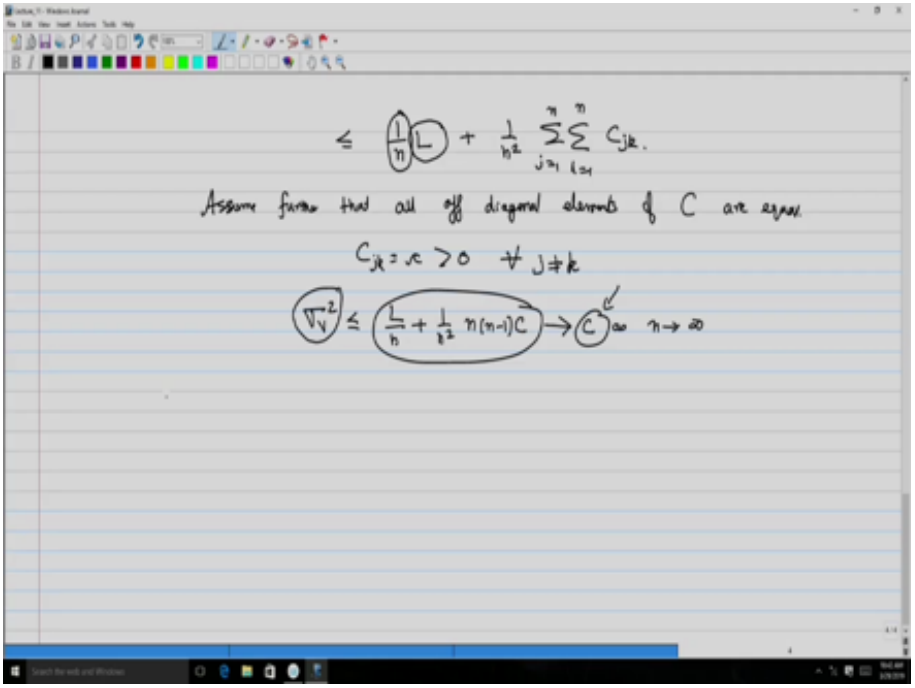

$$\sigma_v^2 \leq \frac{L}{n} + \frac{1}{n^2}n(n-1)C \to C,\ n \to \infty.$$

So this illustrates the point that I was trying to make that you see that the overall variance of the portfolio is bounded by this quantity which tends to C which is contingent only on the covariances of returns of different assets with no rule showing up here explicitly for the variances of each of the individual assets, namely σ_j^2. So taking a cue from this that sometimes the individual variances even though they are suggestive of the risk of the assets they might have a lesser impact on the overall portfolio risk whereas the covariances of the returns of those assets might actually have a more dominant role to play.

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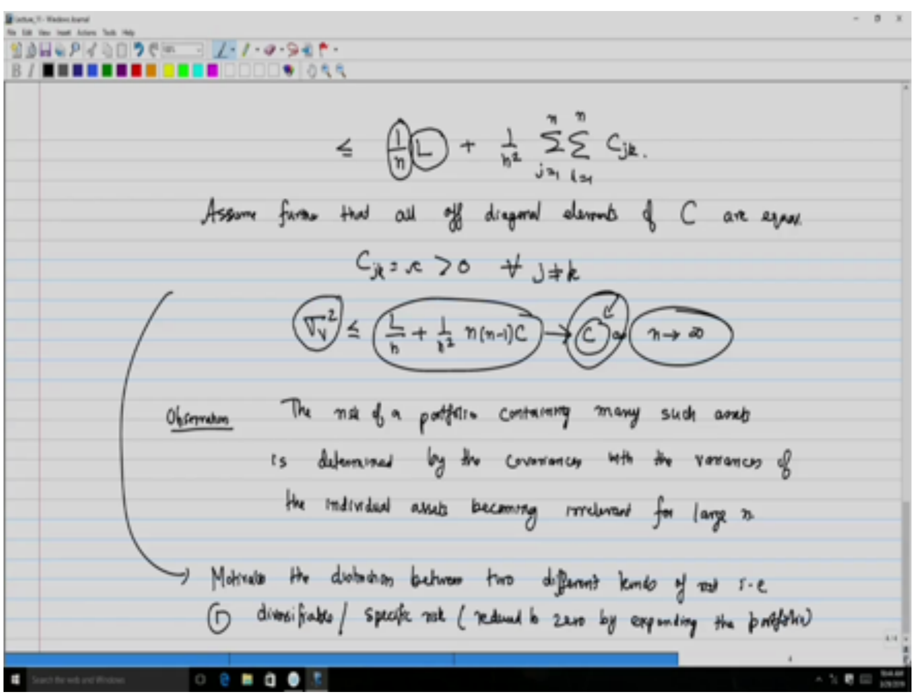

So now we can make the following observation that the risk of a portfolio containing many such assets is determined by the covariances, so in this example it was determined by C with the variances of the individual assets becoming irrelevant for large n, that is for $n \to \infty$ here, so what does this do? So this example now here, this example motivates the distinction between two different kinds of risk that is first one, is what is known as the diversifiable or specific risk and this actually can be reduced to 0 by expanding the portfolio.

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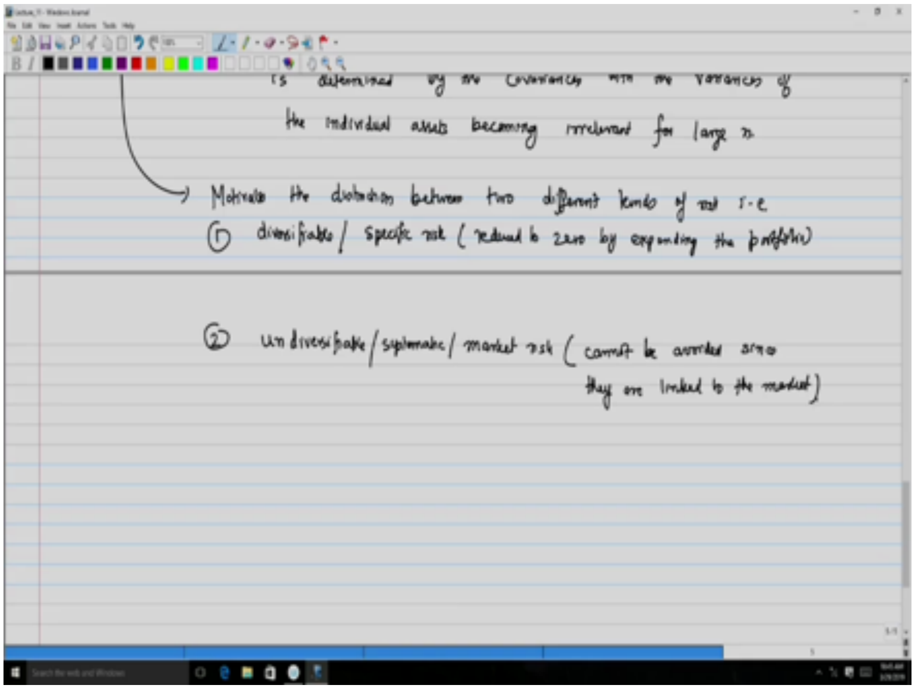

And the second is un-diversifiable or systematic or sometimes which is known as the market risk which cannot be avoided since they are linked to the market. So let me elaborate little bit on this that when you are looking at the risk of the overall portfolio as given by σ_v^2. There are two components, one was which had σ_j^2 which is some sort of a localized or specific risk applicable just to those individual assets and the risk contribution from that particular term can effectively be gotten rid of by taking n to be larger and larger which means that by taking a diversified portfolio, which is the reason why that since you can get rid of this particular term by diversifying the portfolio. Such a kind of risk is referred to as the diversifiable risk.

However, in the second case irrespective of how large you make your n to be and in the particular example we saw that, that particular term which had the covariances the term did not vanish irrespective of how large you make your n to be. So this means that whatever effort is put into diversifying the portfolio, making large n, that is in effective in getting rid of this particular term and that is the reason why this is called as a non-diversifiable risk.

And this means that this is something like a market risk because it captures the correlation of the returns of the each individual assets with every other different assets, so on a broader spectrum we can view these as that this is reflective of the overall collective behavior of the market.

Okay, so now let us move along a little more on CAP M, so we begin by recalling that the capital market line is tangent to the efficient frontier of all risky assets at the point of tangent C (σ_M, μ_M), representing the market portfolio with weights W_M. Now consider any other portfolio V which is represented in the mu sigma plane by the point (σ_v, μ_v) in the (σ, μ) plane.

Next we consider all portfolios constructed from M and V, that is a combination of the portfolio M and V, in which case they form a hyperbola which will be tangent to the CML at the point of market portfolio, namely (σ_M, μ_M). If this hyperbola were to intersect the CML this would contradict the fact that the slope of the CML is the maximum.

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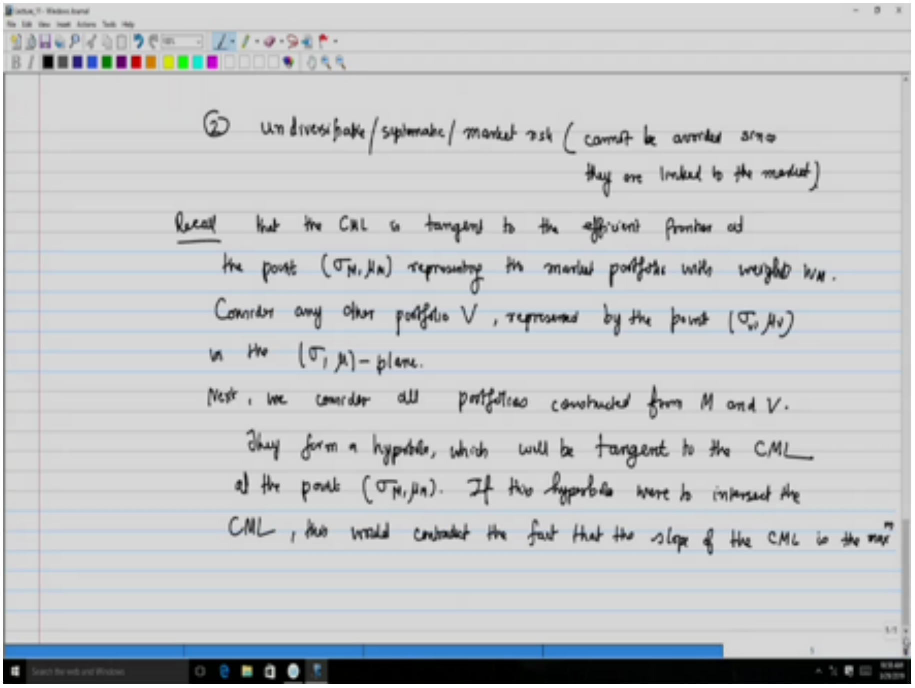

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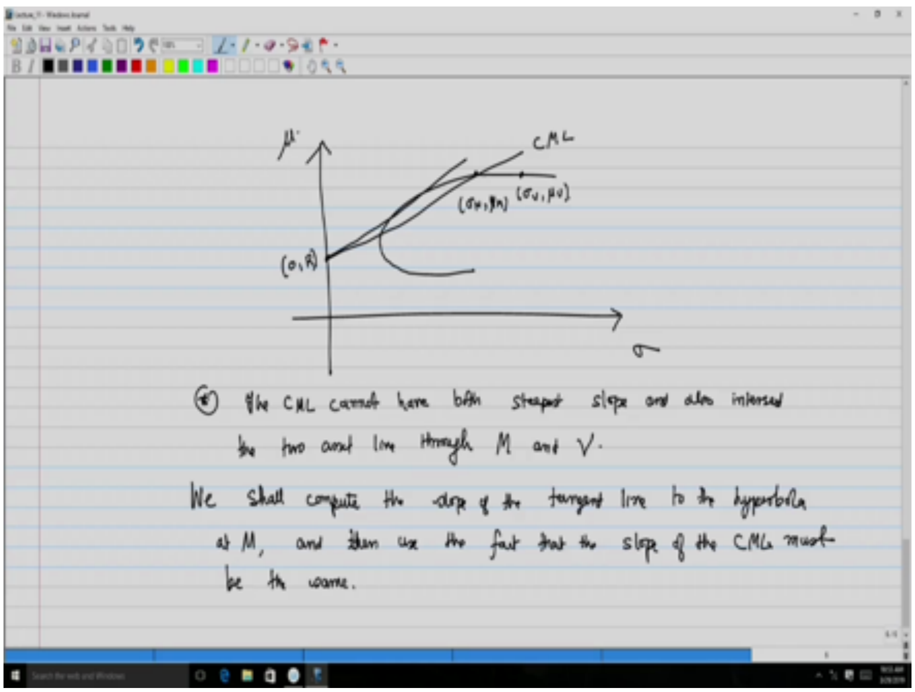

So let us see how this looks graphically, so this is my (σ, μ) plane and this is my completely risk free portfolio and this is going to be the hyperbola and this is the tangent line and this is the CML. So here the CML intersects the hyperbola at the point (σ_M, μ_M) and this is the portfolio (σ_v, μ_v). So just recall the observation, the CML cannot have both steepest slope and also intersect the two asset line through M and V. So then how do we resolve these issues? So we shall now compute the slope of that tangent line to the hyperbola at M and then use the fact that the slope of the CML must be the same.

So accordingly let the weights, so we look at the portfolio comprising of this portfolio V and M and we will choose the weights of V and M accordingly. So let the weights of V and M in the portfolio, so the

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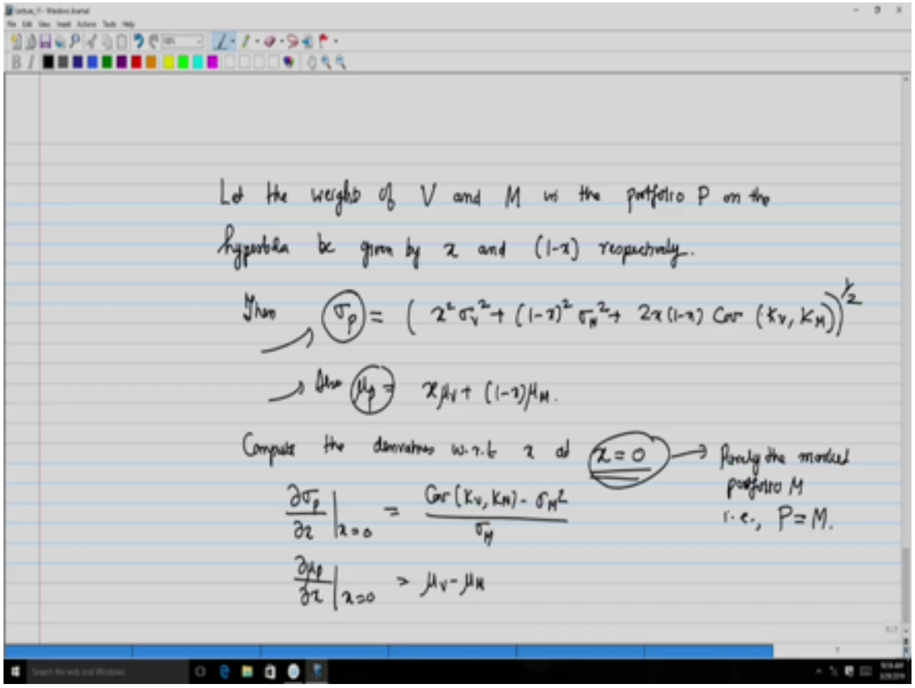

resulting portfolio I will call this as portfolio P on the hyperbola be given by x and $1-x$ respectively. Then what is going to be σ_P or the risk of this portfolio? This is simply going to be $x^2\sigma_v^2 + (1-x)^2\sigma_M^2 + 2x(1-x)Cov(K_V, K_M)$, which are the return of the portfolio V and the return of the portfolio M, raised to half. So this is σ_P.

Also $\mu_P = x\mu_v + (1-x)\mu_M$. We then now need to compute the derivatives with respect to x in both the cases at $x = 0$. Why $x = 0$? Because $x = 0$ will comprise of just purely the market portfolio M that is your portfolio P is the simply going to be equal to the market portfolio. So that means the slope in this particular case will be same as the slope of the CML.

So accordingly we calculate first the derivative of the risk with respect to x at $x = 0$ and this turns out to be $\frac{Cov(K_V,K_M)-\sigma_M^2}{\sigma_M}$ and similarly, the derivative of μ_P with respect to x at $x = 0$, this turns out to be $\mu_v - \mu_M$.

So therefore the slope, so from these two derivatives you can calculate the slope of the tangent line, is the ratio of these derivatives which must be equal to the slope of the CML. What is the slope of the CML? Recall the slope of the CML was $\frac{\mu_M - R}{\sigma_M - 0}$ and this must be the same as the slope of this tangent line evaluated $x = 0$, the reason being that I have already mentioned that at $x = 0$ we will have a just a market portfolio. So then this will become $\mu_v - \mu_M$ divided by covariance, this term divided by this term, $\frac{Cov(k_v,k_M)-\sigma_M^2}{\sigma_M}$.

And this gives you upon simplification $\mu_v = R + \frac{Cov(k_v,k_M)}{\sigma_M^2}(\mu_M - R)$ and here this particular term which captures the covariance of the portfolio with respect to M this is nothing but the beta factor of portfolio V.

So we now summarize the discussion that we have done so far in deriving this particular result as a theorem. So the theorem is as follows: And this is the theorem of the capital asset pricing model or CAP M, it states the following, suppose that the risk-free rate R is lower than the expected return mu MVP of the minimum variance portfolio and we need this so that the market portfolio M exists, then the expected return μ_v on any feasible portfolio μ_v or other portfolio V is given by $\mu_v = R + \beta_v(\mu_M - R)$.

Now again this term $\beta_v(\mu_M - R)$ is called the risk premium and it represents the additional return required by an investor because of the risk. And because of the risk due to the link between the portfolio and the market.

So in sense basically what this does is that, it gives you that you invested a particular portfolio whose expected return is μ_v and this is the risk premium and this is basically the additional return that is expected

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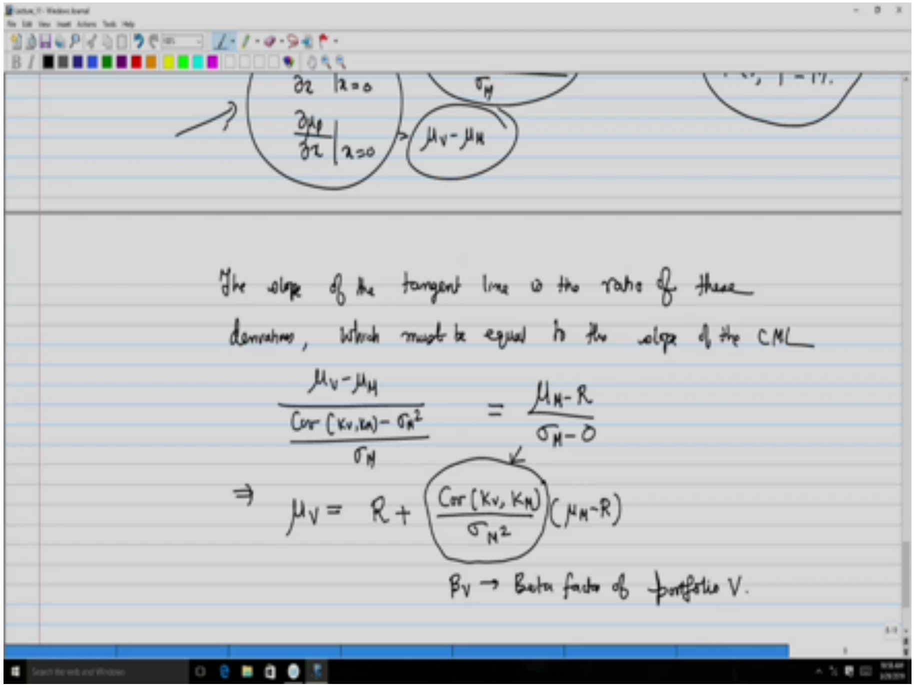

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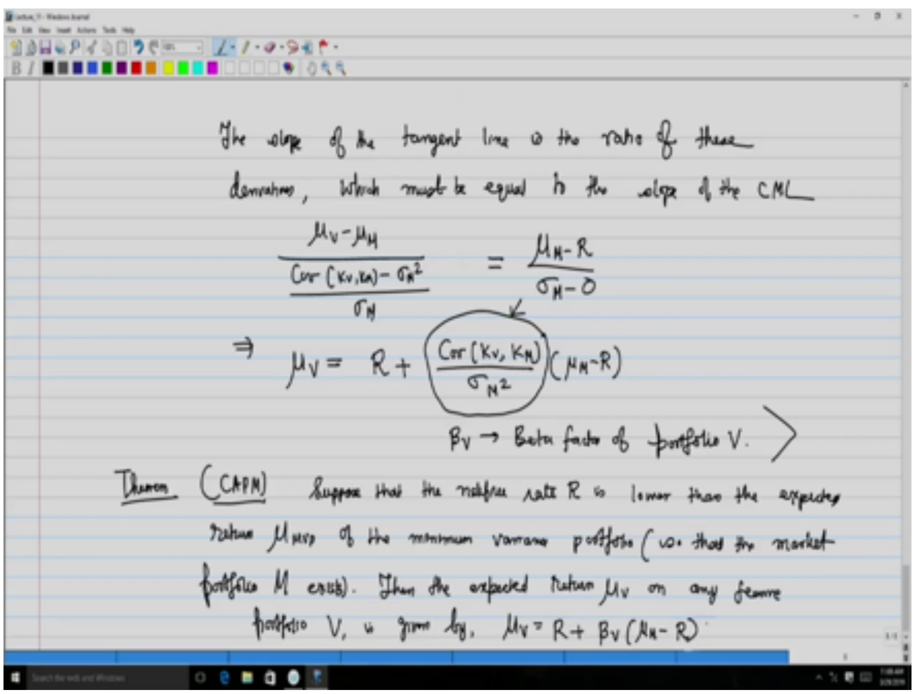

by the investor because of the risk taken and this risk here can be attributed to the factor of β_v and if you look at the expression for beta v this is basically the covariance of the portfolio with respect to the return of the market portfolio. Now since the factor sigma M square is fixed, so in some sense this term is reflective of the risk that is post the portfolio as a result of its interaction with the market.

Okay. So now suppose we want to approximate the return K_V on a feasible portfolio V by a known function $\beta(K_M + \alpha)$, remember K_M was the return of the market portfolio then the error in this approximation is, what is going to be this error? This error is going to be K_V which is the actual observed return minus this approximated return, $\beta K_M + \alpha$ and I will call this error to be ϵ and this approximation is called

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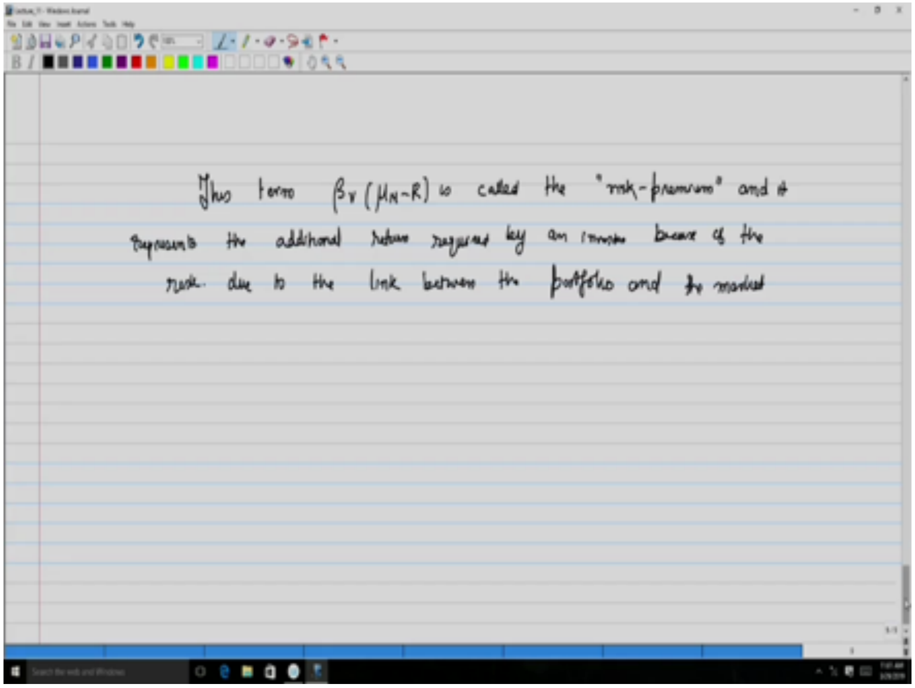

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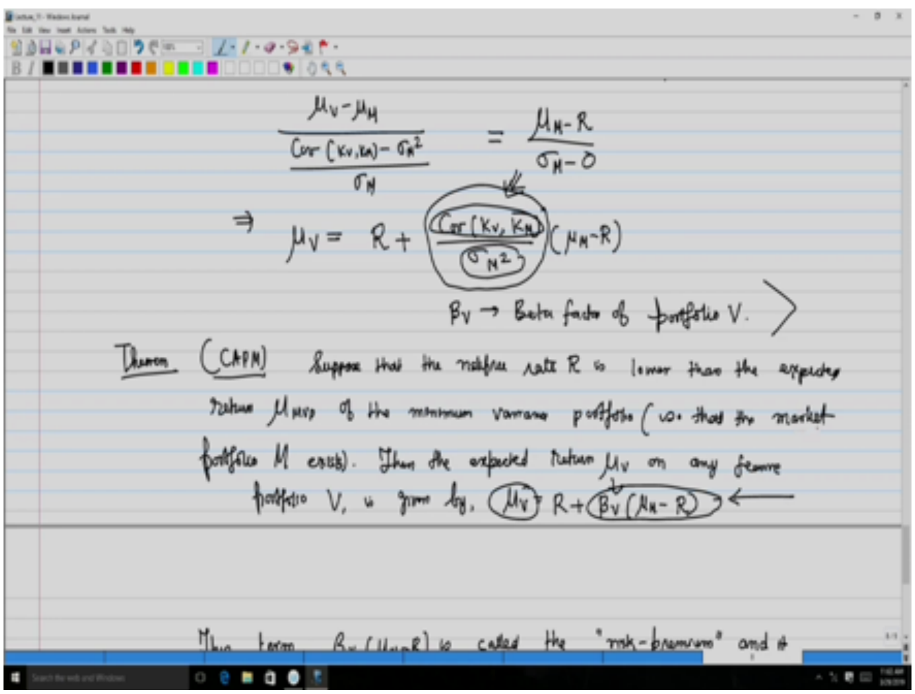

the residual error.

So essentially what I am saying here is that you want to figure out what is going to be a model for the return K_V of the portfolio V by a model of $\beta K_M + \alpha$ and you want to assert in what your β and α are going to be and accordingly you define this particular error and the goal is to minimize this particular error that you have between this value of K_V and then this known function that you have asserted.

So the best approximation in the sense of minimizing the variance far ϵ and this approach is known as linear regression is achieved for $\beta = \beta_v$ and $\alpha = \mu_M - \beta_v \mu_M$, which is the same as $R(1 - \beta_v)$.

So what we are doing is, basically we are making an approximation to K_V making use of this linear

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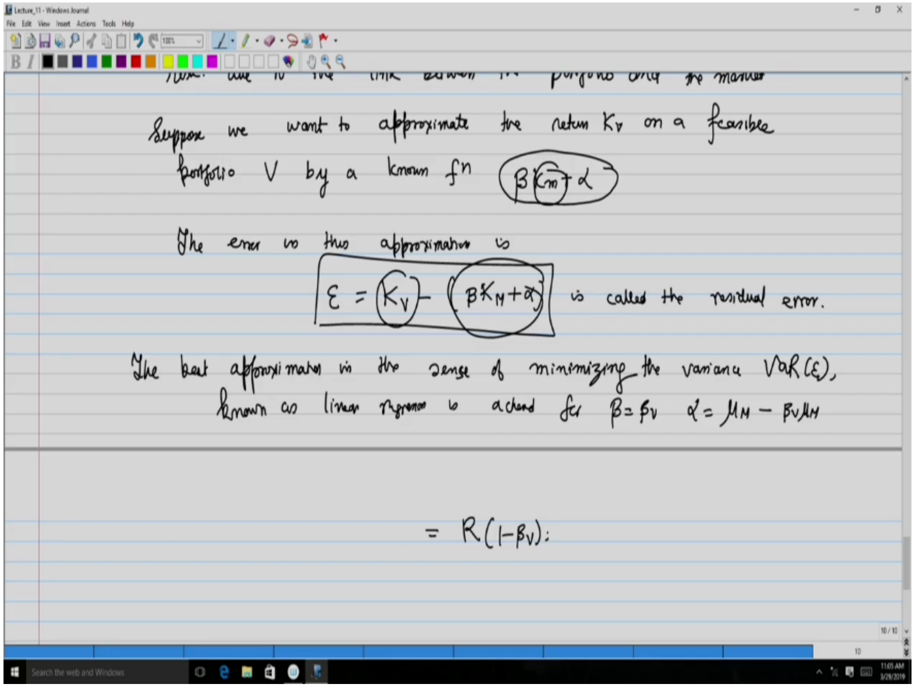

relation and we define the error ϵ as the difference between K_V and the linear approximation and then we take the variance, so this is the residual error and what you want to do is that in order to obtain the best approximation, what we will do is, that we will minimize the variance of this error ϵ and take the derivative with respect to α and β and set it equal to 0 and it turns out that this is achieved for $\beta = \beta_v$ and $\alpha = \mu_M - \beta_v\mu_M = R(1 - \beta_v)$.

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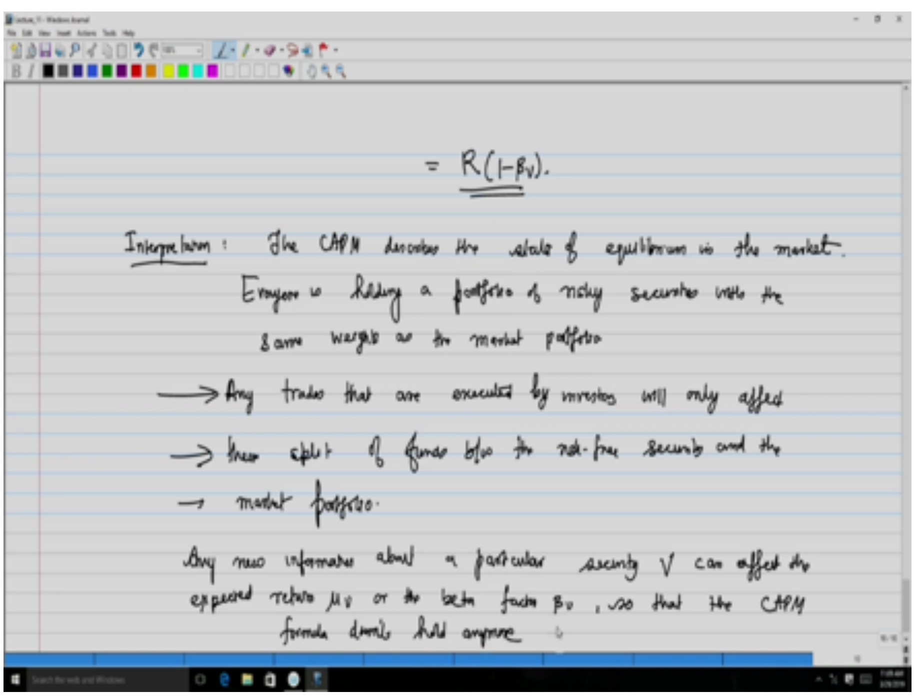

So what does the CAP M do? So let us just make an interpretation of this, of the result that we have done in case of CAP M. So the CAP M describes the state of equilibrium in the market. So basically everyone is holding a portfolio of risky securities with the same weights as the market portfolio and any trades that are executed by investors will only affect their split of funds between the risk-free security and the market portfolio.

So what this means is the following: Initially everyone is at equilibrium and they have invested in the risk-free asset and the market portfolio in some proportion. So which means that everybodys risky investment is identical in nature. However if a disturbance is caused to this equilibrium and by this I mean that any trade that is executed by the investor, that means the investor decides to buy some additional asset or add some additional asset from the risky component namely the market portfolio, then it will affect only their split of the funds between the risky security and the risk-free security.

And so as a consequence of this any new information about a particular security V, so that means the portfolio V might just comprise of only 1 security, can affect the expected return μ_v or the beta factor, β_v, so that as a consequence the CAP M formula that we have derived, this CAP M formula doesnt hold anymore.

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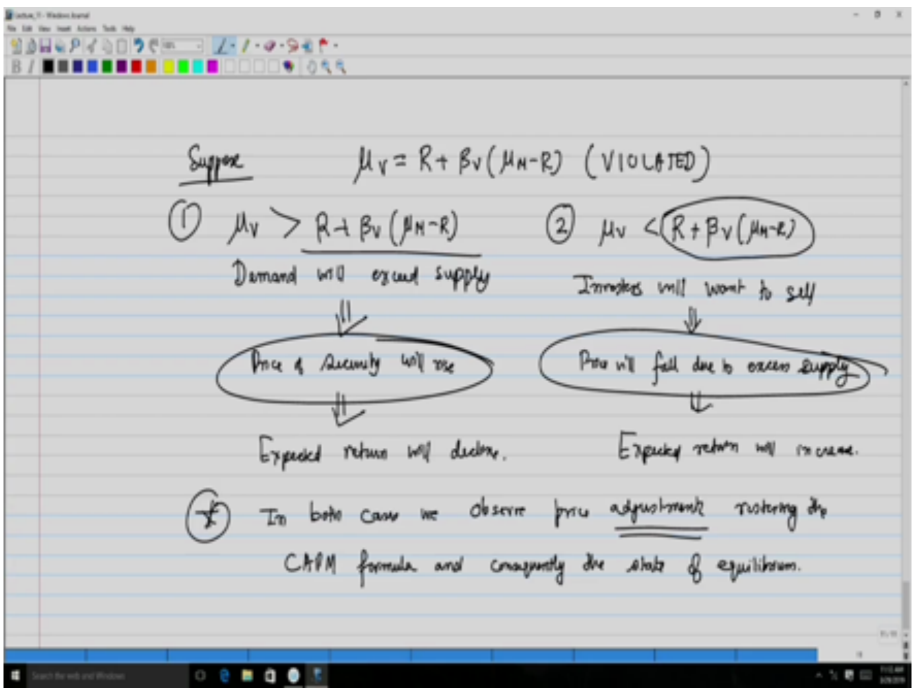

So suppose let us just look at two cases to illustrate this point. So first of all suppose that and so what I am assuming here that this disturbance has caused the $\mu = R + \beta_v(\mu_M - R)$, this has been violated. So then let us look at the two consequent possibilities. So $\mu_v > R + \beta_v(\mu_M - R)$, so in this case the demand will exceed supply because μ_v is greater than the performance given by the CAP M. The consequence of this is when the demand will exceed supply, is that the price of security will rise and when the price rises, so obviously the expected return will decline.

Likewise, let us look at the other case when $\mu_v < R + \beta_v(\mu_M - R)$, so in this case investors will want to sell and the reason they want to sell is that because μ_v is less than what you would get from the CAP M formula, so investors will now want to sell, so when there are large number of investors that are trying to sell, the consequence of this is that the price will fall due to excess supply.

And finally what is going to happen is that, the expected return when the prices fall then the expected return will increase.

So, to sum it up, in both cases we observe price adjustments, right? That means in this case the price of security rose and in this case the price of the security fell, so in both the cases we observe price adjustments restoring the CAP M formula and consequently the state of equilibrium.

So, this brings us to the conclusion of the discussion on the capital market line and CAP M and in particular what we have looked at is that, we have looked at how the market risk can actually play a much more vital role as manifested by the term beta v to the overall risk of the portfolio as compared to the risk that is being contributed to or the risk contribution of each of the individual assets.

Finally, we will go ahead and conclude this discussion on portfolio theory by discussing more about performance analysis in the next class. Thank you for watching.