Complex Analysis

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Module - 1
The Arithmetic, Geometric and Topological Properties of the Complex Numbers
Lecture - 4
Topology of the Complex Plane Part-II

Hello viewers, in the last session, we defined what a limit point is, we were discussing the topological properties of complex plane. We will continue with the discussion of limit points and isolated points. So, a limit point, recall from last time, is such a point that in the deleted neighbourhood of that point, or in a B prime a r, you find at least one point from the set or the intersection with the set of the deleted neighbourhood is non empty.

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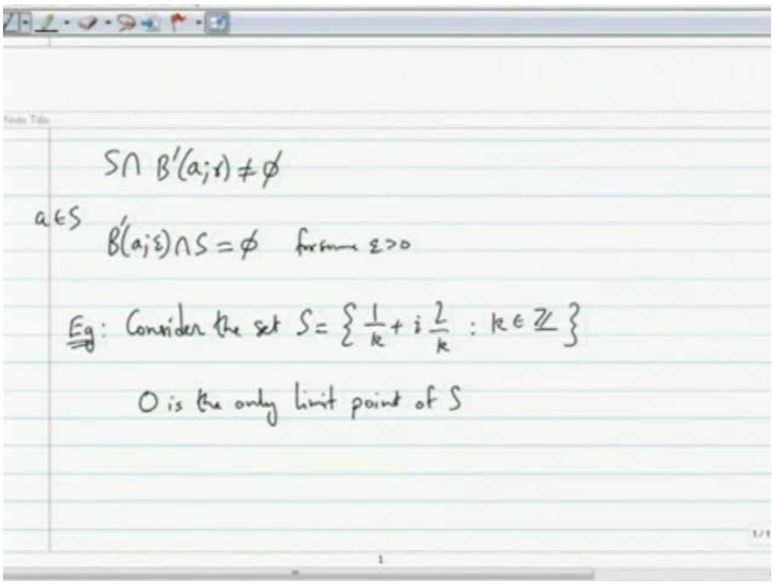

S intersection B prime a r, if that is non empty for every r, that is the point, for every r then, we say a is the limit point. So this is I mean, very similar to what I have defined before and we also saw what an isolated point is. If a point belongs to the set and if there is a neighbourhood of this point, there is a epsilon such that B a epsilon; a belongs to S and B a epsilon intersection S is empty for some epsilon. Or I should say B prime, because a itself is in epsilon. So, you remove the point a and consider the ball of radius epsilon around a. With the point a removed, intersection S if that is empty for some epsilon positive, then a is said to be an isolated point. We saw some examples last time, and we will see one more example here. So, consider the set S equals 1 by k plus i times 2 by k, such that k is integer. So, this is our set. What we are going to show is that 0 is the only limit point of S.

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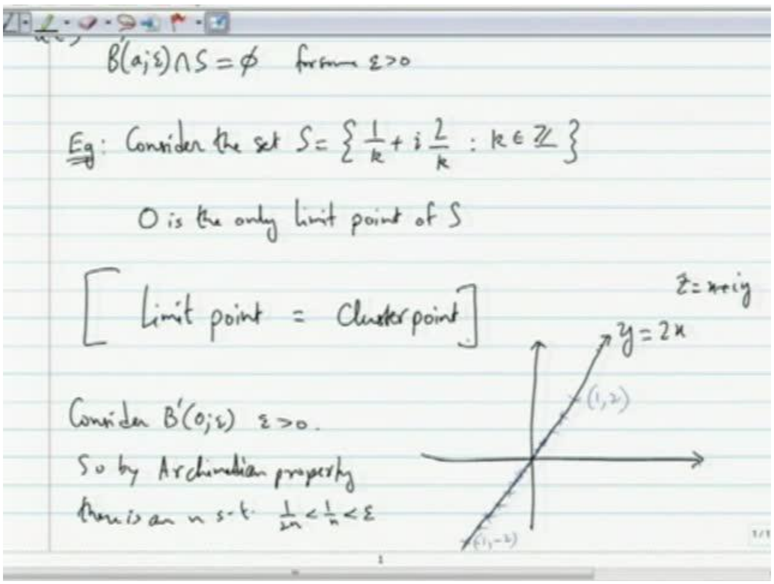

So, please note in literature sometimes a limit point in textbooks, some textbooks called limit points as cluster point. So, there is another name for this limit point. So, we will show that 0 is the only limit point of S. So, notice this example is of a set of the form, you know x plus i times 2 x, where x is of the form 1 by k. So, essentially all the points of this set fall on the line y equals 2 x in the complex plane, x actually is subset of this line. 1 comma 2 that point is on the line and then minus 1 comma minus 2 is on the line and all other points fall between these 2 on the line and 1 by k tends to 0.

So there is a clustering behaviour or a limiting behaviour at 0, although the point 0 in the complex plane itself is not in the set. So firstly, you know we want to show that 0 is a limit point. So, consider B prime 0 epsilon, epsilon positive. So, by Archimedean property of real numbers there exists n, there is an n such that 1 by 2 n is strictly less than 1 by n, is less than epsilon. 1 by 2 n is of course less than 1 by n but, there is an n such that 1 by n is less than epsilon.

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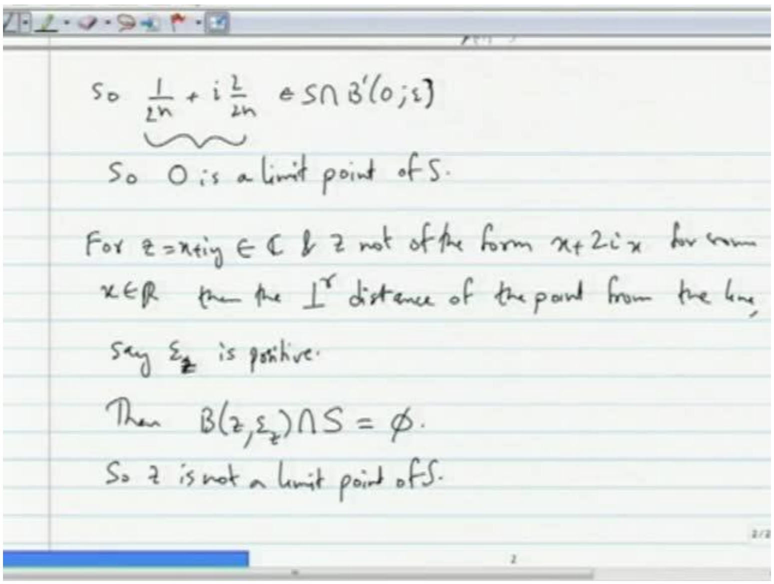

So 0, is indeed, is a limit point of S. So, every neighbourhood of 0 has some other point in S than 0, well 0 itself does not belong here but, there is some other point in there. That is the point. So, then also we will show that any other Z in the complex plane is actually not a limit point. So, the idea is simple. If you take any point half of this line then, you can find a ball around that point, which will stay clear of the line. So, there is no chance of intersecting with the set S, which lies completely, which is the subset of the line.

So, for points lying outside, the idea is simple. So, for Z belongs to Z equals x plus i y belongs to C and Z naught of the form x plus 2 i x for some x belongs to r, these are essentially points which lie outside the line, they are not of the form x plus 2 y x. Then what happens is, then the perpendicular, the short form for perpendicular distance, of the point from the line, say call that epsilon 1, epsilon Z depending on, Z epsilon subscript z, is positive, is strictly positive because the point does not lie on the line.

Then, B Z epsilon Z will, I mean will, completely stay half of the line. So, in particular half of the subset of the line, namely S; here that is empty. So, Z is not a limit point of S. Likewise, if you take a point on the line well if it is away from the point 1, 2 or minus 1, minus 2. Or, what I mean to say is, if it is half of this line segment between 1, 2 and minus 1, minus 2. Even if it lies from the line what we can do is we can take the distance of that point to the point 1, 2 and take a ball of that radius. So that will stay clear of the set S. So, that is the idea.

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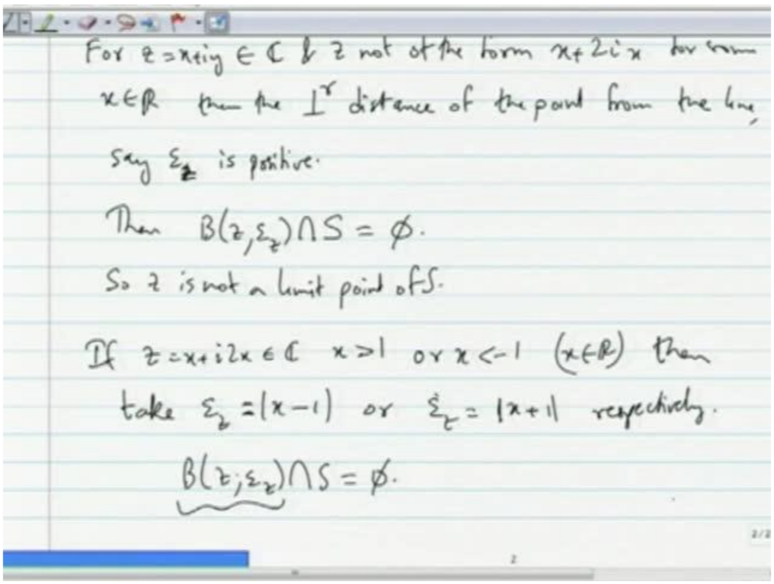

So, to make it concrete, if Z is of the form, if Z is equal to x plus i 2 x belongs to C x greater than 1, or x strictly less than minus 1 x, belongs to r, of course. Then, what you can do is take epsilon Z equals the modulus of x minus 1. So, what we are doing is, we are calculating the distance or taking this x plus i y projecting it on to the x axis. Then we are taking 1 or minus 1 and we are taking the modulus of x minus that.

So, actually this pertains to x greater than 1 for x less than minus 1, we can do something similar. We can take x plus 1, epsilon Z x plus 1. So, if that is x greater than 1 or x is less than minus 1 then, take epsilon Z is that or epsilon Z is modulus of Z plus 1 respectively. Then, what we can guarantee is that B Z epsilon Z intersection S is empty, because I mean if you take a ball of this radius, ball of this radius that definitely is going to stay clear of the set. So that intersection S is empty. So, notice that if you, I mean this ball does not contain points, which are at distance epsilon Z from Z, so that is the definition of ball, the inequality there is strict.

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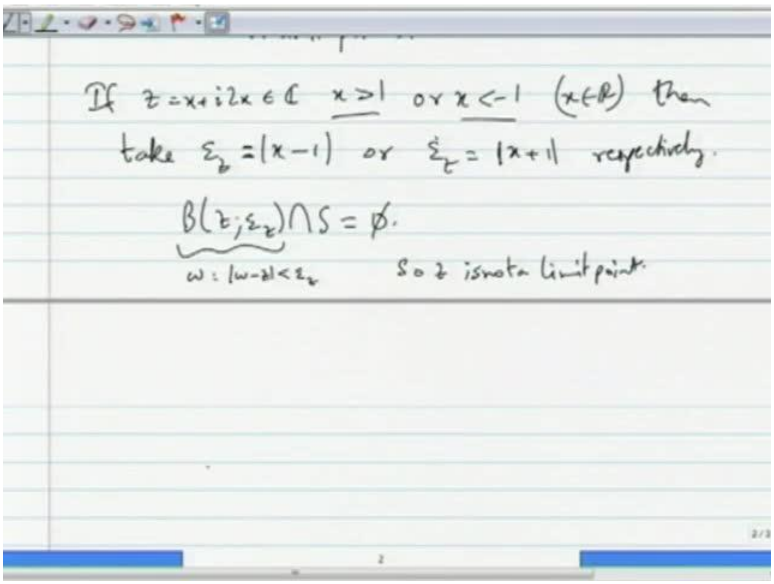

So, the modulus of it is all such points w, I mean the set is set of all points w, such that modulus of w minus Z is strictly less than epsilon Z. So, we do not allow equality, so in particular that point 1,2 or minus 1, minus 2 do not lie in B Z epsilon Z; whatever the cases they do not lie there. So, the intersection with S is definitely empty, so that is the idea. So, any point on that line with x greater than 1 or x less than minus 1 is also not a limit point. So, Z is not a limit point. There is a third case. Now, what about points in between?

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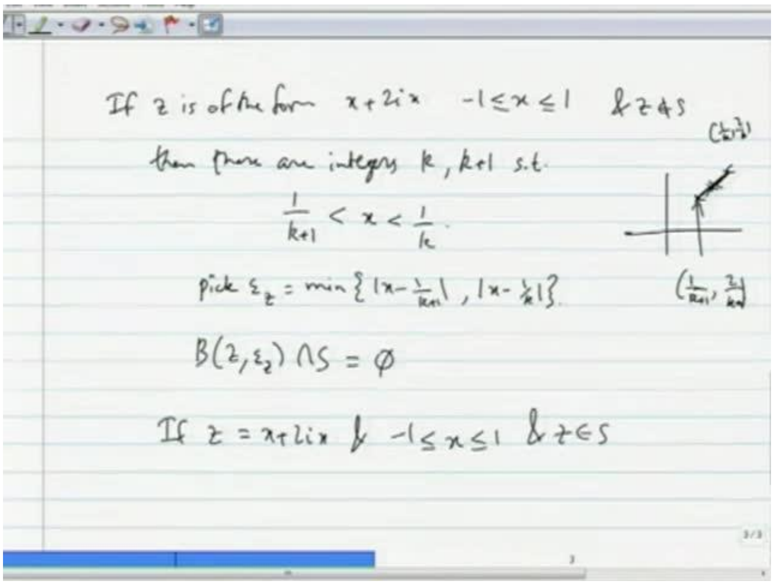

So, if Z is of the form x plus 2 i x. First, let me assume that Z is of this form and Z does not belong to S. In this case there are integers such that x strictly lies between 1 by k plus 1 and 1 by k. Pick epsilon Z is equal to minimum of modulus of x or absolute value of x minus 1 by k plus 1 comma x minus 1 by k. The picture is there is a x here and then there is 1 by k plus 1, plus i times 2 by k plus 1 here.

So, that point is 1 by k plus 1 comma 2 by k plus 1 and then there is that point 1 by k comma 2 by k here. So, you measure the distance to either of these points, x lies between 1 by k plus 1 and 1 by k, and then you take the minimum of these 2. So, B Z epsilon Z intersection S will be empty. B prime, so B prime Z epsilon Z, of course, intersection S will also be empty.

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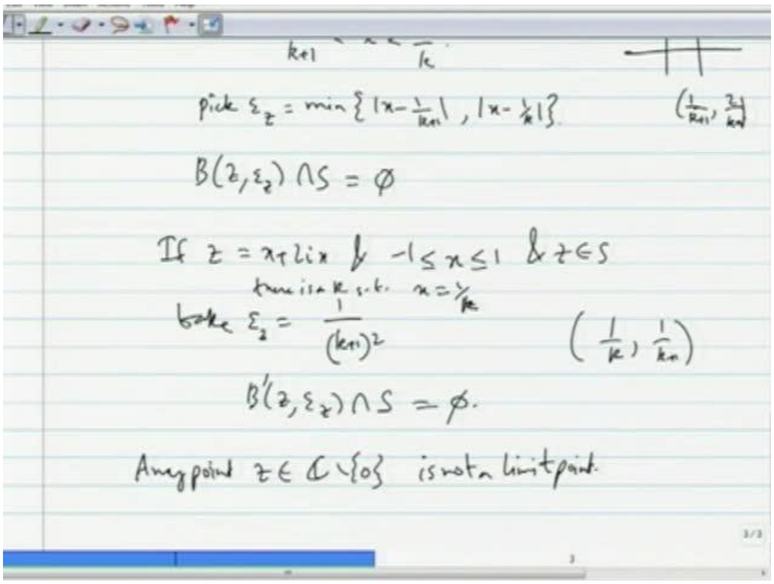

Now, in the case that if Z is equal to x plus 2 i x and minus 1 less than or equal to x less than or equal to 1 and Z belongs to S, then what you do is you take a epsilon, take epsilon is equal to 1 by k plus 1 squared. You could take 1 by k plus 1, that also or 1 yeah, 1 by k plus 1 squared. So, it is the distance between 1 by k and 1 by k plus 1. But, I should say what is k. There is a k such that x is equal to 1 by k, because x belongs to the set.

So then, take epsilon is equal to epsilon Z is equal to 1 by k plus 1 square. So, B Z epsilon Z, B prime Z epsilon Z now; Z of course belongs to the set. So, we need to delete the point itself that intersection S is empty. So, in any case any other Z, any point Z belongs to C minus 0, we have shown, is not a limit. So, every such point has a neighbourhood which stays clear from the set at every point.

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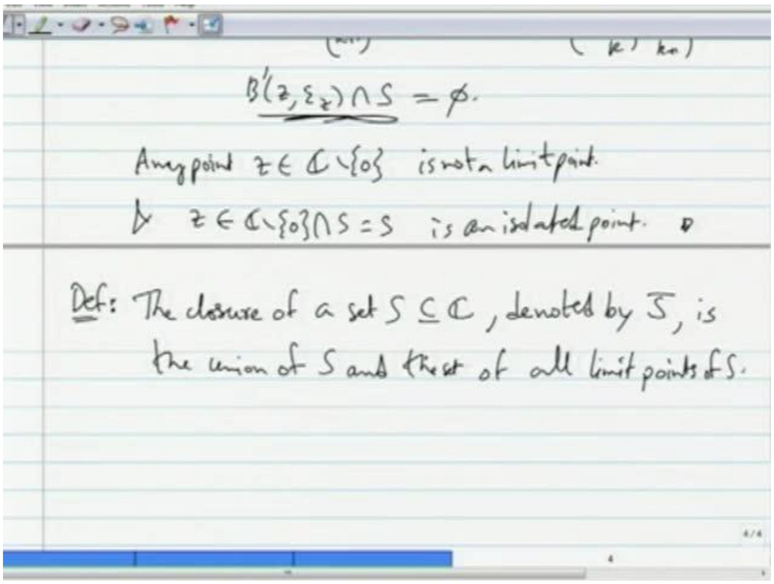

So, what we have shown is that Z belongs to C minus 0 intersection S, which is equal to S. We have shown that S is completely contained in C minus 0; 0 is not a point in this thing and every point in S is an isolated point.

Next, we are going to look at the notion of a closure of a set. So, recall that closed set is something whose complement is open. So, in the context a set is said to be closed in C in the complex plane if its complement is an open set in the complex plane. Now, we have a set which may or may not be a open and may or may not be closed either and then there are points which are limit points to this set and which means may or may not be contain in the set.

So, when you add all the limit points of a set to the set, we are going to show that that will become a closed set. To that end we make the following definition the closure of a set, so we are sort of closing the set with its limit points. So, these limit points are like holes if they are not already present in the set. So, we will close the set, close those gaps or holes with the limit points.

So, the closure of the set, of a set S contained in C denoted by S bar, S bar is closure of S, is the union of S and all the limit points of S. The limit point set, I should say this as the set of all limit points of S. So, that is the closure. So, the first thing we want to know is whether S bar by this definition is a closed set. We are calling it a closure but, we want to know whether it is really a close set. So, I will prove a proposition which says that S bar is closed but, first an example.

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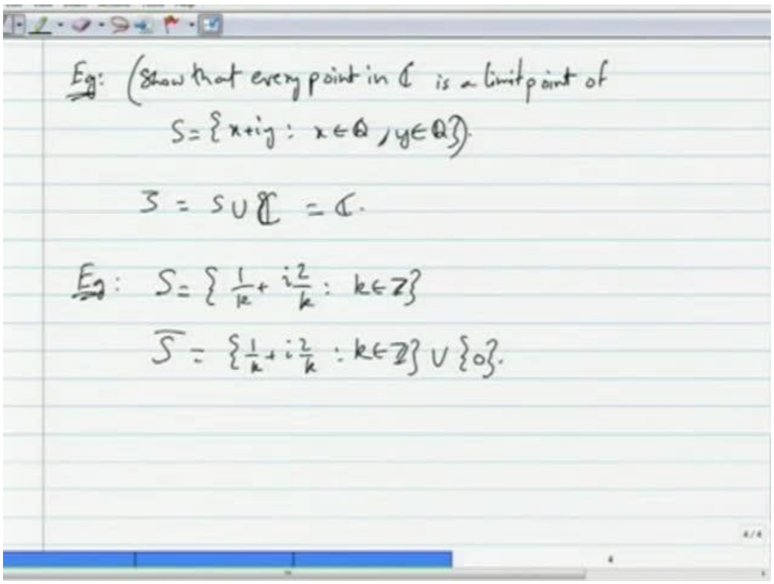

At the end of last session we solved the following exercise. Show that every point in C is a limit point of the set x plus i y, such that x belongs to Q, the rational numbers, and y belongs to Q.

So, what that means is if I call this set S, then what a this means is S closure now is the union of S with set of all limit points. Well, this itself is C, set of all limit points is C. So this is going to be C. So, the closure of a set this all of C, of this set is all of C. In the example that we just saw, so in the earlier example S was the set of all 1 by k plus i times 2 by k such that k belongs to Z. The closure of S is, there is only 1 limit point to S namely 0 that we have shown, so, S closure is going to be 1 by k plus i times 2 by k such that k belongs to Z and union the point 0, that is S closure. So, those are some examples.

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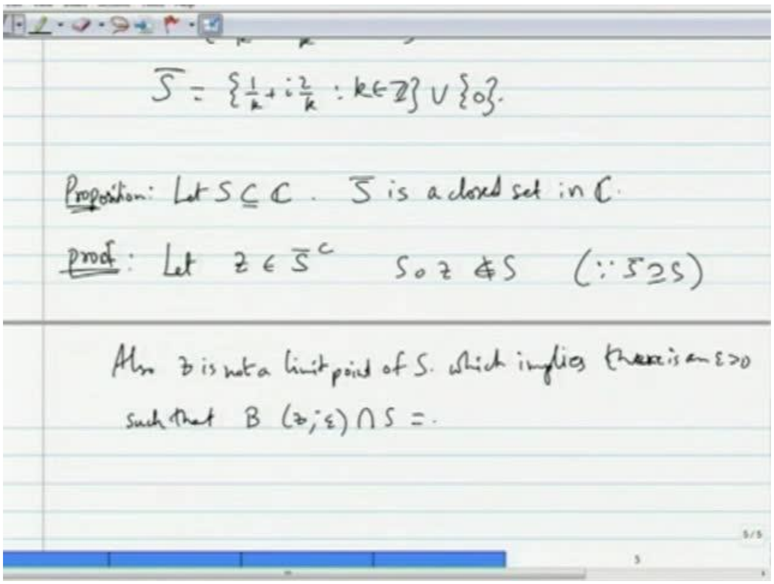

We, immediately want to show the following proposition that let S contained in C be a subset, S closure is a closed set in C, recall. By closed set we mean the compliment is open. So, what we want to show is that the compliment of S closure is open. So, let Z belongs to the compliment of S closure. So, Z cannot belong to S because S closure contains S, since S closure contains S, so in particular Z cannot belong to S. Also, Z is not a limit point of S because limit points are also contained in S closure. So, it implies there is a epsilon positive, there is an epsilon positive such that B Z epsilon intersection S is empty.

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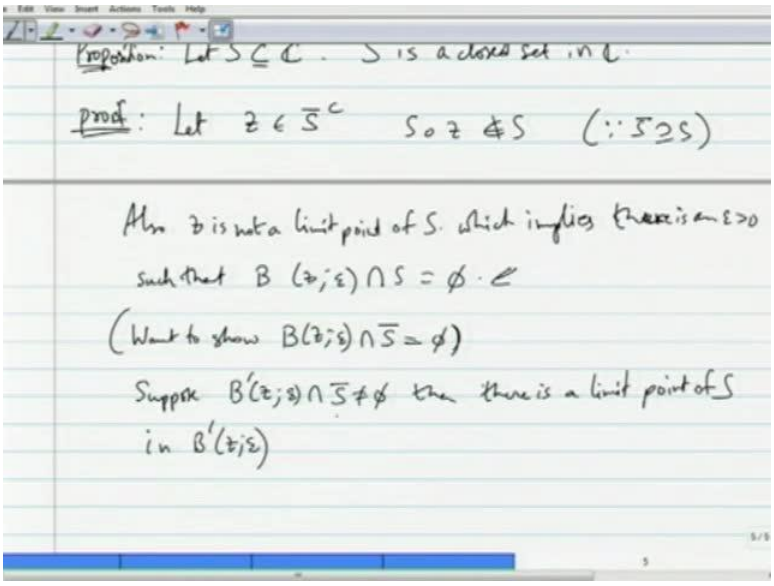

So, what we want to show is that, so we want to show that B of Z epsilon intersection S bar S closure itself is empty. We are able to conclude that Z is not a limit point of S and Z does not belong to S. So, B Z epsilon intersection S is empty that much we know because Z is not a limit point there is some epsilon such that that happens.

Now, we want to show that B Z epsilon intersection S closure itself is non empty or is empty. So, in order to show that suppose it is non empty, we will arrive at a contradiction. So, suppose B Z epsilon, well we already knows Z is not in S closure. So, suppose B prime Z epsilon intersection S closure is non empty, since we already know this then there is a limit point and since S closure is the union of S and limit points there is a limit point of S in B prime Z epsilon.

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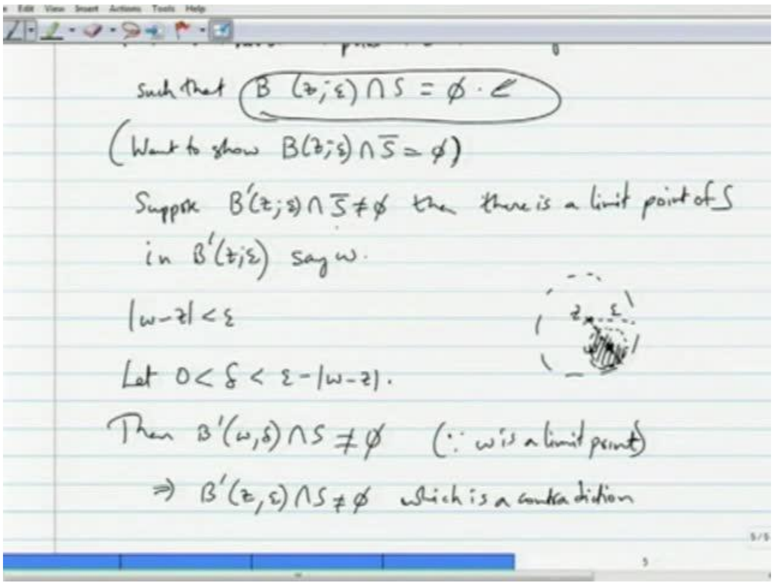

So here is the picture, here is B Z epsilon. Suppose, w is the limit point, Z here and w is the limit point of the set S. So, say I will not call that w. Now, modulus of w minus Z is strictly less than epsilon because it is in the ball of radius epsilon, around Z. Let 0 strictly less than delta, strictly less than epsilon minus modulus of w minus Z. So here is ray or a radius line passing through Z and w and the distance between w and Z is some number which is strictly less than epsilon.

So, epsilon minus modulus of w minus Z is a strictly positive number. So, pick a number between that strictly positive number and 0. So, by properties of real numbers you can do that, then B prime of w delta, so essentially we are talking now delta is this remaining distance, so that is your B prime of w delta, without w. So, this intersection S has to be non empty because w is a limit point of S by assumptions. w is a limit point, means that every neighbourhood of that point, every deleted neighbourhood of that point contains another point of the set. So, this implies that B prime of Z epsilon contains, so this neighbourhood contains, the hash neighbourhood contains a point from S. Which means B prime Z epsilon intersection S also contains the point, which is a contradiction, which is a contradiction, so our assumption that that is non-empty, which is a contradiction.

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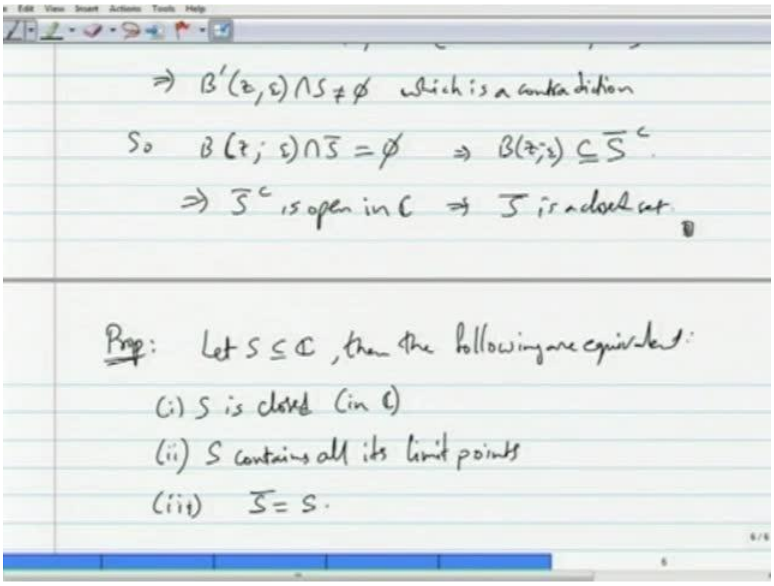

So, B Z epsilon intersection S closure has to be empty. I am including Z because Z is picked from compliment of S closure. So, this implies B Z epsilon there is some epsilon such that for every point Z in the compliment of S closure. There is a corresponding epsilon such that this ball is contained in S closure compliment, which implies S closure compliment is open in C, which will imply that S closure is a closed set. So, it is the proof. So the closure is indeed a closed set.

Now, what we want to show is the following equivalence, just as a further extension. Let S contained in C then, the following are equivalent. First S is the closed set, S is closed. So the context is C, so I will not say inside C. So, S is closed in C. S contains all its limit points and 3, S closure is equal to S. so these 3 are one and the same; these 3 statements are one and the same.

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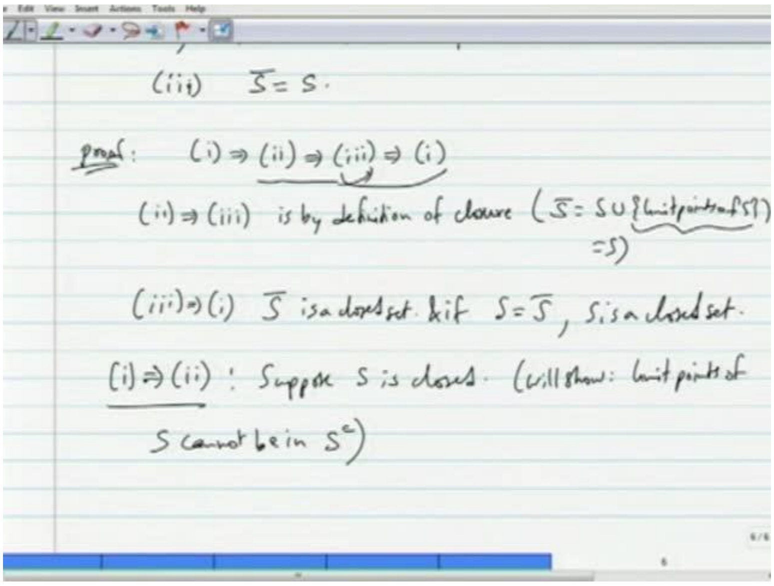

So what is the proof of this fact So what we will show is that, we will show 1 implies 2; implies 3; implies 1. So, then all three are equivalent. So, 2 implies 3 is simply by definition. If S contains all its limit points then, the union of S with its limit points has to equal S, so is by definition of closure. If S contains all its limits points, S closure is S union limit points of S. But, if the latter is contained in S then, the union is simply S. So that is by definition.

Then, 3 implies 1 is by previous proposition. So, S closure is a closed set, S closure is a closed set, and if S is equal to S closure, S is a closed set, so 3 implies 1. We have shown 2 implies 3 and 3 implies 1. We have shown both these. So, we have to show 1 implies 2, that if S is a closed set then it contains all its limit points. So, suppose S is a closed set, S is closed I will say I short then, we will show that no limit point of S can be in S compliment. So, limit points will show, limit points of S cannot be in S complement. That is what we will show, which means every limit point has to be in S.

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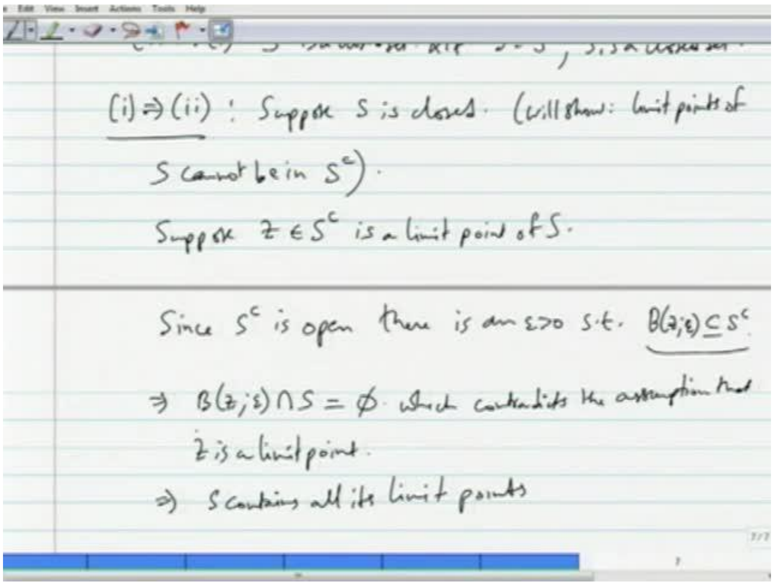

So, suppose Z belongs to S compliment, is a limit point of S. But, we know something about S compliment; since S compliment is open, there is an epsilon positive such that, for open sets we know that there is room around the point present in the set. So, since this is open, there is an epsilon positive such that B Z epsilon is sitting completely inside S compliment. What that means is B Z epsilon for that particular epsilon positive intersection S if that is contained in S compliment this intersection S has to be empty, which contradicts the definition of the assumption that Z cannot be a limit point. I mean if that intersection S is empty for some epsilon, that Z cannot be a limit point of S.

So it contradicts assumption that Z is the limit point. So it has to be that no limit point of S is in S compliment. So this implies S contains all its limits, which is 2. So, that is how 1 implies 2 and the statements are equivalent. So, that is a discussion about closure of a set and then we should also be aware of following notion of interior and boundary points. So we have seen limit points and isolated points there is also something called interior point.

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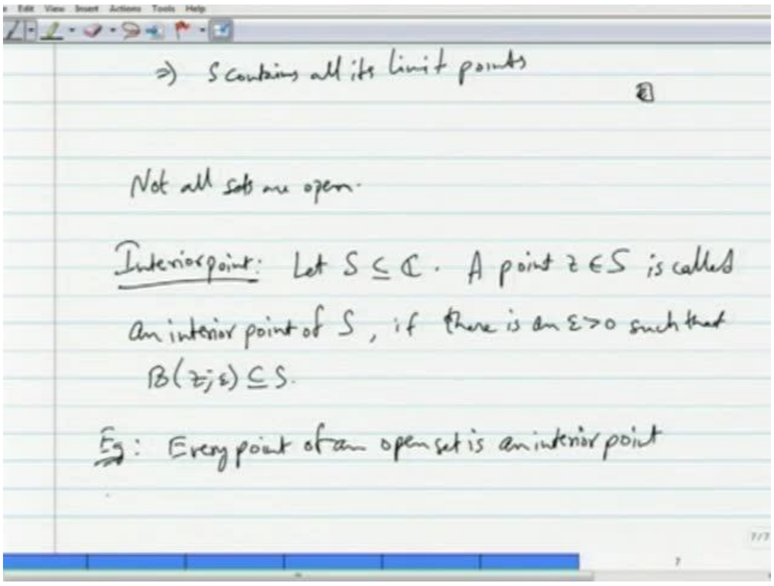

So firstly, we know that not all sets are open, for some points in the set there could be room in it, some points there could be no real room around the point. So, we want to identify those points of a set and label them which have room around themselves. So, that is an interior point, interior point. Let S contained in C, so this is a property about points of a set. So, a point Z belongs to S is called an interior point of S if Z has room around itself, if there is an epsilon positive such that B Z epsilon is completely contained in S. If this condition is met for every point in the set then, the set is open. But that need not be the case. Nevertheless a set could be not open and still have interior points.

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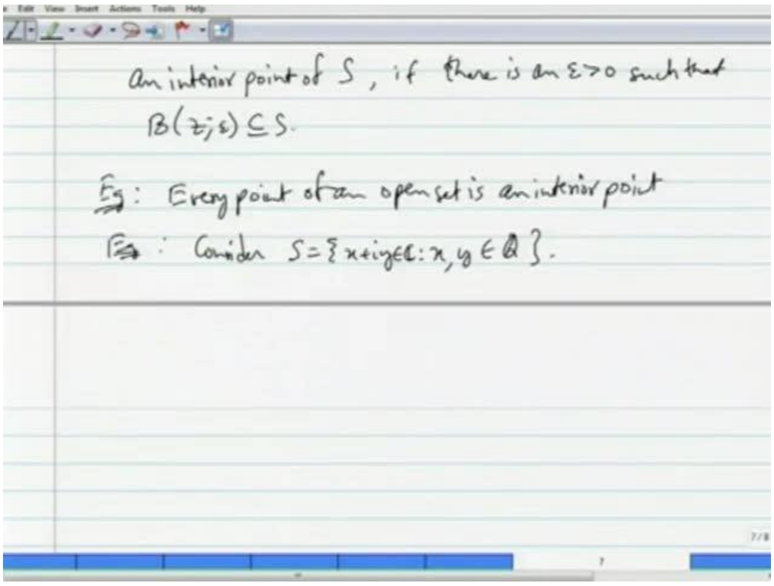

So, first example, well, an open set—every point of an open set is an interior point. Another example is x plus i y in complex plane such that x y are both rational numbers.

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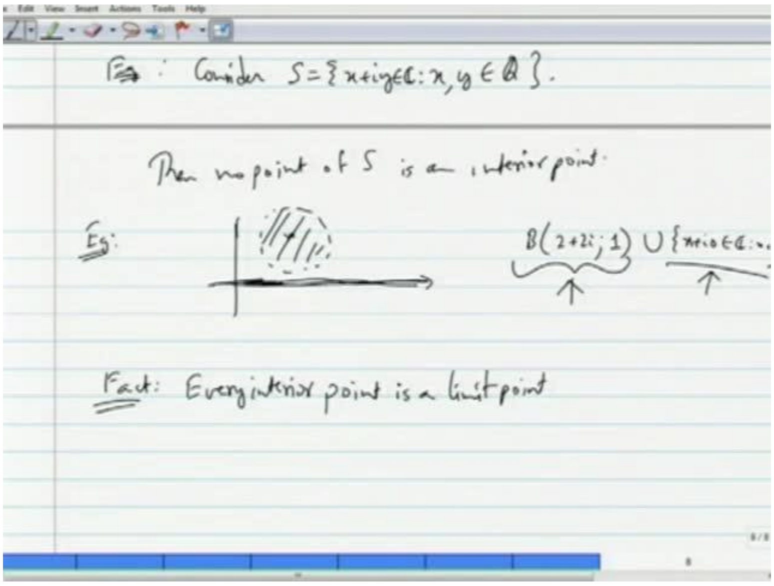

Then, no point of S is an interior point. How do we know this? Well, if you take any epsilon ball around a point in S then, there are bound to be points which have irrational coordinates in 1 or the other component, by component I mean x and y the real and imaginary part, so in every neighbourhood this is true for a point.

So in particular, it is true for every neighbourhood of points in S. So they cannot be interior points, the ball has holes. So, holes in S, so it cannot be an interior point. So, that is a vague proof of why this is true. Of course, there could be sets which have some interior points and some are not interior points. For example, if I take the union of real line, union ball of radius 1 around 2 comma 2. So, it is B 2 plus 2 i, 1 union the real line; set of all x plus i 0 belongs to C, x belongs to r.

So of course, all the points in this, that is an open set—that component, that piece is an open set. So, all point here are going to be interior points and all points here are not going to be interior points. So that is an example, where set contains some interior points and some which are not interior points. So there are other scenarios possible as well. The viewer is asked to construct examples and see various scenarios where sets have some interior points and some which are not interior points.

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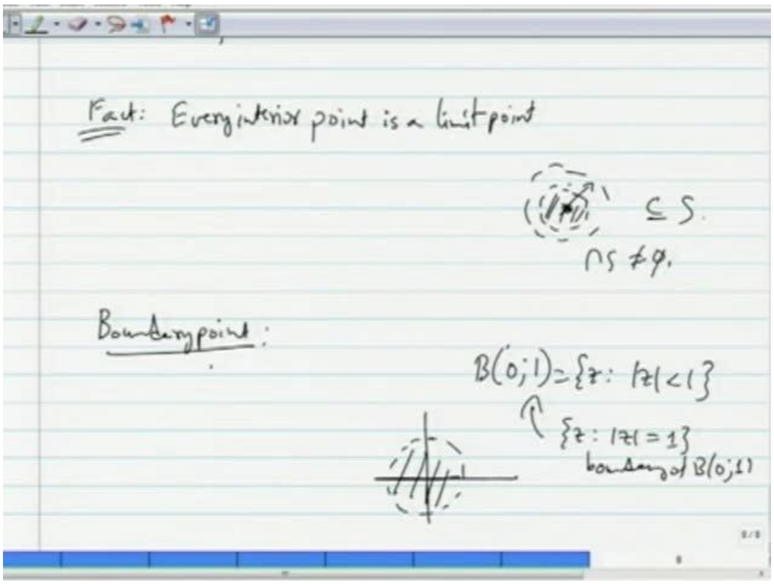

We are going to state a fact that every interior point is a limit point that is clear by the definition of interior point. If something is an interior point there is a B Z epsilon such that this intersection S is non-empty, something is an interior point this ball is completely contained in S. So if you take any epsilon less than this particular initial epsilon, any delta less than epsilon then, that ball intersection, deleted ball—so you delete that point, intersection S is going to be non-empty. So that point is a limit point. So that is why every interior point is a limit point.

Now, there is a notion of boundary and boundary points. So first, I will define a boundary point. So, there are interior points and then there are points which we will call as boundary points by the following notion. So, roughly I will give you the intuition.

Suppose you consider B 0 1. You would want to call all the points on the unit circle in the complex plane as the boundary of this. So, B 0 1 is the points set of all Zs such that modulus of Z is strictly less than 1 and so, you would want to call set of all Z such that modulus of Z equal to 1 as the boundary of this set, boundary of B 0 1. There could be more complicated scenarios.

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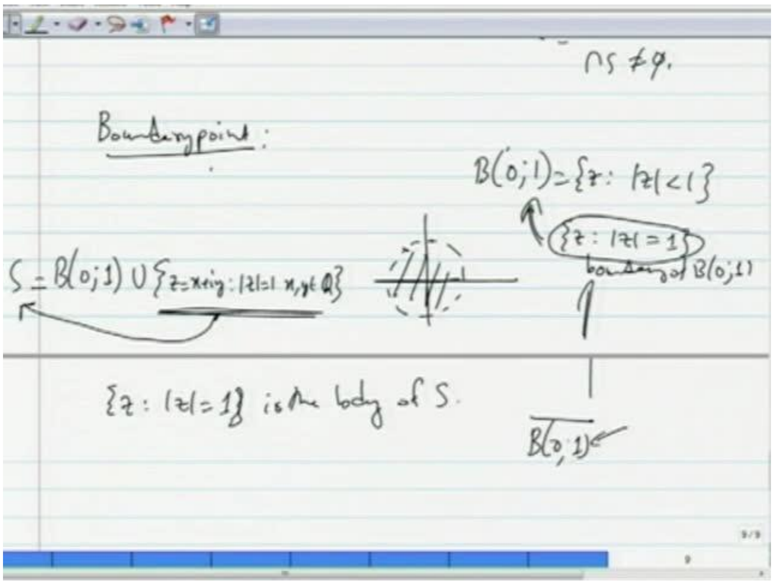

Well, you could take B 0 1 and then may be throw in some points with modulus 1. So let us say you throw in all the points Z equals x plus i y such that modulus of Z is equal to 1, x comma y belong to q; for example. So, you are throwing in some points not all points with modulus 1 have rational coordinates like that - so you are throwing in some points to this but, despite throwing in some points into this set S, you would still want to call set of all Z such that module Z is equal to 1 as the boundary of S. B d y short for boundary of S.

So that is the intuition behind this boundary notion. So you could have points which are contained in this set and you know would still be called as boundary points and you could have point which are completely non contain in the set and still you would want to call them boundary of the set. The contrasting scenario to this example is where you take B 0 1 bar and you would still want to call set of all Z such that mod Z is equal to 1 as the boundary. Mans this contains all its boundary points for example, this set. So, more general definition is as follows.

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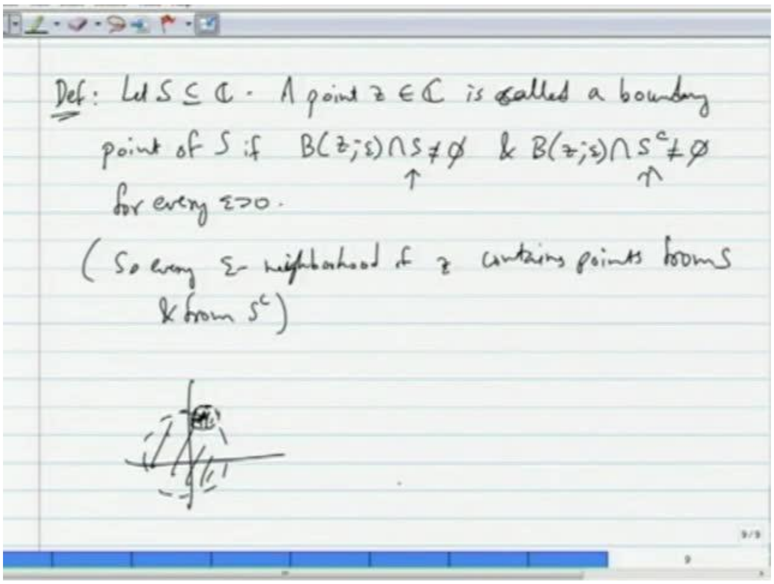

Let S contained in C, definition, let S contained in C. A point Z belongs to C is called a boundary point, notice Z does not need to belong to S, so a point Z in the complex plane is called a boundary point of S if B Z epsilon intersection S is non-empty and B Z epsilon intersection S compliment is also non empty, for every epsilon positive.

So, every epsilon neighbourhood, said otherwise, so every epsilon neighbourhood of Z contains points from the set and from outside the set. So, contains points from S and from S compliment. So this example tells you or this definition tells you that your old familiar examples are true. So, if you take B 0 1 any point here, any point on the unit circle and if you take any neighbourhood of that there are points outside the unit disc and inside the unit disc. So these examples will make all those sets, I mean, all those points such that mod Z is equal to 1 as boundary points of those set. So, that is an example of this definition.

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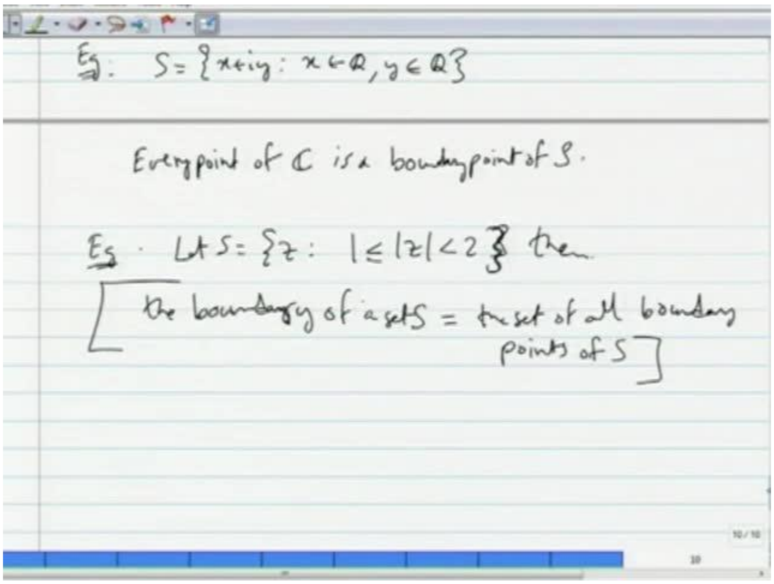

Another example is, let us revisit our set S equals set of all x plus i y such that x belongs to Q and y belongs to Q. What are the boundary points? We saw that no point of this set is an interior point and every point of the complex plane is actually a limit point of this set. So it is a strange set and then what is also true is a every point of the complex plane is actually a boundary point.

So every point of the complex plane of C is a boundary point of S. That is because if you take any neighbourhood of any complex number, it will contain points with rational coordinates and it will contain points which do not have rational coordinates. So, then every point in the complex plane is actually a boundary point of the set. So that is another example. So, here is a yet another example. Let S equal set of all Zs such that 1 less than or equal to mod Z less than 2. Then the boundary, so that is the curly bracket, then what is the boundary of S? Then the boundary of S. So I will not as a part of this example but, more generally let me define the boundary of a set S. Usually this refers to the set of all boundary points of S. I defined the boundary point - a boundary, simply a boundary of the set is set of all boundary points of S. So then the boundary of this set S is boundary of S.

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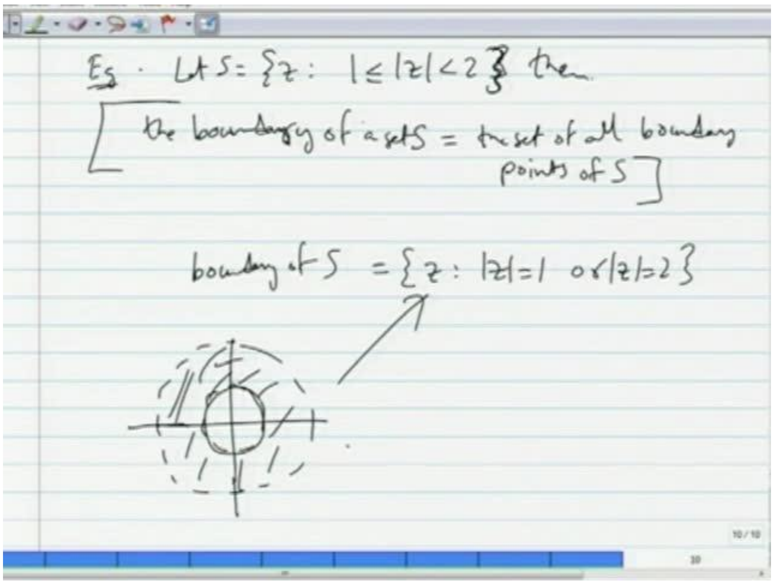

For this example, the set is equal to set of all Z such that the modulus is either 1 or the modulus is 2. That you will identify as the annular region between 1 and 2. So you have that mod Z is equal to 1 mod Z is between 1 and 2, but not equal to 2. So set of all such points. So, boundary is that, so that is a boundary notion of a boundary of a set.

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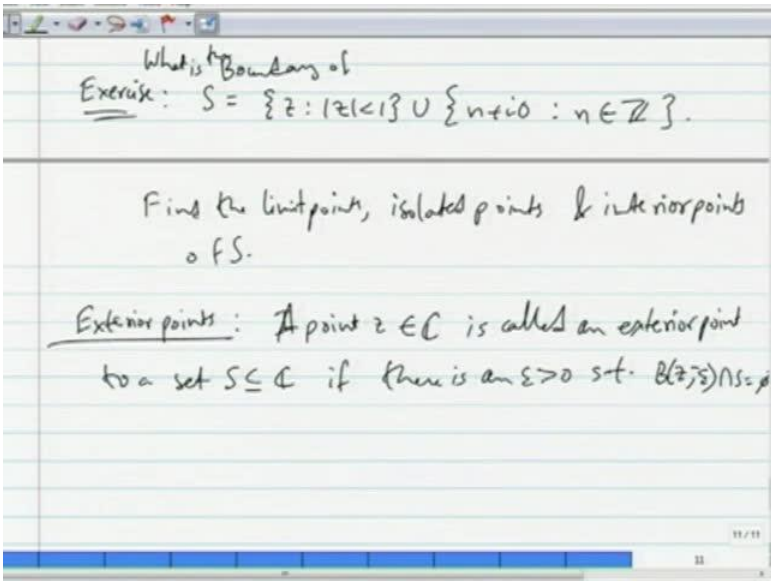

So, exercise to the viewer, what is the boundary of the set? So, what is the boundary of the set such that mod Z is strictly less than 1 union set of n plus i 0, such that n belongs to Z. Calculate the boundary of this.

So, it is the open unit disc union the integers. So, also further find the limit points, isolated points and interior points of S. So, we have seen limit points, isolated points, interior points, boundary points; there is also a notion of exterior points. These are points which lie completely outside the set in the following sense, a point Z belongs to C is called an exterior point to a set S contained in C if there is an epsilon positive such that B Z epsilon intersection S is empty. So, there is a neighbourhood, there is a ball around the point Z which is completely half of the set S. In that event you call Z as an exterior point of the set S.

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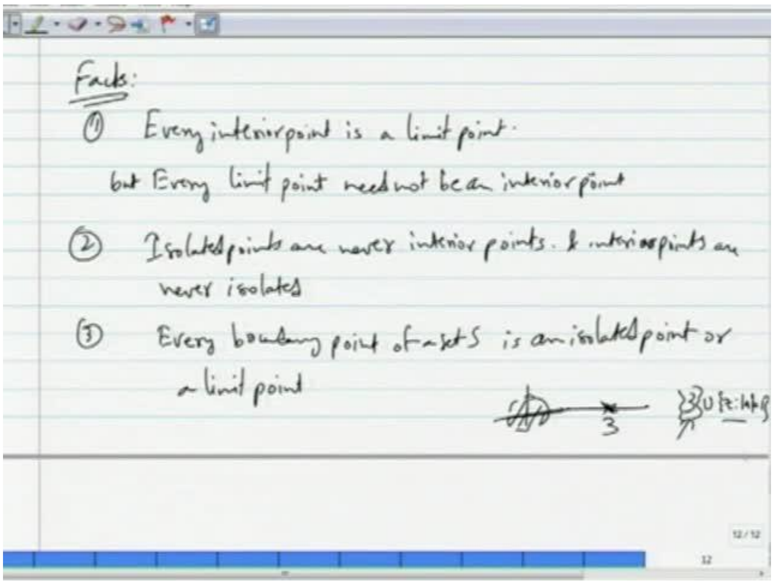

So, here are some quick facts. Please check all these facts, these are easy to check from the definitions. But, this is summary; I mean this is summary of all these limit points. So, 1 needs to internalise these facts. So, every interior point is a limit point. So, this is the relation between limit points and interior points. Every interior point is a limit point, but every limit point, but every limit point need not be an interior point. In fact the limit point need not even belong to the set, so it cannot be interior point sometimes.

So that is 1. Second statement I want to make is that the limit between isolated points and interior points. So, isolated points are never interior points. That is because there is no neighbourhood of those points contained in the set. And interior points are never isolated. Interior points enjoy a whole ball around them which is contained in this set, so they are not isolated. So, interior points are never isolated; that is the relation between isolated and interior points. So, show this fact. Every boundary point of a set S is either an isolated point or a limit point, is an isolated point or a limit point.

For example, if you take the unit disc, the open unit disc union some point here, let us say 3. Then 3 is a boundary point, the boundary of this set is going to be 3 union set of all Z such that mod Z is equal to 1. So, the boundary here is 3 and then the unit circle. So, boundary can have isolated points and then there are other points mod Z is equal to 1 and every point mod Z is equal to 1 is a limit point of the set S. So, boundary points are either isolated points or interior point or limit points. So try to show this fact more generally; that is an exercise.

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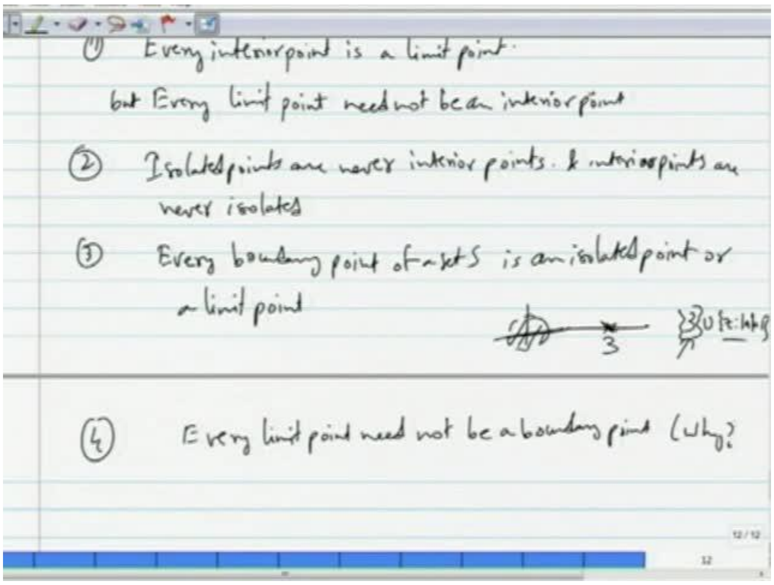

I also want to say that every limit point need not be a boundary point, a boundary point. Why? That is a question for the viewer; need not be a boundary point, why? Think about it, and try to prove the ones, which I have not proved here. Those are exercise for the viewer, I will stop.