Complex Analysis
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Module - 1
The Arithmetic, Geometric and Topological Properties of the Complex Numbers
Lecture - 5
Topology of the Complex Plane Part – III

Hello viewers, we will continue the study of the topological properties of the complex plane. So, far we have seen until what limit points are and what interior points, boundary points and exterior points are. So, today we will start with compact sets, so firstly bounded set.

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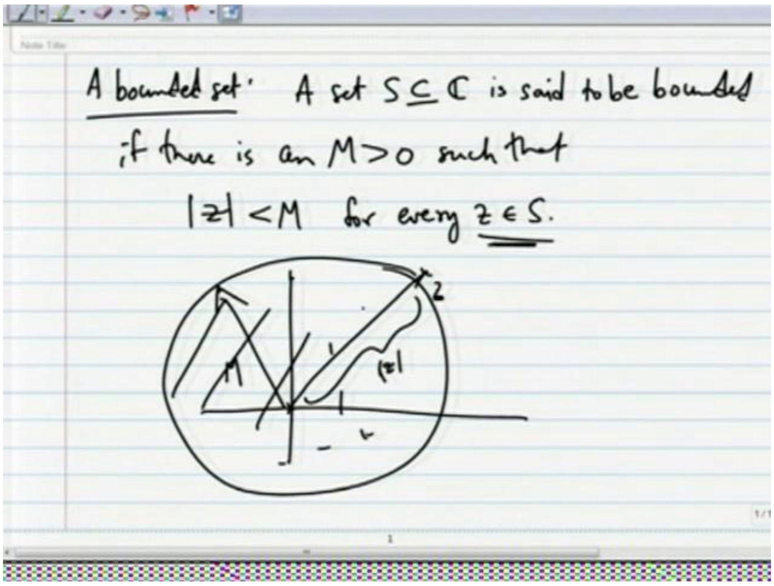

So, a set S contained in C is said to be bounded, if there is an M positive such that the modulus of z is strictly less than M for every z belongs to S. So, what that means is a bounded set physically means that the modulus of z which is the distance of the point z from the origin is bounded for every z belongs to S by the same number M.

So, if you if you draw a circle of radius M around the region then every element of S is contained within this circle of radius f, and so that is a bounded set. So, in total it fits the the picture that you know the the points of S are contained in the large circle. So, that is a definition of a bounded set. We will see that the property of boundedness together with closeness or the property of a set being closed in the complex plane play an important role many times in complex analysis.

So, you will recall from the calculus of functions of one real variable that you had the extreme value of theorem where a function on a closed and bounded interval assumed its maximum value or minimum value on a closed and bounded interval. So, such properties are exhibited by closed and bounded intervals and functions on closed and bounded intervals in real analysis. So, an equivalent concept here is that of closed and bounded sets in the complex plane which we are going to call compact.

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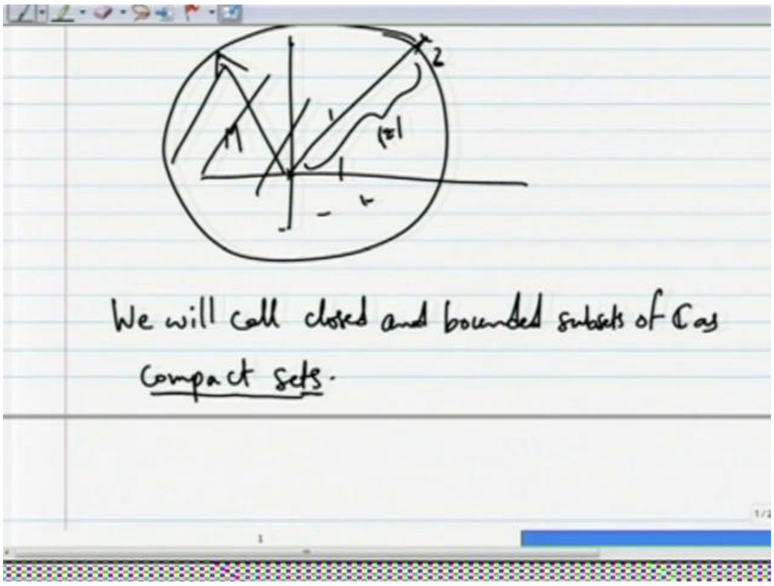

So, we are going to call, we will call closed and bounded sets, subsets of C as compact sets as compact sets. So, that is not a definition. We are going to give a definition now, but firstly, compact sets play the role of closed bounded intervals in real analysis. So, these sets are to complex analysis what closed bounded intervals are to real analysis or functions of one real variable.

And so firstly I want to define compact sets. We will call these kinds of sets as compact sets, but I had not really defined. It is defined in terms of open sets so that it fits a more general setting of an arbitrary topological space, that is not our point of discussion here, but we will give the definition, we will show that it is (( )) or we will state a theorem which says that it is equivalent to the statement here.

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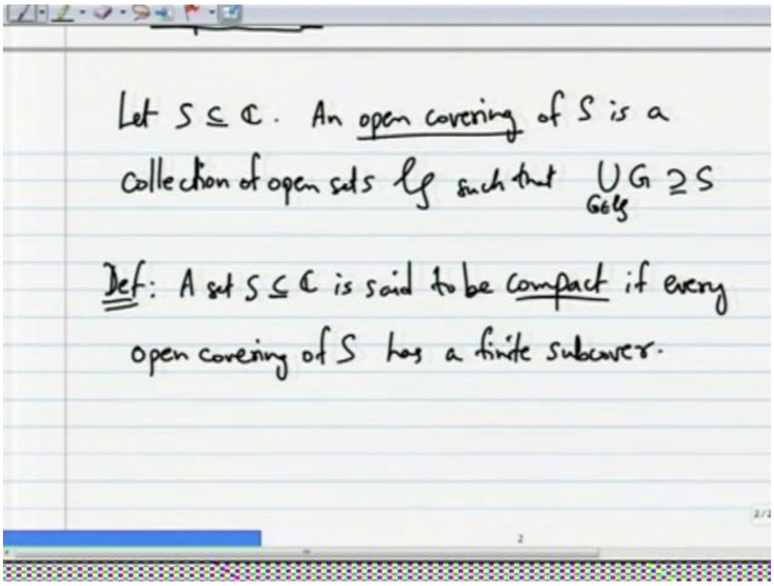

So, firstly an open covering, let S be a sub set of C, let S be a sub set of C, could be empty, does not matter. An open covering of S is a collection of open sets. Let us call its script G. So, the elements of G are open sets and G is the collection, script G is the collection and it is a collection of open sets such that the union of G, G belongs to the script G contains S. So, whose union contains S. Such is an open covering. So, it is essential open sets which cover really the set S.

So, now we will define a compact set. So, a set S contained in C is said to be compact if every, what is important is if every open covering of S has finite sub cover. So, a sub cover is a sub collection of this collection of open sets. So, if, if there is finite collection, finite sub collection of this collection of open sets which is enough to cover the set S, then we say that the set is compact. And this property should be exhibited to, exhibited for every open covering that you can bring for this set S and in that case we call the set compact. So, that is the definition.

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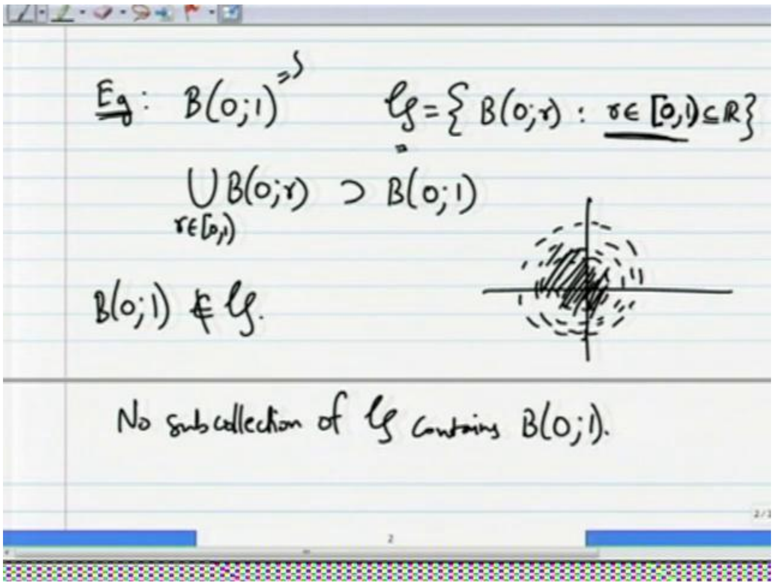

And you will see I mean an example or a non example will actually help us see that such a phenomenon not exhibited by every sub set of complex plane and easy example is B 0 1, the unit disk in the complex plane. So, what you can do is you can form the union of so first let me define a collection of open sets. This is B 0 r such that r belongs to 0 1, this is the interval 0 1. So, this this is interval contained in r, contained in the real numbers. So, this sorry I can sorry I think this is 0, the the interval closed at 0 and open at 1. And the union of actually that does not matter whether I close it at 0 or not.

The union of B 0 r, r belongs to this 0 1. Is, this will definitely contain all of the ball 0 1. So, what I have done is I have that is, this is the following is the picture, here is the unit disk and this collection of open sets here is essentially a set of disks which are growing in size, you can think of them as growing in size and they will have radii. So, it is the inside here. So, they will have radii in the interval 0 comma 1. So, as this radius grows so these, these, this collection of open sets tend to cover the whole ball of radius 1 centre at 0, but unfortunately there is no finite sub collections of this collection which will actually cover all of B 0 1.

Notice, that B 0 1 itself is not included here. So, this B 0 1 does not belong to, this does not belong to the collection G, because 1 is excluded from this. That is an example where we cannot have a finite sub collection which will contain the whole set S. So, in this case set S under consideration is this B 0 1.

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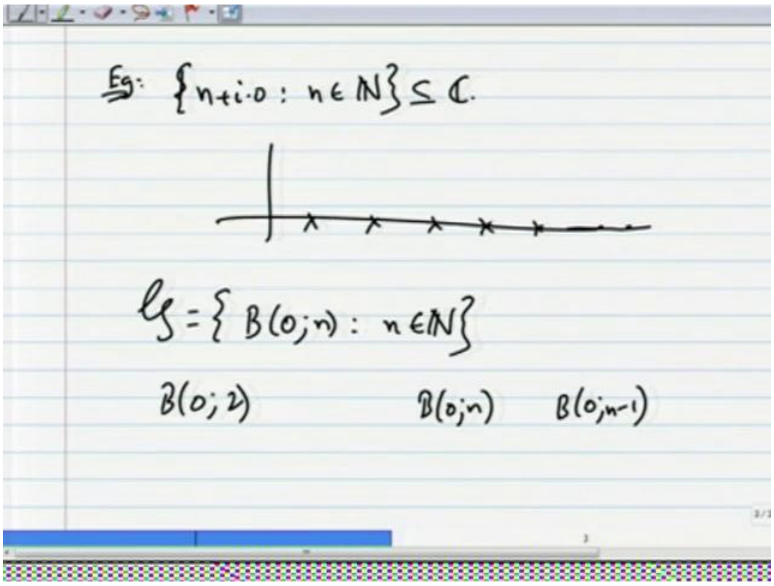

Yet another, yet another example or a non example of compact set is the set of natural numbers or integers contained in… So, let us take the set of all n plus i times 0 such that n belongs to N contained in C. So, these are of course, points on the real line which stand for the natural numbers. So, you can consider the collection G of B 0 let say n such that n belongs to N.

So, each of these in turn starting with B 0 2, starting with B 0 2 is going to contain one additional natural number than the previous B 0 n minus 1. So, B 0 n will contain one more a natural number than B 0 n minus 1, but all in all you can never have a sub collection of this which will cover all the natural numbers, all the set S solely because if you take a sub collection or or a finite sub collection even worse then I should not say just sub collection, but I should say finite sub collection. If you take a finite sub collection then you will have to stop at a point and beyond that point there will still be a natural number which will not a fall in here.

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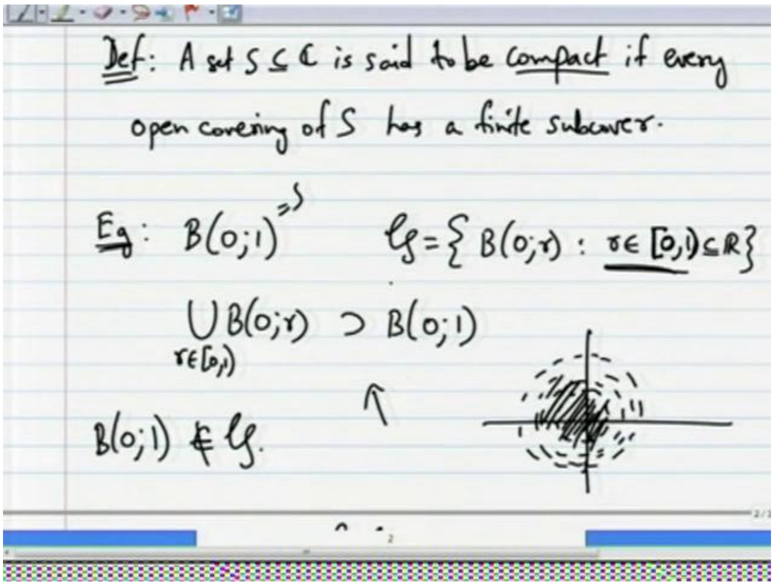

So, I apologise. I should see a finite sub collection in this, in this non example also there is no finite sub collection of G which actually a span all of B 0 1, you can always get a sub collection, that is not the point. So, there is no finite sub collection. So, these are two examples of sets which are not compact and then what are examples of sets which are compact.

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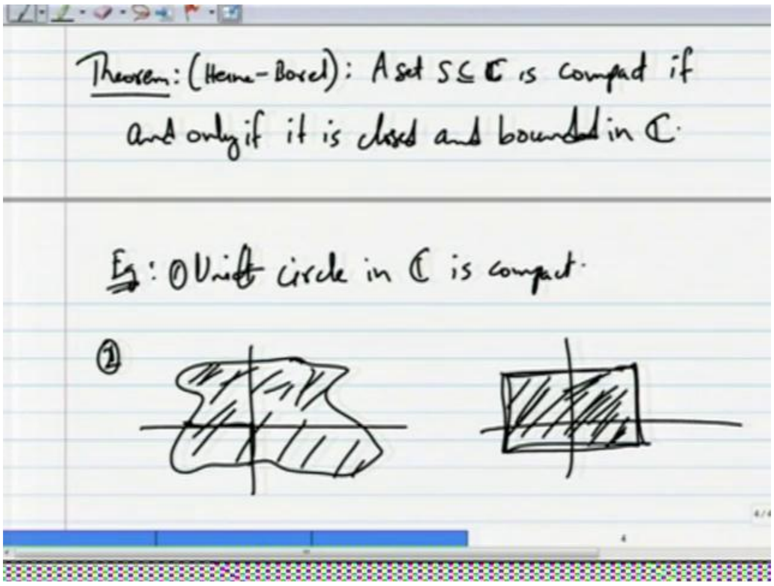

Well, if you will, I mean we will give a theorem. We are not going to prove this. This is the Heine Borel theorem which will give us many examples of of compact sets. So, a set

S contained in C is compact if and only if it is closed and bounded in C. So, we will skip the proof of this theorem, but we will use its conclusion to give examples of compact sets. So, examples of compact sets, the the unit circle circle in C is compact, its closed and its bounded. It is closed because the complement of it is the unit disk without its boundary and everything outside the closed unit disk and we saw that both those sets are open sets.

So, this set is closed and also it is bounded, well the modulus of any number on the unit circle is 1. So, it is bounded definitely. So, this is a closed and bounded set and hence it is compact. So, in general you can take, I give a pictorial example two. You can take any any box like that and consider the set of points inside that box. This this is a curve of some sort, you can I mean if you do not like this you can take some polygon and then consider the box obtained by taking the points inside and may be even points well points inside and on the polygon.

So, this set is compact because it is because it is closed and it is also bounded. There is no reason why this this box should be symmetric. It could be asymmetric, it could be placed completely in the first quadrant for example, or it could be a it could be a little skew, but nevertheless it is going to be a bounded set, because it is a box and and it is closed definitely, because we are including the points on the boundary and and so we are including all its limit points in addition to the interior points. So, such a set is called compact.

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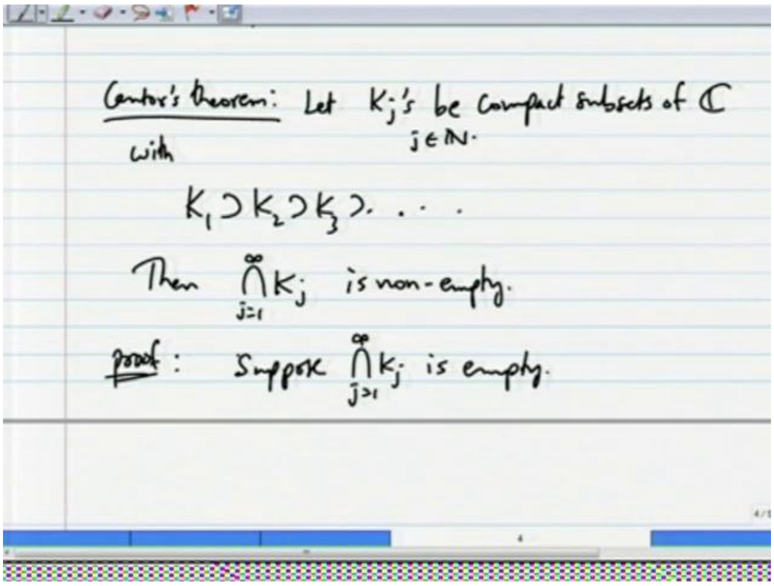

Some properties of a compact set, we are going to see that these compact set are, sets are going to play an important role in complex analysis. And here are some properties. Well, I will give at least one important property. So, it is a Cantor's theorem. So, let let K j's be compact sub sets of the complex plane with… So, this k j's j belongs to N. So, with K 1 containing K 2 containing K 3 etcetera. So, it is a nested sequence of compact sets like that, then the intersection of K j, j equals 1 through infinity, is non empty.

If you take the intersection of the all these nested sequence of compact sets then it has to be non empty. We will see why. A proof, well K 1… So, what we can do is suppose it is empty. Suppose, this inter section is empty. So, if the inter section is empty then the complement of the intersection is all of the complex plane. So, what we can do is we can consider the complements of each of these compact sets. A compact set is closed.

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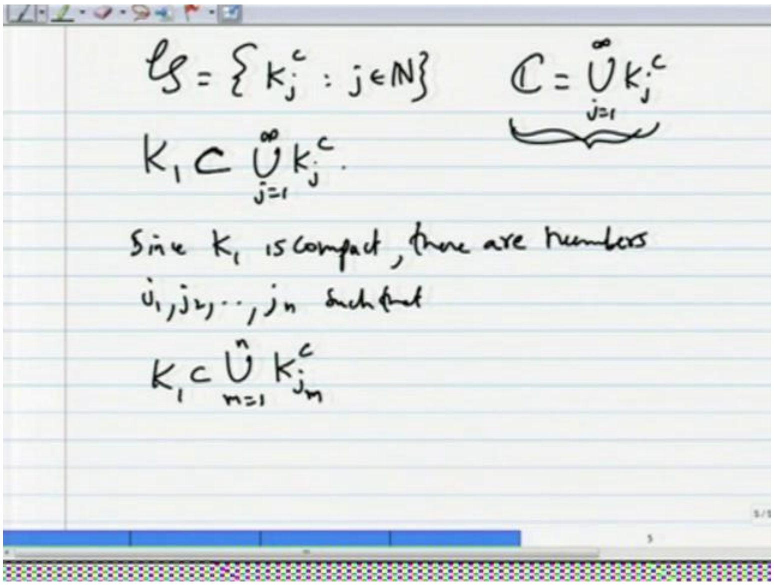

So, its complement is going to be open and so you can, what you can do is if you consider this collection G of the complements of K j's, the complement in the complex plane of these K j's such the j belongs to N. Then you know that the whole of the complex plane is actually covered by this K j complement, j equals 1 through infinity. Why? Because of course, the inter section of all this is empty. So, the the complements when you take the union of all of them, then you are going to get the whole of the complex plane.

In particular K 1 since this is the whole complex plane K 1 is definitely covered by G. G is an open covering for K, K 1. So, K 1 is contained in the union of j equals 1 through infinity K j complement. And here is where, here is one instance where we are going to use the particular definition of compact sets that we have given, that every open covering will have a finite sub cover.

So, there is since K 1 is compact there are there are numbers j 1, j 2, so on until j n let us say such that so there are finitely many indices such that the union of j equals 1 through, I should say m equals 1 through n of K j sub script m complement is going to contain is going to contain K 1, I apologise this is K 1 is contained in, this is the whole complex plane. So, K 1 is contained in the union of this. So, K 1 is contained in here. So, since K 1 is contained in here, these are finitely many and by this condition here that these are nested like this K 1 contains K 2 contains K 3 etcetera.

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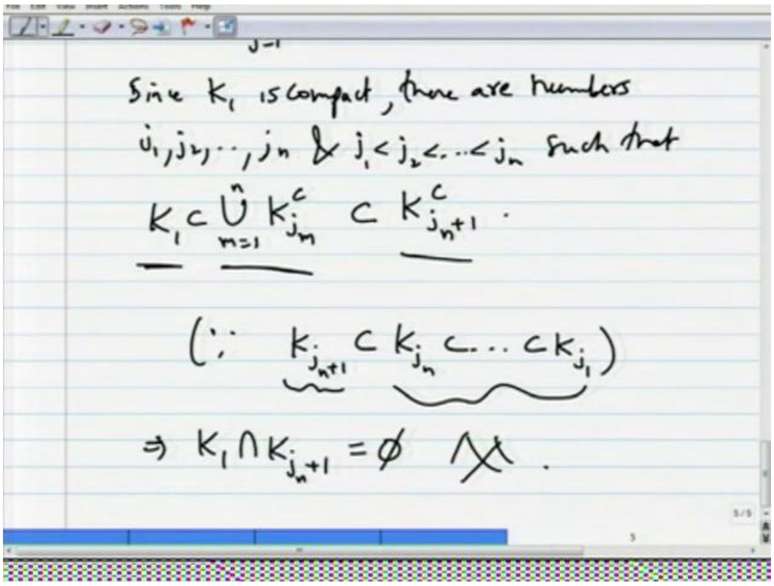

The union of all of these will be contained in K j n plus 1 complement. So, here I am actually assuming more so I will say that j 1 is less than j 2 is less than so on until j n such that. So, I mean when there are finitely many integers I can of course, order them. Finitely many natural numbers I can order them so I will, without loss of generality assume that j 1 is less than j 2 less than etcetera until j n. So, j n is the largest integer and so K j n complement the the union of all of these m equals 1 through n K j, j m complement is going to be contained in K j n plus 1 complement because the union of m equals or or K j n plus 1 is contained in K j n is contained in etcetera contained in K j 1 if you wish.

So, since this is true the union of the the complements of all of these will definitely contained in the complement of this L j and plus 1 and so, this tells you that K 1 contained in the complement of K j n plus 1 which implies that K 1 intersection K j n plus 1 whatever that induce is is empty, but this is the nested set any point in K j n which occurs for the down has to be contained in K 1, so this is the contradiction. So, this contradicts the given hypothesis and so the intersection has to be non empty.

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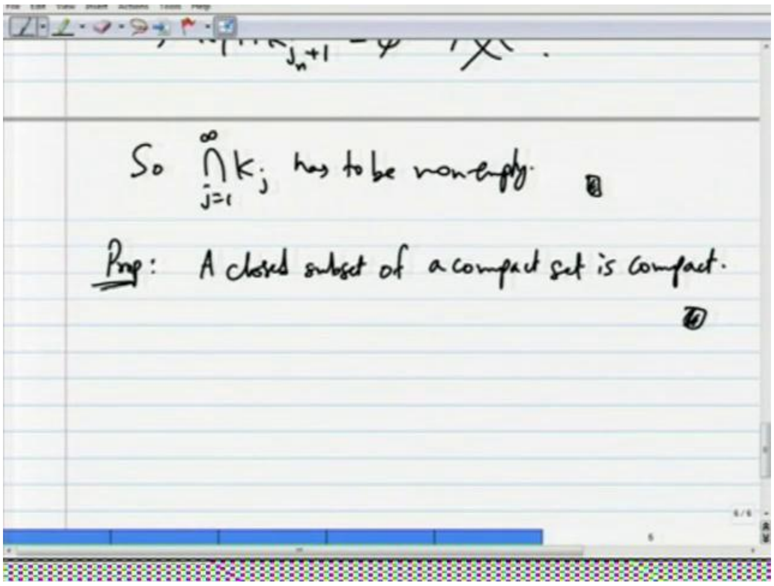

So, the inter section of K j, j equals 1 through infinity has to be non empty. So, it is one property of compact sets and yet another proposition is as follows. Here is another property which we might have some use for. So, a closed subset this, the proof of this is very clear, a closed subset of a compact set is compact. Any subset of a compact set is bounded because the whole set itself is bounded.

If you have a closed subset of a compact set then you have a bounded set and it is also closed by hypothesis. So, it is compact. So, the proof is just directly there in the statement. So, that is another property which we might have some use for. So, then we will see the concept of connectedness. So, this is a different property and we will have use of this property as well from time to time.

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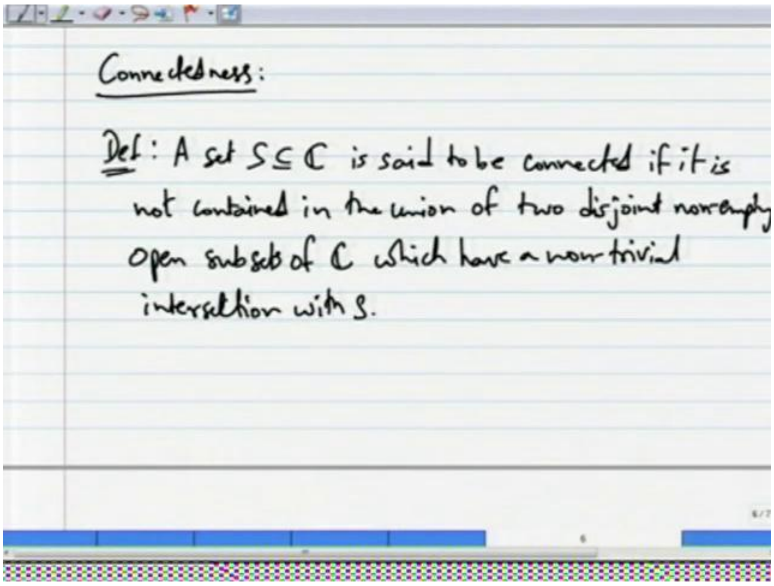

So, connectedness so like compactness, connectedness is also defined in terms of open subsets of the complex plane. So, a set S contained in C is said to be connected if it cannot be expressed or if it is not contained not contained in the union of two disjoint non empty open subsets of C, which have a non trivial intersection with S. So, I mean that is a mouthful.

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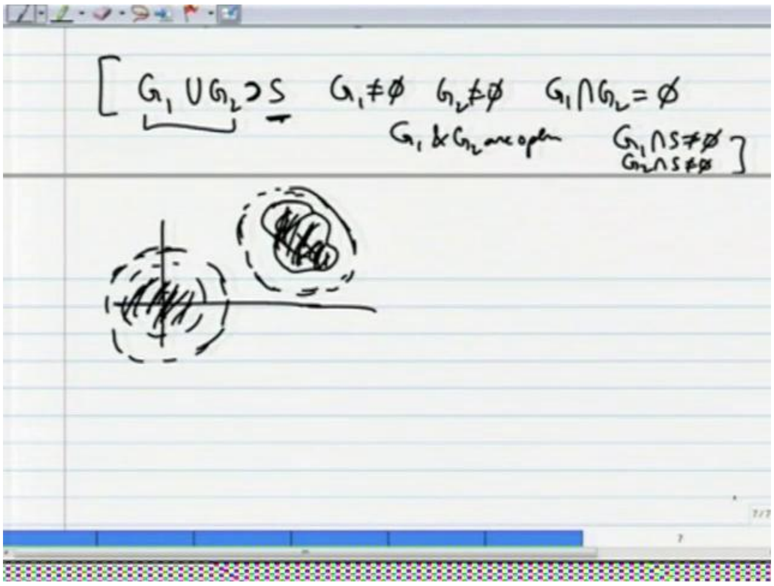

Let me explain. So, if there are, if you can never write S to be contained in G 1 union G 2 where G 1 is non empty, G 2 is non empty and G 1 intersection G 2 is empty and G 1

G 2 are open with G 1 intersection S is non empty and G 2 intersection S is non empty. If you can never write S to be contained in such union, then S is connected. So, what that means is I know that sounds like a bunch of conditions. So, what you want to avoid is that is the is the following. So, for example, you look at the following intuitive example.

Suppose, you have the unit disk and you have yet another, you know set like that there. So, what you can do with the complex plane is you can come up with one open set which contains that piece and yet another open set which contains this piece and so you can separate these two blobs here. And we exactly want to avoid this situation here. So, this this set which is the union of the unit disk and this piece here is we want to call that disconnected.

So, a connected is the opposite, you can never write this S to be contained in G 1 union G 2 where G 1 and G 2 are disjoint open sets like that. We want them to be non empty as well and then they are open sets and they should have some non trivial intersection with S that is also important. So, that is the definition in terms of open sets. So, we will not consider here all the intricacies of this definition here because what we are going to see are mostly open connected sets which we will call as regions or domains.

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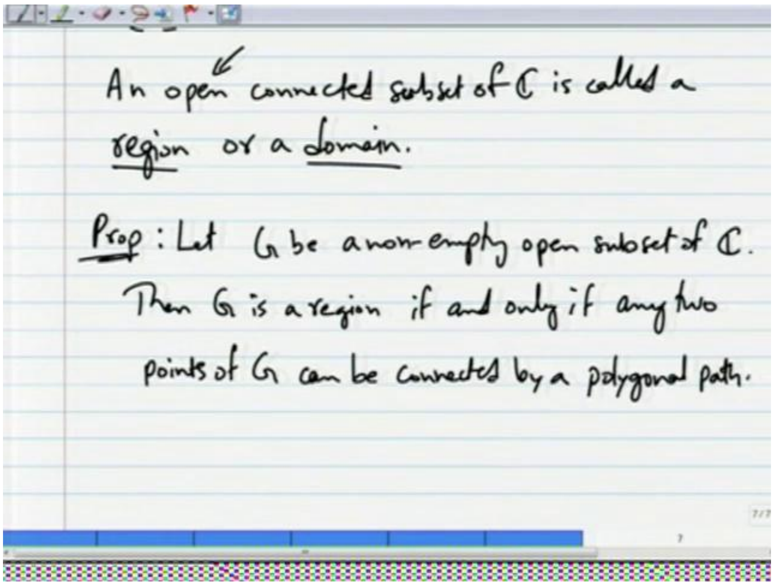

So, an open connected set, subset of C is called a region or a domain. I will interchangeably use these words. Sometimes, I will call open connected subsets of C as domains and sometimes I am going to call them regions. They both refer to open connected subsets of C and we will see that with that additional condition open this connected sets behave in better fashion.

So, we can sort of use the property of open connected sets and do away with this definition itself. But I am giving this definition for completeness. Let G be a non empty open subset of C, then G is a region which means it is an open connected subset of C, if and only if any two points of G can be connected by a polygonal path. So, a polygonal path is a finite union of straight line segments in the complex plain. So, that is the polygonal path and if you take any two points in an open connected set, we will show that they can be joined by polygonal path. And if an open set is such that you can join any two points by polygonal path then that that open set will be connected. So, it has to be connected.

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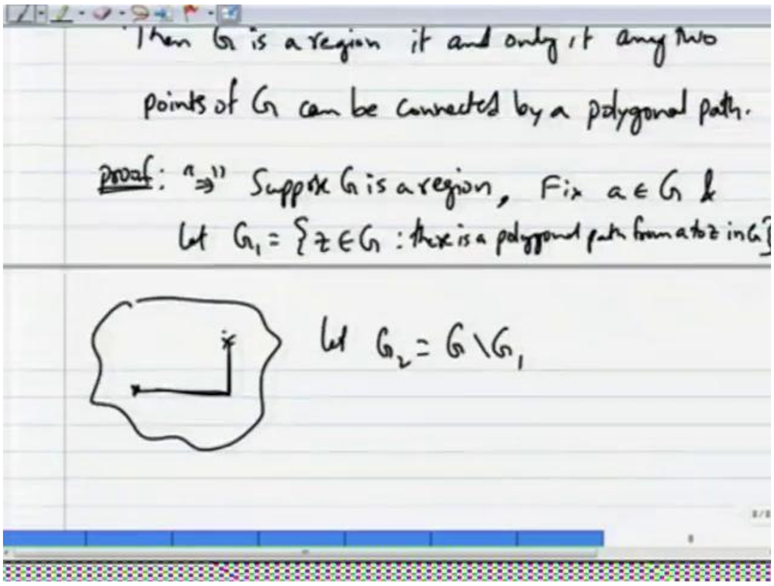

So, proof once again here we will use this property and we will use the property in the direction that if we have an open connected set then any two points can be connected by the polygonal path. That is the direction we will frequently use this property in. So, I will proof only the direction and and skip the proof of the converse all though this is an if and only if statement.

So, I will prove one direction that if G is a region then any two points can be joined by a polygonal path. So, suppose G is a region. So, fix a point a belongs to G. So, G is non empty. So, a belongs to G and a 1 or let G 1 be the set of all points in G such that there is a polygonal path from a to z which is completely contained in G. If you have some set like that of polygonal path is a path like that. So, let G 2 is the complement of G 1 in G. So, we are going to show that G 1 and G 2 the complement of G 1 and G, both of them are open sets. So, by the definition of connectedness. So, if you write…

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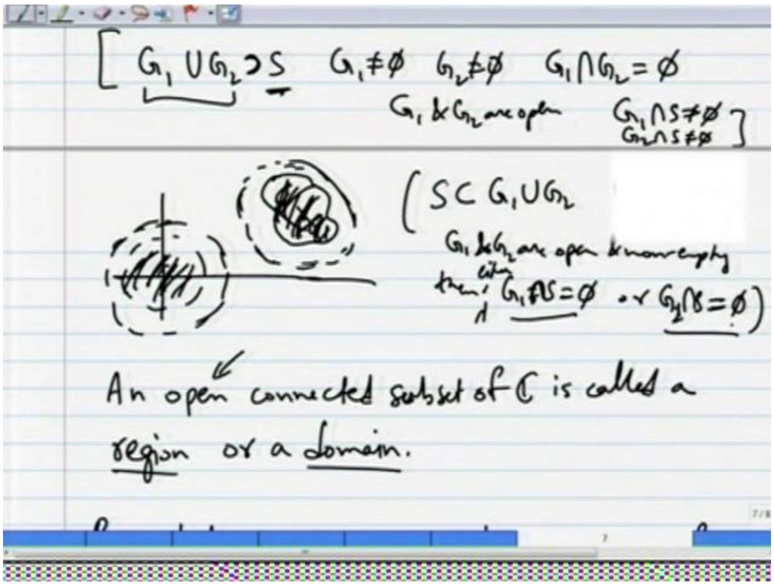

So, another implication of definition or another way to say this definition is that if you are able to express S with all these conditions if you are able to express S contained in G 1 union G 2 wherein G 1 G 2 open, non empty are open and non empty then then either G 1 is, either G 1 intersection S is empty, then either this is empty or G 1 intersection G 2 intersection S is empty. What that means is S is completely contained in either in G 1 or G 2 if you are able to write a connected set. So, if S is connected and you are able to write it to be contained in G 1 union G 2 where G 1 and G 2 satisfy all this conditions then one of this has to be empty. It is the same; it is the same thing saying all this.

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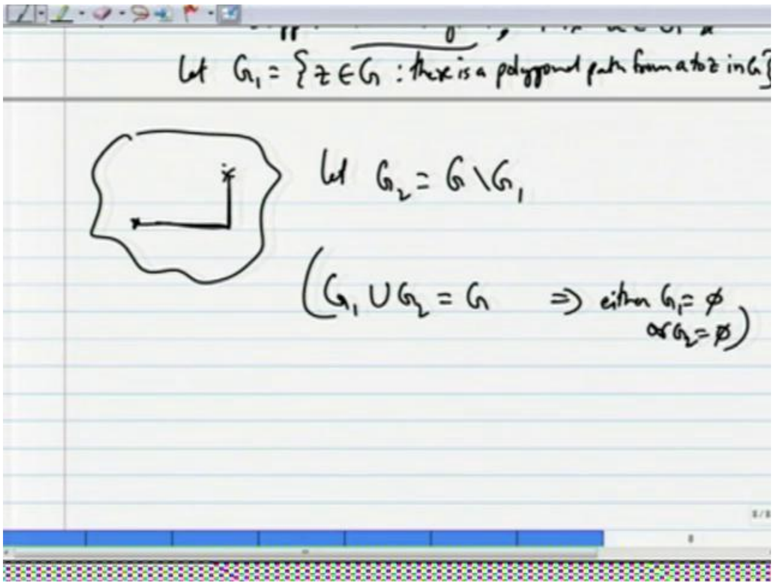

So, connected set when if you show the G 1 and G 2 are are open then G 1 union G 2 is G which implies that and G is connected, G is the region. So, this will imply that either G 1 is empty or G 2 is empty, that is the strategy. So, we want show that either G 1 is empty or G 2 is empty. So, this is the standard way of using connectedness. So, normally the, we will encounter proofs where we will use the property of the region, that it is connected and we will use it in the following fashion. We will split it into two open sets and then which are disjoint and since the set is connected, given set is connected it has to be that one of these open sets is empty. So, we might encounter such proofs in this course.

So, here is one such. So, here is G 1. G 1 is set of all points in G which are polygonally connected to a. So, we are able to make way from a to z in the region G, we are able to connect a to z. And so what we miss out is all those points in G which cannot be connected from a. So, G 1 is open.

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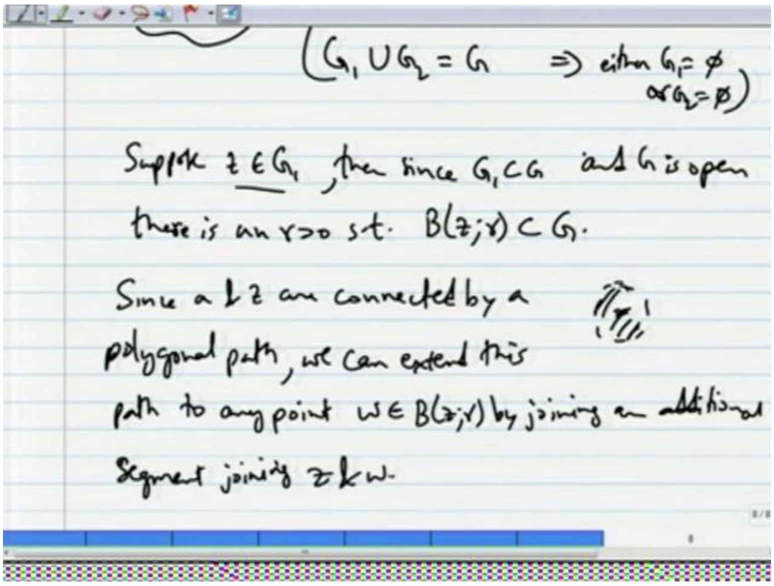

So, if suppose z belongs to G 1. Then since G 1 is subset of G. So, since G 1 is contained in G there is and G is open and G is open, being an open set there is an r positive such that B z r is contained in G. So, if z is contained in G there is boll of radius r around z which is completely contained in G because G is open set.

Now, what happens is since z belongs to G 1 since a and z are connected by a polygonal path, we can extend this path to any point w belongs to B z r by joining perhaps an additional line segment, additional segment containing or additional segment joining z and w, z and w.

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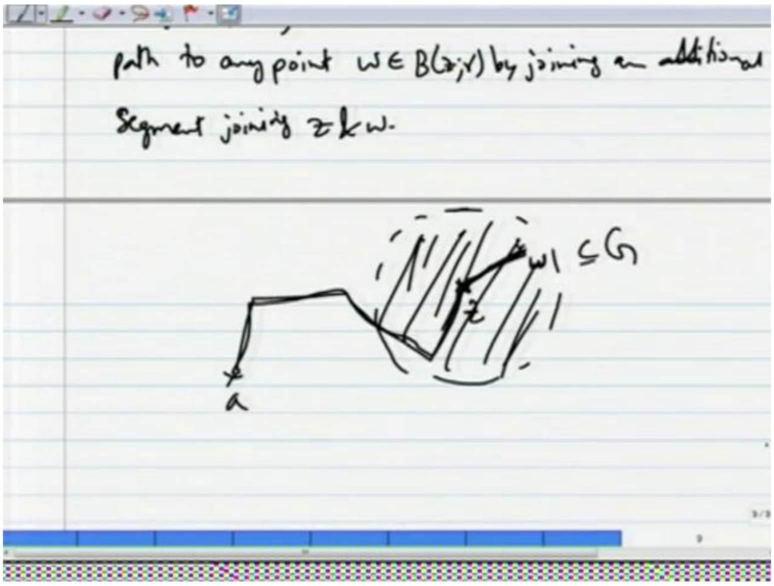

Well, so if you have this boll of radius r. Let me draw a bigger figure picture and w is any point here. So, there is a path from well I should say polygonal path. So, there is a path from a to this z here. z is the centre of this boll, by joining one additional piece like that we are not disturbing the finiteness because we are, we have added one more segment and this is, this joining of additional z to w we still have a polygonal path starting from a ending at w. And notice that if this path is completely in G then so is this path because this open set, so is this new path which is obtained by the joint because this whole set is, opens sorry boll is contained in G. So, this this additional line segment is also contained in G. So, the path joining a and w is also contained in G.

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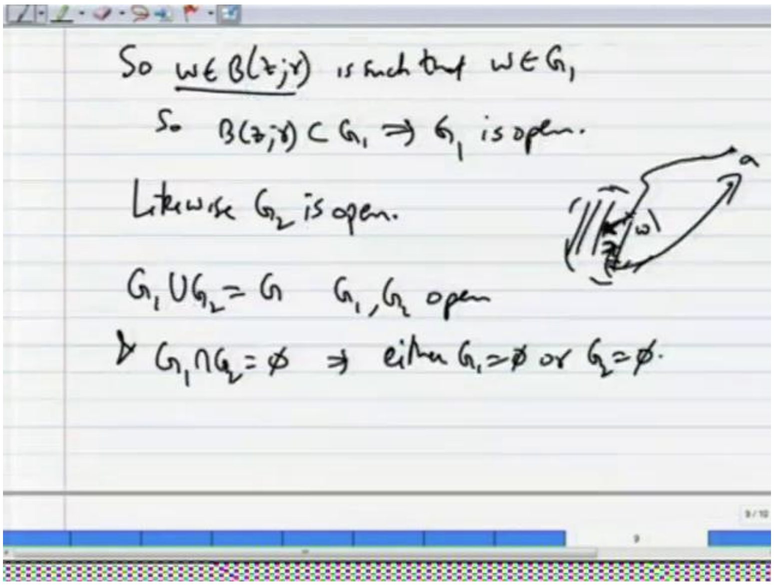

So, w belongs to B z r is such that w belongs to G 1 because now a and w can also be polygonally connected via a path which is completely contained in G. So, in conclusion this is an arbitrary in B z r. So, B z r is completely contained in G 1. So, G is open which implies or rather G 1 is open. Likewise the same applies to G 2. So, likewise I will just say likewise G 2 is open. It is the complement and if you are unable to connect a and anything in the complement via polygonal path, the path joining z and w where w is the point in a boll around z, that also will not be contained in G. So, you cannot polygonally connect any point in the neighbourhood of z and so G 2 will also be open.

And now or said otherwise, if you have a point in a neighbourhood of z which can be polygonally connected to a, then you can extended by a segment to join z as well to a polygonally. So, what that does is it puts z in G 1, but by assumption it belongs to G 2. So, that is why G 2 has to be open. G 2 is open. So, G 1, G 1 or union G 2 is all of G, G1 and G 2 open and G 1 intersection G 2 is empty which implies either G 1 is empty or G 2 is empty which is what we want if if one of them is empty, then you would have shown. Well, G 1 is non empty.

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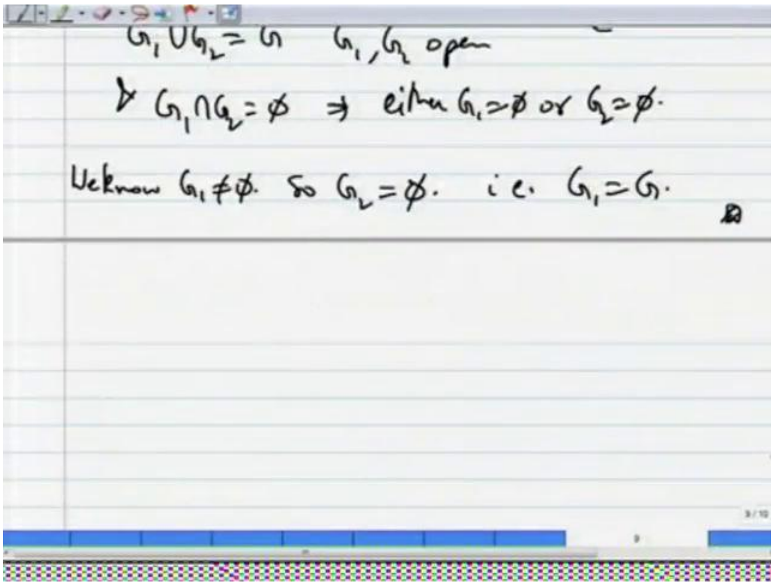

We know that, we know G 1 is non empty. So, so G 2 is empty. So, every point what that tells is that i e G 1 is equal to G. So, every point in all of G can be polygonally connected to this fixed point a. So, that completes the first portion of the proof. Well, I am going to skip the other direction like I mentioned. So, that is the proof of this proposition and it is this property of open connected sets that we keep on revisiting. So, we will use the fact that any two points in an open connected set can be joined using a polygonal path which is completely connect, contained in the open connected set.

So, for technical reasons I am sorry I will go back to this definition of the region. I will call an open connected non empty subset, I will always take a non empty subset of C and that is what I will call as region or a domain. So, please note that, that has to be non empty subset of C for it to be called a region or domain. By convention an empty set is connected. So, I insist that it been non empty.

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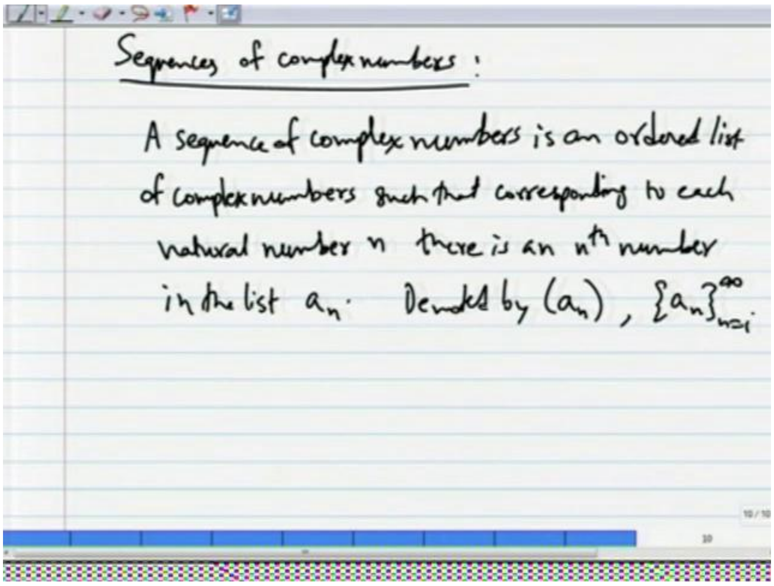

Next, we will look at sequences of complex numbers. So, a sequence of complex numbers is is an ordered list of complex numbers such that corresponding to each natural number n there is an nth complex number, number in the list. We will call it a subscript n and so since the viewer I am assuming is familiar with sequences of real numbers. So, sequences of complex numbers are similar.

So, it is it is a list where there is a starting starting complex number, the first complex number, the second complex number etcetera. So, it is the starting point then you have a list and then this is denoted by, denoted variously by a n and sometimes even by a n n equals 1 through infinity. So, there are different kinds of notation.

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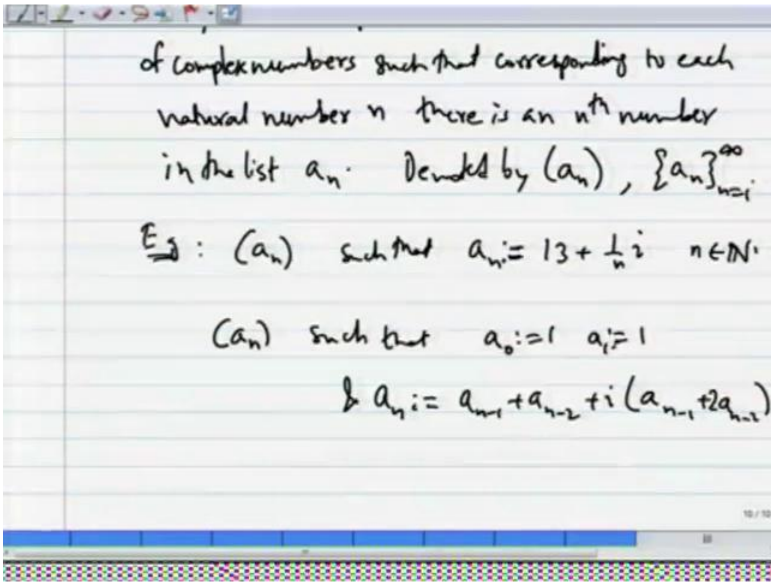

And an example, well a quick example a n such that I will define the nth term, the nth term is defined by 13 plus 1 by n times i. So, I have given a formula in terms of n where n is a natural number. So, that is an example of a complex sequence and then you could also define a sequences recursively like the Fibonacci sequence in case of real numbers.

So, you could have also recursive definitions like this a naught is 1, a 1 is defined as 1 and a n is defined as a n minus 1 plus a n minus 2 plus perhaps imaginary part is a n minus 1 plus a n minus 2 times a n minus 2, something like this. So, this is the recursive definition of a sequence. So, it is a, it helps you build the sequence. So, these are sequences of complex numbers and we can talk of their convergence in the topology of the complex plane.

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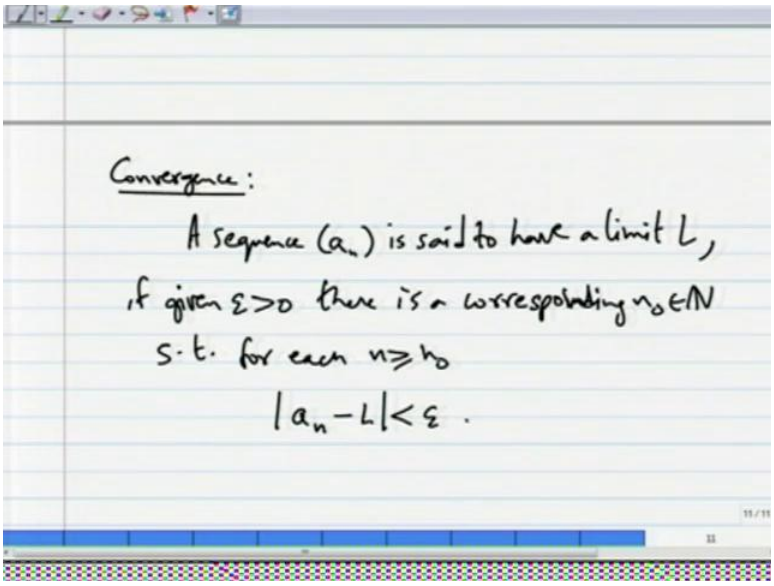

So, we say so that convergence sequences can converge. A sequence a n is said to have a limit l, a complex number l. So, if given epsilon greater than 0 there is there is a corresponding corresponding n naught belongs to n such that for each n greater than or equal to n naught the modulus of a n, the nth term minus l is strictly less than epsilon for each n greater than or equal to n naught.

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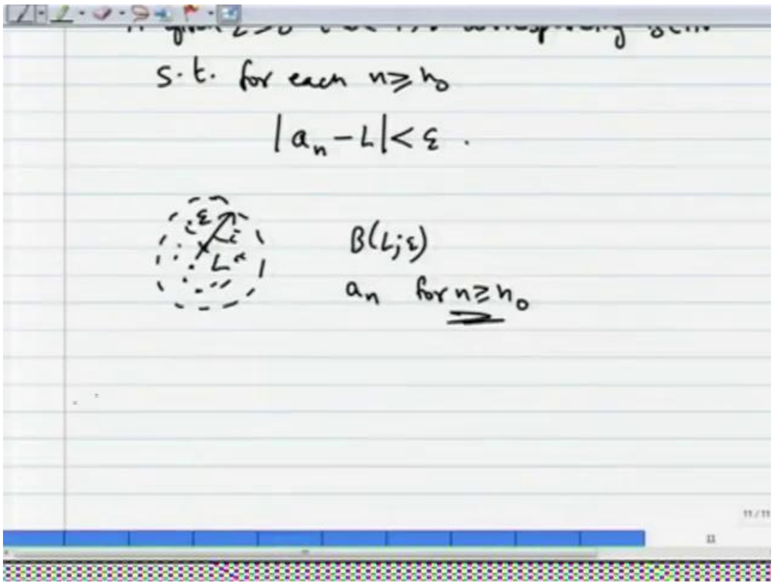

So, what that means is that a complex number l is said to be the limit of a sequence a n, if for any given epsilon. So, that epsilon is provided. So, if for any epsilon like that positive given you consider the boll of radius epsilon around l and then all the all the a n’s for n greater than or equal to certain n naught depending upon epsilon. So, all these a n’s will now be inside this boll. So, after a certain stage n naught, stage as in you think of this order list.

Somewhere down the list starting from some points all numbers down in the list will be contained in this boll. So, the, that is when you say that, that is the meaning of modulus of a n minus l is strictly less than epsilon. Then you say that and if this this behaviour is exhibited for every epsilon you can find a corresponding n naught, then you say that sequence converges to the limit l.

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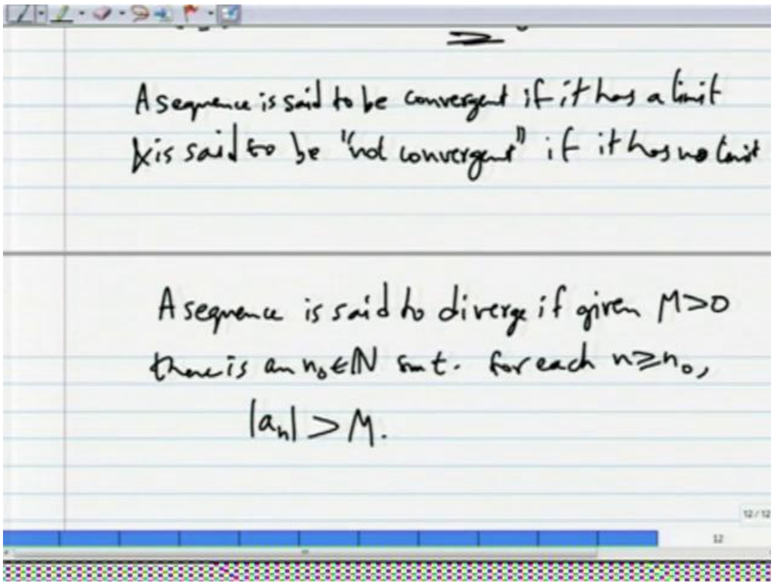

And a sequence is said to be convergent if it has a limit and is said not to converge or is said to be not convergent, is said not to converge, is said to be not convergent if it has no limit. It is said to diverge if the modulus increases arbitrarily. So, a sequence is said to diverge a complex sequence of course, is said to diverge. If given any large positive number there is an n naught belongs to N such that for each n greater than or equal to n naught the modulus of a n is quite large. It is greater than the provided M.

So, you give these circles, you imagine the circle centered at the origin and then large circles. So, if if there is an n naught corresponding to each of these circles such that modulus of a n is larger than the radius of the circle. So, which means these a n's all of them fall outside a circle of radius M. Then you say that this sequence diverges.

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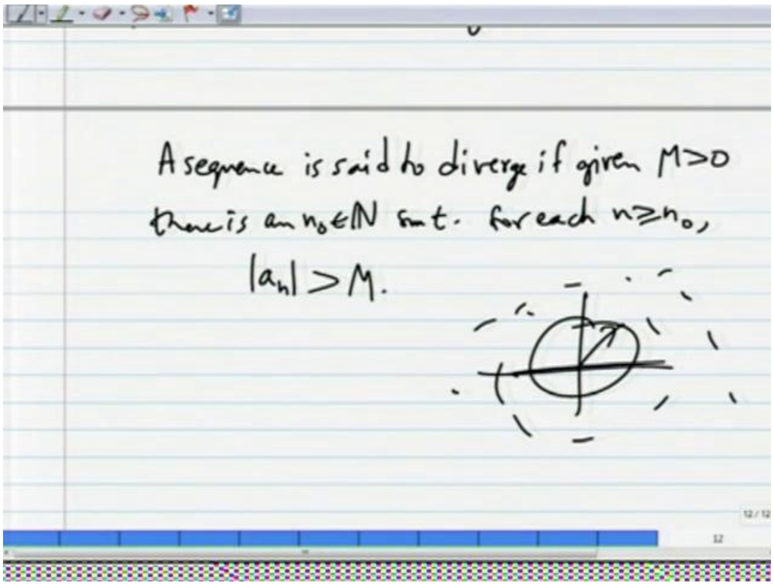

So, if I give you an M and circle of radius M around origin then all your a n's are outside this circle of radius M and this behaviour if it is exhibited for every positive real number M, if you can bring a corresponding n naught then you say the sequence diverges. So, that is about sequences converging and diverging.

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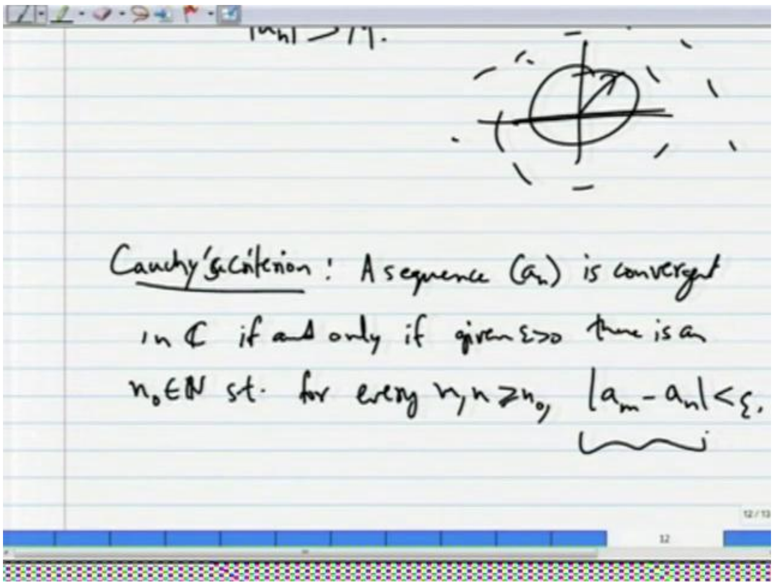

So, it is similar to the notion of convergence and divergences of sequence of real real numbers and there is a Cauchy's criterion for convergence. So, a sequence a n is convergent in C which means it has a limit if and only if given epsilon greater than 0 there is an n naught belongs to n such that for every m comma n greater greater than n naught or greater than or equal to n naught the modulus of a m minus a n is strictly less than epsilon. So, that is similar to the Cauchy's criterion for real sequences except here we considered the modulus of the difference a m minus a n. And then we have the, for compact sets we have following Bolzano Weierstrass property.

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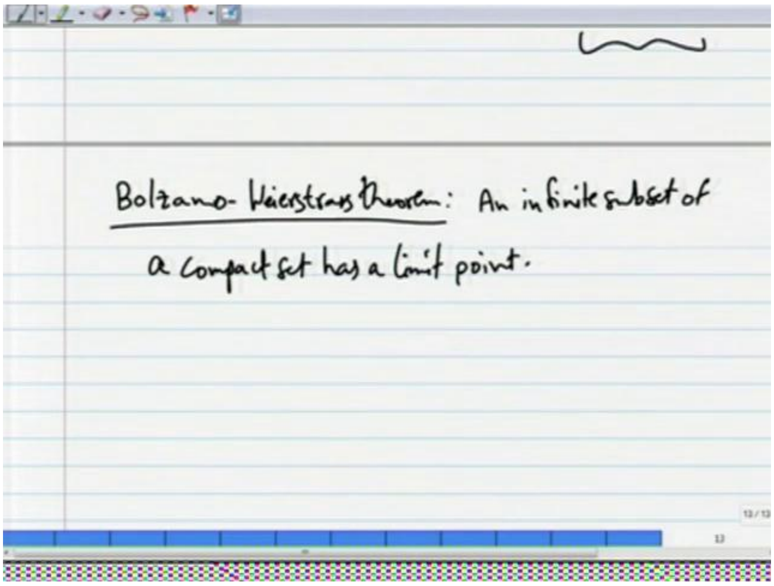

So, Bolzano Weierstrass theorem or a version of it actually. So, an infinite subset of a compact set has a limit point, it is interesting. Once again, we see that appearance of the compact set here and this in general of course, it is not true if if you do not consider a compact set. So, excuse me. So, this is not true for example, if you consider the natural numbers infinite sub set of the complex plane that may not have a limit point.

So, that is the Bolzano Weierstrass property which we might use, property of compact sets and with this we will conclude the topology or the study of the topology of the complex plane. And we will see that the topological properties of the complex plane have a, have an important role to play in the study of analytic functions. And that thing we will see all throughout the course.