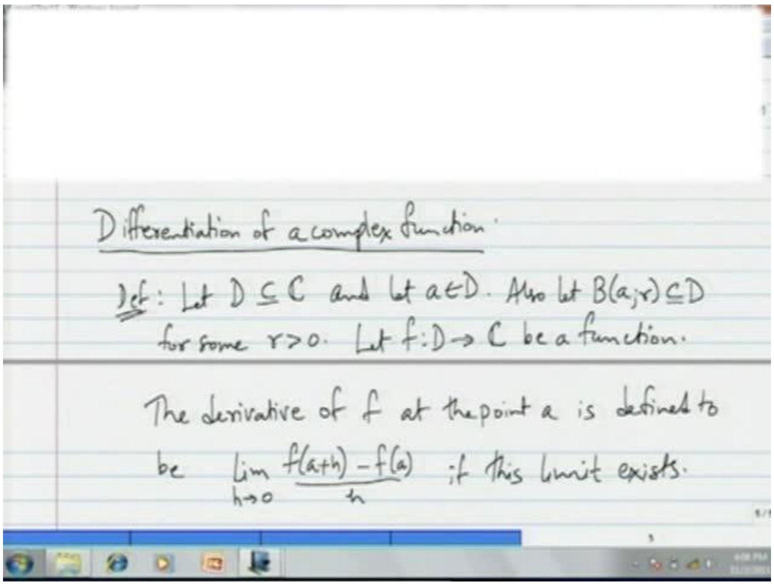
Complex Analysis
Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 2
Complex Functions: Limits,
Continuity and Differentiation
Lecture - 3
Differentiation

(Refer Slide Time: 00:27)

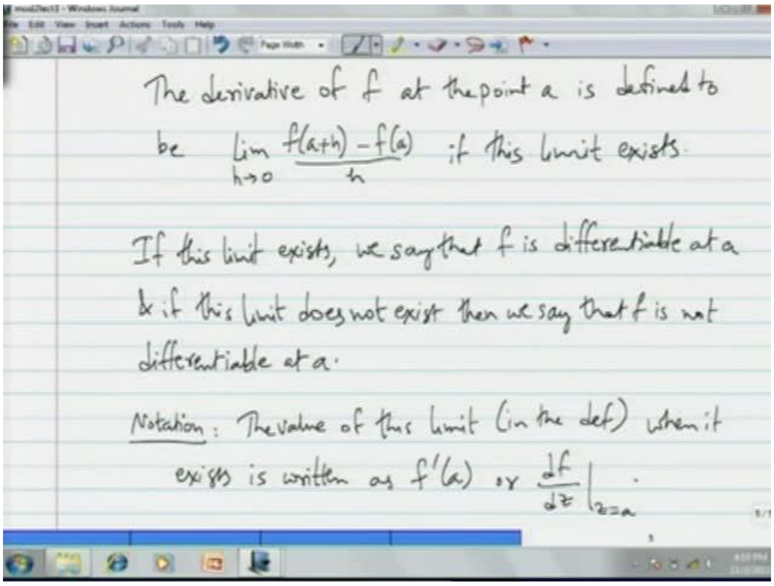


Let us look at differentiation of a complex function. So, the concept of of the derivative of a complex function is motivated by well the concept of derivative of a real valued function of a real variable and is defined very similarly, so here is the definition. Let D be contained in C and let a be a point in set D. Also, let the ball, the open ball of radius r be contained in the set D around a open ball of radius r around a be contained in D for some number r, real number r greater than 0, strictly greater than 0.

Now, let f from D to C be a function. Now, we will define the derivative of f at the point a. Under these assumption the derivative of f at the point a is defined to be or defined to be the limit as h approaches 0 of f of a plus h minus f of a divided by the complex number h which is approaching 0 if this limit exists, this limit exists. So, this is the difference quotient which is familiar to the viewer from functions of one real variable.

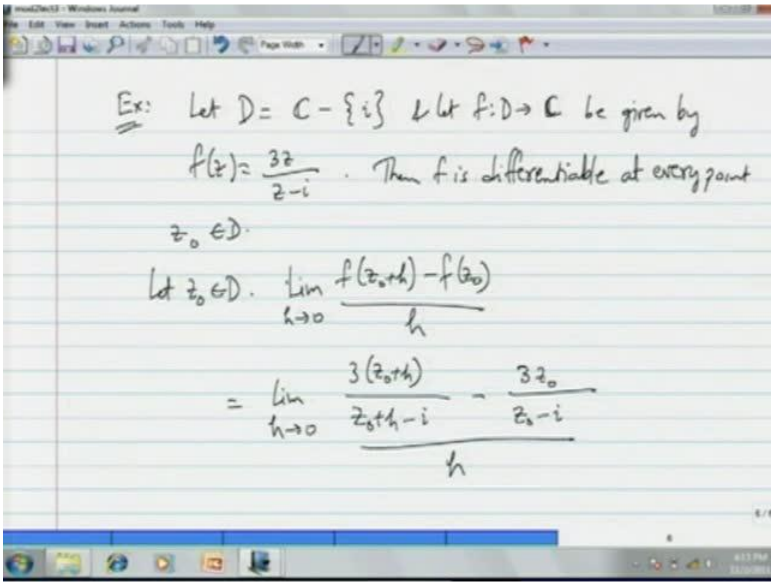
So, the definition must be familiar to the viewer as well, but here the context is that we have complex function and the difference quotient is complex quotient. So, f of a plus h is a complex number, f of a is a complex number, the difference is a complex number and we divide by other complex number h which is approaching 0.

(Refer Slide Time: 03:27)



And quickly let me say that if this limit exists we say that f is differentiable at a and if this limit does not exist then we like wise say that f is not differentiable at a. And we use the following notation for derivative, inspired by function of one real variable. The value of this limit in the definition when it exists is written as f prime of a or as d f by d z at z equals a. We will more often use the notation f prime of a rather than d f by d z at z equals a.

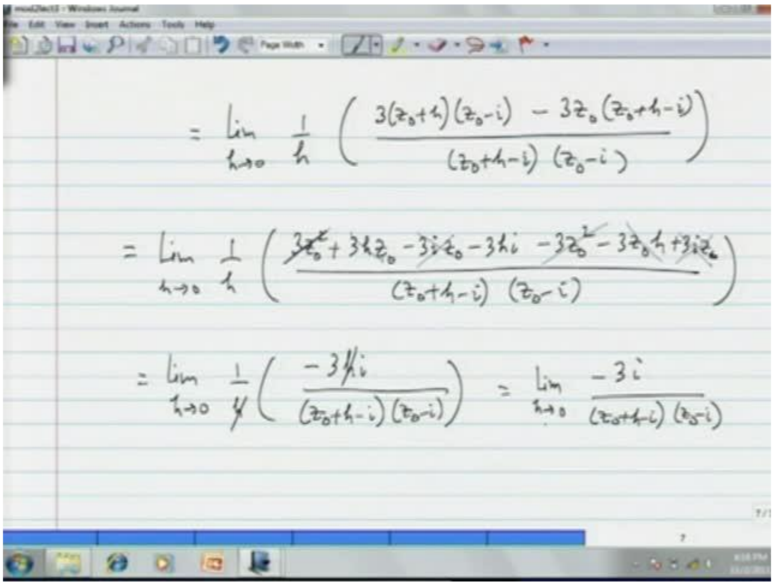
(Refer Slide Time: 05:17)



So, let us see an example of of a function which is a differentiable. So, let D equals C minus the complex number i. So, it is all of the complex plane except the point i and let f from D to C be defined by, can be given by f of z is equal to 3 z by z minus i. So, it is a rational function and then f is differentiable at every point z naught and D. So, let us see why, let us see why f is differentiable at every point z naught and D.

So, let z naught belong to D like it is given and let us compute the limit of the difference quotient f of z naught plus h minus f of z naught by h. So, this is equal to, from the definition of the function this is this is equal to limit as h goes to 0, 3 times z naught plus h divided by z naught plus h minus i that is your f of z naught h minus 3 times z naught by z naught minus i. That is your f of z naught divided by h.

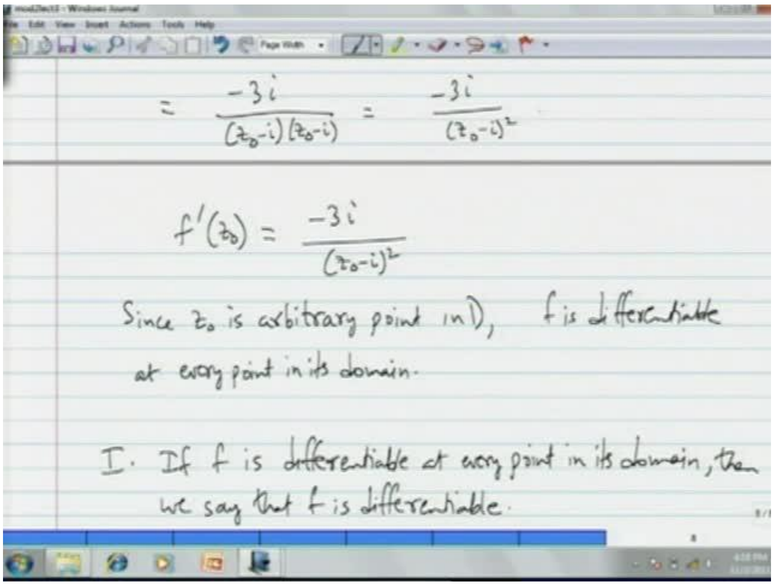
(Refer Slide Time: 07:23)



So, this in turn is the limit as h goes to 0, 1 by h times. Let us clear the fraction and in the numerator and we get 3 times z naught plus h times z naught minus i minus 3 z naught times z naught plus h minus i divided by z naught plus h minus i times z naught minus i. So, this gives me the limit as h goes to 0. Let us now multiply everything out in the numerator so that I get 3 z naught square plus 3 h z naught minus 3 i z naught minus 3 h i. And then the second term here is minus 3 z naught square upon expansion minus 3 z naught h plus 3 i z naught divided by z naught plus h minus i times z naught minus i.

So, this is the limit as h goes to 0. Then we have 1 by h times. So, let us observe cancelations. So, 3 z naught squared cancels with 3 z naught square. Just a minute, let us pick a colour. So, 3 z naught square cancels with 3 z naught square and then 3 h z naught cancels with a 3 z naught h minus 2 z naught h and likewise 3 i z naught cancels with 3 i z naught and finally, we are left with minus 3 h I. So, minus 3 h i divided by z naught plus h minus i times z naught minus i. So, since h approaches 0 and h does not equal 0 you can cancel the h in the numerator and the denominator and then this is limit as h goes to 0 of minus 3 i divided by z naught plus h minus i times z naught minus i. So, now we can let the limit as h goes to 0 come into the picture.

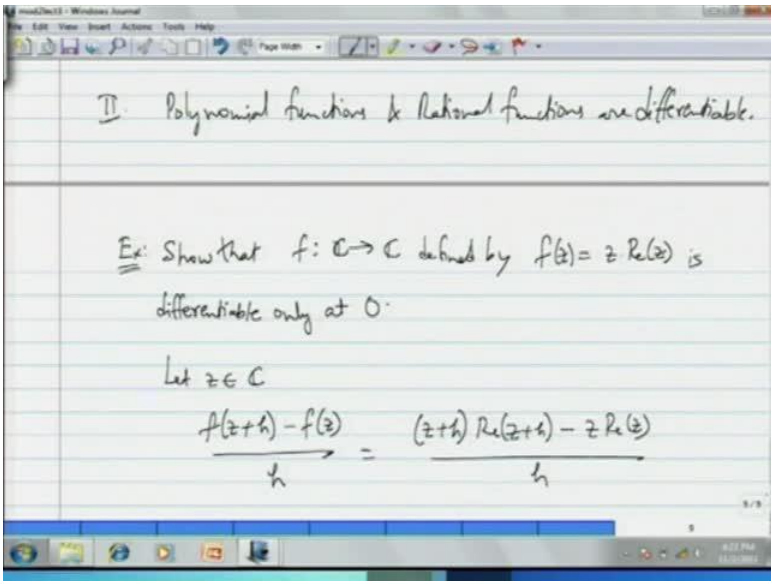
Refer Slide Time: 10:16)



And notice that this is a rational function of h assumptions where z naught is, z naught and i and other things 3 etcetera are constants. So, the value of this rational function as h goes to 0 is obtained by substitution h is equals 0 like I commented earlier. So, then this is minus 3 i by z naught plus now h can be assumed to be 0 so this is z naught minus i times z naught minus i. So, you get minus 3 i by z naught minus i square and since so, this limit exist not only does this limit exist we have found out what f prime of z naught is.

So, f prime of z naught is minus 3 i by z naught minus i square and since z naught is arbitrary point in the domain f is differentiable at every point in its domain. And so when a function is differentiable at every point in its domain we simply say that f is differentiable. So, like for continuity where we say the f is continuous simply when it is, when it is continuous at every point in its domain. So, remark one, if f is differentiable at every point in its domain, then we say that f is differentiable. f is differentiable without any difference to the point where it is differentiable.

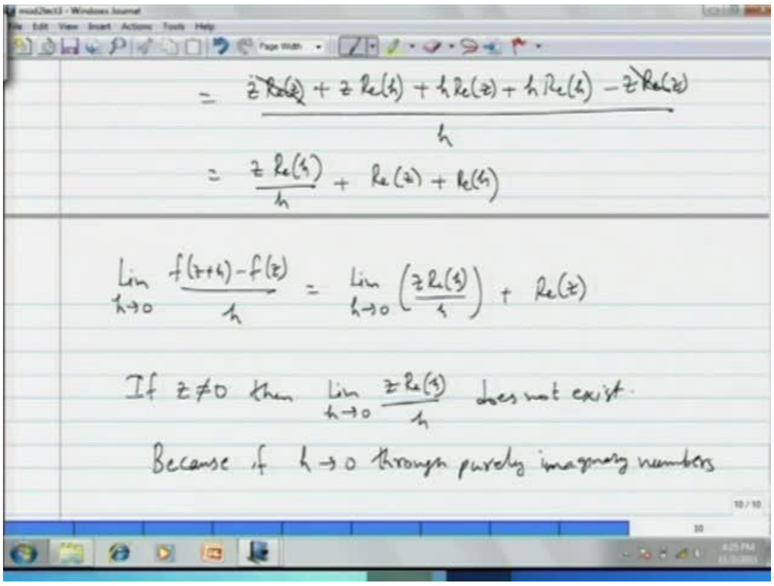
(Refer Slide Time: 12:52)



And so the second remark is polynomial functions and rational functions are differentiable. So, the phenomenon exhibited by the f in our example is actually characteristic of any rational function and and any rational function is differentiable at every point in its domain and likewise polynomials it is easy to show are differentiable. Let us look at yet another example. Here is an example of a function which is peculiarly differentiable only at a single point. So, show that the function defined from complex numbers to complex numbers defined by f of z equals z times real part of z.

So, it is a product of the complex number z with the real part of the z is differentiable, show that this is differentiable only at 0, only at r origin 0. So, let let us show this. Let z belong to C and let us examine the difference quotient f of z plus h minus f of z divided by h. And this is by the definition of the function z plus h times the real part of z plus h minus z times the real part of z divided by h. So, when a…

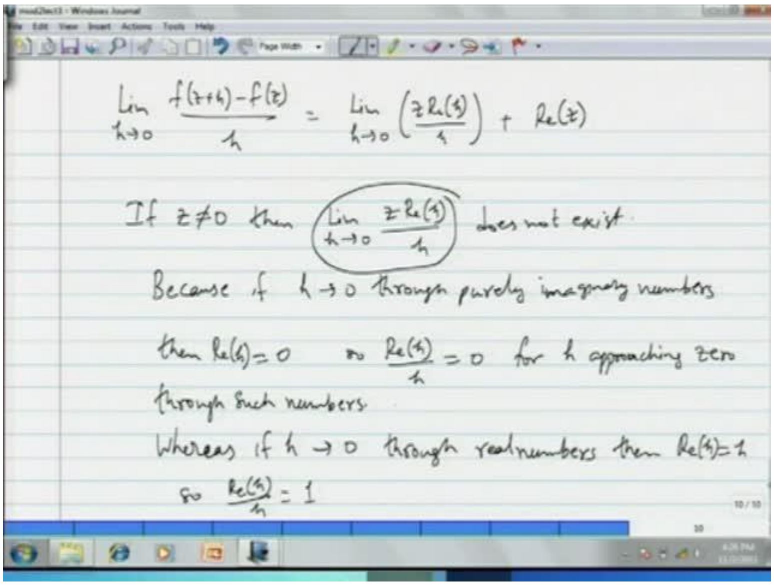
(Refer Slide Time: 15:07)



Well, further this is actually equal to z times the real part of z because the real part of z is plus h is the real part of z plus the real part of h. So, this is the z times the real part of z, z plus z times real part of h plus h times the real part of z plus h times that real part of h minus z times the real part of z divided by h. So, we thus get z times the real part of h after cancelation here and here divided by h plus the rest of terms have an h in them. So, we can cancel with the denominator as long as h is not 0.

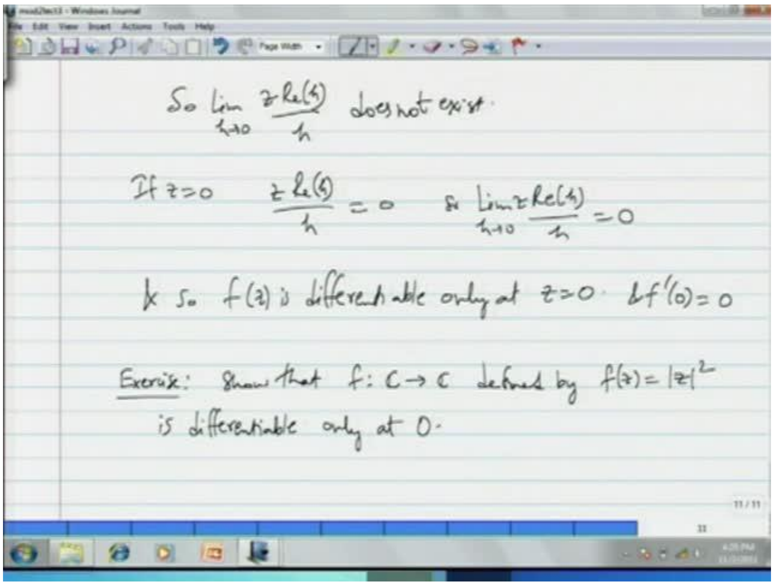
Well, in the first place we cannot divide by h if h is 0. So, this is plus real part of z plus real parts of, real part of h. Now, the limit let us now look at what happens when we try to take limit as h goes to 0 of f of z plus h minus f of z by h, this is the limit as h goes to 0 of z real part of h by h plus well the real part of z plus the real part of h will be simply the real part of z as when h approaches 0 the real part of the h approaches well 0. So, we just have real part of z. Now, but if z is not equal to 0 then the limit as h goes to 0 z times real part of h by h. This limit does not exist. Why? That is because if h tends to 0 through purely imaginary numbers. So, through numbers which are, which have the real part as 0, then then the real part of h is 0.

(Refer Slide Time: 17:58)



So, the limit is well so then that happens. So, real part of h by h is 0 for h, for h approaching 0 through such numbers whereas, whereas if h tends to 0 through purely real numbers, through real numbers which means the imaginary part of such numbers is is 0. Then the real part of h is h itself so that real part of h by h which appears here in the limiting process is actually equal to 1 and there is… So, then there are two ways of approaching 0 which produce different z real part of h by h. So, one produces z time 0 constantly and the other producers z times 1 constantly, z times 1. So, this limit does not exist.

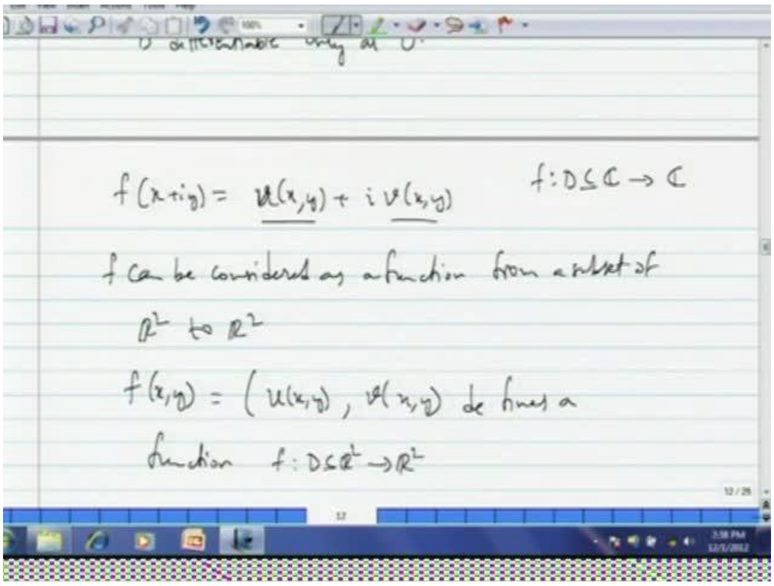
(Refer Slide Time: 19:36)



So, limit h goes to 0 z times real part of h by h does not exist, but at the other hand we are lucky if z is equal to 0. If z is equal to 0 we do not have to worry about how h approaches 0 already the the quotient z times real part of h by h is equal to 0. So, this limit exists. So, let us let us go back to, allow me to go back a little bit and look at the difference quotient here. You see that the limit as h goes to 0 of this difference quotient boils down to this limit we have been examining and plus real part of z.

And we showed that this limit exists only when z is equal to 0. So, when z is equal to 0 this function f of z is differentiable and when z is not equal to 0 this said limit does not exist and so the function is not differentiable. So, let me conclude that and so f of, f of z is differentiable only at z equals 0 and we know that f prime of 0 is 0. So, that is an example, peculiar example we have to bear in mind where the function is differential only at single point and here is an exercise for the viewer. Show that f from C to C defined by f of z is equals the modulus of z square is differentiable only at the origin and where else of course. If f is a complex function firstly, it can be considered as a function from subset of R 2 to R 2 that we already saw.

(Refer Slide Time: 22:20)

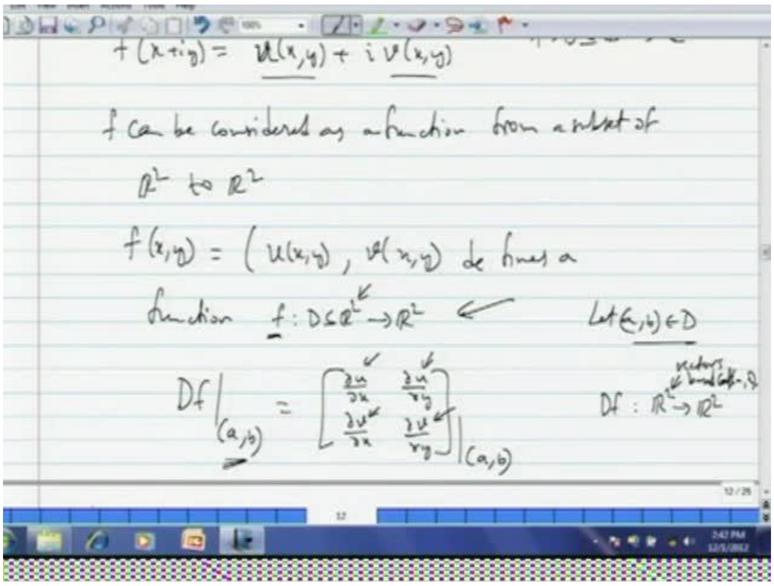


So, if f of x plus i y is v of x sorry u of x comma y plus i times v of x comma y where u and v are the real and imaginary parts of f throughout its domain. So, f is from D contained in C to C. So, then so u and v are the real and imaginary parts of f. Then f can be considered as a function, f can be considered as a function from a subset of R 2 to R 2. So, by forgetting the complex structure what we can do is we can say f of x comma y so this is a really a different function, but let me use the sane notation f, f of x y is equal to u of x comma y comma v comma x comma y defines a function f from D contained in R 2 now to R 2, this we have already seen.

So, then one can ask that well we have the notion of differentiability for function from R 2 to R 2. And then now we have a new definition for complex differentiability so then are the notions same or are the notions are different and to what extent. So, I will remind the viewer that if a function from R 2 to R 2, if if f is the function from R 2 to R 2 then the differentiability of f is defined in terms of a matrix.

So, the f is, the differentiation of f near a point or at a point a b is a linear approximation of the function near that point and so if you need a linear, linear approximation function from R 2 to R 2 you need to consider linear transformation which then becomes a matrix. Whereas here we have the differentiation of a complex function as a complex number. So, there is difference between these two notions, but we will try to reconcile the difference in a, in the short while.

(Refer Slide Time: 25:06)

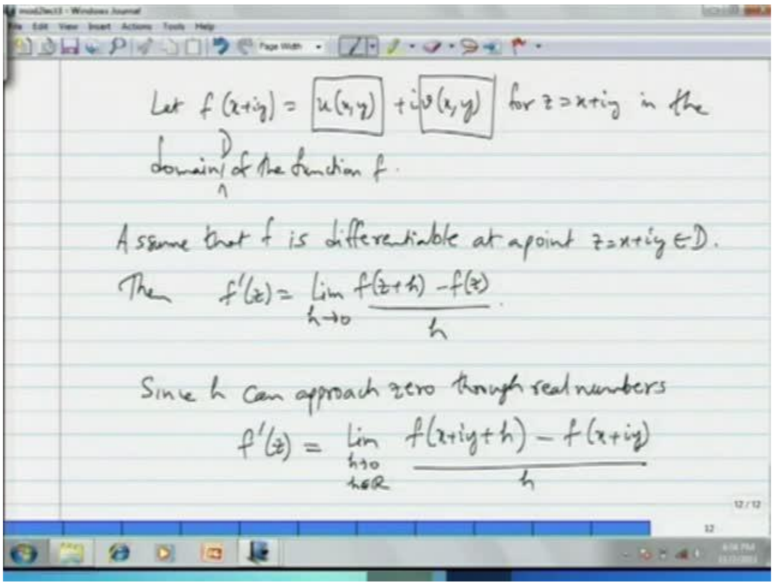


So, in this, in this (( )) of f. D f, I will remind the viewer d f at a point a b is going to be, well a b is point in the domain and I am assuming that f is differentiable at a b. Then D f at a b is a linear approximation of f at the point a b and it is dou u by dou x dou u by dou y dou v by dou x and dou v by dou y all these partial derivatives of u and v evaluated at the point a b.

So, the linear approximation of the function f or the differentiation as we defined it exists if and only if the partial derivatives of f, all of them, all these dou u by dou x dou u by dou y dou v by dou x and dou v by dou y exist and are continuous at the point a b. And when they exist and are continuous we have, D f is this. If you take a point, well a vector in R 2 based at the point a b then D f takes vectors in R 2 near vectors in R 2 to vectors in R 2. So, R 2 vectors this is vectors near the point a b or based at the point a b, based at the point a comma b.

That was a sitting for function of two real variables and whose range is subset of R 2 and we see, we will first see that the partial derivatives dou u by dou x dou u by dou y and dou v by dou x and dou v by dou y satisfy certain relations amongst themselves. And these are called Cauchy Riemann equations which we are now going to examine.

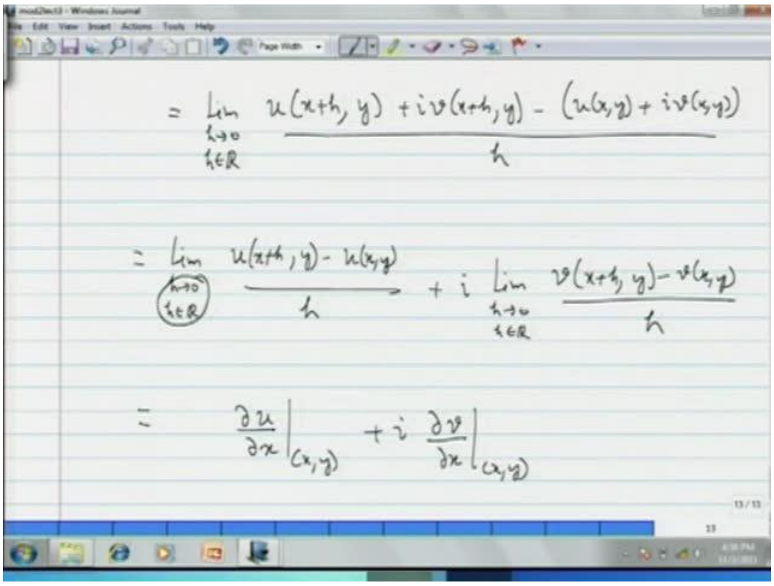
(Refer Slide Time: 27:19)



Now, assume that so under under this assumption assume further that f is differentiable at a point z is equal x plus i y belongs to D. Then of course, the limit f prime of z is the limit as h tends to 0 f of z plus h minus f of z by h. So, like in the example we saw preceding this discussion h can approach 0 through various ways or at least through two different ways, one is through purely a real numbers and one is through purely a imaginary numbers. So, let us examine what happens as a consequence.

Then, since h can approach, h can approach 0 through real numbers, f prime of z is equal to of course, now we are assuming that this limit exists. So, the value of this limit is also equal to limit as h goes to 0 through real numbers. Here, I am writing h belongs to R, what that means is h is a real number and h approaches 0 of f of x plus i y. Allow me to write z as x plus i y by our assumption plus h. Here, h has no imaginary parts so it is simply h as a real number minus f of x plus i y divided by h.

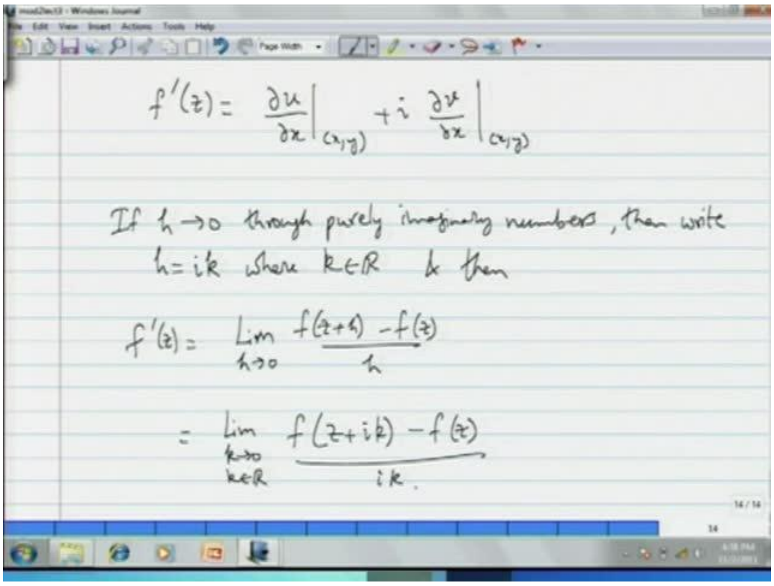
(Refer Slide Time: 29:43)



So, when we see this when we write this in terms of u and v. So, this is equal to limit as h goes to 0 h belongs to R. Now, I will simply consider the limit as h goes to 0 as a real limit of u of x plus h because x plus h now is the real part of x plus i y plus h comma y plus i times v of x plus h comma y minus u of x comma y plus i times v of x comma y divided by the number h. So, this is the limit as h goes to 0, h belongs to real number. So, after separation we get u of, let me write this, u of x plus h comma y minus u of x y divided by h and we are assuming that this limit exists.

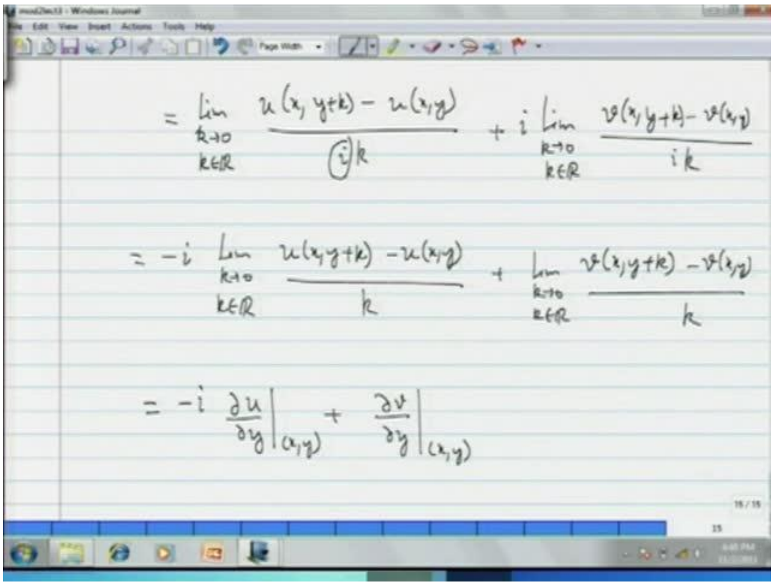
So, I will write plus i and separate the limiting process h goes to 0 h belongs to R. v of x plus h comma y minus v of x y divided by h. So, we readily recognise that h is through real numbers. So, we readily recognise that this, these limits are nothing but the partial derivatives of u and v with respect to x. So, there are various notations so let me use the dou u by dou x notation of of the function u at the point x comma y plus i times dou u rather dou v by dou x at the point x comma y.

(Refer Slide Time: 32:02)



So, when h approaches 0 through purely real numbers we see that f prime of z should really equal dou u by dou x at the point x comma y plus i times dou v by dou x at the point x comma y. So, now let us suppose that h approaches 0 through purely imaginary numbers. So, if h approaches 0 through purely imaginary numbers then write h as i k. So, where k is real now, where k belongs to R. So, there is no real part to h now. It is purely imaginary. So, we can write it as i k and assume that k is a real number which approaches 0. So, and and then f prime of z will now be limit as well its, by definition this is limit as h goes to 0 f of z plus h minus f of z by h. Now, it has equal the limit as k goes to 0, k is a real number of f of z plus i k minus because h is i k by assumption, f of z by i k.

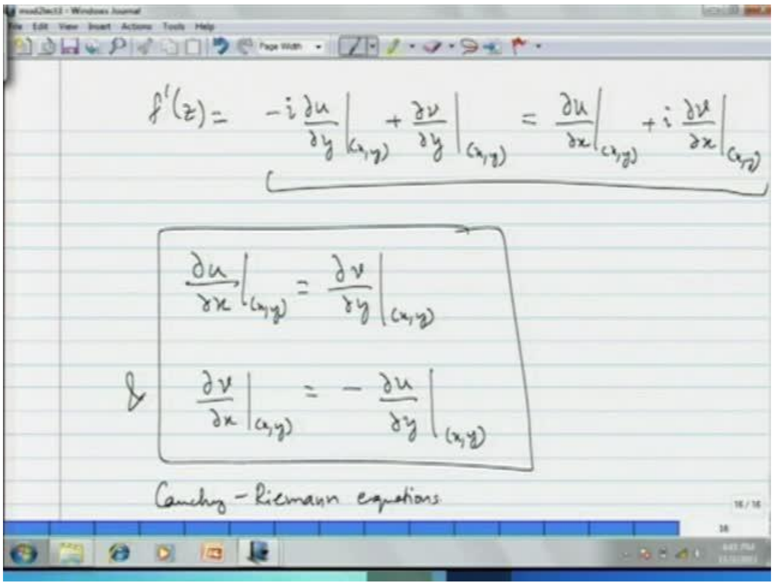
(Refer Slide Time: 34:02)



So then when we once again we write this in terms of u and v what we get is, this is the limit as k approaches 0, k belongs to R, u of x comma y plus k minus u of x comma y divided by i k minus or rather plus limit plus i times the limit as k goes to 0 k belongs to R v of x comma y plus k minus v of x comma y divided by i k. So, really I, I will eliminate this i here in the denominator because when we jump to the context of functions of two variables the complex number appearance there is not desirable.

So, let me write this as minus i times the limit as k goes to 0, k belongs to R because 1 by i is minus i. u of x comma y plus k minus u of x y divided by k and plus i cancels here in the second term. So, we get limit as k goes to 0 through real numbers, v of x comma y plus k minus v of x y by k, which we recognise as the partial derivatives of u and v with respect to y. So, this is minus i times dou u by dou y at the point x comma y plus dou v by dou y at the point x comma y.

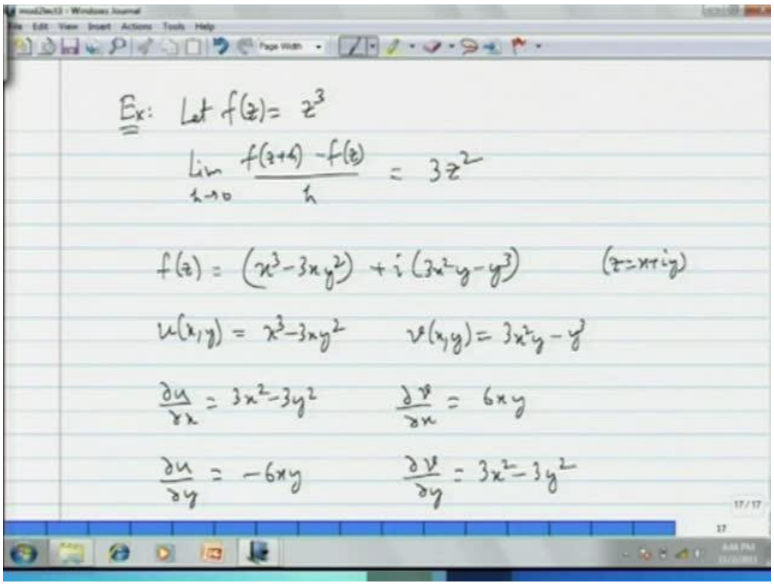
(Refer Slide Time: 36:08)



So, in summary f prime of z is minus i dou u by dou y at the point x comma y plus dou v by dou y at the point x comma y and from a earlier expression this is equal to dou u by dou x at the point x comma y plus i times dou v by dou x at the point x comma y. Now, equating the real and imaginary parts in this in this equation what we get is the following.

We get dou u by dou x at the point x comma y is equal to dou v by dou y at the point x comma y and dou v by dou x at the point x comma y is minus of dou u by dou y at the point x comma y. And these two equations are called the Cauchy Riemann equations after the famous mathematicians Cauchy and Riemann. And so when a function is differentiable at point z equals x plus i y the real and imaginary parts satisfy the Cauchy Riemann equations. So, let us see a quick example here.

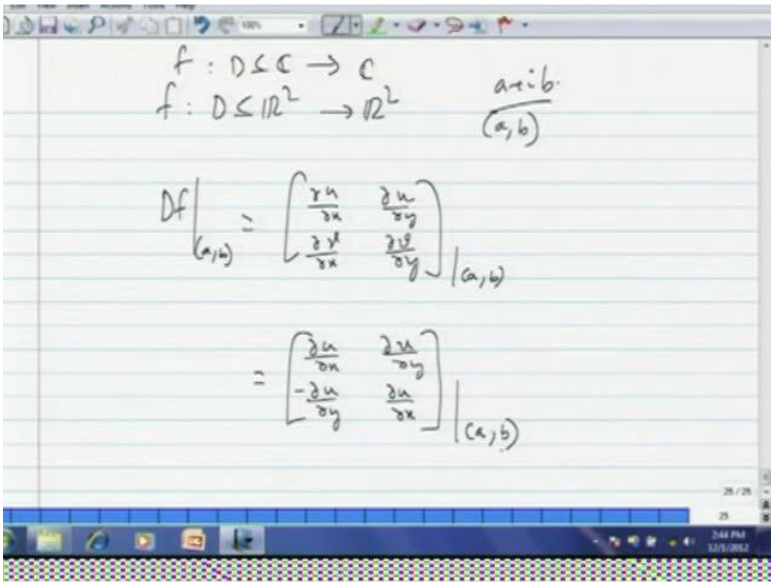
(Refer Slide Time: 37:55)



So, let us look at this example. Let f of z equals z cube which is a polynomial function. Then the limit as h goes to 0 well of course, f of z plus h minus f of z by h. One can calculate this comes out to be 3 z square and when we try to write f as its real and imaginary parts what we get is x cube minus 3 x y square plus i times 3 x square y minus y cube where z is x plus i y. And so here u of x comma y is equal to x cube minus 3 x y square and v of x comma y is equal to 3 x square y minus y cube and one can check that dou u by dou x is 3 x square minus 3 y square and dou v rather dou v by dou x is 6 x y dou u by dou y is minus 6 x y and dou v by dou y is 3 x square minus 3 y square.

So, f is a function which is differentiable at every point in its domain and as you can see dou u by dou x is equal to dou v by dou y and dou u by dou y is minus of dou v by dou x. So, f satisfy, u and v satisfy the Cauchy Riemann equations. Now, I will make some remarks about differentiability, complex differentiability.

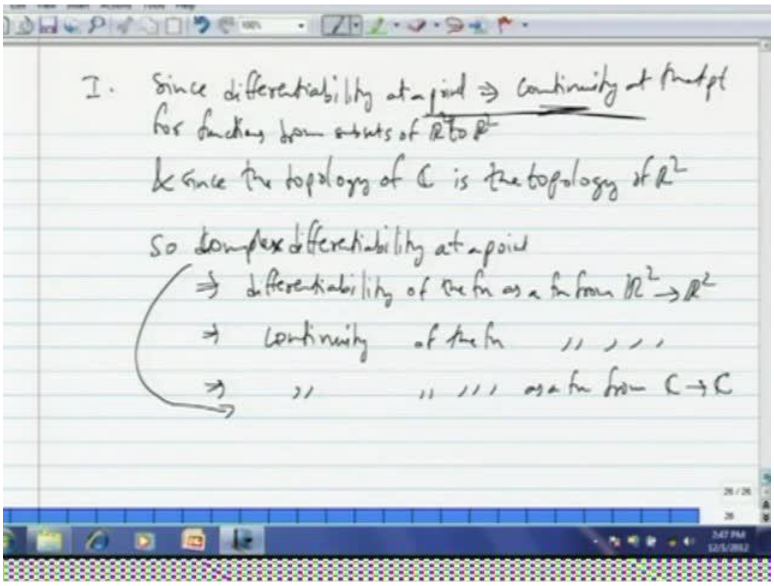
(Refer Slide Time: 40:01)



So, early on we saw that f can be considered as a function from a subset of R 2 to R 2 if f is a function from complex, subset of complex plane to complex plane and when f is differentiable at a point a plus i b it is also differentiable at point a comma b and D f at the point a b has the form dou u by dou x dou u by dou y and dou v by dou x dou v by dou y. It is given by this matrix at the point a b.

Now, since f is complex differentiable at a plus i b the partial derivatives satisfy the Cauchy Riemann conditions. So, this can be rewritten as dou u by dou x dou u by dou y and dou v by dou x is a minus dou u by dou y and dou v by dou y is minus dou u by dou x at the point a comma b. So, so it is it is differentiable. So, f is differentiable in this sense, in the R 2 to R 2 sense and there are further conditions, further restrictions on the partial derivatives. So, at least it is differentiable function from R 2 to R 2.

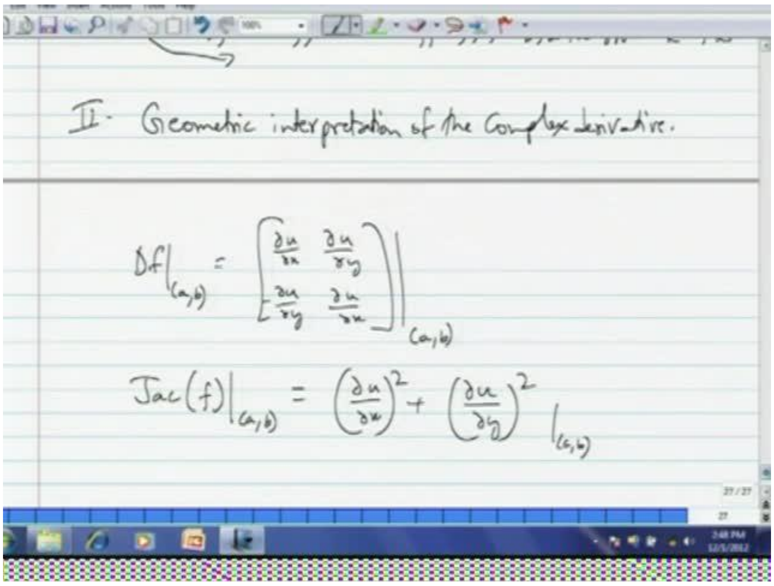
(Refer Slide Time: 41:34)



So, what we can conclude is that since differentiability at a point implies continuity at that point for functions from subset of R 2 to R 2. And since the topology of C is the same as the topology of R 2, is the topology of R 2, is the same as the topology of R 2. What I mean by that is epsilon bolls in R 2 correspond to the epsilon bolls in C and open sets in R 2 correspond to open sets in C, precisely they are equal, the topologies are equal so and since continuity is just a topological property.

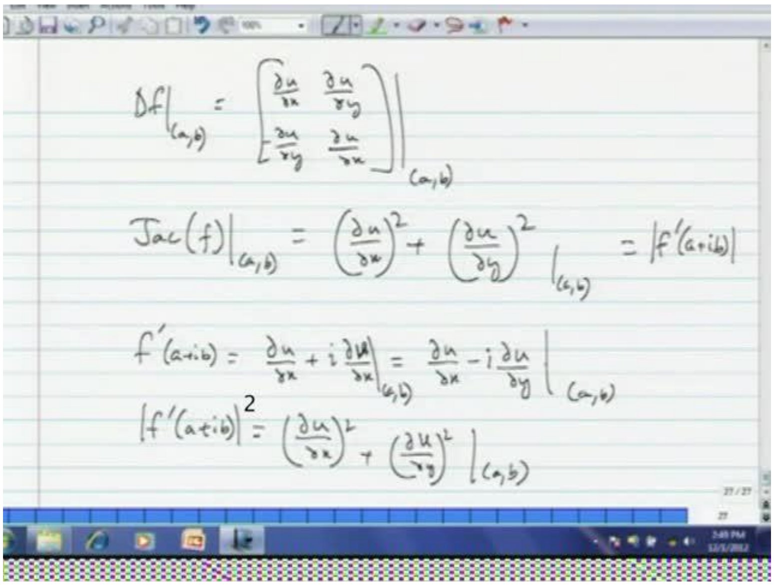
So, differentiability, complex differentiability implies differentiability at a point, I should say at a point like we saw about implies differentiability as a function from R 2 to R 2 at that point which implies the continuity of the function as function from R 2 to R 2 at that point. And this implies the continuity of the function as a function from R 2 to R 2 or from subset of C to C once again at that point. So, differentiability, complex differentiability implies complex continuity. So, that is true even in even for complex functions. So, that is a vague explanation of that point.

(Refer Slide Time: 44:04)



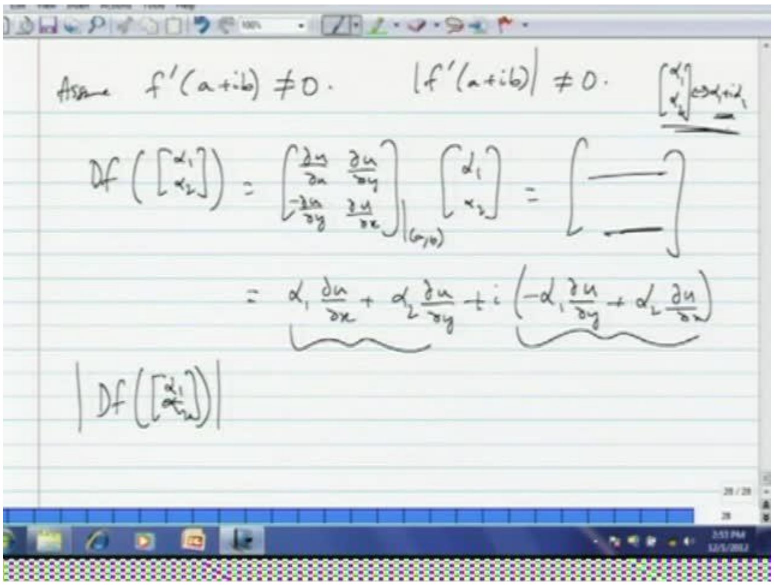
And the second note is as follows, we are going to see the geometric interpretation of the derivative, of the complex derivative. So, you see that D f has this special form when f is complex differentiable and if we consider f as the function of R 2 to R 2, it has this form dou u by dou x dou u by dou y and dou minus dou u by dou y and dou u by dou x at the point a b. So, the Jacobean of f at the point a b, at the point a b is the determinant of this matrix which is dou u by dou x square plus dou v dou u by dou y square at the point a b, at the point a b.

(Refer Slide Time: 45:25)



And f prime like we saw above the derivation of Cauchy Riemann equations. So, f prime of z a plus i b is going to be dou u by dou x plus i times dou v by dou x which is equal to dou u by dou x by the Cauchy Riemann equations this is minus i times dou u by dou y. So, that minus does not matter what I, what I want to say is that the modulus of f prime of a plus i b is now dou u by dou x. All this evaluated of course, at a b, this set a b. So, this is, the modulus is dou u by dou x square plus dou u by dou y square at the point a b.

(Refer Slide Time: 46:23)



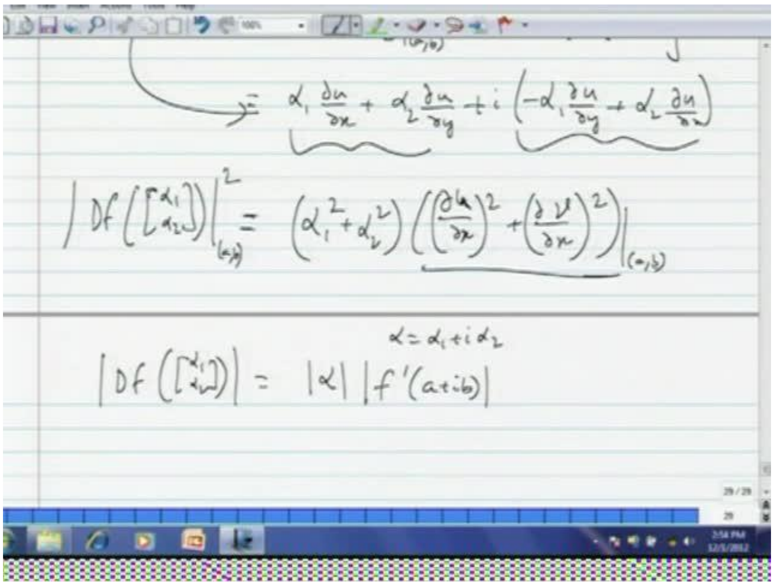
And so we will give a geometric interpretation when so assume for the time being that f prime of a plus i b i mean D f collapses some of the vectors at the point a b in R 2. So, when f prime of a plus i b is non zero it is interesting to see what happens. Then the modulus of course, is non zero, is non zero and then D f which is now a linear transformation of a vector alpha 1 comma alpha 2 or or vector let me write vectors as alpha 1 alpha 2 in R 2. So, the the vector alpha 1 alpha 2 correspondents to a complex number alpha 1 plus i alpha 2.

We are going to treat this vector as a small increment to a point or a small increment to the point a comma b in the complex plane or a plus i b in the complex plane. So, since D f is a linear approximation of the function near the, near the point, near the point a b. So, if we take a small increment to a plus i b namely alpha 1 plus i alpha 2 which is you know small in modulus then D f transforms this vector to the the range and then we can add f of a plus i b to this, to the transformation of this vector alpha 1 plus i alpha 2 to get a linear approximation of the function.

So, that is the idea of differentiation any way. Now, so we will consider the transformation of alpha 1 plus i alpha 2 namely the corresponding vector alpha 1 alpha 2 under D f. So, what that gives us is D f of alpha 1 and alpha 2 is of course, dou u so I will multiply the matrix dou u by dou x dou u by dou y minus dou u by dou y dou u by dou x all this at the point a b of course, times alpha 1 alpha 2 which gives me dou u. So, alpha 1 dou u by dou x plus alpha 2 dou u by dou y plus i times minus alpha 1 dou u by dou y plus alpha 2 dou u by dou x.

So, the modulus of the transform vector is going to be so I I mean I wrote the result as a complex number, but really they should be you know, this portion going in the first component and this portion going in the second component. I am using this correspondence.

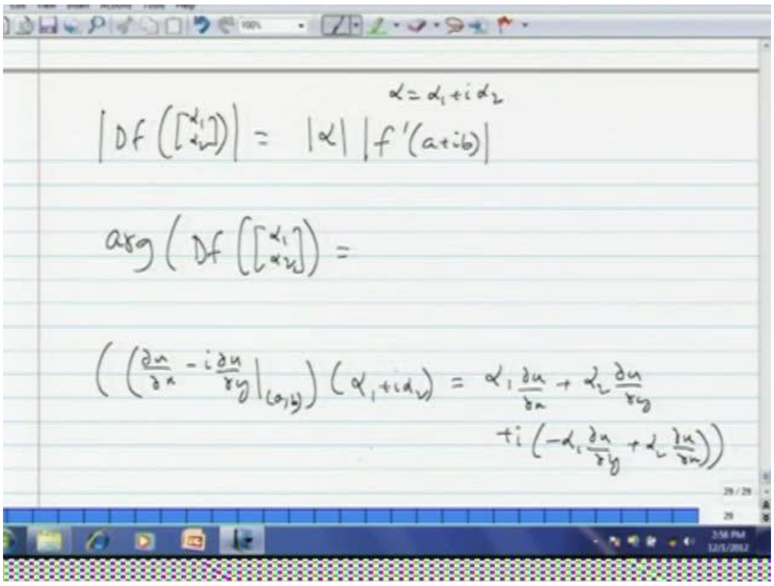
(Refer Slide Time: 49:40)



So, so the modulus of this in this form, the modulus of D f of alpha 1 alpha 2. So, I am using, abusing notation a little bit, but we are trying to see what the geometric interpretation here is. So, this is going to be after some calculation alpha 1 square so the modulus square is the alpha 1 square plus alpha 2 square times dou u by dou x that is dou u by dou x whole square plus dou v by dou x square at the point a b. So, this is at the point a b and so the modulus is really the modulus of alpha itself, alpha here is alpha 1 plus i alpha 2, the modulus of alpha itself and this we recognise as the modulus of f prime a plus i b.

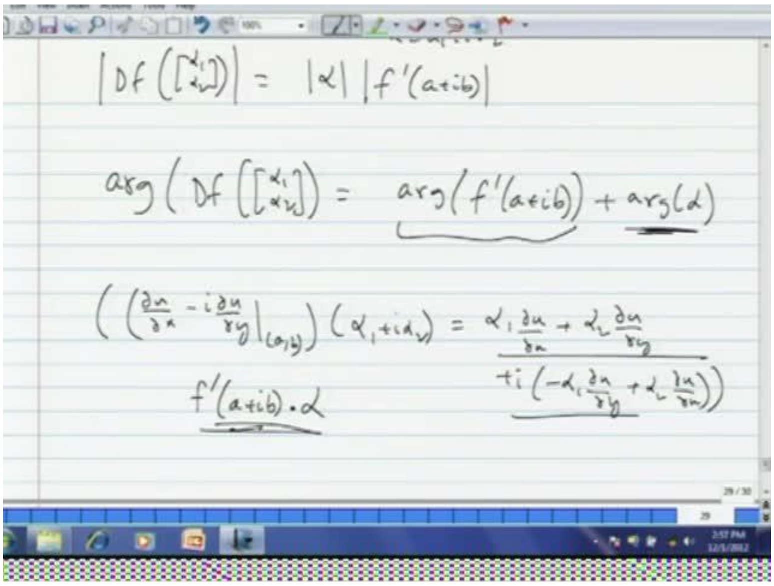
So, if you take a vector which is alpha 1 plus i alpha 2 which is a increment to the point a plus i b which you can consider as an increment to, complex increment to the point a plus i b. Then what f prime does is takes this vector alpha 1 plus i alpha 2 to a vector which is, whose modulus is modulus of alpha times the modulus of the derivative of f. So, picture is in order. So, here is… So, I will, I will draw the picture after I complete the other part namely the argument.

(Refer Slide Time: 51:31)



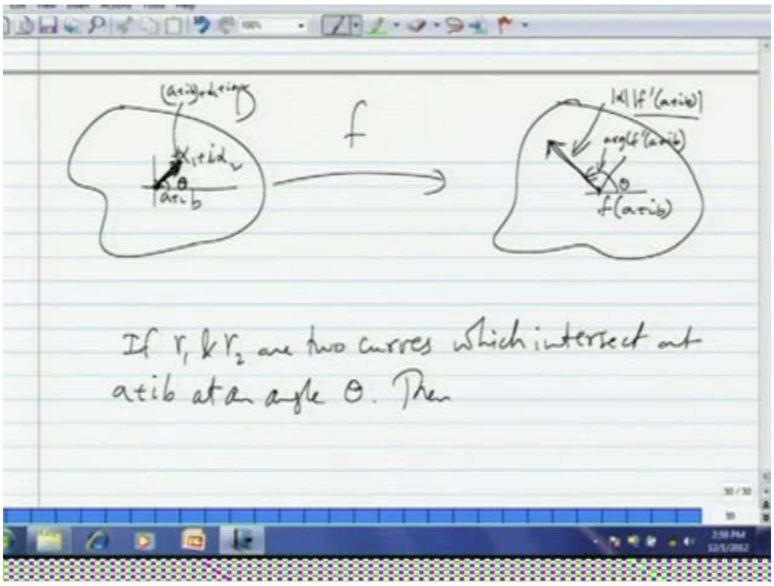
So, likewise what we can say is that the argument of D f of alpha 1 alpha 2, what is this? So, we will look at this computation here. The argument is what is it going to be, so before I compute the argument let me, let me show that dou u by dou x minus i dou u by dou y which is your f prime at the point a b. So, this at the point a b times alpha 1 plus i alpha 2 actually gives you this alpha 1 dou u by dou x plus alpha 2 dou u by dou y plus i times minus alpha 1 dou u by dou y plus alpha 2 dou u by dou x which is nothing but this expression is nothing but what we have here.

(Refer Slide Time: 52:48)



So, D f of alpha 1 alpha 2 is your, is your f prime at the point a plus i b times the vector alpha 1 plus so the vector alpha, the complex multiplication of these two complex numbers. So, the argument of D f of this is nothing but argument of f prime of a plus i b plus the argument of alpha because we know that the argument of product of two complex number is the sum of the arguments of those two complex numbers. So, using that principle I have this here. So, what I can say is that, what I can say from here is that the argument of alpha is augmented by argument of f prime of a plus i b.

(Refer Slide Time: 53:43)

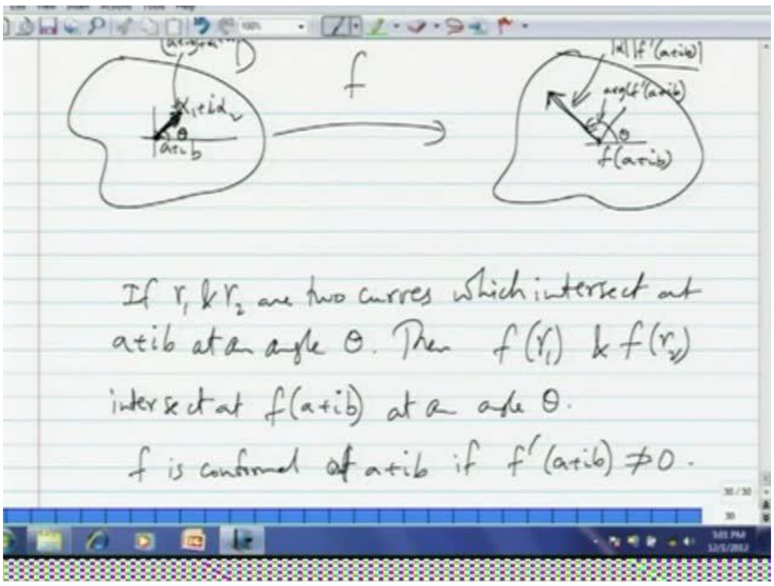


So, now I have a picture of the derivative. So, here is the domain D and then here is the point a plus i b and if I consider a small complex increment to that alpha 1 plus i alpha 2 then this is transformed via f to, this point goes to f of a plus i b and this vector alpha 1 plus i alpha 2. So, this is actually the complex point, you can say that is the complex point a plus i b plus alpha 1 plus i alpha 2. So, that is, that is why I am calling this increment vector. So, this increment goes to a vector so this this argument let me imagine that, that is theta that is theta which is the argument of alpha 1 plus i alpha 2.

So, this vector goes to a vector like that whose length is now modulus of alpha times the modulus f prime at a plus i b. So, it is multiplied by modulus of f prime of a plus i b and then it is turned by an additional angle of, so this is theta and then this is argument of f prime of a plus i b. It is… So, the effect of derivative is, it takes an increment vector and scales it up by the modulus of f prime and then turns it further by the argument of f prime.

So, that is the picture of the, that is the geometric interpretation of what f, the derivative of f does to increment vectors when f prime is non zero. So, that is a geometric meaning. So, what this this this results in is the following. So, if gamma 1 and gamma 2 are two curves which intersect at a plus i b at an angle angle theta so theta.

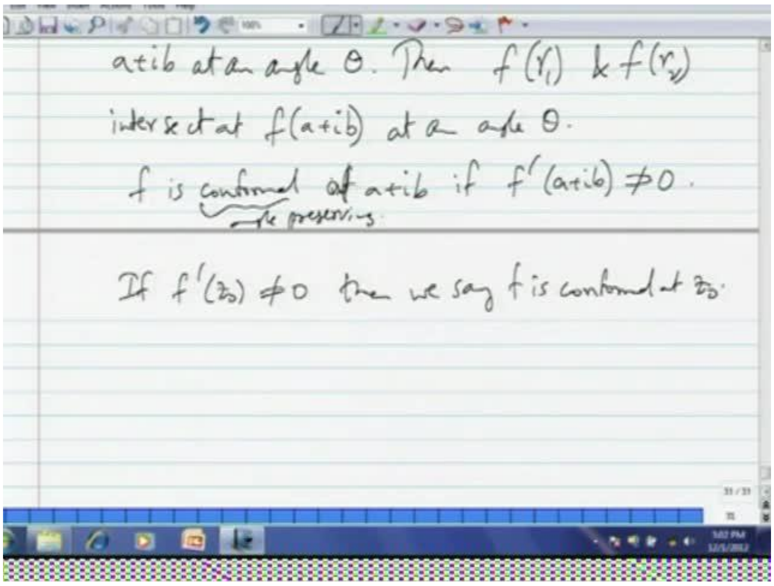
(Refer Slide Time: 56:19)



Then f of gamma 1 is a curve in the in the range plane. f of gamma 1 and f of gamma 2 intersect at f of a plus i b at an angle theta. That is because it has moved the tangent vector to gamma 1 by an additional angle of argument of f prime and it has moved the tangent vector to gamma 2 by an additional angle of argument of f prime.

So, when you compute the difference between these two, the arguments of these two tangent vectors it is going to be the same whether you compute difference in the range or whether you compute the difference in the domain because both of these two vectors have been, both of these vectors have been moved by an additional (( )), additional angle in there range. So, what happens is f when it differentiable at a plus i b preserves angles. So, such a map is called conformal. So, f is conformal if at a plus i b, if f prime of a plus i b is non zero. You want this this thing.

(Refer Slide Time: 57:44)



So, so owing to this we will just say that if f prime of z naught is non zero. So, conformal means angle preserving, conformality means angle preserving. So, f is angle preserving at a plus i b then derivative is non zero. So, if f prime of z naught is non zero, then we say f is conformal, because of this property we say f is conformal at z naught. It is angle preserving, and so we say f is angle preserving or conformal at z naught. So, it is a geometric interpretation of f prime, when f prime is non-zero at a point.

So, we will consider the converse to the statement that f is complex differentiable implies the real and imaginary parts of f satisfy the Cauchy Riemann equations in the next session. So, in particular we, we are going to see if f satisfies Cauchy Riemann or the real and imaginary parts of f satisfy Cauchy Riemann equations, then is the function complex differentiable. So, the short answer to that is no and if we impose one little additional condition that if the partial derivatives are continuous in addition to satisfying the Cauchy Riemann equations, then f becomes complex differentiable. So, we will prove that in our next, next session and I will stop here.