Complex Analysis

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Module - 2
Complex Functions: Limits, Continuity and Differentiation
Lecture - 1
Introduction to Complex Functions

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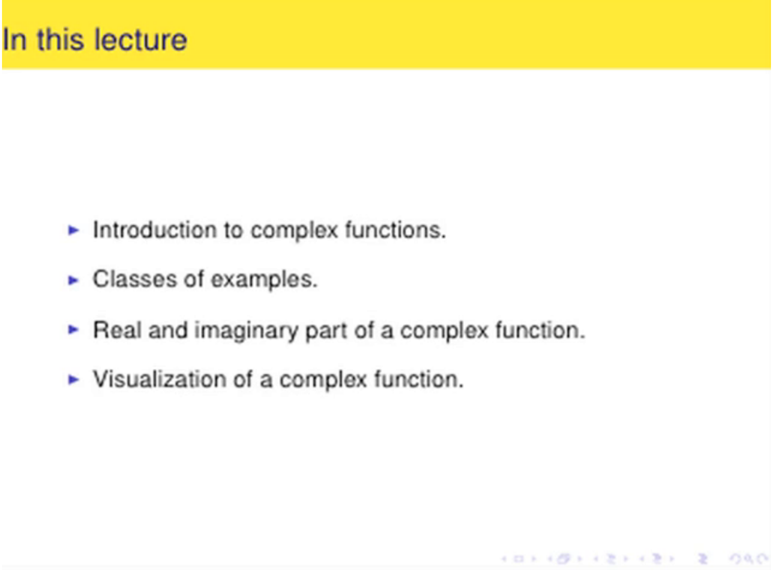

Hello viewers, in this session, we are going to learn about complex functions.

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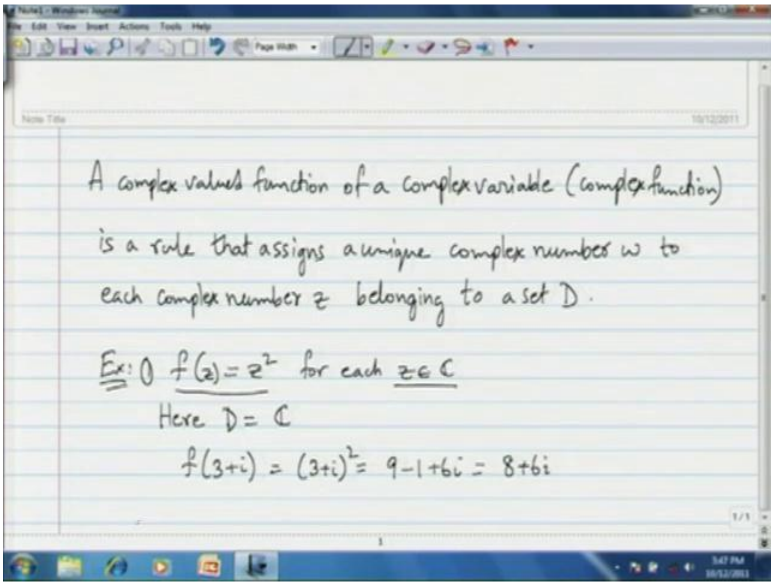

So, I will start by defining what I mean by a complex function. So, a complex value function of a complex variable or a complex function, for short is a rule that assigns a unique complex number w to each complex number z belonging to a set D. So, that is the definition of a complex valued function of a complex variable or a complex function in short. So, an example is an order, so an example is as follows; f of z usually use symbols f g h etcetera to denote a functions, complex functions. So, f of z given by z square for each z in the complex plane is an example of a complex function.

So, here the set D is all of C, because we are giving this rule, f of z equal z squared for each z in all of the complex plane. So, the set D here is C and let us do an example computation f of 3 plus i 3 plus i is a complex number. So, f of 3 plus i is going to give you 3 plus i squared; so this is 9 minus 1 plus 6 i which is 8 plus 6 i. So given a complex number we can compute what this rule f does to that given complex number.

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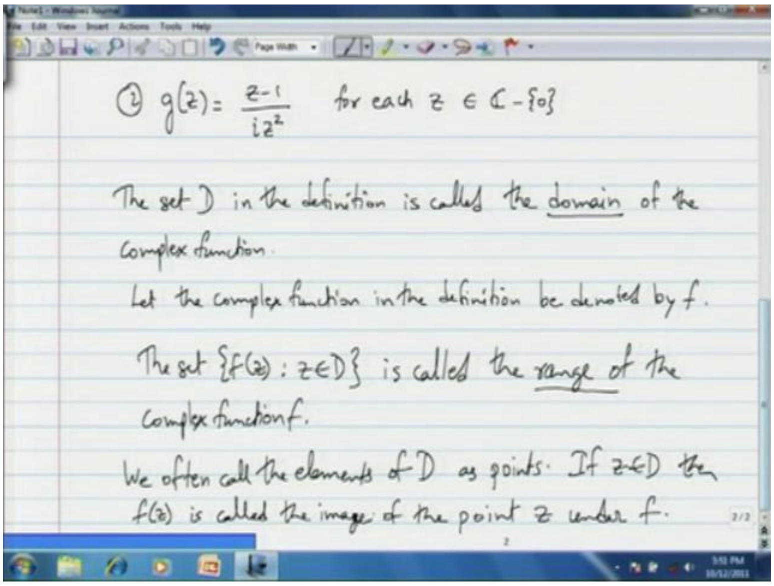

So, another example, so let me call this example one, another example can look like g of z equals z minus 1 divided by i z squared, where this rule is for each z in c minus the set 0. So, if you remove this 0 in the complex plane then this rule g tells you what to do with any given number z in the complex plane minus 0.

So, this is another example of this of this function and now the set D which appears in the definition, the set D in the definition is called the domain of the complex function of the complex function and the set so before I say this let me let me give the function a name so let the complex function in the definition be denoted by a symbol f, let us say. Then the set f of z such that z is in the set D the set of all f of z such that z is in D is called the range of the complex function f. So, we understand the domain and the range and we often call the elements of D as points and if z belongs to D if z is a point then f of z is called the image the image of the point z under the function f. So, these are some terminology we keep using.

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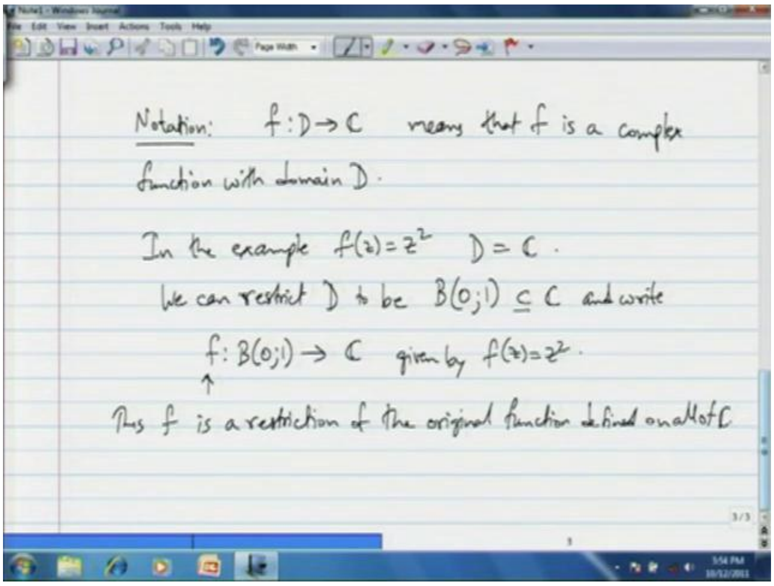

This is some terminology we will keep using and then we will also follow the following notation. So, if we write f colon D to C, we will mean, this means that f is a complex function with domain D. So, that is the notation we will keep using. And if a complex function is provided to you with a certain domain we can always reduce the domain in some sense and artificially restrict the given complex function to that reduced domain.

So, here is what we can do in the example f of z equals z squared we saw that D is equal to C. So, this rule f of x equals z squared was given for every complex number z. Now, what we can do is we can restrict D to be let us say the ball of radius one around the origin, which is definitely contain in the complex plane and write f from B 0 1 to C given by f of z equals z squared. So, this technically, this f now is a restriction it is a new function, it is a restriction of the given function f of z equals z squared. So, do this so can say this f this f is a restriction of the original function defined on all of C. So, that is that is something we can do we can always restrict the domain of a given function.

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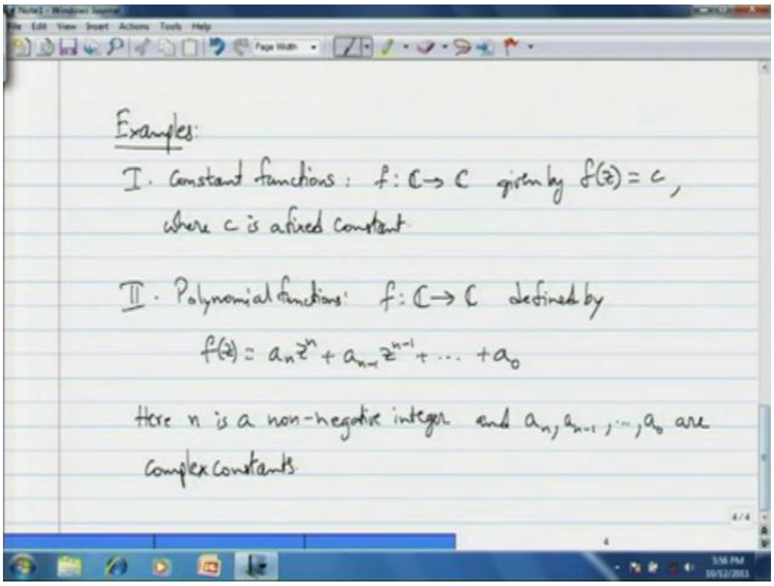

Now, we will see some more classes of examples of complex functions which will be which we will keep using in the lectures to follow. So examples; the first class of examples is constant functions. We will often use the word constants to mean the constant functions whenever appropriate. So let us take f from C to C given by f of z is equal to C, where C is a fixed constant. So, these kinds of functions are called constant functions.

And the second kind of functions are polynomial functions. So, these are functions from C to C defined by f of z is a n z power n plus a n minus 1 z power n minus 1 plus one plus a a 0. So, here n is a positive integer or let me say it is a non negative integer. It could be 0 and a n a n minus 1 etcetera, until a 0 are complex constants. So, for each choice of n and for each choice of these constants a and through a 0 you get a different polynomial. So, you get a polynomial and so you can construct different polynomials this way.

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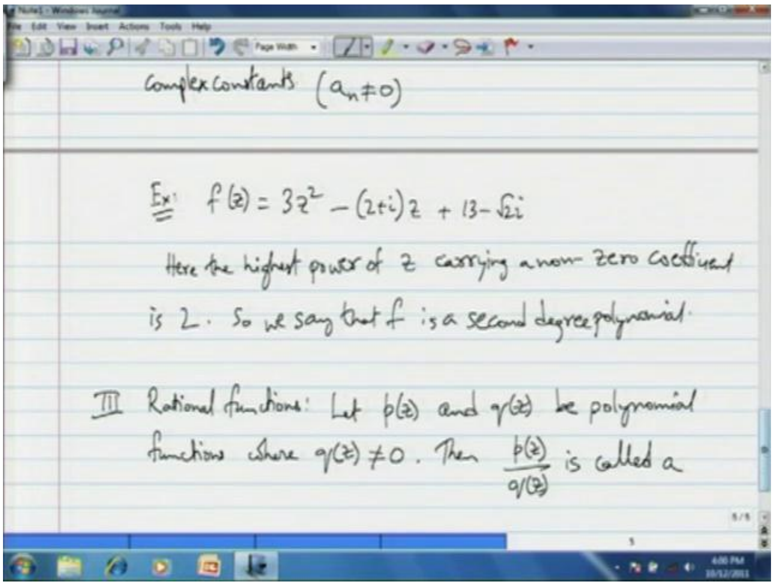

So, in general we try to assume that a n is not equal to 0 otherwise, it is its useless to gives less to write the first term a and z power. So, an example of polynomial function is as follows: f of z given by 3 z squared minus 2 plus i times z plus 13 minus root 2 i. So, this is an example of a polynomial. So, here the highest power of z carrying a non-zero coefficient is 2. So, we say that f is a second degree polynomial.

So, in general the highest power of z which has a non-zero coefficient is called the degree of that polynomial. So, in the case of this generic presentation f of z, here we say that the degree of this polynomial is an... So, notice that the constant functions can be considered as polynomials they are polynomials of degree 0.

So, the first class strictly speaking is a part of this second class of these examples namely polynomial functions. So, the third class of functions, that I want to talk about are a is the rational functions. So, these are functions of this of the sort p by q where p and q are polynomial functions.

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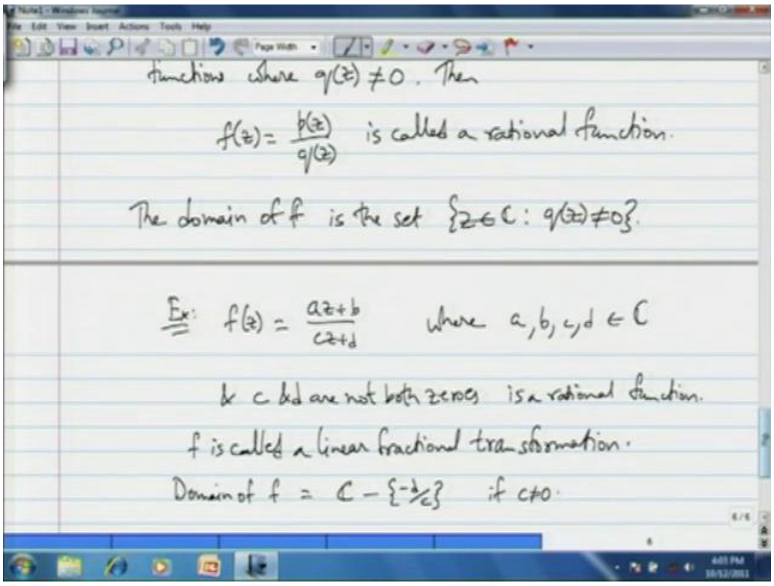

So, let p of z and q of z be polynomial functions, where q of z is not equal to the constant 0 polynomial, it can be any other than the constant 0 polynomial. Then p of z by q of z is called a rational function. So I will rewrite this, I will say.. Then f of z given by p of z by q of z is called a rational function. So, here the domain is not really all of complex plane necessarily, but it is its all those points in the complex plane which skip the 0's of the polynomial q of z.

So, the domain of f is the set z in complex plane, such that q of z is not equal to 0. Because, at those points where a q is 0 the denominator assumes the value 0 and at such points f is not defined the way it has been presented. So, that is a rational function. So, simple examples can be as follows; f of z equals a z plus b by c z plus t where a b c d are complex numbers and c and d are not both 0's, is a rational function and these are called linear fractional transformations. The word transformations, we will see why that is being used and under the condition that a d minus b c is not equal to 0, we will give them a name called a Mobier's transformation and we will study their properties in detail later.

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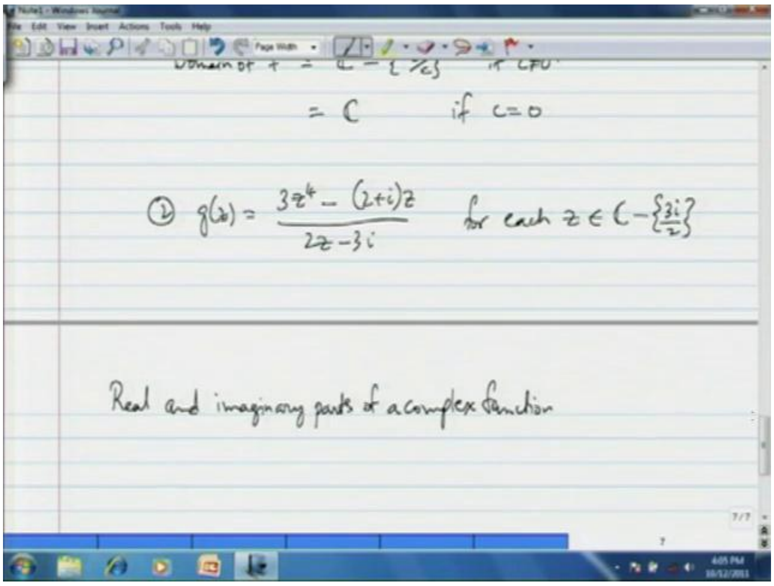

So, this is called, so let me say that f is called a linear fractional transformation and the domain of f is essentially all of the complex plane minus the set minus d by c, if is not zero. So, and the domain is all of c if c is indeed 0. So, that is a rational function.

So, another kind of example of this rational function is as follows; f of z let me use a different symbol, example; 2 g of z given by 3 z power 4 minus 2 plus i z by 2 z minus 2 i. So, this is a rational function defined on for each z belongs to a complex plane minus the 0.3 i by 2. So, at 3 i by 2 the denominator assumes a value 0. So, we remove the point 3 i by 2 and the rest is the is the domain for this g. So, so next we have seen three classes of these examples and we will build on this repertoire of these example, we will construct more and more examples classes of examples.

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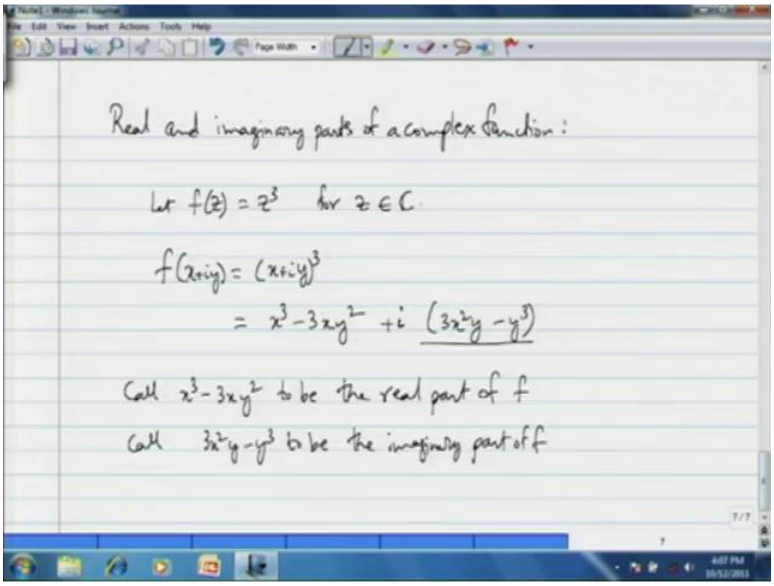

So before moving on we will now talk about the real and imaginary parts of a complex function. So, real and imaginary parts of a complex function. So, let let us see let us start with an example that f of z b the polynomial z cube f of z is z cube for for z belonging to the complex plane. So, we we generally write a complex number z as x plus i 1. So, writing z as x plus i y, we see that we can calculate x plus i y cube using the complex number arithmetic. So this gives us x cube minus 3 x y squared plus i times 3 x squared y minus y cube. You just expand the cube and substitute i squared equals minus 1 to get this expression and from from here we want to call, so call x cube minus 3 x y squared to be the real part of f.

So, we see that this expression of f as x cube minus 3 x y squared plus i times etcetera, is valid for every complex number x plus i y. It has been calculated generically. So, this x cube minus 3 x y squared is always going to denote the real component of the image of a point z under f. And then similarly, call 3 x squared y minus y cube which is essentially the coefficient of i here to be the imaginary part of f.

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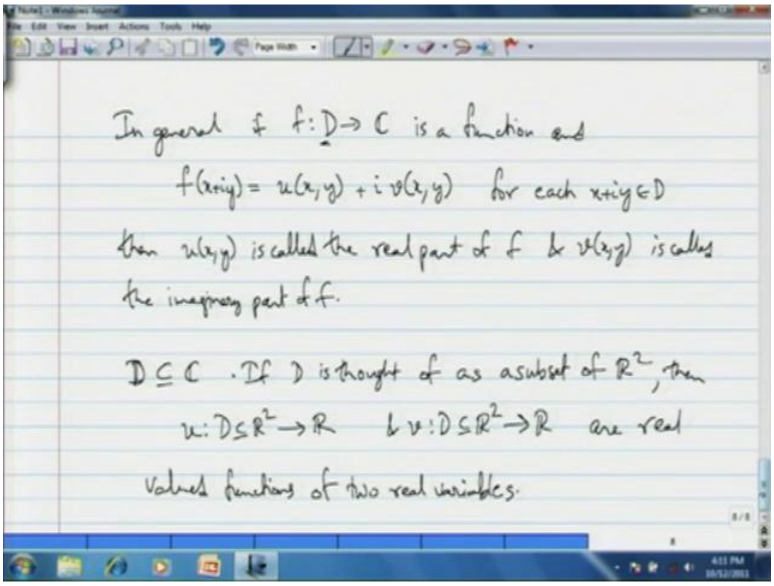

So, in general... So, this is an example of f of z equals z cube. So, in general if we take f complex function in general if f from D to C a domain D to C is a function and f of x plus i y is equal to u of x comma y plus i times v of x comma y. So, in the example, just presented notice that the real part was an expression in terms of x and y. And likewise the imaginary part of f was a an expression in terms of x and y.

So, in general if f of x plus i y is a function with u with this u of x y plus i times v of x y presentation for each x plus i y belonging to D. Then u of x y is called the real part of f and v of x y is called the imaginary part of f. So, notice that u is can be construed as a function a real valued function of two real variables.

So, let me say this in the following way if D is a subset of the complex plane. So, D which appears to be the domain of f, is a subset of C. So, if D is thought of as a subset of R 2. So, we have we have given geometry to the complex numbers it is a complex plane, we represent it as a complex plane. So, if we think of numbers indeed, the complex numbers indeed has points x comma y in R 2. Then u is a function from D contained in R 2 to R. So, u of x y after all is an expression which gives you a real number.

So, u is such a function and v from D contained in R 2 to R are real valued. Of course, real valued functions of two real variables, two independent real variables x and y. So, so in general it is a good exercise to practice with a few functions what its real and imaginary parts are going to be. So, the viewer is advised to pick a few functions and find the real and imaginary parts.

So, next we want to talk about visualising these complex functions what I mean by that is, we had a nice picture for real valued functions of real variables. So, in calculus we draw graphs of functions of one real variable. So, in the case of complex functions we are we are not that lucky to draw graphs directly because in the case of one variable real functions we had needed one axis for input. So, the x axis was the input axis.

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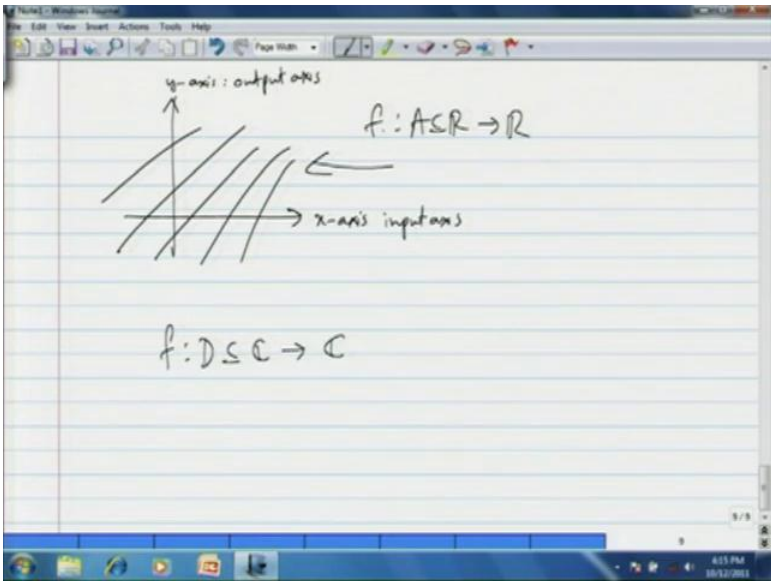

The picture for a real numbers is a real line and the y axis is was the output axis. So, we we plot the inputs on the x axis and the outputs which are the real numbers on the y axis. So, all in all we needed a plane to picturize, to visualize functions from R to R. So, f from a contained in R to R; so the graph is some subset of the plane, but in the case of functions from some subset of the complex plane to complex plane, we need two dimensions essentially the plane to draw the domain, to picturize the domain and two more dimensions to picturize the the range.

So, all in all we need four dimensions to picturise this graph in one piece. Since, we cannot visualize four dimensions at least on on the board so what we will do is we will try to draw one piece of complex plane for visualizing the domain and another piece of a complex plane for visualizing the range of this function f.

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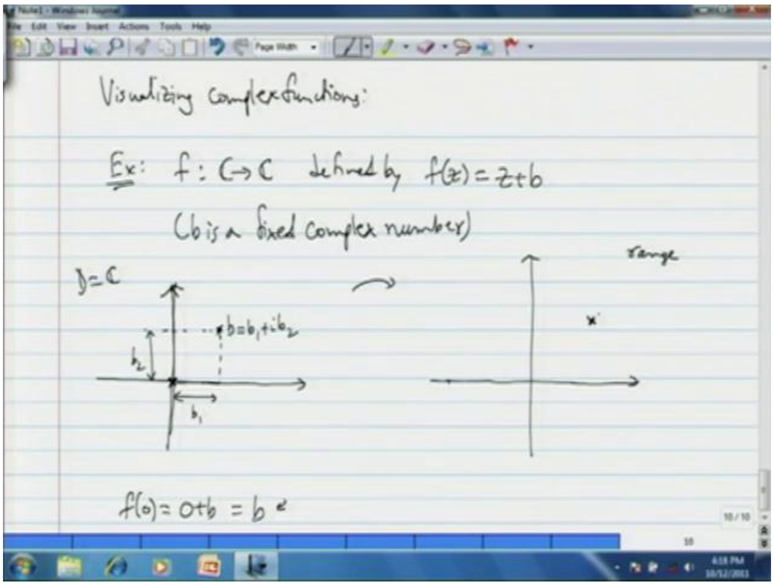

So, let me explain with an example. So, here is my first example; visualizing complex functions. So, let us consider f from C to C defined by f of z equals z plus b, a simple example z plus b. So, b is a fixed complex number. So, this function the domain is of course, all of the complex plane. So, this picture here of a plane is that for the domain equals the complex plane. This is the real axis and that is the imaginary axis and I need another piece of this complex plane to picturize the range of this function so this is the this is further range or the output of this function.

So, if I take origin, I will use cross mark to denote this origin. So, I can calculate where origin goes to f of 0 is 0 plus b it goes to a complex number b. So, right here in the domain itself let us imagine that b is some number over there, let me put a dot for b. So, b this is b and b let me call this b 1 plus i b 2, here it is a fixed constant. So, we know that this distance. So, this distance right here is b 1 and the vertical distance of b from the x axis or the real axis is b 2. So, the image of a origin now is going to be that exact point like we calculated here, it is going to be that exact point b. So this cross mark here in the domain goes to this cross mark here in the range.

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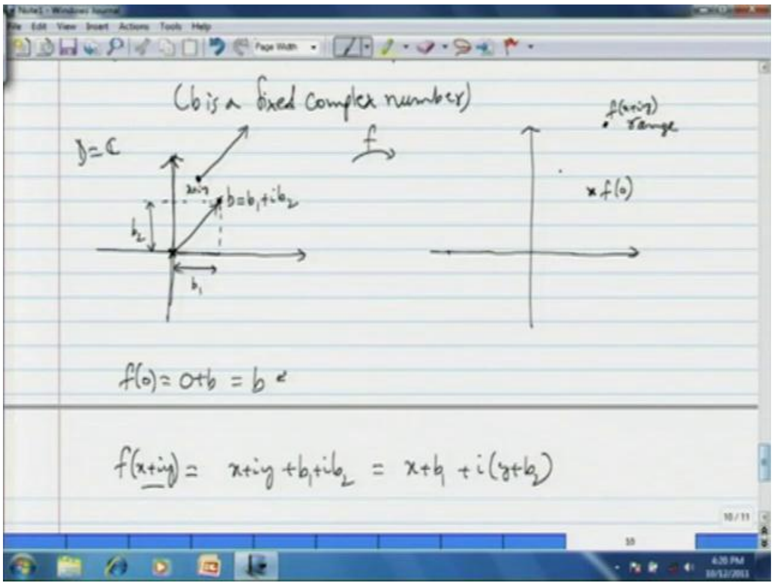

So, in general, so in general if you pick a a point x plus i y, let us calculate what is f of x plus i y. This is going to be x plus i y plus b which is b 1 plus i b 2. So, this is going to be x plus b 1 plus i times y plus b 2. So, what this means in in terms of pictures is that this point x plus i y is translated via this this number b. So, what that means is that if you imagine this point b in the domain, here to be the n point of a vector it starts at the origin. Then take any other point, let me say that x plus i y.

So, this is x plus i y so imagine the vector b starting at the point x plus i y. So, here is the vector b starting at the point x plus i y. So, the image of the point x plus i y is going to be exactly that the n point of that vector, which originated at x plus i y. So, here is your image f of x plus i y. So, the image of x plus i y is going to be that f of x plus i y f of 0.

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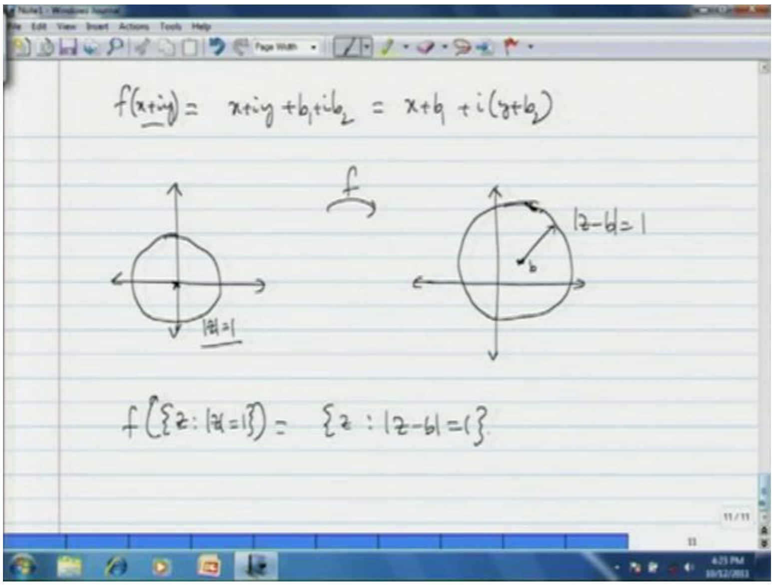

So if you if you in general consider any contour here let me draw another picture. So, here is a picture for the domain and here is a picture for the range. So, generalising what we have done if you take any contour let us say a, let us say a circle of unit radius gives circle of radius one centred at 0. So, this is a absolutely modulus of z equals 1.

So, the image of this circle under the given function is a essentially the translation of this circle by b units. So, this circle moves to a position like that, where we saw that so although the circle looks enlarged it is not really enlarged. So, here is the origin or the image of the origin so the centre moves to b and and the image of this modulus of z equals 1 is all such points such that modulus of z minus b is equal to 1.

So, the image of set of all z such that modulus of z equals 1 is going to be set of all z such that modulus of z minus b now is going to be 1 etcetera. So, any any contour or any region in the domain is is moved by the vector b. So, you can imagine f to be a transformation it is a it is it moves any portion in the domain to some portion in the range. So, it is a transformation that way.

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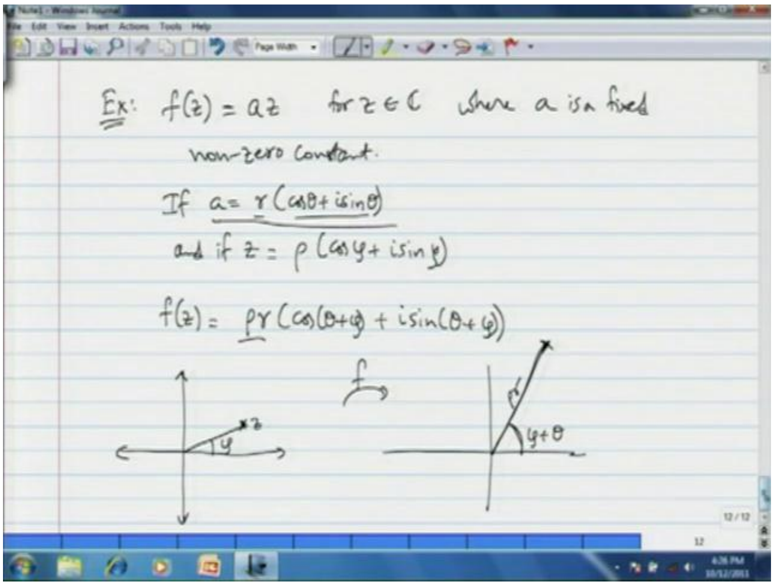

So, we will see another example of visualization. So, here is an example; f of z equals let us say a z where for z belongs to c where a is a fixed non-zero constant. So, f multiplies a given complex number z with the complex number the fixed complex number a. So, for visualizing this function it is better that we use a polar coordinates. So, if a is written as a r times cosine theta plus i sine theta and if b or if z equals some rho times cosine phi plus i sine phi. Then f of z is going to be rho r times, we saw this earlier cosine theta plus phi plus i times sine theta plus phi.

So, what this does is that if this function does is that, if we take a a complex number z any place. So, it has a certain modulus and a is a a is a complex number with modulus r. So, the image of this point z is going to be a complex number with modulus r times the modulus of z. So, let me first also say that z is a number z is a number with an angle of phi with the positive x axis it is an argument. So, let us assume, let us pretend that a phi the angle of opening with the positive x axis.

So, then here a times z is going to be a times z is going to be a complex number whose modulus is going to be rho times r. So, the length of this vector is rho times r and the angle of opening with the positive x axis is going to be phi plus theta. So, this theta and r of course, are from a. So, what we can I mean the way to visualize f is that it elongates a given vector z or contracts it appropriately depending on whether r is greater than 1 or less than 1 and rotates the whole of the complex plane. So, rotates the complex plane by an angle theta where theta is this fixed angle coming from a. So, that is the visualization of this function a z.

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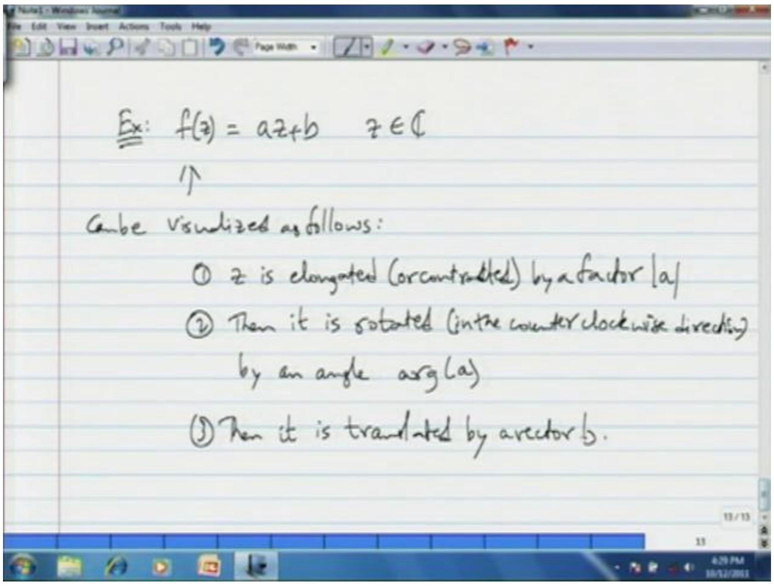

So, in general if I or combining these two examples if I give you a function f of z equals a z plus b. So, this function it it actually elongates a given complex number by the modulus of a and then it rotates by the angle which is the argument of a and then translates the complex plane by the constant b.

So, f of z can be visualized as can be visualized as follows. So, it is it is a combination of three actions. So, first f or let let me say a three actions so the first of the actions is z is elongated or contracted by a factor the modulus of a and second then it is rotated in the counter clockwise direction direction by an angle argument of a a then it is translated by a a vector b. So, this function f can be pictured as the combination of these three actions in that order, which which is outline there, that is another example.

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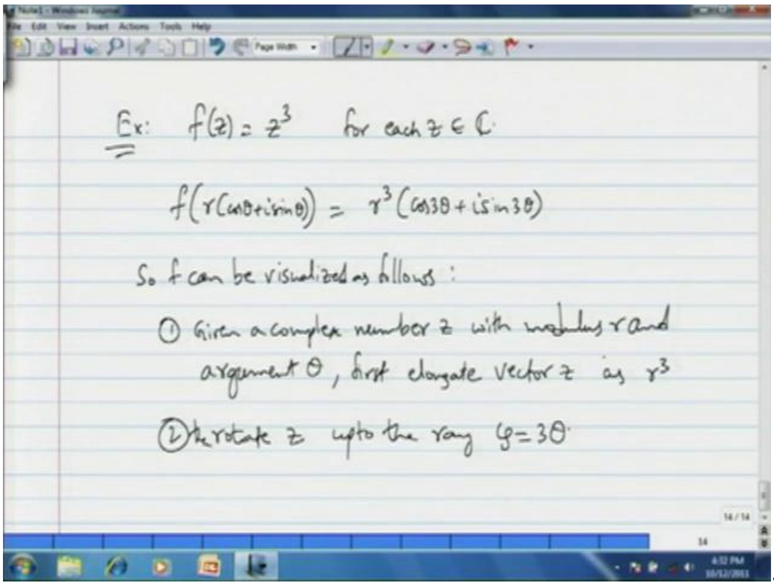

So, let us see yet another example of of visualizing this function. So here is the example of f of z equals z cube, this is a third degree polynomial simple polynomial for each z belongs to c. So, in order to visualise this function a, what we will do is we will once again write z in polar coordinates. So, f of r times cosine theta plus i sine theta. So, if i take a point r times cosine theta plus i sine theta which is z then the image of this is going to be r cube times cosine 3 theta plus i sine 3 theta.

So, f can be visualised as follows; so given a complex number z with modulus r and argument theta, first elongate z. So, this is performed in the complex plane these kind of operations are performed on on the points on the complex plane. So, first elongate z the vector z as r cube and then secondly rotate z to an angle or a then rotate z up to the line up to the ray, phi equals 3 theta.

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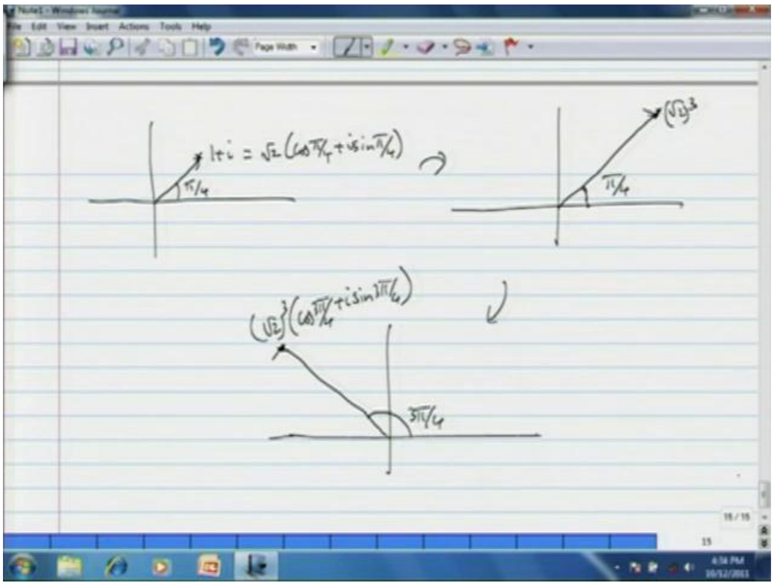

So, let us see an example of...So, let us see an example of how to do this. So, if I take a z equals 1 plus i, this can be written as root two times cosine pi by 4 plus i sine pi by 4. So, this can be thought of as a vector like that. Then what f does is it first elongates this point z to the length root 2 cube, which is essentially 2 root 2 along the same line so this angle is pi by 4. So, first keep the angle as pi by 4 and elongate this vector to root 2 cubes followed by then followed by...

Now, rotate this point about the origin until you reach three times this angle pi by 4, which is 3 pi by 4. So, here is 3 pi by 4 so you rotate this about the origin until you reach this point root 2 cube times cosine 3 pi by 4 plus i sine 3 and now this is the image of the point you started off with, that is how you can visualize z cube.

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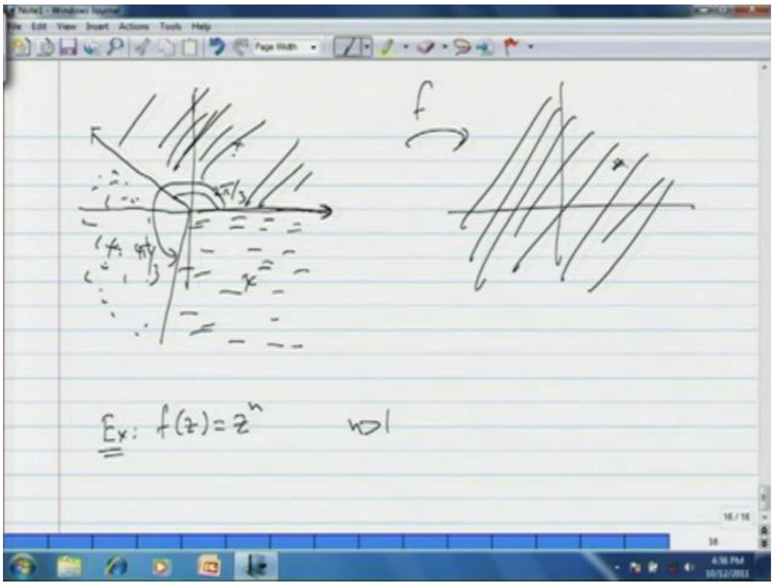

So, all in all if you if you take this wedge, so this angle is 2 pi by 3. So, this so this infinite wedge, so all complex numbers which range in their argument from 0 to 2 pi by 3. Then the image of this you can convince yourself is all of the complex. So, the image of this is going to be of the f applied on this p this is going to be all of the complex plane. So, when you want a when you want to apply f on this piece let us say, this is 4 pi by 3 this angle is 4 pi by 3.

So, when you want to apply f on the piece on the infinite wedge between 2 pi 2 pi by 3 and 4 pi by 3, then you will once again get the whole of the complex plane and likewise when you want to apply f on the wedge between 4 pi by 3 and 2 pi. So, you will once again get another copy of the complex plane for the range. So, the image... so, if you pick 1 point here in the complex plane there are three pressure images here. So, one lying here; one lying in this infinite wedge and another lying in this infinite wedge; so this is how we visualize this complex function. So, in general if you consider f of z equals z power n for n greater than 1.

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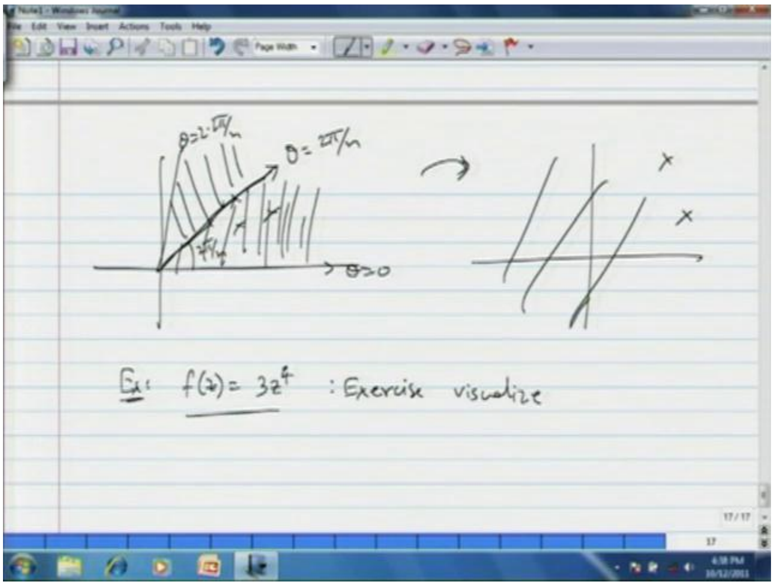

Then this function takes the wedge between theta equals 0, this can be thought of us theta equals 0, and this can be thought of as theta equals 2 pi by n. So, this is an infinite ray standing for theta equals 2 pi by n. So, this angle is 2 pi by n. So, this function takes all the points between these two infinite rays including the rays, and sends it to all of the complex plane. So, essentially any complex number here in the, in the co domain or the range has had one pre image in this infinite wedge.

So, that is how we visualize this function f of z equals z power n, and then you can keep going 2 pi by n, and then 2 times 2 pi by n, theta equals two times 2 pi by n etcetera; and once again the image of this is going to be all of the complex plane. So, you can practice and keep a drawing more of these pictures for polynomials. So, for example, try the following, try to picturize f of z equals 3 z to the power 4. So, try to visualise so exercise so visualize f.