Complex Analysis
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Module - 2
Complex Functions: Limits,
Continuity and Differentiation
Lecture - 2
Limits and Continuity

Hello viewers, in this session, we will discuss the concept of limits of complex functions and there continuity and we will begin with limits. The limit of complex function as the variable approaches a fixed complex number a in its domain, is defined in a similar fashion to the limits of a functions from R 2 to R 2. So, limits in fact the concepts are the same.

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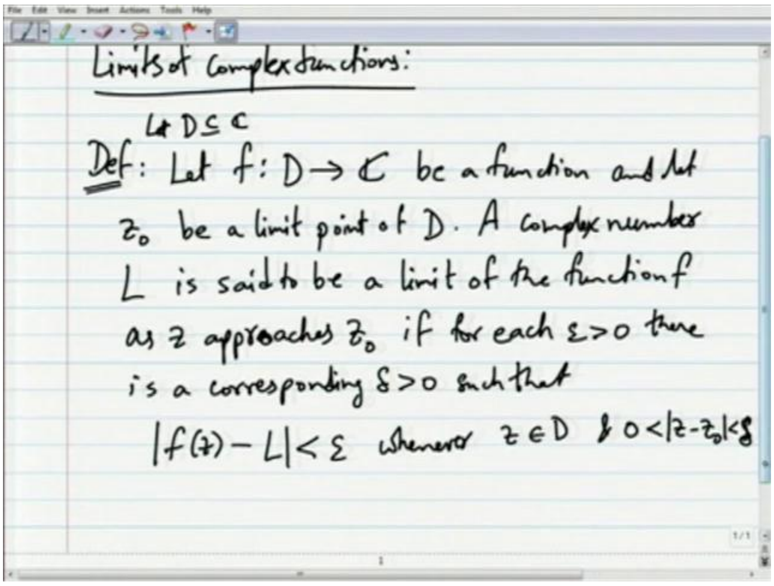

So, we define, firstly limits of complex functions and it is definition. So, let f from D to C be a function, a complex function. So, D is contained in C, let D contained in C, and let f from D to C be a function and let z 0 be a limit point of the set D of the domain of the function. So, a complex number L is said to be well is said to be a limit of the function f as the variable z approaches z 0, if for each epsilon positive there is a corresponding delta positive. Such that modulus of f of z minus L is strictly less than epsilon whenever z belongs to D and 0 strictly less than modulus of z minus z 0 strictly less than delta is less than delta.

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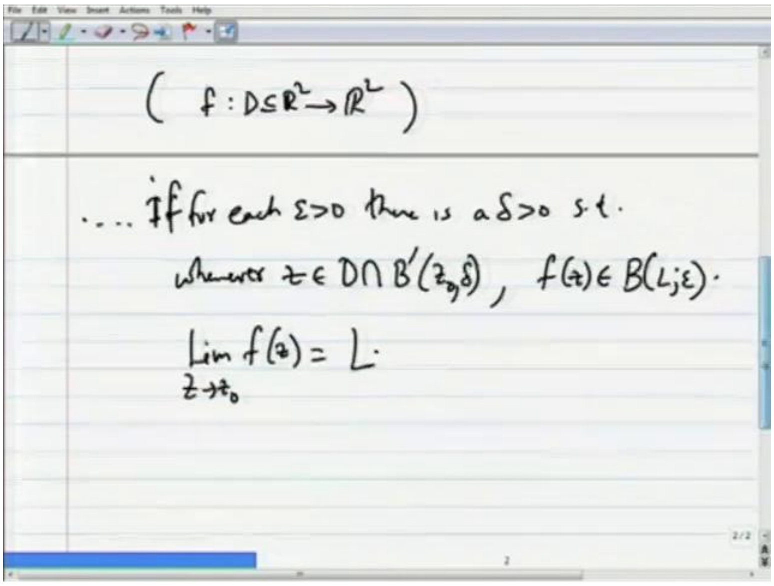

So this definition is similar to the definition of limits of functions from R 2 to R 2. So this is similar to definition of limits of functions from R 2 to R 2 and a instead of the modulus of the complex number in that case we have the norm of a of the number or or the ordered pair in R 2. So, the norm of f of x x y in minus you know the limit l in R 2 etcetera.

So, with that change this definition is the same. In fact, these definitions agree because the topology of the complex plane is the same as the topology of R 2 and and hence, and also since the limits depend upon the concept of limits depends upon the topology of the underlying case the the limits are one and the same. Another way or rephrasing this definition is that L is said to be limit of f as z approaches z 0 for each epsilon greater than 0.

So, I will just write the last statement, here if for each if, so dot, dot, dot, if for each epsilon greater than 0, there is there is a delta positive such that whenever z belongs to D intersection domain intersection the deleted neighbourhood of z 0 of radius delta. We have that f of z belongs to a ball of radius at most epsilon cantered it the complex number L. So, in that event we say that L is the limit of the function f as z approaches z 0 and we write this as limit as z goes to z 0, f of z is equal to L. Owing to the notation we are familiar from real analysis.

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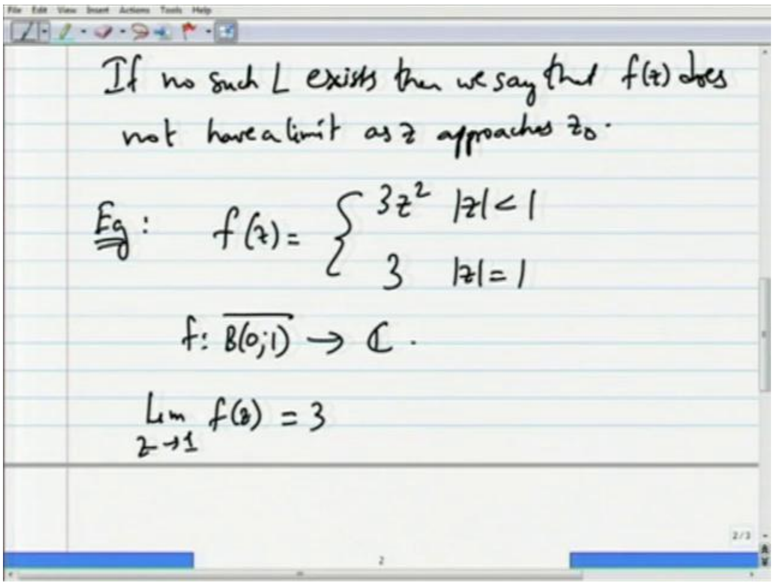

And if no such L exists, then we say that f of z does not have a limit, as z approaches to z 0 or in other words f of z does not the limit of z goes to z 0 f of z does not exist. So, terminology is already familiar to the viewer from real analysis. And here is an example if you have f of z is equal to 3 z square for modulus z less than 1 and 3 for modulus of z equals 1. So, f is a function from B 0 1 bar to C, let us say and in this case the limit as z goes to 1 of f of z is equal to 3. So, it is exists and the limit is equal to 3.

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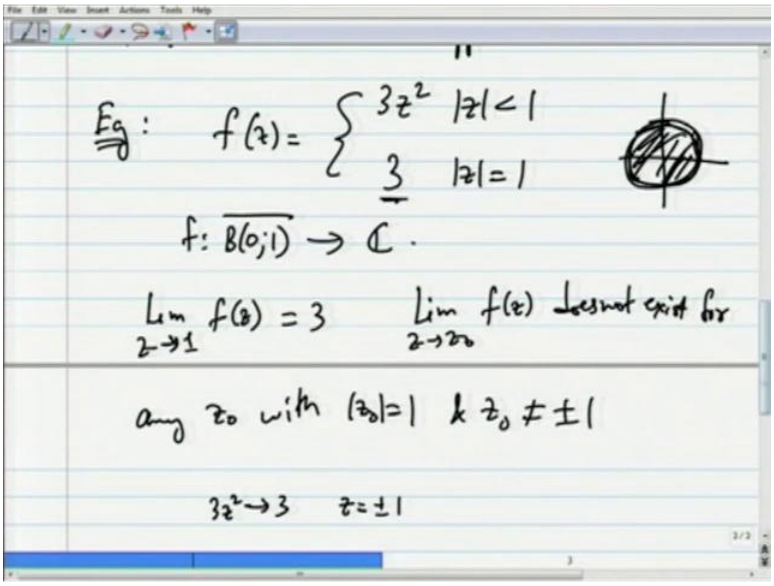

The limit as z goes to z 0 f of z does not exist for any z 0 with modulus of z 0 is equal to 1 and z 0 not equal to plus or minus 1. So, this function is defined on the closed unit disc, so that is the closed unit disc. It is something like 3 z squared on the inside and then it is defined to be the complex number 3 for all of the unit circle. So, of course, we know that 3 z squared tallies with 3, 3 z square approaches 3 only when z is equal to plus or minus 1.

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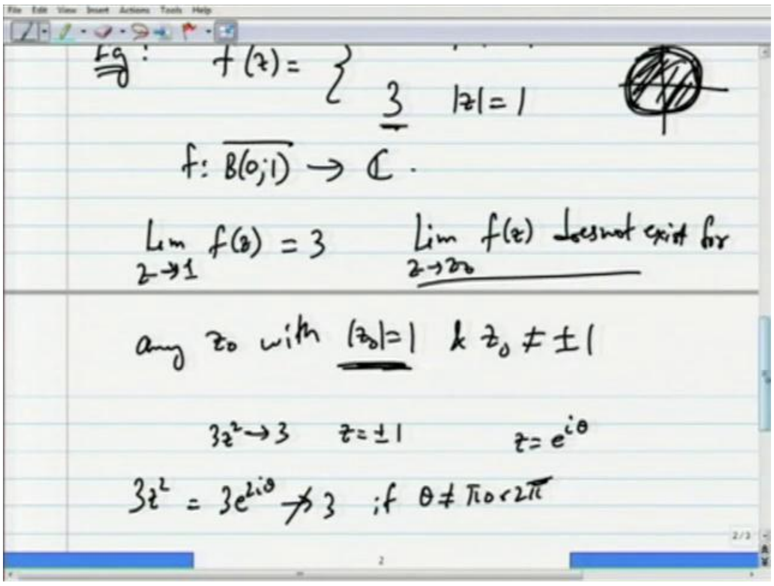

And for others, if if you have a z is e power i theta well if z 0 has modulus one it will look like e power i theta, then 3 z square will look like 3 e power 2 i theta. Which does not approach 3 if theta is not equal to pi or 2 pi. If it is not in other words if it is not a multiple of pi this does not approach 3.

So, so in summary that is why the limit does not exist for z 0 with those properties. So, that is the brief explanation, but one can use this definition to actually prove that limit does not exist. So, it is an exercise good exercise to the viewer to use the definition the epsilon delta definition of limit to prove that prove the (( )) which has been made in this example. So, so since the viewer is already familiar with the concept of limits from real analysis multivariable calculus in particular. I will I will assign in this as an exercise to the viewer.

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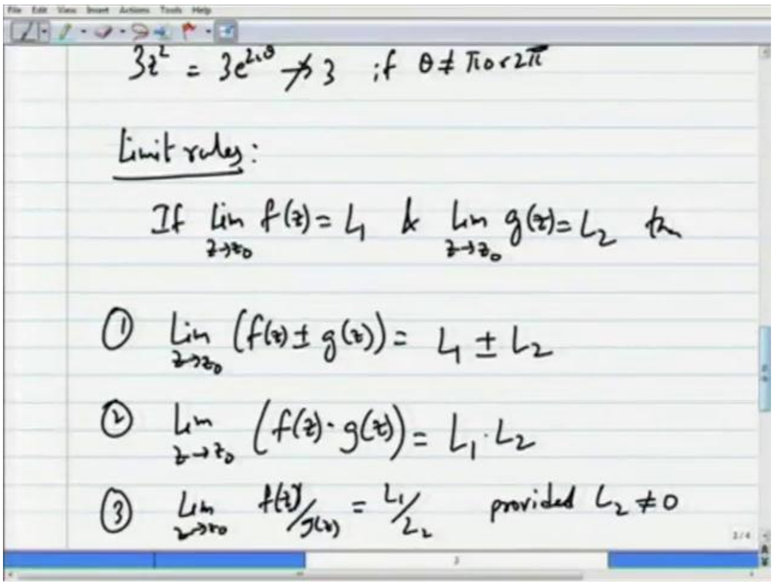

And moving on, there are rules the following limit rules. So, these limits exists limit as z goes to z 0 f of z exists and it is equal to L 1 and limit as z goes to z 0 g of z is equal to L 2, then following hold the limit as z goes to z 0 of f of z plus or minus g of z is equal to L 1 plus or minus z and the second properties that limit as z goes to z 0 of f of z times g of z which is also a complex function the limit of this is equal to L 1 times L 2, third the limit as z goes to z 0 of f of z by g of z is equal to L 1 by L 2 provided L 2 is not 0.

So, under the assumption that L 2 is not equal to 0 the limit of f of z by g of z also exists as z goes to z 0 and that is equal to L 1 by L 2.

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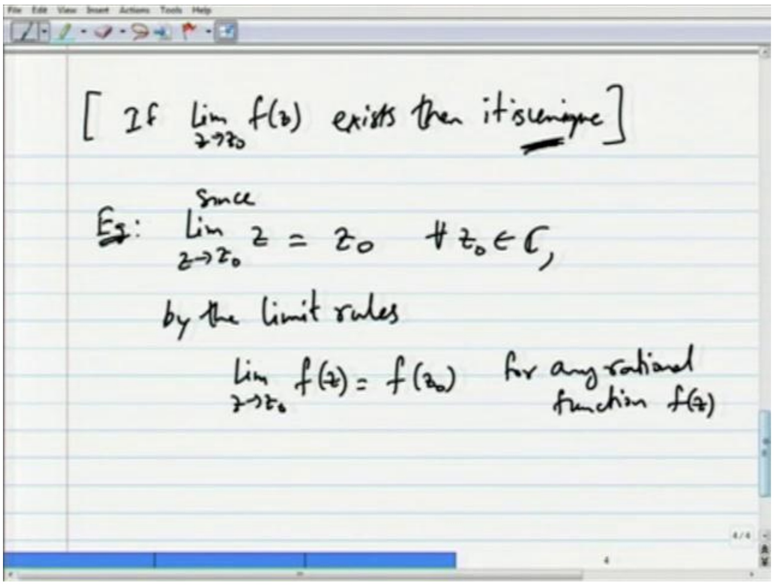

So, I should hasten to mention that, it is true that if limit of z goes to z naught f of z exists. Then then it is unique like in the case of functions from r 2 to r 2. It is also true for complex functions that the limit has to be unique. So, the proof of this statement this fact is also an exercise to the viewer; and using that, these properties, all of these properties, well all of these properties hold, and it is also true that the limit is unique. Now, from this what we can say is that the limit as z goes to z naught.

Since, the limit as z goes to z 0 this is equal to z 0 of course, the limit of the function z itself is equal to z 0 for all z 0 belongs to c, what we can say is that by the limit rules then limit as z goes to z 0 of f of z is equal to f of z 0 for any rational function for any rational function f of z. And for any z 0 belongs to domain of f. If f of z here looks like 3 plus i z lets say z square plus 2 z minus i by 1 plus i z power 4 minus z z q plus 1 something like that.

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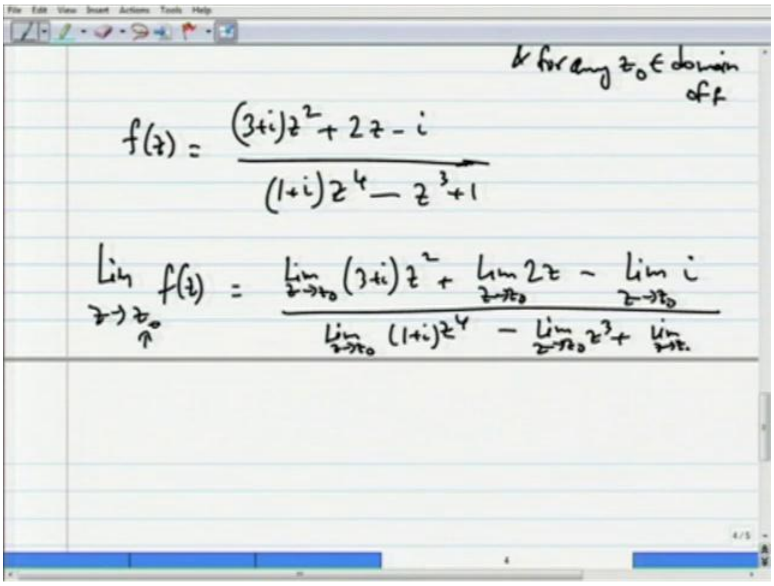

Then, the limit as z goes to z 0 of f of z, where z 0 is the point in the domain of f. This will be by the limit rules this is equal to limit z goes to z 0 of 3 plus i z square plus limit z goes to z 0 of 2 z minus limit z goes to z 0 of i divided by etcetera, limit z goes to z 0 of 1 plus i z power 4 minus limit z goes to z 0 z q plus limit z goes to z 0 of 1.

So, you can use rule three to distribute the limit to the numerator and denominator and then you can use rule one to distribute it over addition to get that and then you can use the rule for multiplication to further distribute this limits and say that this is 3 plus i times limit of z goes to z 0 z times limit z goes to z 0 of z etcetera. I will not write the whole thing.

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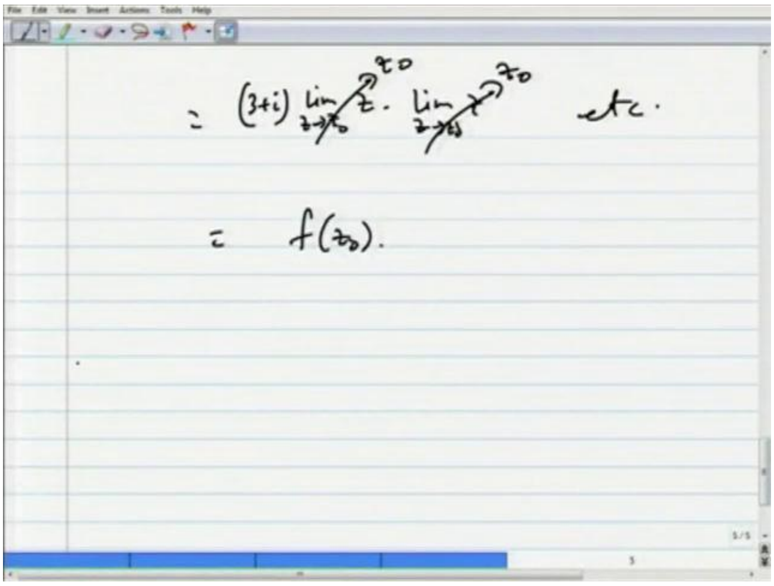

So, ultimately you are going to get this is equal to f of z 0, because each of this is z 0 etcetera; so, you are going to get f of z 0. So, it is true that limit of rational functions as z goes to z 0, where z 0 is a point in the domain of f is going to give you f of z 0, as a consequence of these limit rules.

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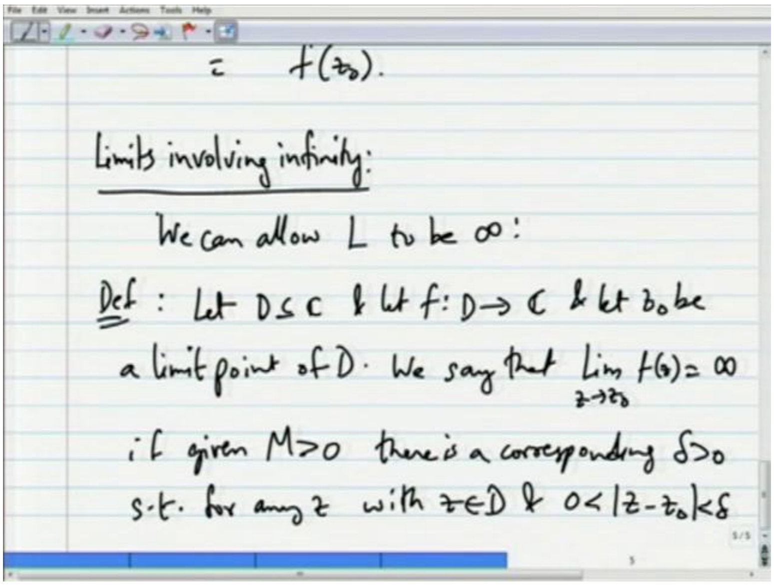

Then after this there is the concept of limits involving infinity. These will be useful to us when we study let us say the behaviour of functions, may be when when the complex variable tends to a certain point, which we will call singularity or the behaviour of functions as a the variable becomes very large in modulus or it tends to infinity. So, in particular when we study Mobius transformations we will be using these limits. So, here is here is when we can allow or what we mean by allowing L to be infinity. So, we can allow the limit the limit L in the definition to be infinity by doing the following.

Here is the definition; let f from so firstly let D b same setup D b contained in C and let f be a function from D to C with.... and let z 0 be a limit point of D. So, same set up as before. We say that the limit as z goes to z 0 of f of z is equal to infinity. So, we are allowing the limit to be infinity, if given M greater than 0 there is a corresponding delta greater than 0, such that for any z with z belongs to D and 0 strictly less than modulus of z minus z 0 strictly less than delta. With any such (( )) the modulus of f of z is greater than M. So, said otherwise by staying close enough to z 0 you can guarantee that the modulus of f of z for every z in the domain and close enough to z 0 the modulus of f of z is arbitrarily large.

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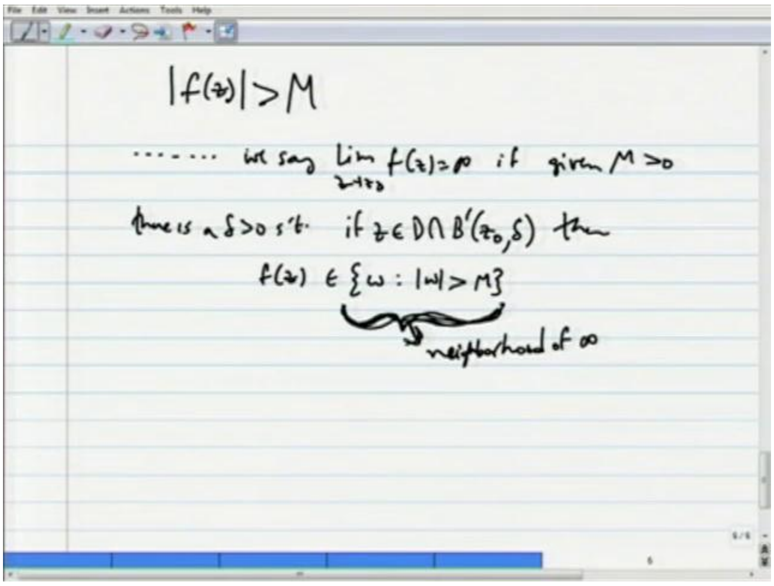

So, another way to say this is that using so another way to say this is that so all that with all that setup we say we say limit as z goes to z 0 f of z is equal to infinity. if given M greater than 0 there exist say there is a delta positive such that if z belongs to D intersection, the deleted neighbourhood of z 0 delta deleted neighbourhood of z 0 of radius delta. Then f of z belongs to the set of all w with the modulus of w greater than M.

I am just writing what I said in the definition in symbols and the point of doing this is that they recognise this to be the neighbourhood of infinity.

So, recall we called such things as neighbourhoods of infinity, when we studied that topology of the complex plane and this is an neighbourhood of infinity. So, in that sense now this definition is similar to this definition, where we said that f of z belongs to neighbourhood of n, to a epsilon ball around L. So, here we are allowing L to be infinity, so the epsilon ball around the L will look like this although it is not really that epsilon, but it is a neighbourhood of infinity.

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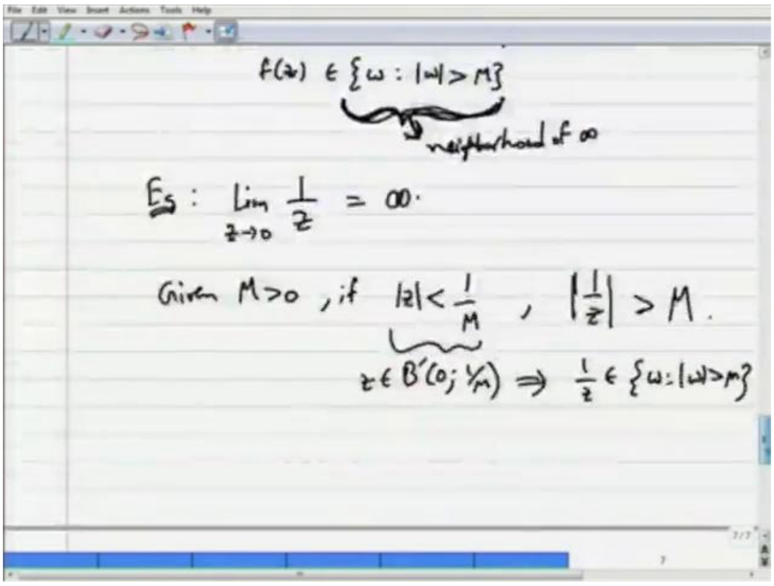

And there is also an example is in order of course, the limit as z goes to 0, for example of 1 by z. So, it is a simple example, because given given M positive, if modulus of z is strictly less than 1 by M, the modulus of 1 by z then will be greater than M. Which is what we want, which is what we want and so limit as z goes to 0 f of z is equal to infinity. This is B prime 0 1 by M. So, if z belongs to this implies f of z one by z belongs to set of all w, such that the modulus of w is greater than M. So, that is an example.

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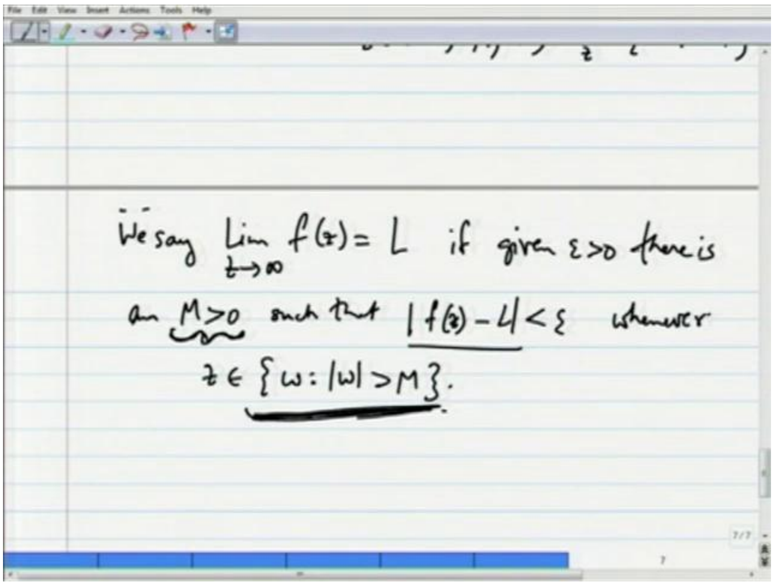

There is also a concept of netting the variable tend to infinity. So, we say limit as z goes to infinity now we are letting the variable tend to infinity of f of z is a complex number L. If given epsilon greater than 0 there exists an M positive. So, there is an M positive, such that the modulus of f of z minus L strictly less than epsilon. Whenever, z belongs to the set of all w, such that modulus of w is greater than M.

So, whenever we are near infinity this is the neighbourhood of infinity, if well the function should firstly be defined there. So, I did not write the setup, but we are assuming that f is defined on neighbourhoods of infinity. So, whenever z belongs to a neighbourhood of infinity if if we are able to control the neighbourhood of infinity using this M such that the modulus of f of z minus a complex number L is strictly less then epsilon for any given epsilon. Then we say that limit as z goes to infinity of f of z is L.

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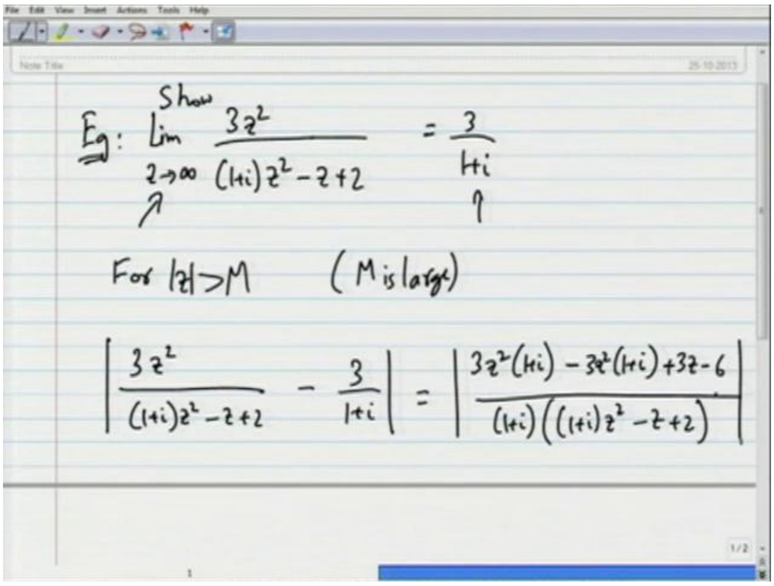

As an example of limit as z goes to infinity, let us consider the following; So, show that limit as z goes to infinity of 3 z squared by 1 plus i times z squared minus z plus 2 is show that this limit exists and is equal to 3 by 1 plus i. So, show so in order to see this for large for complex numbers with large modulus for mod z greater than M, we will assume M is large. So, close to infinity or in a neighbourhood of infinity.

So, what we will do is we will show that the modulus of this expression 3 z square by 1 plus i times z square minus z plus 2 minus 3 by 1 plus i. The modulus of this is actually strictly less than... you know any is is arbitrarily less. So, then we can conclude that the limit as z goes to infinity by definition of this expression is 3 by 1plus i.

So, upon simplification this gives me 3 z square times 1 plus i, I will clear the denominator minus 3 z square times 1 plus i plus 3 z minus 6 divided by 1 plus i times 1 plus i z squared minus z plus 2 and then this we will store well well simplify it further upon cancellation I have 3 z minus 6 by 1 plus i times 1 plus i z square minus z plus 2. I can divide by z square in the numerator and denominator z is arbitrarily large.

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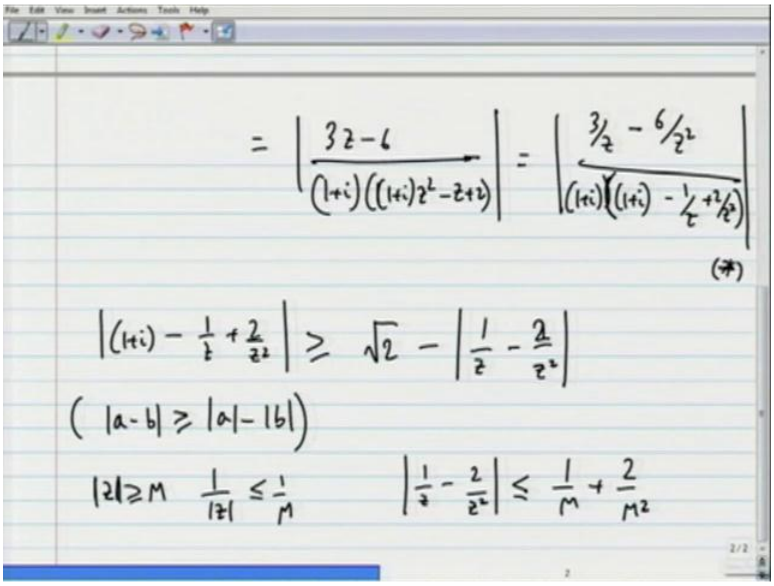

So, let me divide by z square in the numerator and denominator I get 3 by z minus 6 by z square divided by 1 plus i times times 1 plus i minus 1 by z plus 2 by z square in modulus. So, we will preserve this expression which is the estimation that we want to make. So, the modulus of this is equal to the modulus of this we will store this as star.

And then notice that in the denominator we have 1 plus i minus 1 by z plus 2 by z squared the modulus of this is greater than or equal to the modulus of 1 plus i by triangle inequality this is greater than or equal to modulus of 1 plus i, which is root 2 minus the modulus of 1 by z minus 2 by z square.

So, I am using the triangle inequality mod of a minus b is greater than or equal to mod of mod a minus mod b. Now, notice that when z is greater than or equal to M 1 by, I mean when mod z is greater than or equal to M 1 by mod z is less than or equal to 1 by n. So, the modulus of 1 by z minus 2 by z square is less than or equal to 1 by M plus 2 by M squared. So, this is minus of modulus of 1 by z minus 2 by z squared is greater than or equal to minus 1 by M minus 2 by M squared.

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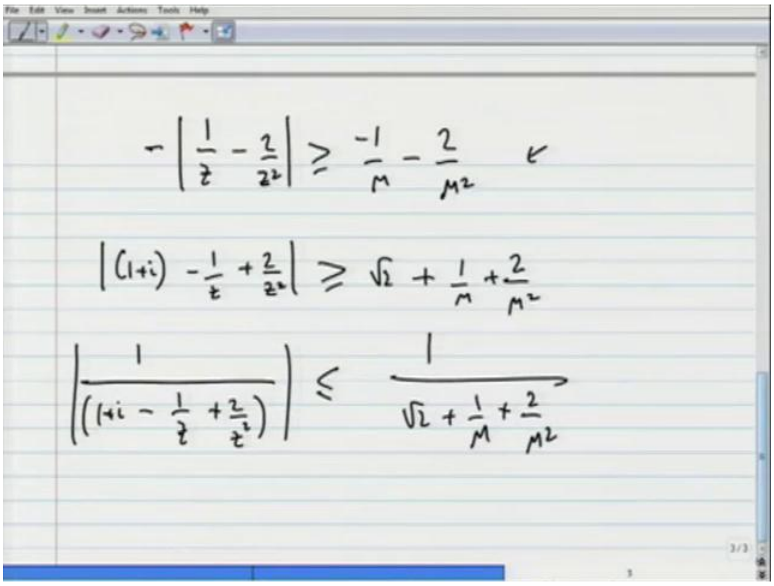

So, we can substitute this in here to say that the modulus of 1 plus i minus 1 by z plus 2 by z square is greater than or equal to square root 2 plus 1 by M plus 2 by M squared and taking the reciprocal 1 by 1 plus i minus 1 by z plus 2 by z square in modulus is less than or equal to 1 by square root 2 plus 1 by M plus 2 by M squared. Now, we will bring in star let me go back to the expression star. So, this piece in the denominator, we have a said that 1 by that is less than or equal to this quantity.

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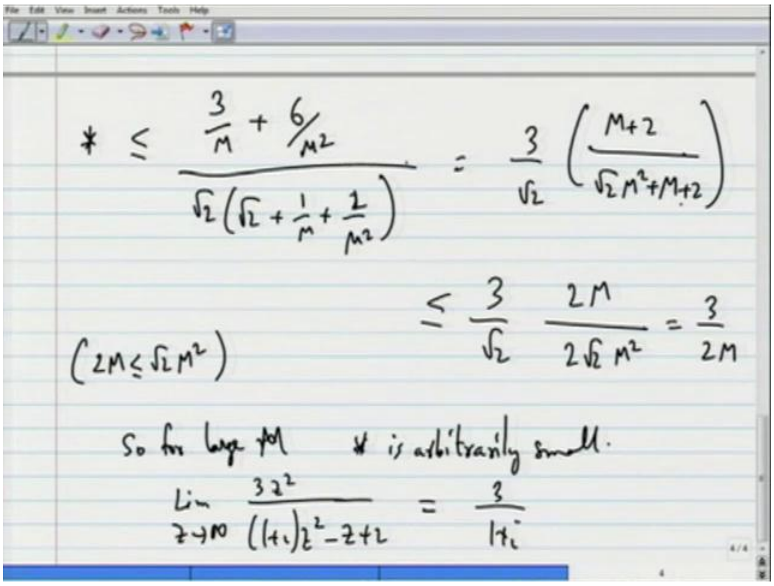

So, modulus of so star the expression star is less than or equal to 3 by M plus 6 by M squared. Because, I have 3 by z and minus 6 by z squared by triangle inequality modulus of a minus b is certainly less than or equal to mod a plus mod b. So, I have 3 by M plus 6 by M squared divided by the modulus of 1 plus i square root 2 and then I have square root 2 plus 1 by M plus 1 by M squared. Due to the 2 by M squared due to this expression. Then this is less than or equal to 3 by square root 2 upon simplification M plus 2 divided by this is actually equal to this 3 by root 2 times M plus 2 by square root 2 M square plus M plus 2.

So, for large M I can assume this is 3 by root 2 this is less than or equal to 3 by root 2 times 2 M, M plus 2 is less than or equal to 2 M. That I can assume for large M and likewise M plus 2 in the denominator is less than or equal to 2 M and further, I can also assume that M 2 M is less than or equal to root 2 M squared for large enough. So, for large enough M I can assume the 2 M is less than or equal to root 2 M squared. So, for such an M for such a large M I get this is 2 root 2 M squared which is 3 by 2 M.

So, this is this expression star is less than or equal to 3 by 2 M. So, for large M for large M, which means if mod z is greater than or equal to a very large number this star is arbitrarily small. So, I can conclude that limit z goes to infinity of 3 z squared by 1 plus i times z squared minus z plus 2 is indeed equal to 3 by 1 plus i.

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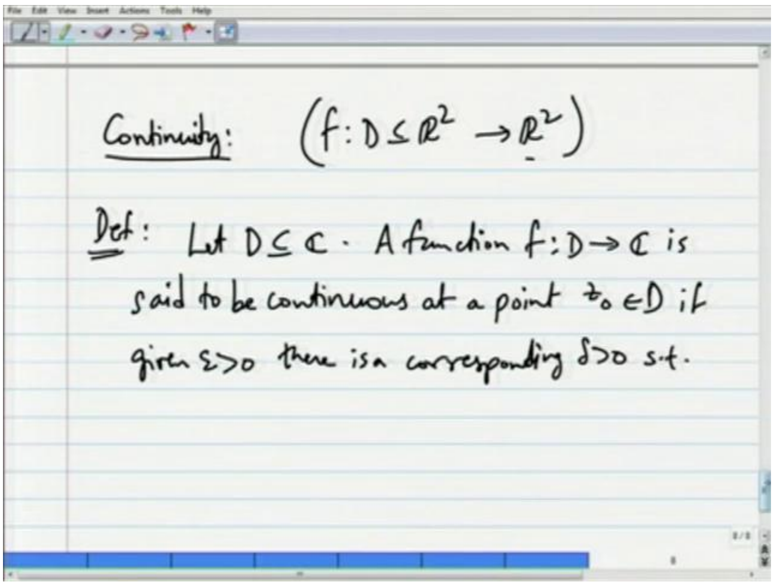

Next let us discuss continuity of complex functions. So, the concept of continuity once again is the same as the concept of continuity of functions from subsets of R 2 to R 2. So, I am assuming that the viewer as seen the definition of continuity of functions from subsets of R 2 to R 2 and I will nevertheless define continuity for complex functions here again. So, the setup once again is similar let D continue in C and let or I will say function f from D to C is said to be continuous at a point z 0 belongs to D, if given epsilon positive there is a corresponding delta positive such that the modulus of f of z minus f of z 0.

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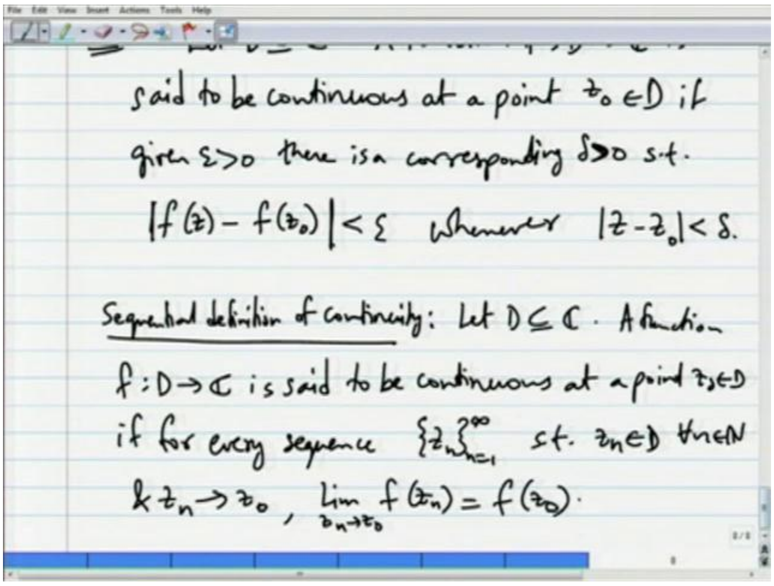

So, in this case we are fixing the limit to be the functional value at z 0. We are not allowing it to be some complex number L, but it is a its a functional value at z 0 and z 0 f of z 0 makes sense, because z 0 is a point in the domain and this is strictly less than epsilon whenever the modulus of z minus z 0 is strictly less than delta.

So, delta should be strictly positive is the point and so if that happens then we say that limit as... we say that f is continuous at a point z 0 belongs to the domain and there is a corresponding sequential definition of continuity. So, it says that D contained in C a function f from D to C is said to be continuous said to be continuous at a point z 0 belongs to D if for every sequence so for every sequence that is important z n n equals 1 to infinity. So, that is the sequence such that z n come from the domain for all n belongs to n and z n converges to z 0 the limit as z n approaches z 0 of f of z n is equal to f of z 0.

If for every sequence which converges to z 0 the limit of f of z n is equal to f of z 0. f of z n is now a sequence and if this sequence converges to the point f of z 0 in the complex plane, then we say that f is a f is continuous z 0. So, notice that this limit here refers to the limit of the sequence f of z n. So, it is a sequence f of z n n equals 1 to infinity and if so that is the sequence in the complex plane and its convergence is written in this fashion.

So, in this event you say f is continuous and the both these are one and the same both these definitions are are the same the equivalence of this is something we are not going to prove here, but these two definitions of continuity are one and the same .

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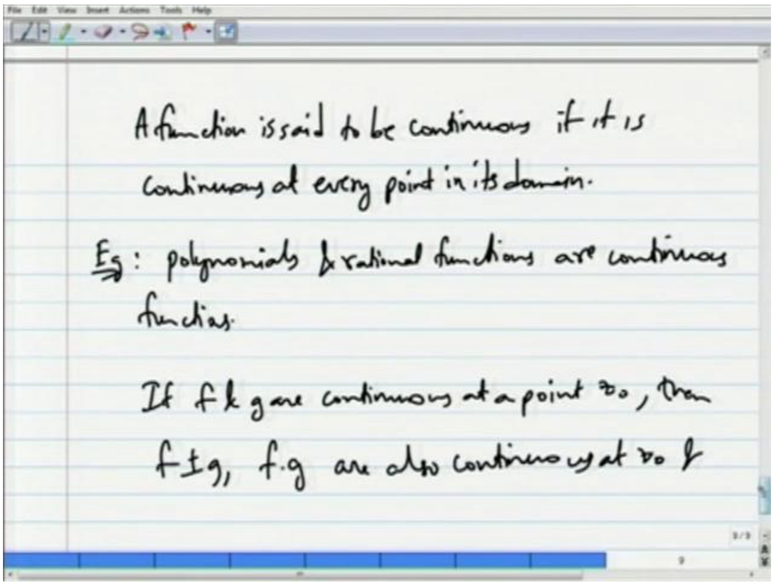

And a function is said to be continuous, if it is continuous at every point in its domain. So, we have defined continuity at a point and if the function is continuous at every point in its domain we simply say that it is continuous without any further adjectives.

So, examples; well we know examples of continuous functions already polynomials of complex variables and rational functions are continuous functions. They are continuous at every point in the domain and so they are continuous functions and it also true that if f and g are continuous at a point z 0, then f plus or minus g f times g are also continuous at z and if f g of z 0 is non 0 then f by g is also continuous at z 0.

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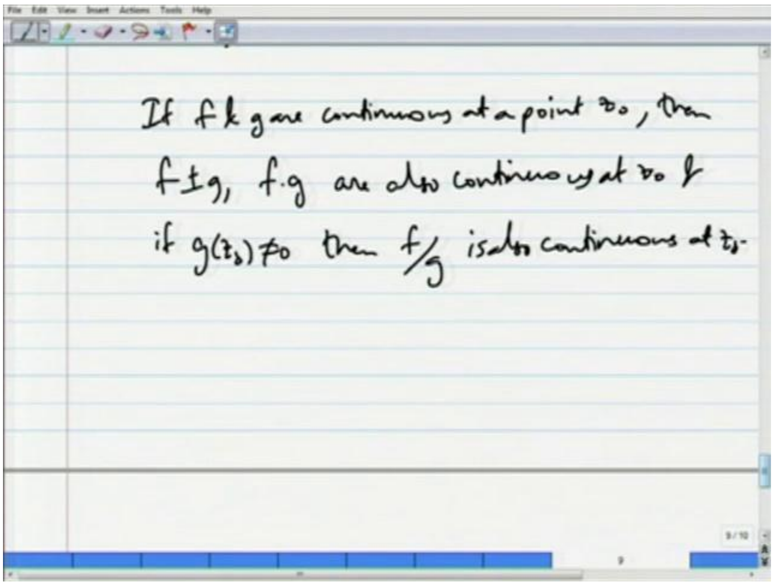

So, next we will look at some important theorems which we for continuous functions, which we are going to use during the course.

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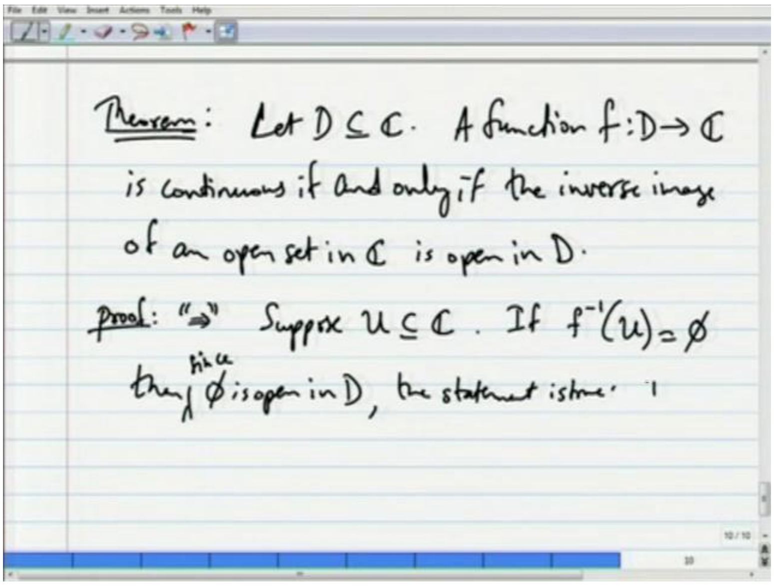

So, the first of them gives an alternative characterisation of continuous functions. So, a function let let D be a subset of. A function f from D to C is continuous if and only if the inverse image of an open set in C is open in D. So, we will have many occasions where we will use this characterisation of continuous functions.

So, function which is continuous at every point in its domain has to satisfy the property that the inverse image of an open set in C is open in the domain with respect to the domain. And likewise if that property is satisfied then the function is automatically continuous. So, here is the proof of a this theorem. So, we will first prove the direction that if a function is continuous then the inverse image of an open set in C is open in D.

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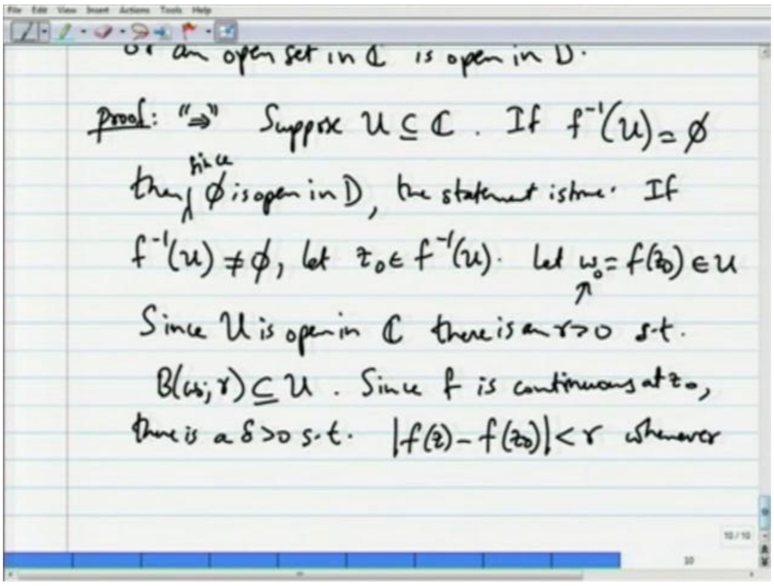

So, suppose u is an open subset of C and if the inverse image of u is empty then of course, empty set is open in D, then since this is open in d the statement is true. Now, if f inverse of u is non empty let you can pick a point in f inverse of u, let w is w 0 is equal to f of z 0. Since, u is open in C, there is an r positive such that for this f of z 0, which now belongs to u because z 0 is in inverse of u this will belong to u.

So, what you can do is since u is open you can find an r positive such that the ball of radius r around w 0 is contained in u. So, we are using the fact that u is an open set in C and by continuity. Since, f is continuous at z 0 there is delta positive such that the modulus of f of z minus f of z 0 namely w 0 is strictly less than r whenever the modulus of z minus z 0 is strictly less then delta. So, I should say whenever 0 strictly less than modulus of z minus z 0 strictly less then delta and z belongs to D, we want z to belong to the domain for f of z to be define.

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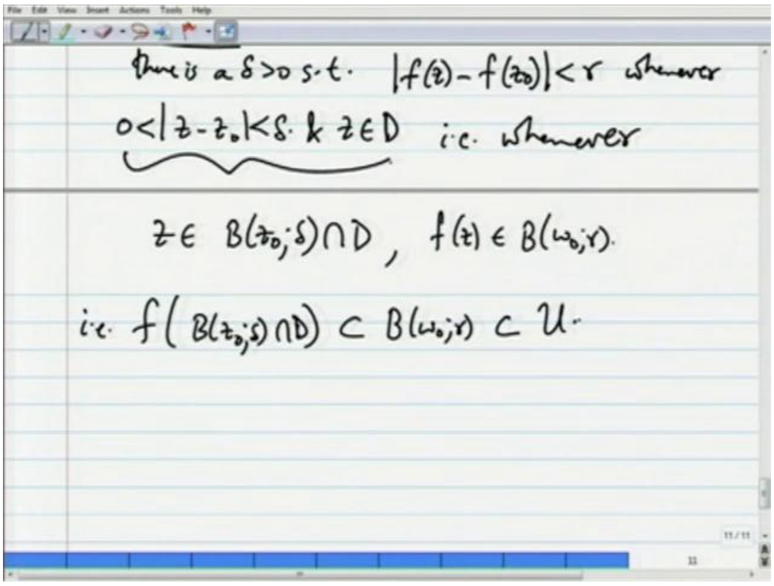

So, f is continuous, we have this property I mean this is the definition of continuity at z 0. So, what this says is that such a z 0 the image of such a z 0 is in B w 0 r. So, i e whenever z belongs to B z 0 delta intersection D, f of z belongs to B w 0 r. What that means is that the image i e the image of B z 0 delta intersection D is contained in B w 0 r which is contained in u. So, that tells that f inverse, f inverse of u contains B z 0 delta intersection D.

So, so it is open so f inverse of u is open in D. It is... this is an open set in i mean this is an open ball and hence, an open set in the complex plane and when you intersect it with D this set is open in D. So, f inverse of u then now is open in D.

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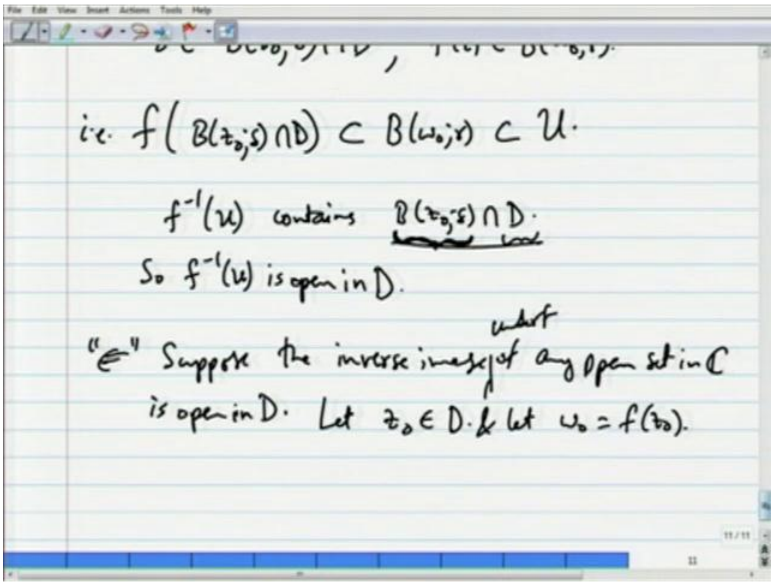

That is the proof of the forward direction and then suppose in the other direction suppose at the properties satisfy we want to show that f is continuous. So, suppose in the inverse image of any open set in in C is open in inverse image under f of course, is open in D. Let z 0 belong to D you want to show that f is continues there and let w 0 belong to f of z 0 or w 0 equal f of z 0.

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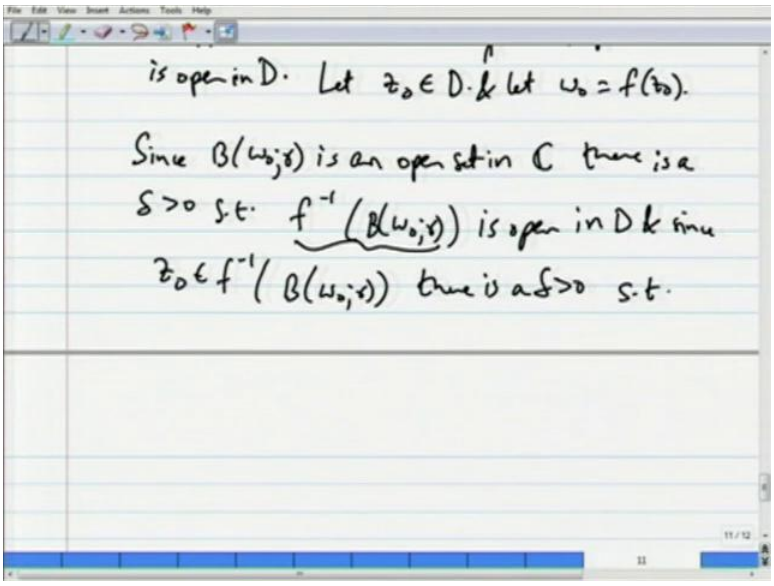

Since, B w 0 r is an open set in C, there is a delta positive such that in the inverse image we are assuming that the inverse image of any open set is open in D. So, the inverse image of B w 0 r, that is an open set is open in D and since, z 0 belongs to z 0 is f inverse of.. is contained in f inverse of w 0. So, it it is contained in this set and since, z 0 belongs to f inverse of B w 0 r, which is an open set now. There is a delta positive there is a ball of radius delta centred at z 0, which is contained in this set.

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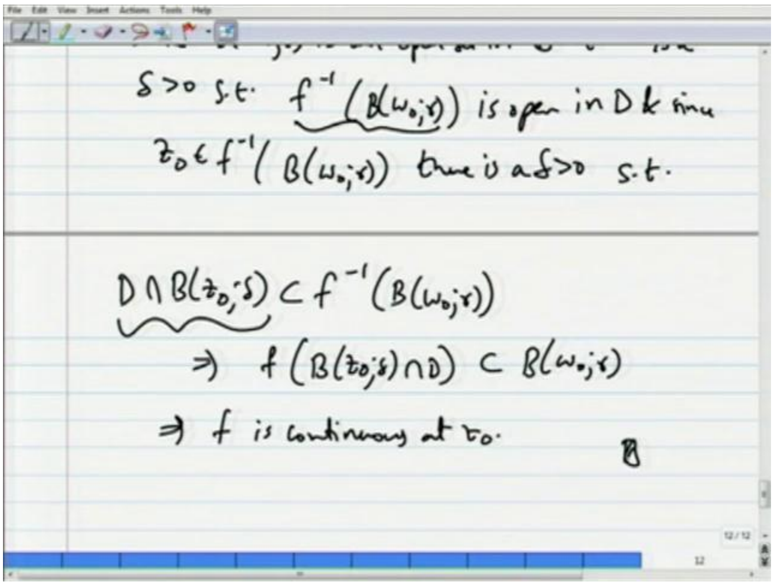

So, there is a delta positive such that the ball of radius delta around z 0 is contained in an open set in this open set. So, I should say D intersection this because we do not know if all of this is in D, so D intersection that is contained in that. So, what that says is that this implies f of B z 0 delta intersection D is contained in B w 0 r, which is nothing, but the definition of continuity at z 0. So this implies f is continuous at z 0 and that proves this theorem.