Mathematical Finance

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Module 3: Modern Portfolio Theory
Lecture 4: Minimum Variance Line (Continued), Market Portfolio

Hello viewers, welcome to this class on this course on mathematical finance. You would recall that in the previous class, we had talked about the mean and the variance of the returns in case of a portfolio. We have already discussed the case of a two-asset portfolio and then we talked about two scenarios, 1st where we seek to obtain the minimum variance portfolio and secondly we showed to determine what is going to be the minimum variance line corresponding to the minimization of the portfolio variance subject to the constraint that the expected return on the portfolio is kept fixed. So we will continue this discussion in todays class.

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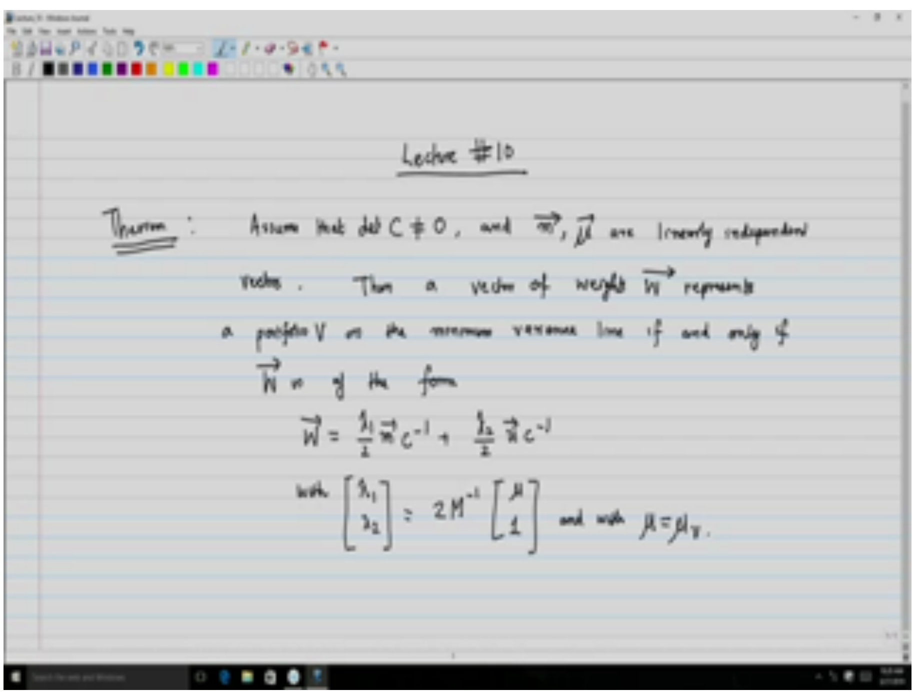

We begin this lecture with a theorem which will summarize the result that we had obtained in the previous class. We assume that the determinant of the covariance of return is not equal to 0. So that C^{-1} exists and that the vector of the mean return and $\vec{u}$ which was the vector of all 1's are linearly independent vectors. Then we can say that a vector of weights W represents a portfolio V on the minimum variance line if and only if this $\vec{W}$ of weights is of the form $\vec{W} = \frac{\lambda_1}{2}\vec{m}C^{-1} + \frac{\lambda_2}{2}\vec{u}C^{-1}$ with the Lagrange multipliers, $[\lambda_1, \lambda_2]^T = 2M^{-1}[\mu, 1]^T$ and with $\mu = \mu_V$.

So now we will move on to a theorem which is known as the two-fund theorem and this will be based on the results that we just stated in case of the variance line and we will look at an implication of this particular two-fund theorem.

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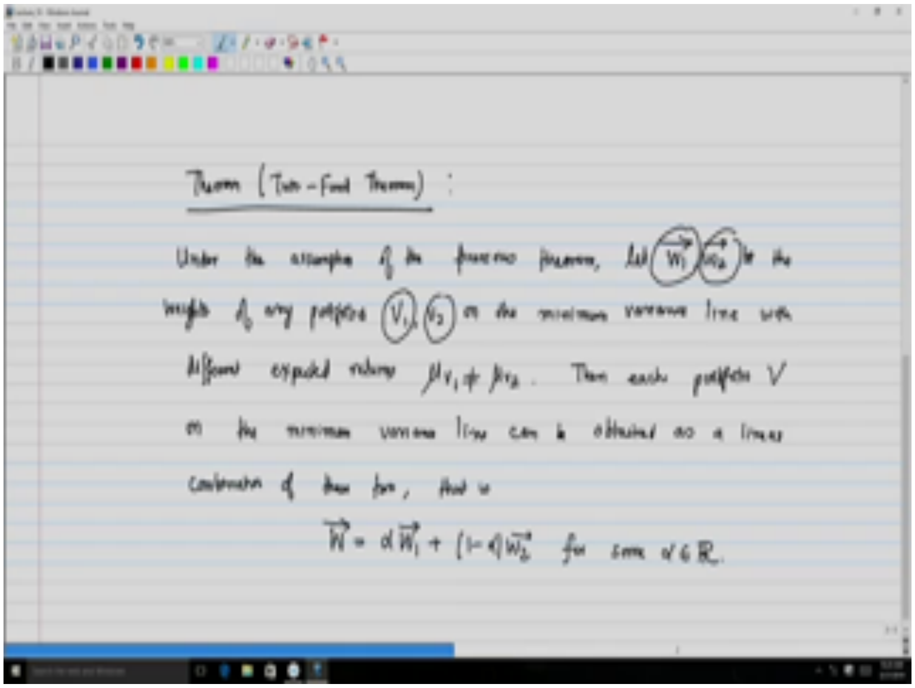

Accordingly, we state the two-fund theorem and as the name suggests, it will basically involve two different funds. So under the assumption of the previous theorem, we let the $\vec{W}_1$ and $\vec{W}_2$ be the weights representing any portfolio V_1 and V_2 respectively on the minimum variance line. So that is why we are considering two portfolios, $\vec{W}_1$ and $\vec{W}_2$ with the corresponding portfolios being identified as V_1 and V_2. And both of them lie on the minimum variance line.

But with different expected returns, that is that the expected return μ_{V_1} and the expected return μ_{V_2} for these two portfolios, they are different from each other. Then each portfolio V on the minimum variance line can be obtained as a linear combination of these two. That is, the weight of this portfolio will be W and that will be a linear combination of $\vec{W}_1$ and $\vec{W}_2$ for some $\alpha \in \mathbb{R}$.

So this means that if I considered two portfolios, V_1 and V_2 with the respective expected returns being denoted by μ_{V_1} and μ_{V_2}, both of which are different from each other, and these two portfolios lie on the minimum variance line, then any portfolio on the minimum variance line can be written as a linear combination of these portfolios. So what it means that you are able to actually generate the entire set of portfolios on the minimum variance line by simply taking a linear combination of these two portfolios, V_1 and V_2 provided that they have different expected returns. So let us look at the proof of this.

So the poof goes as follows. First of all, we find alpha such that mu_V is going to be equal to a linear combination of μ_{V_1} and μ_{V_2} which gives you $\alpha = \frac{\mu_V - \mu_{V_2}}{\mu_{V_1} - \mu_{V_2}}$ and recall that this will exist because $\mu_{V_1} \neq \mu_{V_2}$. So that means you are able to figure out what is going to be the alpha in terms of μ_{V_1}, μ_{V_2} and μ_V. So accordingly, since both the portfolios, remember we chose the portfolios from the minimum variance line.

So since both the portfolios V_1 and V_2 belong to the minimum variance line, they will satisfy obviously that $\vec{W}_1 = \mu_{V_1}\vec{a} + \vec{b}$ and likewise, $\vec{W}_2 = \mu_{V_2}\vec{a} + \vec{b}$. Now consequently $\alpha W_1 + (1-\alpha)W_2$, this is going to be equal to alpha and I replace W_1 with this expression here. So it will be $\alpha[\mu_{V_1}a + b] + (1-\alpha)[\mu_{V_2}a + b]$ and this will become $[\alpha\mu_{V_1} + (1-\alpha)\mu_{V_2}]a + b$. And this is nothing but $\mu_V a + b = W$.

So we have the observation that V belongs to the minimum variance line. So what we have essentially shown here is that we start off with two portfolios, V_1 and V_2, both of which lie on the minimum variance

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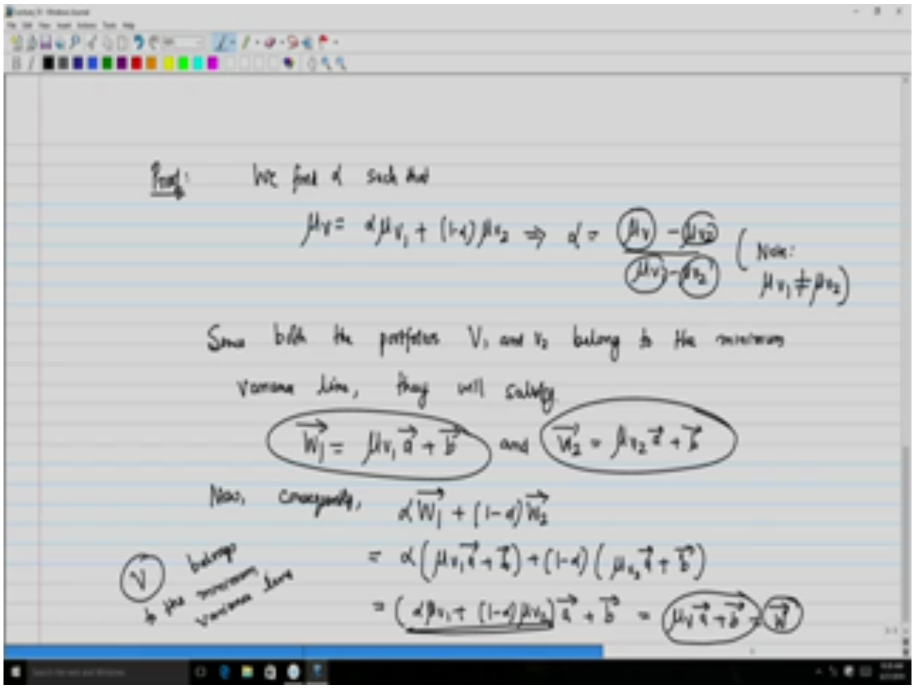

line and then I take a linear combination of these two portfolios resulting in a 3rd portfolio, V. So which means that this portfolio V has the weights given by the linear combination of the weights W_1 and W_2 of the portfolios V_1 and V_2 respectively and it turns out that weight of this portfolio V can be written as a linear combination of $\mu V a+b$. So which means that this weight W here corresponds to essentially a portfolio that lies on the minimum variance line where all the portfolios having weights can be written in this particular form. So let us now look at us the implication of this two fund theorem that we have done.

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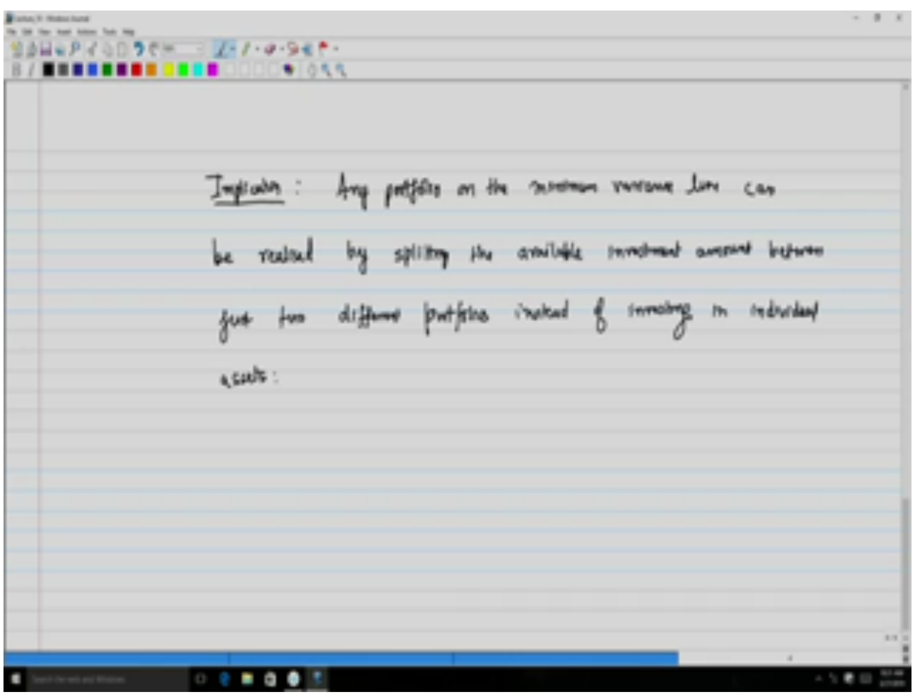

So I will just note down what the implication for this is. The implication for this is the following that any portfolio on the minimum variance line can be realized and by this I mean that you can actually create a portfolio by splitting the available investment amount between just two different portfolios instead of investing in individual assets.

So by this, I mean that if I am trying to invest as driven by a minimum variance line which means that I am seeking to minimize the variance of the portfolio that I will choose while setting my expected return to a certain level then instead of creating a portfolio of individual assets repeatedly, I would rather go and pick two assets on the minimum variance line and depending on what the μ_V is or what I expect out of this new portfolio, I will decide what is going to be my alpha and accordingly decide and invest in those two portfolios V_1 and V_2. So here, the fraction of the amount that I will invest in portfolio V_1 and obviously equivalently that I will invest in V_2, can easily be figured out because it is given in terms of μ_V, μ_{V_1} and μ_{V_2}. So by this, I mean the following.

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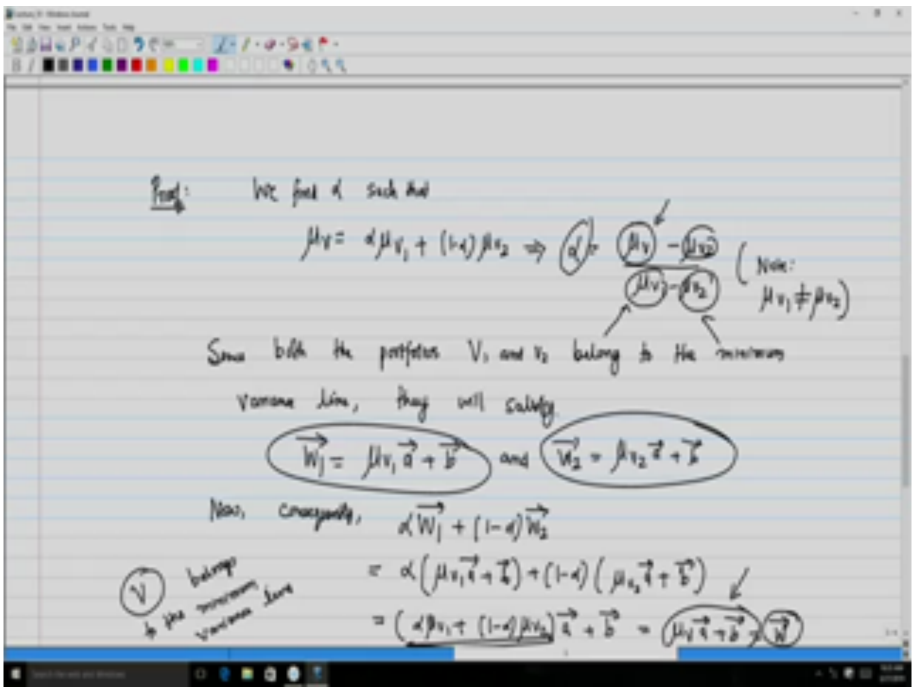

If I go back to this setup, here I have chosen these two portfolios whose expected returns μ_{V_1} and μ_{V_2} are known a priori and then I decide that that I want to create a put for a with this expected return of μ_V and instead of creating a portfolio of several assets, what I decide to do is that I will choose a value alpha, making use of this μ_V and accordingly invest the fraction alpha of my total amount of money in this portfolio V_1 and the remaining fraction of $(1-\alpha)$ in the portfolio V_2 and then the resulting portfolio V that I will get will also see on the minimum variance line but in this case, I have also accomplished the return μ_V that I had set my target for.

Now we state the following proposition and it says the following that the standard deviation σ_V of a portfolio V comprising of a risky security with expected return μ_1 and standard deviation of returns σ_1 and a risk free security with return, say, R and standard deviation zero, depends on the weight W_1 of the risky security as σ_V is equal to absolute value of $W_1\sigma_1$. So this means the following that I am considering a portfolio V. In my portfolio V, there is a risky security and then there is a risk free security.

For the risky security, the expected return is μ_1 and the risk is σ_1 and for the risk free security, the expected return is R which is just a deterministic quantity because it is risk-free and because it is risk-free, the standard deviation in this case is going to be equal to 0. And the corresponding weights that I assign to the risky asset and the risk-free asset are W_1 and $1-W_1$ respectively. Then the standard deviation σ_V of this portfolio comprising of just these two assets, one risky and one risk free, is connected to the standard deviation of the risky asset in the portfolio by this following relation.

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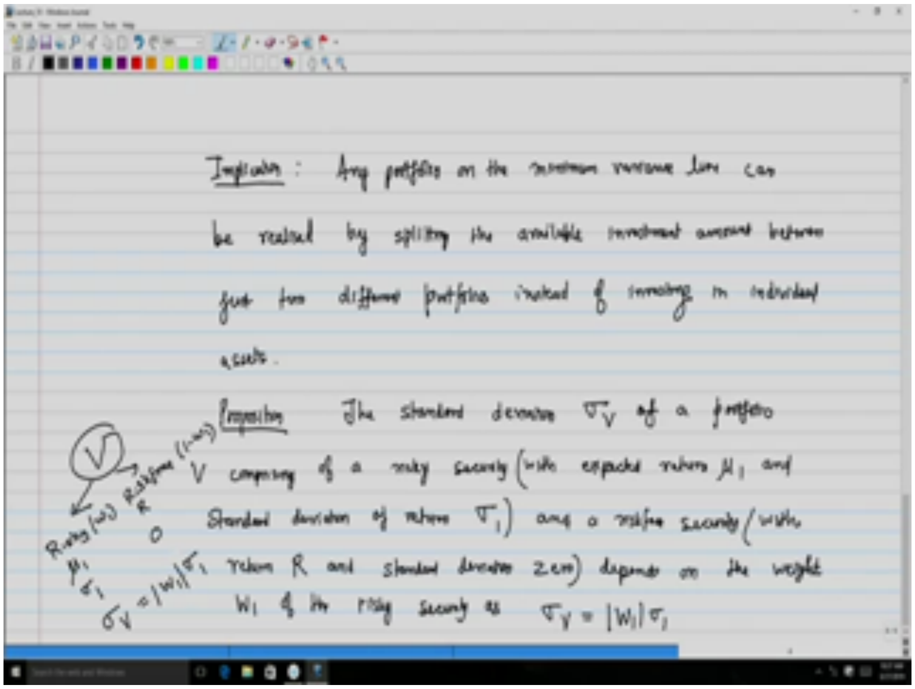

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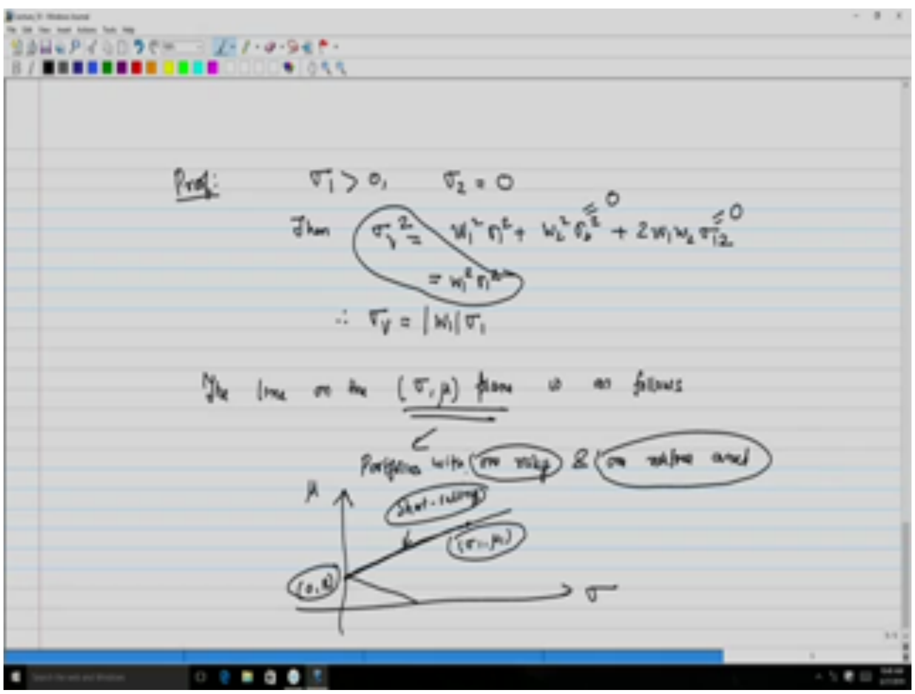

So we can write this proof. It is a straightforward proof. So here obviously, $\sigma_1 > 0$, that is the risky asset has a positive risk and $\sigma_2 = 0$. Then $\sigma_V^2 = W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + 2W_1W_2\sigma_{12}$. So obviously here, $\sigma_2 = 0$ and consequently, $\sigma_{12} = 0$. So we are only left with the term of $W_1^2\sigma_1^2$. So therefore, σ_V is simply going to be absolute value of $W_1\sigma_1$ obtained by taking the square root on both sides of this relation.

So then, the line on the (σ, μ) plane is as follows. And in this sigma plane means this is for portfolios with one risky and one risk-free asset. This will look something like this. This is σ, this is mu, this is $(0, R)$ which corresponds to the risk-free asset or just the investment in the risk-free asset, then this is what the feasible portfolios or all the possible portfolios will look like. And this is basically like a pair of straight lines as opposed to the parabola that we had seen in case of a portfolio just comprising of the risky asset.

And this particular stretch here, this is the region where there is no short selling. So I must all these portfolios, this particular portfolio $(0, R)$ represents just a portfolio which comprises only of the risk-free asset and this portfolio here is a portfolio with no risk-free asset but only just the risky asset. So this brings me to a very important concept known as the market portfolio and market portfolio in this framework of Markowitz or more particularly capital asset pricing model plays a very important role as some sort of a benchmark index because it is representative. In principle, it represents the overall behavior of the market as a whole. So, investing in a market portfolio whose weights we are going to derive now, is almost synonymous to the situation where you are investing in the market as a whole with the weights being assigned to each of the assets available in the market being represented by the proportional market capitalization of all those assets. So we bring, this brings us to the next stage of market portfolio, the next topic.

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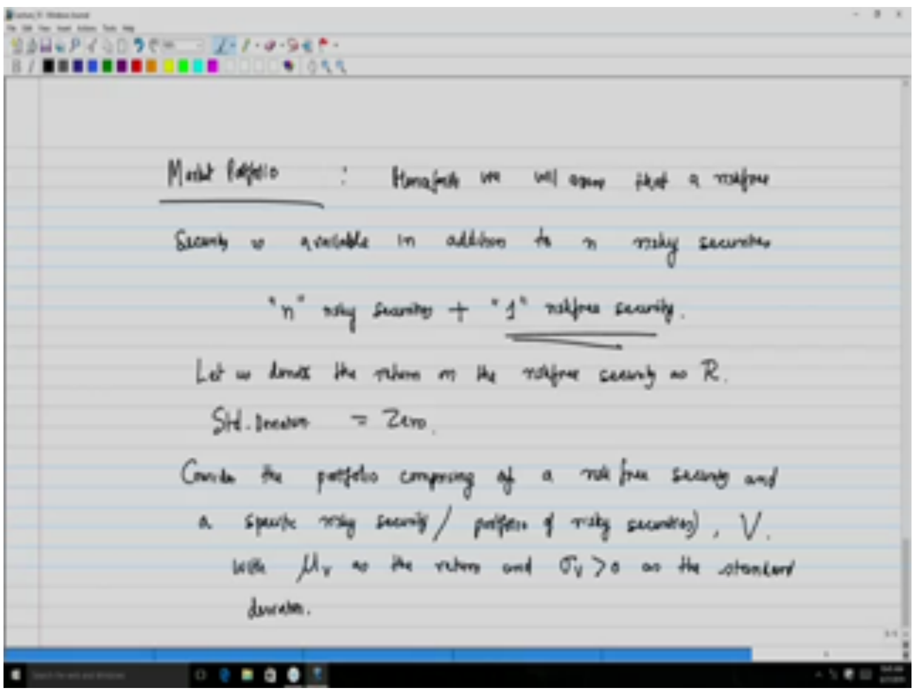

So for the purpose of the discussion from now onwards, so henceforth, we will assume that a risk-free security is available in addition to n risky security. So this means that in the CAPM framework where we had n risky securities, to this we will now add one risk-free security. So now that you are considering a risk-free security, so let us denote the return on the risk-free security as R that you have already seen. So and in this case, the standard deviation is going to be equal to 0.

And we will consider the portfolio comprising of a risk-free security and either a specific risky security or if you want to a more generalized setup, a portfolio of risky securities and I will denote this portfolio as B. So this means my portfolio V will comprise of a risk-free security and then a risky component which can be either in the form of a single risky asset or which will itself be a collection or a portfolio of n number of risky security assets. So then, we take the mean or the expected return of the portfolio as μ_V as the return and $\sigma_V > 0$ as the standard deviation. So this means that μ_V and σ_V are the return and risk for this portfolio which includes both risky and risk-free assets. So what is going to be the efficient frontier in this particular case?

So as we have seen previously in the theorem, we will draw motivation from there and we will see that in the (μ, σ) plane, the set of feasible portfolios, basically will look something like this with this particular portfolio here on the Y axis denoting a portfolio with only the risk free asset, and this is going to be

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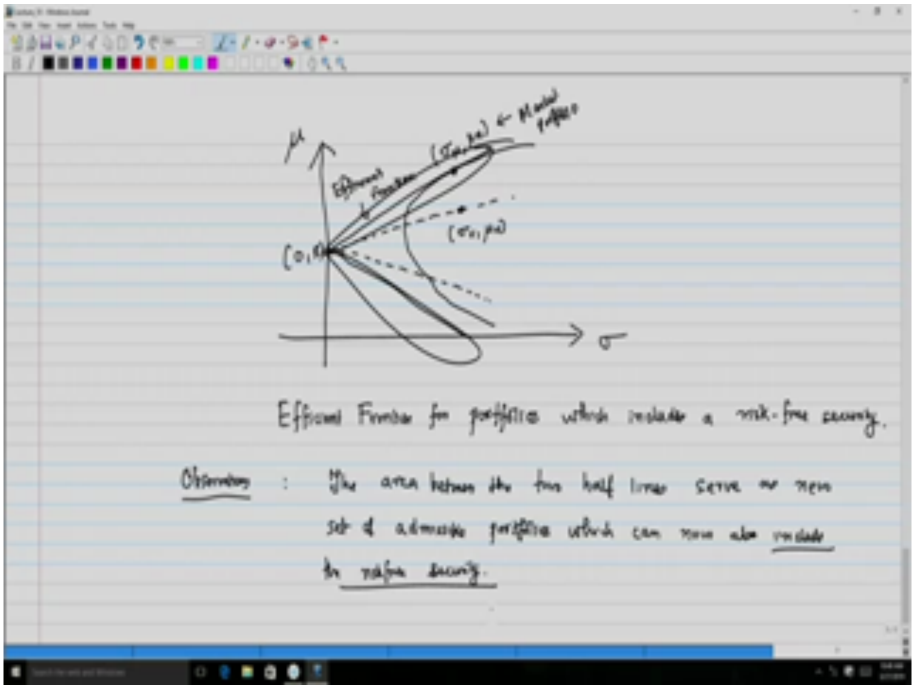

(σ_V, mu_V). And here, this particular portfolio which is at the point of tangency of the parabola and this particular line here, this will have a return and risk which I will denote by (σ_M, μ_M) and this is what will be known as the market portfolio.

And this top line here, this is what is known as the efficient frontier in case of the portfolio V. So this essentially what I have got here is the efficient frontier for portfolios which includes a risk-free security. So I will just make a couple of observations here. The area between the two half lines serves as the new set of admissible portfolios which can now also include the risk-free security. So this means that earlier whatever was below this parabola or to the right of the parabola, that region was the set of all the feasible portfolios and now that and that was the case when you had only the risky assets. And now that when we have moved on to the scenario where we can also include the risk-free assets, then the shape of the feasible portfolios is such that it includes all the portfolios or the region that is between the two lines, namely this line here and this line here. So within that region, whatever lies is known as the or is the collection of the feasible portfolios which includes the risk free security also.

Now let us talk about the efficient frontier. Remember that, in the previous case, we had talked about the efficient frontier after we had defined the feasible set and the efficient frontier was the top part of the parabola which represented either the minimum variance portfolio or the portfolio which gives the minimum variance in case of a fixed level of return or gave the maximum return in case of a fixed level of risk. So likewise, we are going to talk about the efficient frontier in case of this portfolio which includes a risk-free asset.

So accordingly, we can state the following that the efficient frontier of this new feasible set is the half line passing through the exclusively risk-free portfolio, that is $(0, R)$ and a tangent to the hyperbola or the parabola. It looks like a hyperbola or a parabola, representing the minimum variance line constructed from risky securities. So by this I mean, basically that this top part that we have here, this is going to be the efficient frontier and this is tangent to the hyperbola that was obtained from a portfolio which comprises exclusively of the risky asset.

And this particular point of tangency that we have here, this point of tangency is what is known as the market portfolio. So what purpose does bringing this market portfolio into the picture serve and you know, how does it actually help us from the investment point of view? So it helps us in the following way that if you are a rational investor, so every rational investor who is driven or who respects the dominance relations

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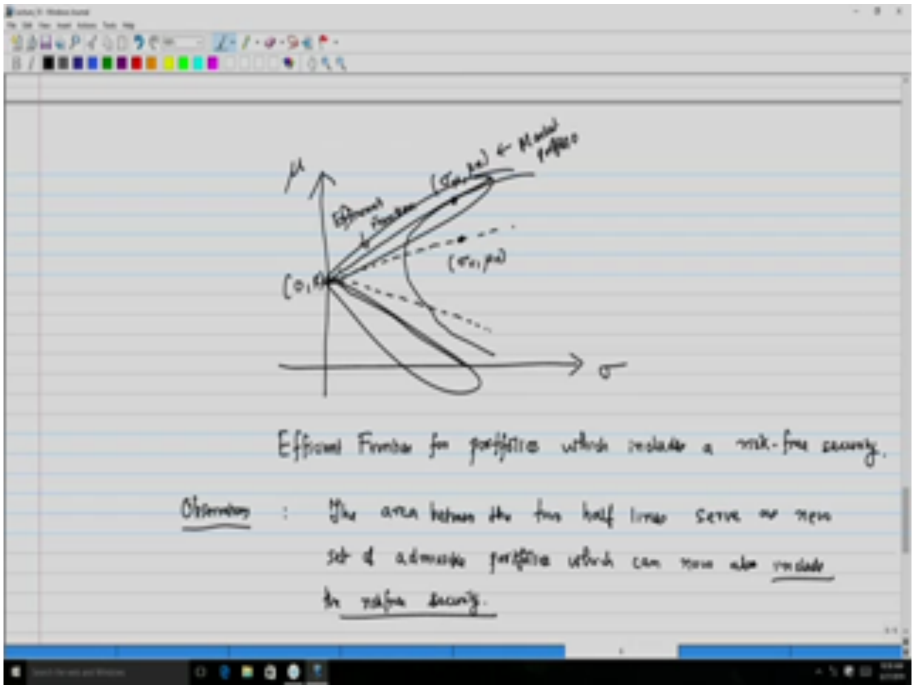

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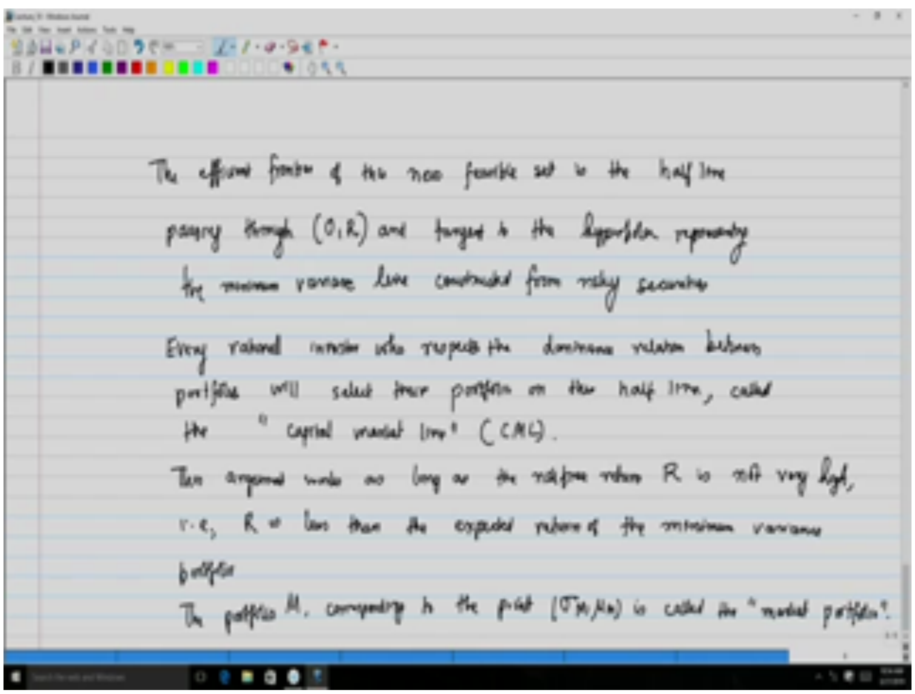

that we have discussed previously, between portfolios will select their portfolio of choice on this half line and this half line is called the capital market line or abbreviated as CML.

So this means that any rational investor who believes in the dominance theory, and decides that they want to invest in a portfolio comprising of risky and the risk-free asset, will then decide to purchase a portfolio from the top half of the line, that means will decide to purchase a portfolio from here. And this particular half line is what is known as the capital market line and we will derive the equation of this at a later stage. Now this particular argument that I have just made will work as long as the risk-free return R is not very high.

That is, by very high means that R is less than the expected return of the minimum variance portfolio. So this market portfolio that we have here, I will denote by M and then I will make my next statement. The portfolio M corresponding to the point of tangency (σ_M, μ_M) is called the portfolio. So the next question naturally that we are trying to address or we will look to address is what exactly is this market portfolio and what are the weights that we are supposed to assign to the market portfolio?

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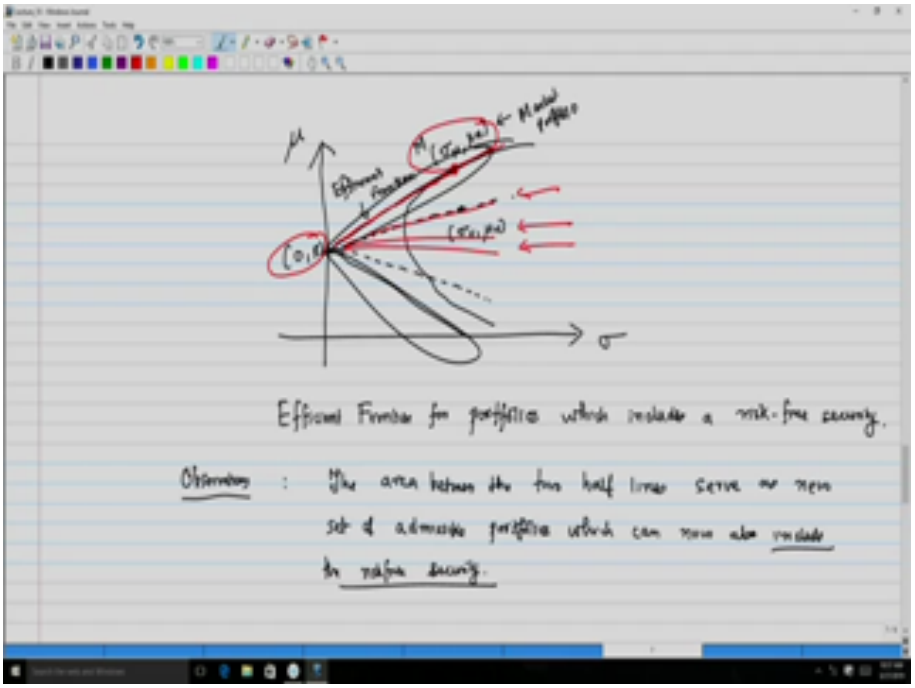

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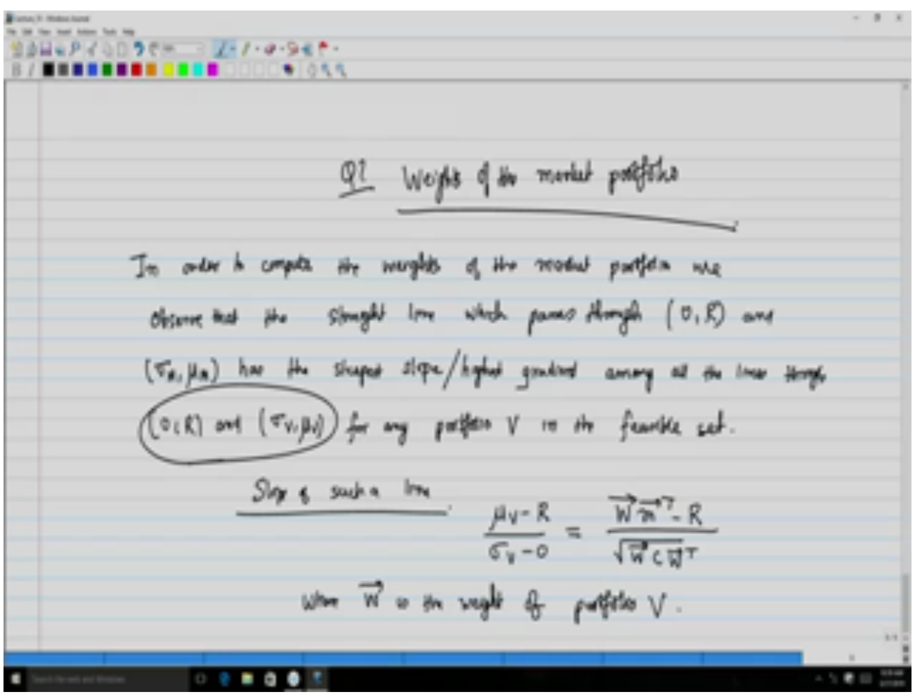

And accordingly, we will address this next and the question that will look at is what are the weights of the market portfolio? So in order to compute the weights of the market portfolio, we observe that the straight-line which passes through the portfolio $(0, R)$, that means the completely risk-free portfolio and the portfolio, the market portfolio (σ_M, μ_M) has the steepest slope or the highest gradient among all the lines through $(0, R)$ and (σ_V, μ_V) for any portfolio V in the feasible set that is constructed from the risky portfolios.

So by this I mean that we observe that we have this particular line which connects with some portfolio, (σ_V, μ_V). There could be many such possible lines and we observe that the line which connects $(0, R)$ with the market portfolio is going to be that particular line amongst all such lines and this means that this particular line which connects $(0, R)$ and the market portfolio will have the highest slope amongst the innumerable different lines which connects $(0, R)$ and (σ_V, μ_V). And so, this is the key factor while we are actually as returning what is going to be the weights of the portfolio M here.

So accordingly, so if I take any line $(0, R)$ and any generic portfolio (σ_V, μ_V), then what is going to be the slope of the line? The slope of that particular line is given as follows. The slope of such a line which connects $(0, R)$ and (σ_V, μ_V) is obviously given by $\frac{\mu_V - R}{\sigma_V - 0}$. What is μ_V?

In the notation, the vector notation that we had used earlier, $\mu_V = \frac{WM^T - R}{\sqrt{WCW^T}}$, where W is the weight of portfolio V. So W is going to be the weight of this particular portfolio in the feasible set of only exclusively risky portfolio. So please keep that in mind. So now, once I have actually given the slope of the lines joining $(0, R)$ and (σ_V, μ_V).

And I am trying to find out what is going to be the weight of the market portfolio and then I mentioned that amongst those lines the market portfolio will be given by that particular line which has the highest slope. So this motivates us to look at the maximization of this particular slope which connects $(0, R)$ with any generic portfolio (σ_V, μ_V).

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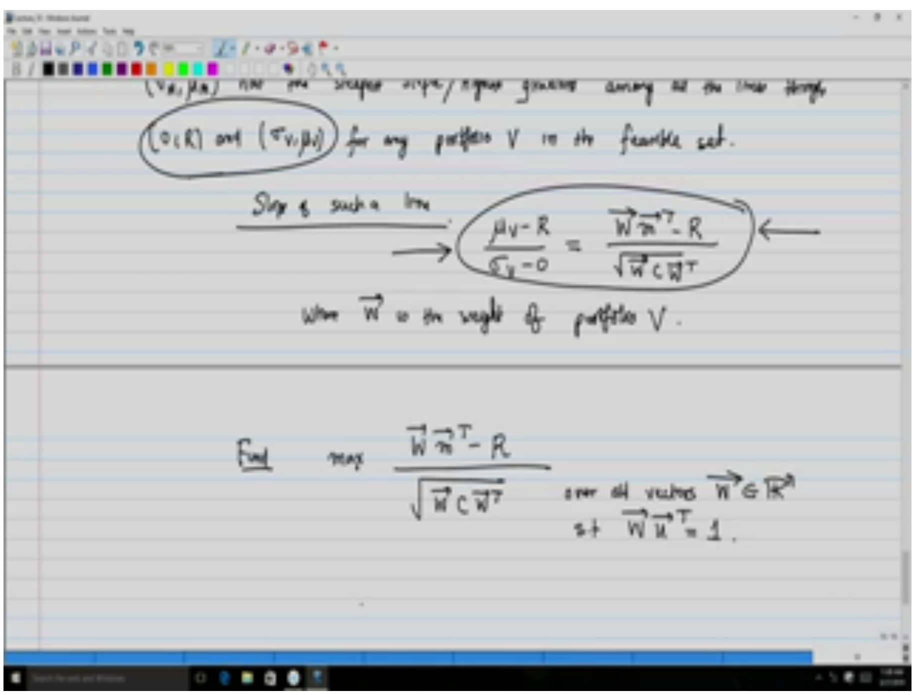

To this end, we then need to find the maximization of this slope $\frac{WM^T - R}{\sqrt{WCW^T}}$ over all vectors $W \in \mathbb{R}^n$ or weight vectors such that the constraint Wu^T awesome of all beats being equal to 1 is satisfied.

Now here, since we have to actually ascertain the weights and these weights are in $\mathbb{R}^n$, so obviously we have to resort to the method of Lagrange multipliers as we have used in certain cases previously. So therefore, we define the Lagrangian $F(W, \lambda)$, there is only one Lagrange multiplier since there is only one constraint, namely sum of weights being equal to 1 this will be given by $\frac{Wm^T - R}{\sqrt{WCW^T}} - \lambda(Wu^T - 1)$.

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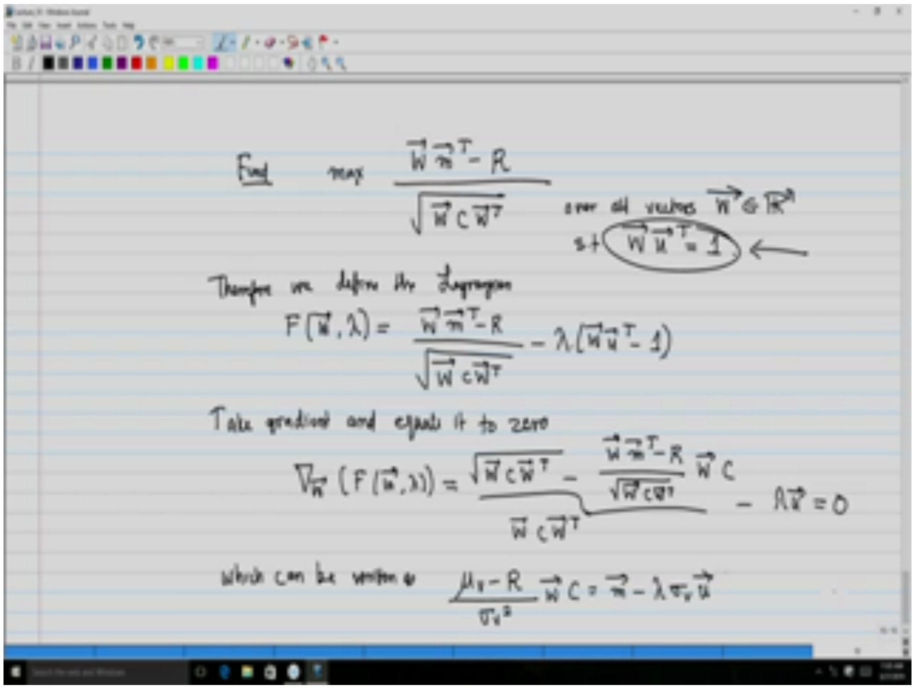

Now in order to maximize this, what we do is that we take the gradient and equate it to zero. Alternatively, you can actually open up the entire expression for $F(W, \lambda)$ in terms of the component weights, W_1 to W_n and take the derivative of F, the partial derivative of F with respect to W_1 to W_n and collate them again and the result will be, and said that equal to 0 and the consequent result would be as follows.

$$\nabla_W F(W, \lambda) = \frac{\sqrt{WCW^T} - \frac{Wm^T - R}{\sqrt{WCW^T}} WC}{WCW^T} - \lambda u = 0$$

and which can be written as

$$\frac{\mu_V - R}{\sigma_V^2} WC = m - \lambda \sigma_V u.$$

So now, we have to further simplify this. So what we do is that we have to figure out what is going to be our λ.

So accordingly, we will multiply by the vector W^T on the right and get

$$\frac{\mu_V - R}{\sigma_V^2} WCW^T = \mu_V - \lambda \sigma_V,$$

which implies that

$$\frac{\lambda = R}{\sigma_V}.$$

So from here, we can conclude that therefore

$$\gamma WC = m - Ru, \quad \text{where} \quad \gamma = \frac{\mu_V - R}{\sigma_V^2},$$

so that

$$\gamma W = (m - Ru)C^{-1}.$$

Now here we make an observation.

Note that gamma is dependent on W. I have the γ here and this is dependent on W because we have the term μ_V and σ_V^2. And so obviously, γ here, up to this point is dependent on something that has not yet been

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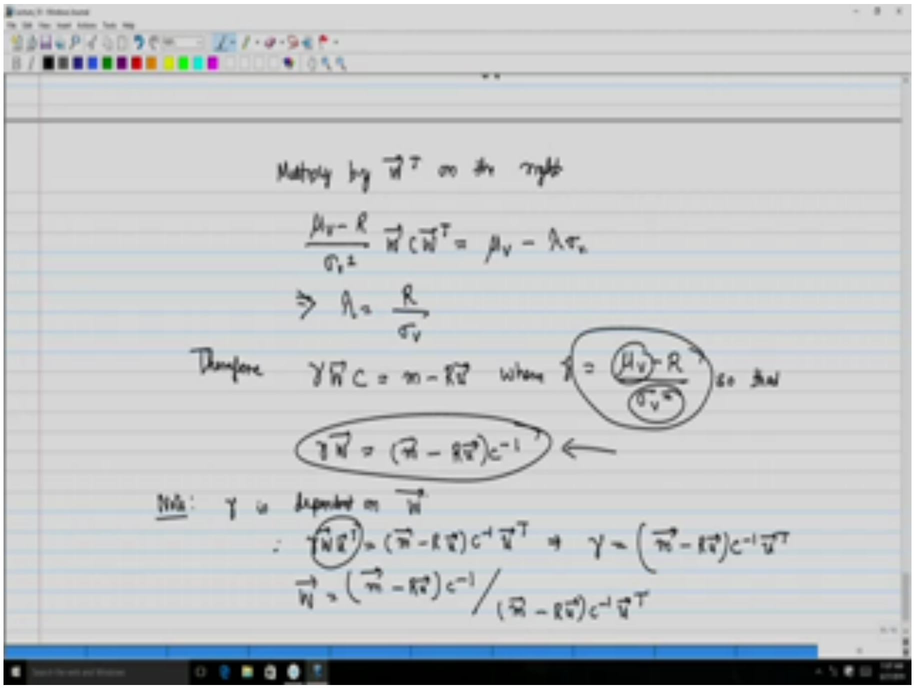

ascertained. So we end up getting some sort of an implicit relation. So in order to ascertain to handle this, we have $\gamma W u^T = m - RUC^{-1}u^T$. So that means we multiply both sides by post multiply both sides by u^T and what this gives you is the following that γ, remember $Wu^T = 1$.

So this will give you $\gamma = (m - Ru)C^{-1}u^T$. So that means, the weight W, so I can now replace the value of gamma here to obtain the weighted $W = \frac{m-RuC^{-1}}{m-RuC^{-1}u^T}$. So now we can sum up the entire discussion as this following theorem.

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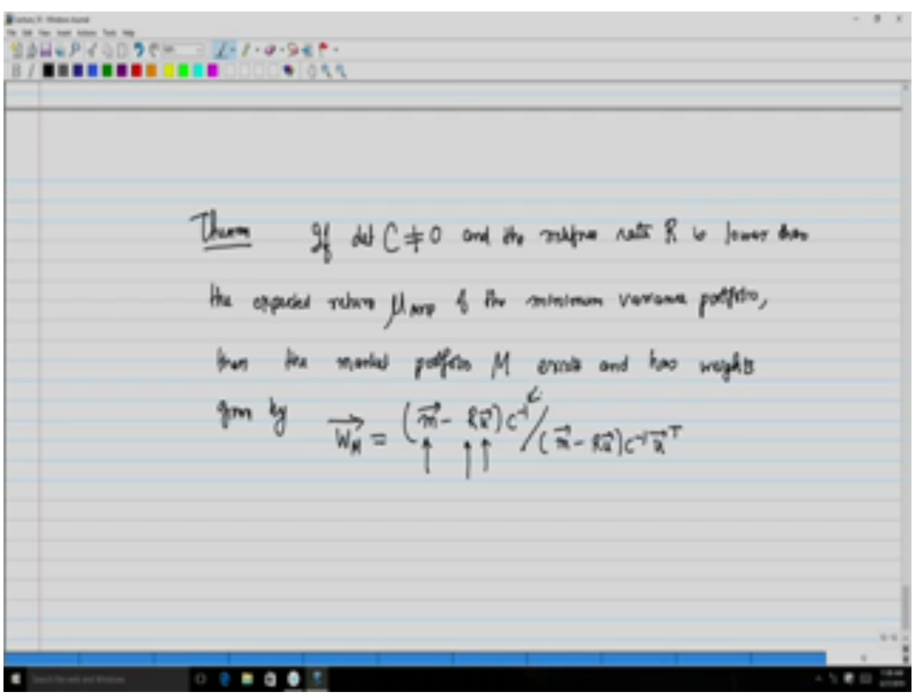

If $detC \neq 0$ and the risk free rate R is lower than the expected return μ_{MVP} of the minimum variance portfolio, then the market portfolio M exists and has weights given by W_m as m. So I just, I have already ascertained what my weights are.

So this is given by $\frac{m-RuC^{-1}}{m-RuC^{-1}u^T}$. So this is how we actually obtain the market portfolio. Here m is the vector of all returns, u is the vector of all 1's, R is the risk-free rate, C is the co-variance matrix and so accordingly, all the quantities are known and we can explicitly obtain the weights for the market portfolio.

So this brings us to the end of todays lecture and in the next class, we will look at for the final topics that we had in portfolio theory, namely the capital asset pricing model, CAPM and some implications of it, especially in the context of the capital market line. Thank you for watching.