Complex Analysis
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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 6
Problem Solving Session

Hello viewers, in this session, we will solve some problems based on the theory we have seen. So, let me begin with the calculation of a residue, so the viewer once again is asked to pause after each the problem, and try to solve the problem by himself or herself. Then I am anyway going to present the solutions to those problems.

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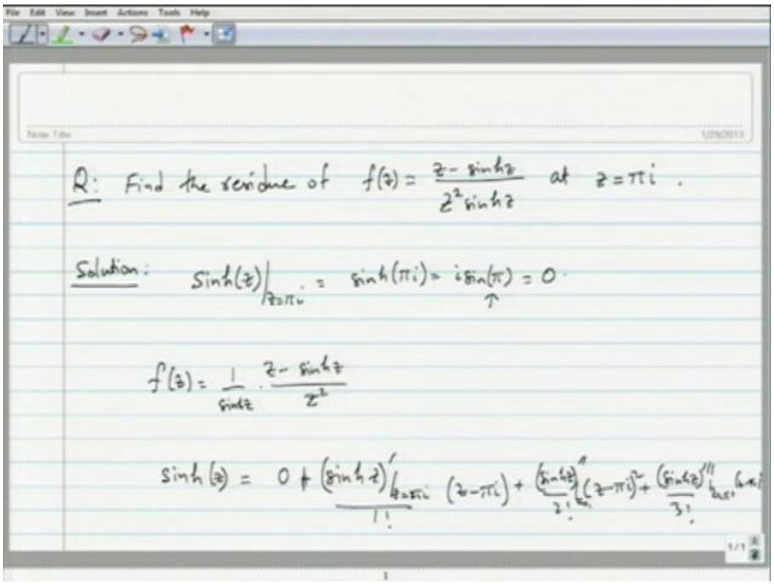

So, let me start with a question on residues, so calculation of residues. So, find a the residue of the function f of z equals z minus sine h z divided by z squared sine h z at the point z equals pi i here by sin h z, I mean the hyperbolic sine z. Now, try to solve this problem and here is the solution to this question, okay? So, first notice that this function f has a 0 at z equals I mean the denominator has a 0 at z equals pi i. So, sine h z at pi i z equals pi i is sine h of pi i, which is sine h sin of pi i is sin pi. We know this formula from our earlier considerations, sine h of i z is i sine z, so this gives us a 0.

So, the denominator has a 0 at pi i. Also notice that, the 0 of sine pi of the sine function at pi is a simple 0. So, what that means is that the sine h z has a simple 0 another z equals pi i. Now, using that fact we know that if we write f of z is equal to z minus sine h z by z squared and times 1 by sine h z. So, sine h z has a simple 0 at z equals pi i, so f has a simple pole at z equals pi i. So, in order to find the residue, we just need to find the coefficient of 1 by z minus a, in the Laurent series expansion for f of z. So, how do that? We will resort to the power sees expansion first of sine h z.

So, notice that sine h z the power series of this around z equals pi i is going to be sine h at pi i itself is 0 plus the derivative of sine h z, the first derivative of this at the point z equals pi i divided by 1 factorial times z minus pi i, plus sine h z the second derivative of sine h z by 2 factorial times z minus pi i squared, plus the third derivative of sine h z at the point z equals pi i. So, all this at z equals pi i at z equals pi i divided by 3 factorial times z minus pi i cube. So, for lack of space I will write below z minus pi i cube, plus so on. There are higher order terms.

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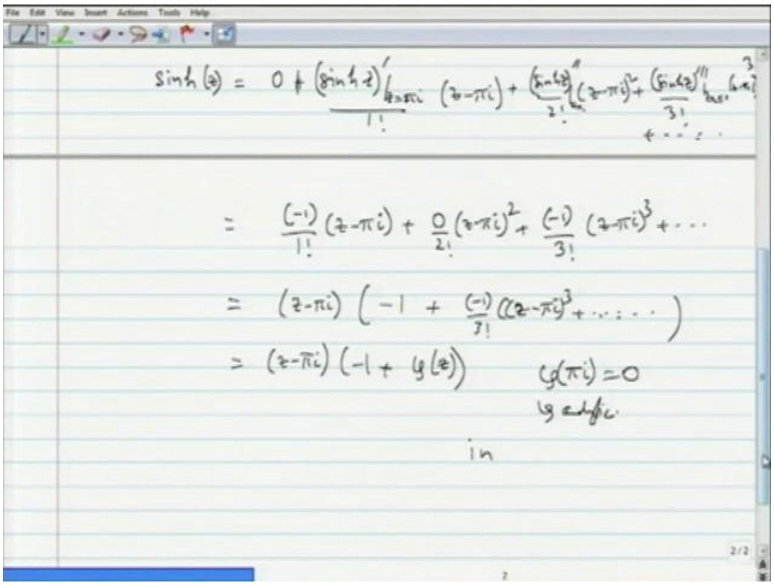

So, notice that the first derivative of sine h z, itself is a cos h z. So, that will give us cos h z at equals pi i which is cos pi, so that gives us minus 1 by 1 factorial times z minus pi i plus, well the second derivative of sine h z at at the point z equals pi i is a sine h z itself at z equals pi i, which gives us 0 by 2 factorial times z minus pi i squared, plus the third derivative is once again cos h z at z equals pi i gives us minus 1. Once again divided by 3 factorial times z minus pi i cube plus so on.

So, this is this can be written as z minus pi i times minus 1 plus well minus 1 by 3 factorial times z minus pi i cube plus, so on and all these so on terms will have a a z minus pi i power something positive integer. So, this can be I will write this as z minus pi i times minus 1 plus a function phi of z, which is 0 at pi i phi of pi i what is important is phi of pi i is 0 and phi phi is analytic function, phi analytic. So, this is definitely valid this Taylor's series expansion of sine h z is definitely valid in some in some neighbourhood of pi i r, such that r is strictly positive.

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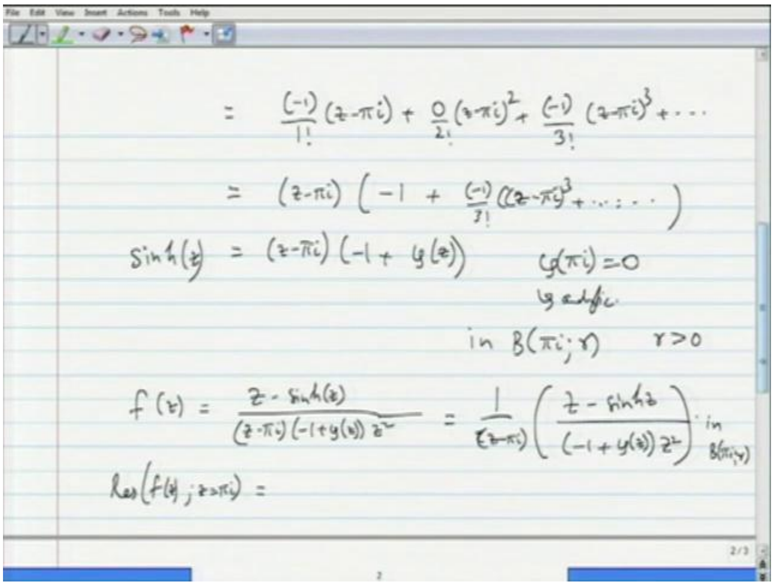

So, all this expression is definitely valid in some some disk around pi i, okay? So, now we will use this expression in f of z f of z is equal to z minus sine h z divided by z minus pi i times minus 1 plus phi of z times z squared. So, notice that this part is analytic at z equals pi i for f of z. So, since we know that the pole of f at pi i is simple we know that the residue of f of z at z equals pi i, we just simple has to be, has to be 1 by...

Well, before I write that this is now equal to 1 by z minus pi i times all this function z minus sine h z divided by minus 1 plus phi of z times z squared at least in a neighbourhood of a b pi i in a b pi i r. So, locally around pi i at least this expression is valid. So the residue of this is going to be well here is a analytic function in b pi r. So, the value of this at the point pi i itself will give us the residue.

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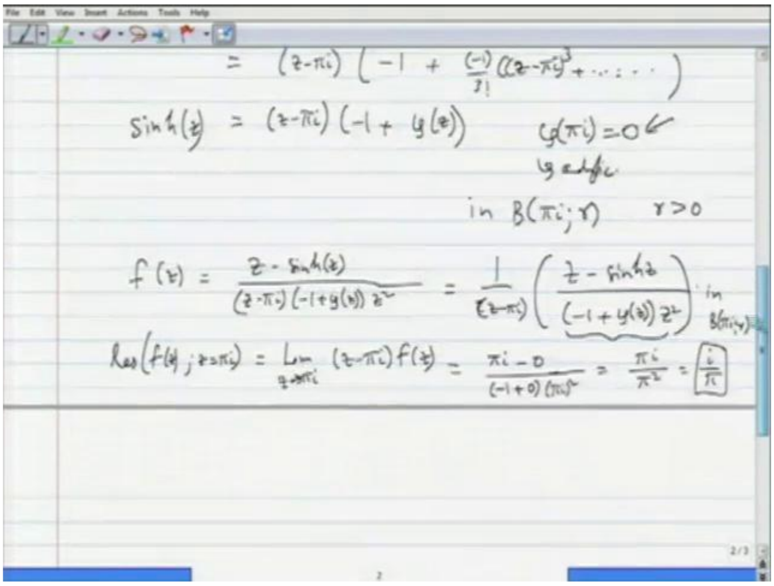

So, or in other words this is the limit as z goes to pi i of z minus pi i times f of z. So, this factor of z minus pi i cancels with this and this gives us pi i minus sine h pi i, which is 0 divided by minus 1 plus 0 remember phi at pi i is 0, like I mentioned here. Then z squared is pi i whole squared which gives us pi i by minus minus pi squared, so it is pi squared so this is i by pi. So, that is the residue of f of z at the point z equals pi i. So, this is an example in calculation of residue. So, let us see another example of this sort.

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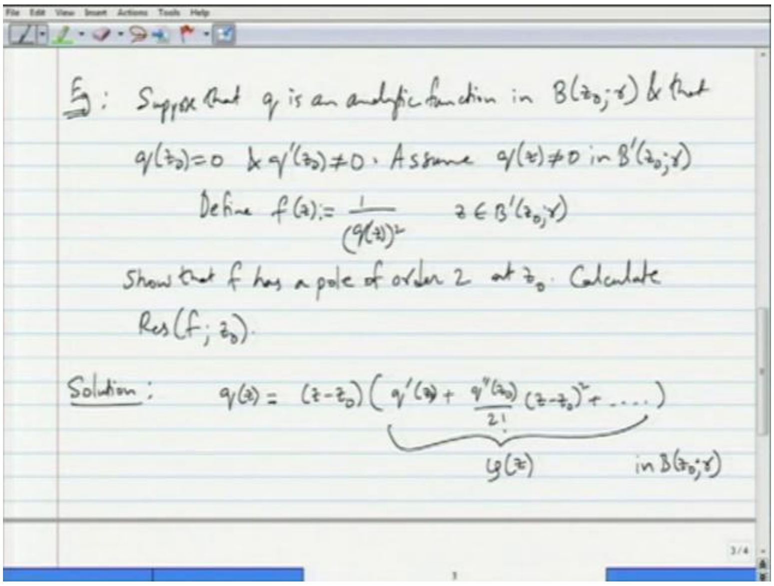

So, um example two, suppose that q is an analytic function in B z naught r. So, and that q of z naught is 0 and q prime of z naught is non zero, define f of z is equal to 1 by q of z squared, okay? So, also you probably need to assume that assume q of z is not equal to a 0 in B prime z naught r. So, z naught is the only 0 of q in B z naught r so define f of z as 1 by q of z squared for z belongs to B prime z naught r. So, show that f has a pole of order 2 at z naught calculate the residue of f at the point z naught. So, the viewer can try to answer this problem and here is the solution to this problem.

So, firstly note that q is analytic at z naught and has a 0 at z naught, so q of z is equal to z minus z naught times using Taylor's theorem. Let us say this is times Q prime of z plus q prime of z naught, sorry divided by 1 factorial plus q double prime of z naught divided by 2 factorial times z minus z naught plus q higher derivatives of q. So, allow me to call this function as phi of z for convenience and this expression is valid definitely in B z naught r, which is the domain of analyticity of q. It is given that Q prime of z naught is non zero, so q has a simple zero at z naught.

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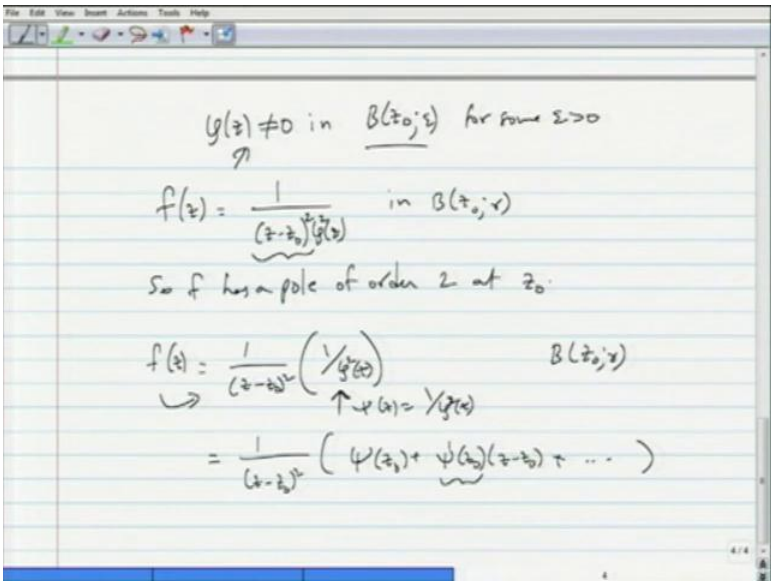

Now, f of z, now I have to remark here that T of z is non zero in B z naught epsilon for some epsilon positive k so b phi of z naught itself is non zero because Q prime of z naught is non zero. The, then by continuity of phi you can say that phi of z is non zero in b z naught epsilon, okay? So Q has a simple 0 at z naught. So, f of z is equal to 1 by z minus z naught times phi of z in B z naught r and I apologise a squared phi squared because f is 1 by q of z whole squared. So, f is that and phi of z is non zero in B z naught epsilon. So, f has a simple or a pole of order 2 at...

So, just by looking at this expression we can conclude that f has a pole of order 2 at z naught. So, if we break the expression for f of z as 1 by z minus z naught squared times the function 1 by phi squared of z, then we can write the power series expansion of this expression in B z naught r 1 by phi of z is non zero q is non zero in all of B prime z naught r. So, a phi is not going to be 0, there in B z naught r. Then have 1 by phi squared z can be expanded as a power series or by Taylor series. So, the the first derivative of 1 by phi squared of z is going to give us coefficient of 1 by z minus z naught for the expansion for the Laurent series expansion of f of z and that will tell us the residue of f. Remember the residue is just a coefficient of 1 by z minus z naught in the Laurent series expansion for f of z.

So, all we need to do is find the first derivative here the reason, I am saying that is the first derivative is that, if I call this function some psi 1 by phi squared of z is psi. Then 1 by z minus z naught squared times psi of z naught plus. Similarly, prime of z naught times z minus z naught etcetera is the power series expansion of psi, psi is this function 1 by phi squared of z. So this coefficient will give me the the coefficient of 1 by z minus z naught for f of z.

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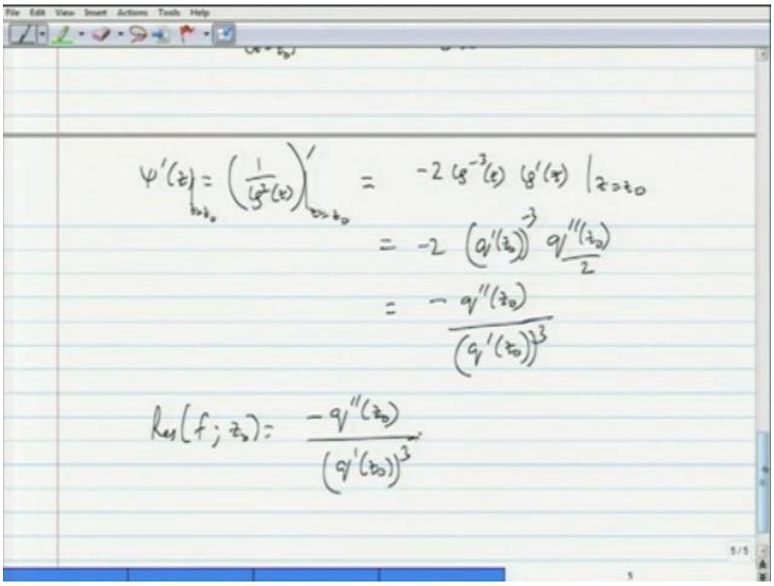

So, let us find that the first derivative of psi psi prime of z is going to be the derivative of 1 by phi squared of z at the point z naught at the point z equals z naught. So, this is going to be minus 2 phi power minus 3 of z times phi prime of z. So, this is minus 2 phi power minus 3 of z times phi prime. So, you just have to look at what phi of z is so phi of z itself has this kind of expansion around z naught. So, phi of z phi prime of z is going to be your q double prime of z naught, so phi of phi prime of z is q double prime of z naught divided by 2 factorial phi power minus 3 of z 1 by phi cube of z at z naught is q prime of z naught power minus 3.

So, this is I forgot the 2 factorial, so this is divided by two so this gives minus q double prime of z naught divided by q prime of z naught cube. So, the residue of f at the point z naught is going to be minus q double prime of z naught divided by q prime of z naught cube, so it is the residue and f has a pole of order 2 at z naught, that is the solution to this problem.

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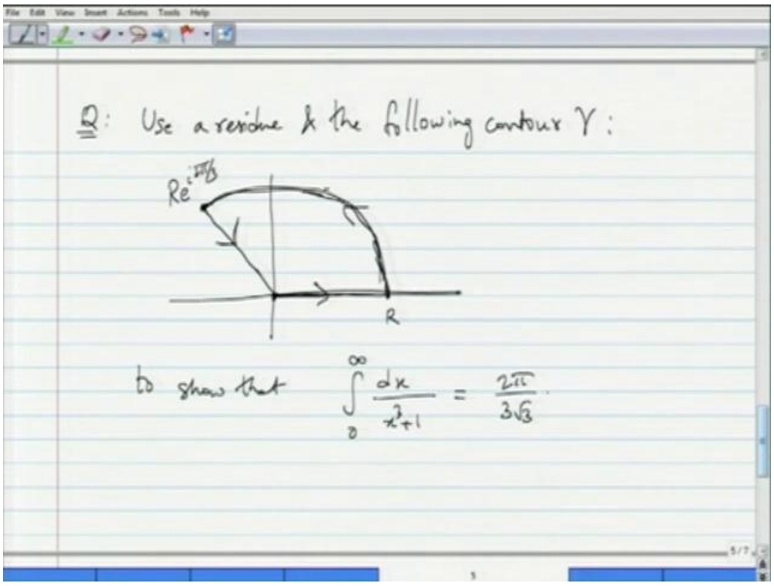

Now, we will move on to the next problem, this question is a contour integral of certain sort. So, use a residue and the following contour, so the contour is as follows. So, you start at the origin go until some point R on the real line and then you go along a circular contour. So, that is the portion of a circle. Then this is the ray theta equals 2 pi by 3, so this is the angle 2 pi by 3, so you start from origin, go in this direction, go along the circle and come back to the origin along the ray theta equals 2 pi by 3. So, this point itself is Re power i 2 pi by 3. That is this point, so let us call this this whole contour a gamma, okay?

Use a residue and the following contour to show that integral 0 to infinity of d x by x cube plus 1 is equal to 2 pi 3 root 3. So, let me pause here to mention that we can calculate indefinite integrals of these sorts. So, real indefinite integrals by using Complex analysis and Cauchy's residue theorem. So, this is one example form the set of examples, which can be solved thus, so the viewer is advised to look into text books which have been recommended to solve more problems of this sorts.

And some of the text books actually give a whole tool kit in order to a technical tool kit in order to solve, a integrals of this sort of find indefinite integrals of these sort using Cauchy's residue theorem, okay? So, we are not going to present the whole tool kit here, but what we are going do is study couple of examples in that direction. So, here is one example in that direction, so given this contour show that this indefinite integral has that value.

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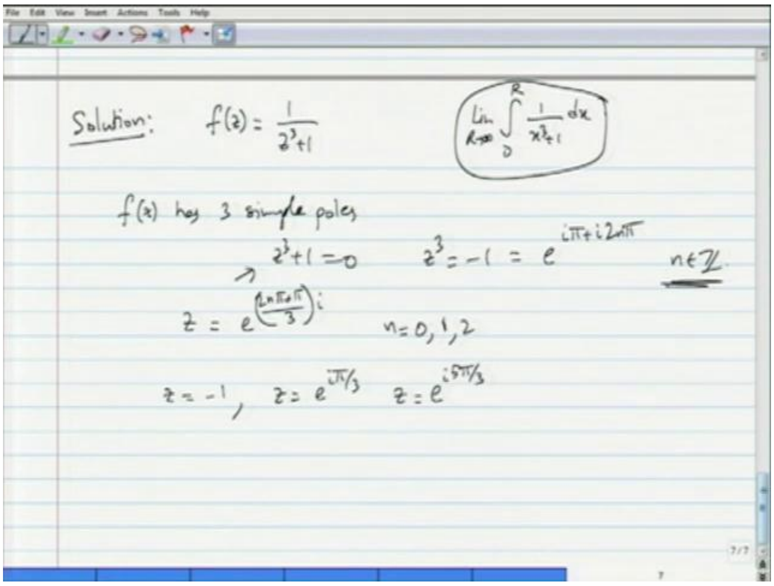

So, I will present the solution to this problem here, so looking motivated by this function here d 1 by x cube plus 1 which is the integrand, so I am sort of considering motivated by that, I a m considering the function 1 by z cube plus 1. So, this function notice that on the real line from 0 to R 0 to r f of z is a real number on the real line on 0 to R it gives me 1 by x cube plus 1 d x when i integrate f of z, okay? So, and if I take the limit as R goes to infinity there is hope that I will be able to get what I want. I I will able to evaluate this indefinite integral hopefully.

So, motivated by that I am considering this function f of z equals 1 by z cube plus 1. Notice that f of z has 3 simple , 1 by z cube plus 1 is a polynomial or z cube plus, 1 is a polynomial. So, 1 by z cube plus 1 has 3 simple poles. Since, we know the the solutions to z z cube plus 1 is equal to 0. We have seen examples of these sorts, we want to solve z cube is equal to minus 1. So, if we write minus 1 as e power i pi plus 2 pi i. So, i 2 pi or 2 n pi rather n belongs to z integers. Then we know that, z can be can take the values e power 2 n pi plus pi divided by 3 times i and all the values of i means m belongs to z are going to be (( )). So, we will just use 1 0 1 and 2 from 3 onwards, we are going to get back solution. We already got, so we have seen this procedure earlier. z equals minus 1 is one such root and z equals e power i pi by 3 is 1one root of this of this equation.

The third one is going to be z equals e power i phi pi by 3, so n equals 0 corresponds to z equals e power i pi by 3 z equals n equals 1 is going to give us, 2 pi plus pi 3 pi by 3, which is z equals minus n equals 2 gives us, phi pi e power i phi pi by 3, okay? So these are the 3 roots of z cube plus 1 equals 0 and hence these are the three simple poles of f of z. These three points and notice that given this contour given this closed contour given simple closed contour, there is only one among these three, which is in the inside of this contour namely e power i pi by 3, which occurs on the line theta equals pi by 3 on the half ray theta equals pi by 3. The others are on the negative real axis and in the fourth quadrate.

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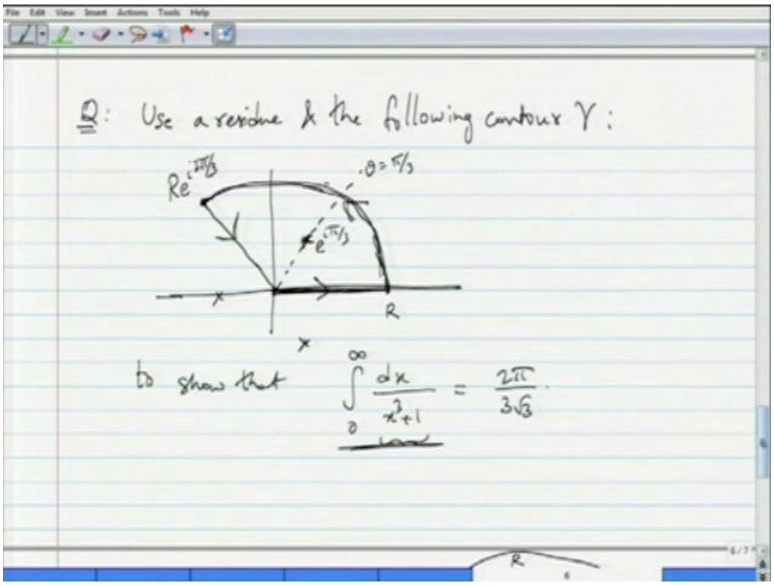

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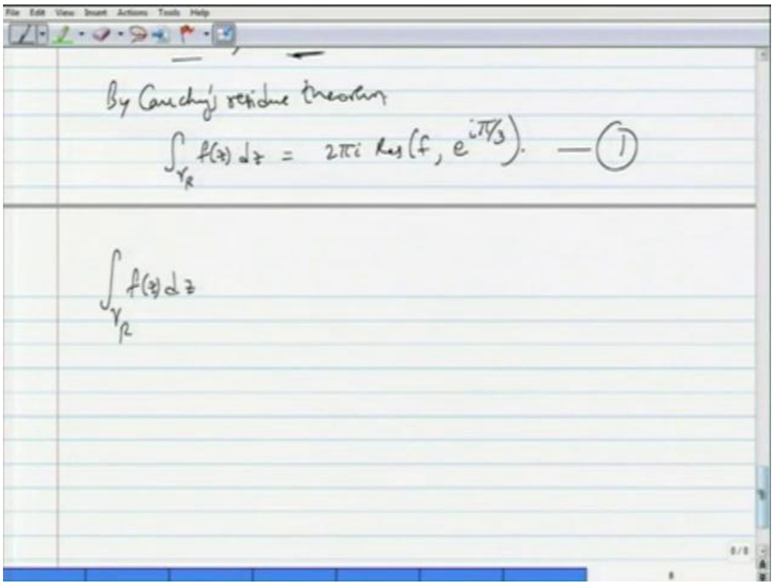

One this is a conjugate, this is a conjugate of this, so those are the three poles of f of z. So, the contour integration form by Cauchy's residue theorem. We know that the contour integration of f over gamma. So, let me suggestively call this contour gamma r because I have a variable R here capital r here. So, the integration over gamma r of f of z d z has to be 2 pi i times the residue of f at the point e power i pi by 3, but that is only a half of the story. Of course, I want to know, how I can get this integral or at least integral 0 to r 1 by x cube plus 1 d x out of this equation?

One, let me call that 1, now for in order to see, how I get that expression. Now, let me break the contour into its natural parts and let me consider the integral form 0 to r on the real line, okay? So let us concentrate on gamma r of f of z d z the integration of f of gamma r. So, it constitutes of three parts.

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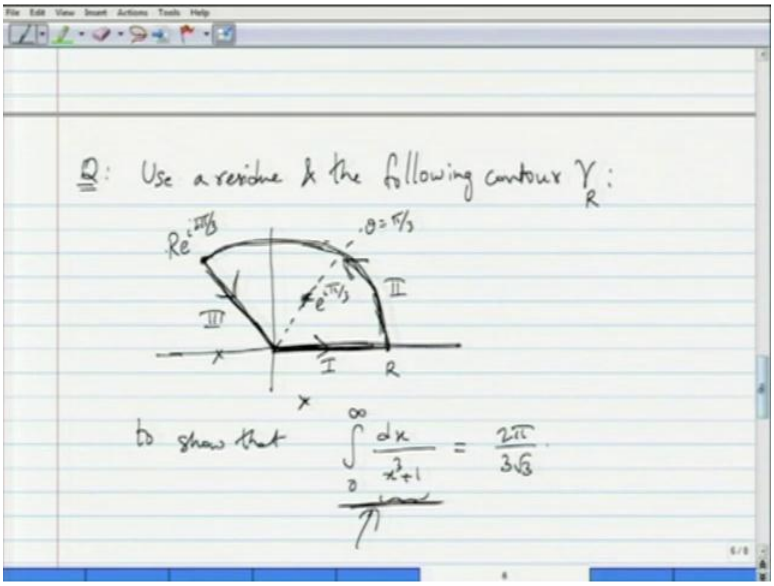

One is this real line part two is this semicircular part and three is this half ray part. So, let us consider each one of them separately.

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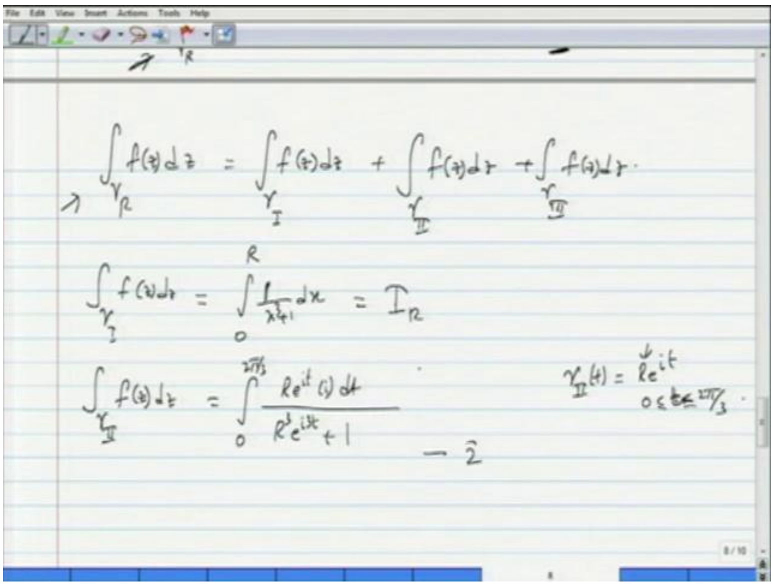

So, let me call that integration over gamma roman 1 f of z d z plus integration over gamma roman 2 f of z d z plus integration over gamma roman three, f of z d z with their provided orientations. So, when we consider integration over gamma 1 f of z d z. So, this is equal to well integration from 0 to r of f of 1 by x cube plus 1 d x like we want because f of z is 1 by z cube plus one and z is real on gamma 1. So, that is good and then integration over gamma 2 of f of z d z is going to be on gamma 2.

Let us parameterize gamma 2 as gamma 2 of t is equal to r e power i t, where t ranges from 0 to 2 pi by 3. So, this is a portion of a circle, so that is how we can parameterize its is t a circle of radius r? So, this is integration from 0 to 2 pi by 3 of d t d z is r e power i t times i times d t. Then in the denominator we have z cube plus 1 z cube on gamma 2 of t is r cube e power i 3 t plus 1. So, we will preserve this expression as a equation two and on gamma 3 f of z d z by using the parameterization gamma of t is equal to gamma.

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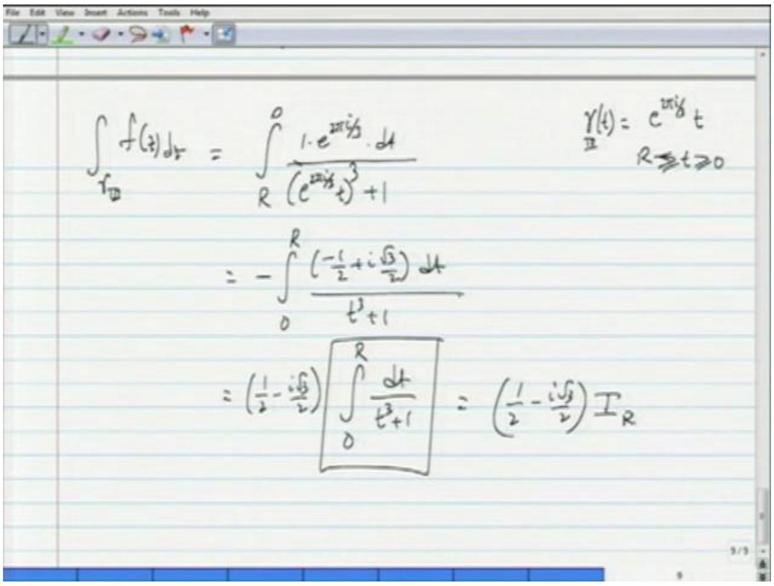

Three of t is equal to e power 2 pi i by 3 times t where t goes from r to 0. So, notice that the orientation on this is from this point towards the origin. So, I have to use t goes from r to 0. So, this is equal to integration from R to 0 of gamma prime gamma 3 prime is going to give me 1 times e power 2 pi i by 3 times d t divided by gamma, sorry z cube on gamma 3 is going to be e power 2 pi i by 3 t whole cube times t whole cube plus 1.

So, this is by changing the limits of integration to 0 to 0 to R, I get a minus sign. 2 pi i by e raised to 2 pi i by 3 is cos 2 pi by 3 plus i sine 2 pi by 3 d t divided by e power 2 pi i by

3 cube gives me 1. So, this is t cube plus 1. So, by absorbing the minus sign into the complex number I get, half minus i root 3 by 2 times integration from 0 to R d t by t cube plus 1, but notice that this of looks like the integral that we need, so this is this complex number half minus i root 3 by 2 times the integral that we need with the variable R. So, let me call that i r.

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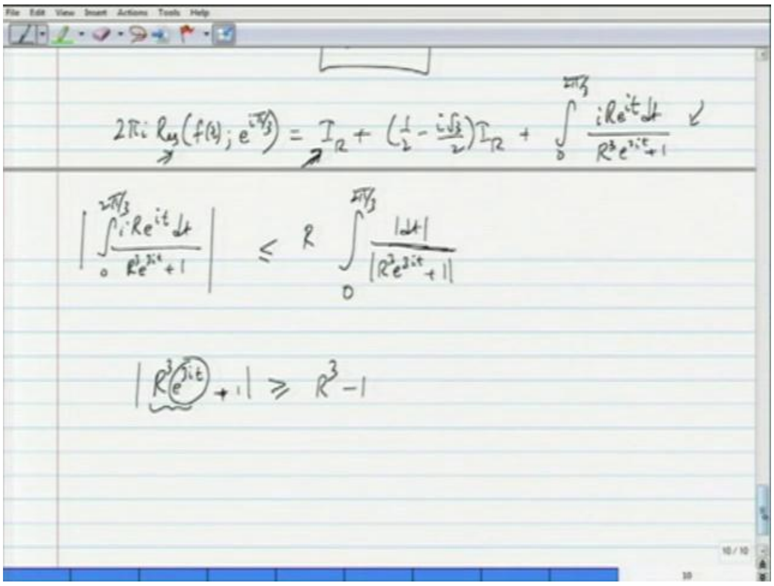

So, the residue 2 pi i times the residue of f of z at the point e power i pi by 3 from equation one. We notice that from equation 1, we notice that 2 pi times residue of that is equal to integration over gamma R of f of z d z. So, this is in turn equal to i r plus this complex number half minus i root 3 by 2 i r which are the integrations on gamma 1 and gamma 3 respectively plus the integration on gamma 2 is this expression here, which we preserved integration from 0 to 2 pi by 3 of r e power i t i r e power i t d t by R cube e power 3 it plus 1.

So, limit as R goes to infinity is what we are interested in because i r as R goes to infinity gives us the definite integral or the improper integral that we want. So, if we can calculate we can calculate the residue here, so if we can calculate this integral or we if we can estimate this integral and say something, then we will able to conclude something about i r as R goes to infinity. So, with that as motivation let us examine modulus what this integral comes to i r e power i t d t by R cube e power 3 i t plus 1 in modulus this is less than or equal to well in the numerator the modulus of r i e power i t is going to be, simply r R is a positive real number. Then integration from 0 to 2 pi by 3 of modulus of d t divided by the modulus of R cube e power 3 i t plus 1.

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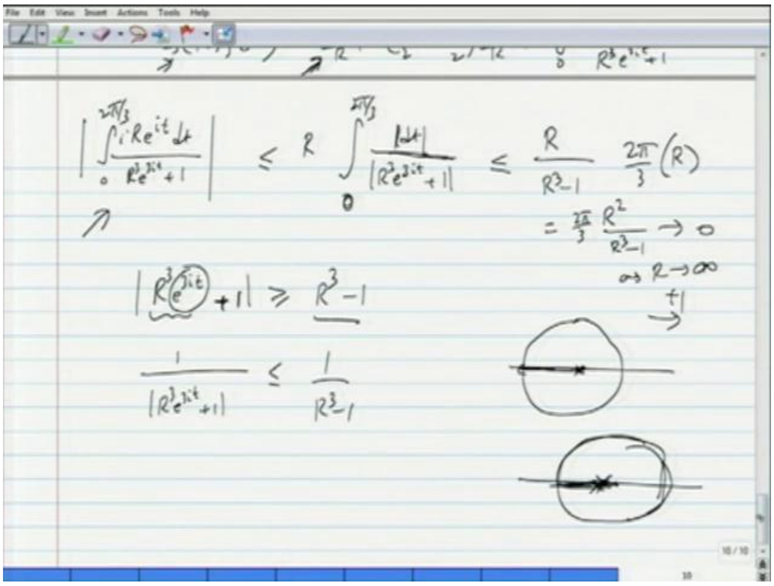

So, notice that the modulus of R cube e power 3 i t e power 3 i t is a point on the unit circle and R cube tells us that it is this complex number is a point on the circle of radius R cube. And then plus 1 tells us that this is in modulus, so this in modulus is at least at least r cube minus 1, so that is because if you take the circle of radius R cube around the origin and shift it to the right by adding 1. Then this circle goes to some circle like that, so the origin is now off centre its off centre and this distance is at least R cube minus 1. So, any point on this circle now on this shifted circle, so transformation is by plus 1 on the shifted circle at least has modulus R cube minus 1, okay?

So, it is the idea, so 1 by the modulus of R cube e power 3 i t plus 1 is going to be less than or equal to 1 by R cube minus 1. So, this is less than or equal to this integral is less than or equal to R by R cube minus 1 times the length of this the this curve 0 to 0 pi by 3 of modulus of d t, which is a portion of a circle which is pictured in the contour. So, this is 2 pi by 3 times R. So, that is the length of that curve, now you see that this is equal to R squared 2 pi by 3 by R cube minus 1. So, which tends to 0 as R tends to infinity. Remember we are interested in the integral as R tends to infinity.

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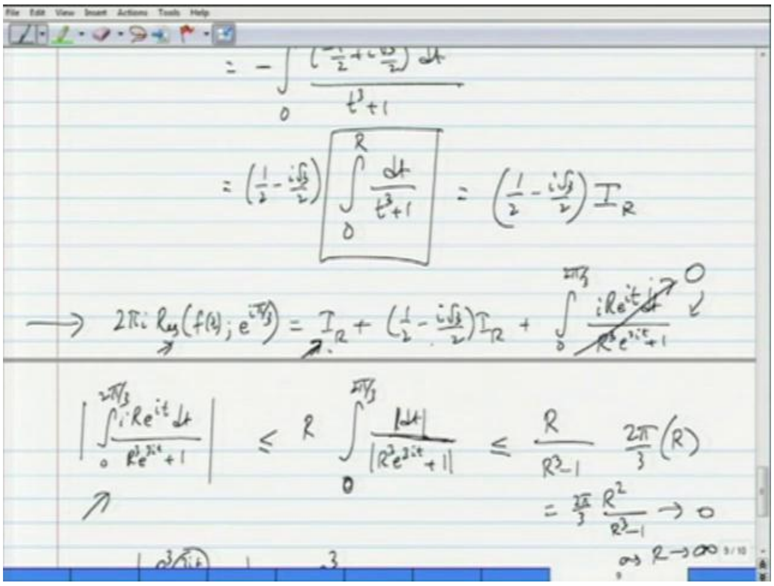

So, that is good for us because when we take the limit as R goes to 0 on both sided of this equation this integral tends to 0, this integral tends to 0, so we are left with just these two pieces so in summary.

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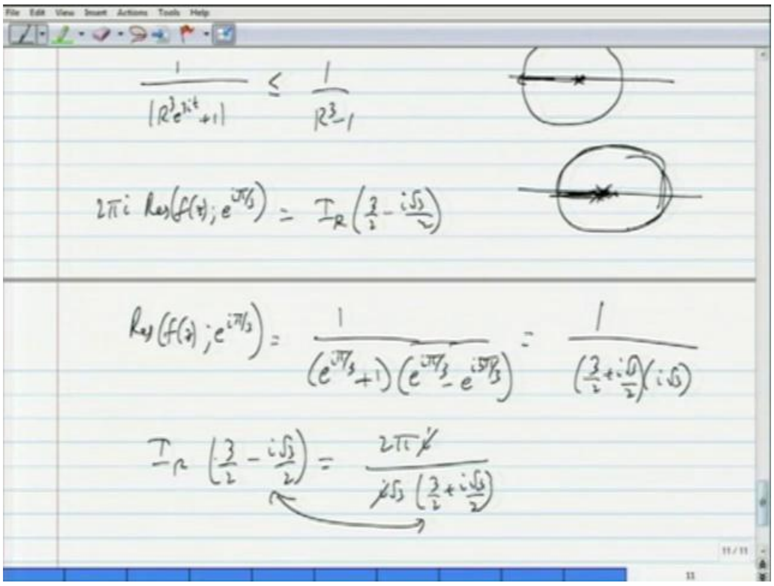

If we two pi i times the residue of f of z at the point at the point e power I pi by 3 which we can calculate is I R times 1 plus half minus i root 3 by 2, which is 3 by 2 minus I root 3 by 2. So, it is easy to calculate the residue of f at e power i pi by 3, I will just give the value the residue is going to be 1 by e power I pi by 3 its just 1 by e power I pi by 3 minus. So, plus 1 times e power i pi by 3 minus e power i phi pi by 3 and that is going to be 1 by 3 plus 3 by 2 plus I root 3 by 2 times i root 3.

So, I R from by substituting this over there we get I R is equal to I R times 3 by 2 minus I root 3 by 2 is equal to 2 pi i by i root 3 times 3 by 2 plus I root 3 by 2 notice these are conjugates. So, we get the modulus when we multiply then I cancels here and then, so you get I R is equal to 2 pi by the modulus of this complex number here is 3.

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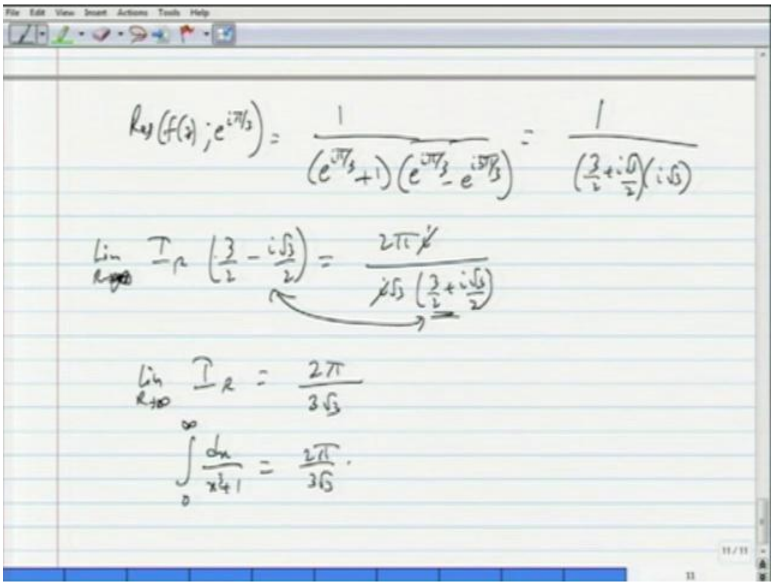

So, you get three root 3 so that is the value this is I R in limit. So, limit as R goes to infinity, which is what we want. Since, we have already taken the limit for the other integral, so this is integration from zero to infinity of d x by x cube plus 1 is equal to two pi by 3 root 3, okay? So, that is a contour integral an example of contour integral. So, we can use the residue theorem, and and some calculations like this estimations like this in order to calculate real indefinite integrals of some sorts. So, the in this example, I have given the contour already in the problem in the statement of the problem normally the challenge is lies in choosing an appropriate contour for given problem, so here is another example of that sort.

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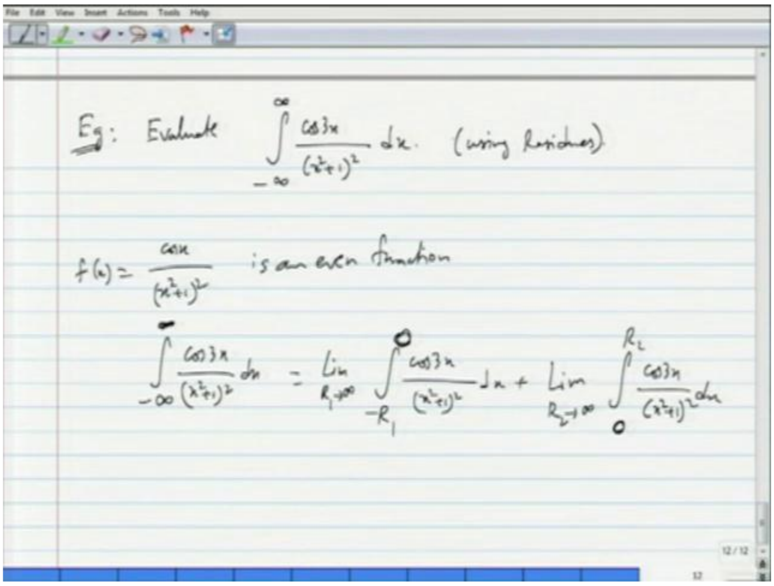

So, evaluate integration from minus infinity to infinity of cosine 3 x by x squared plus 1 whole squared d x. So, the idea is using residues, so this function cosine x by x squared plus one whole squared notice is f of x equals this is an even function.

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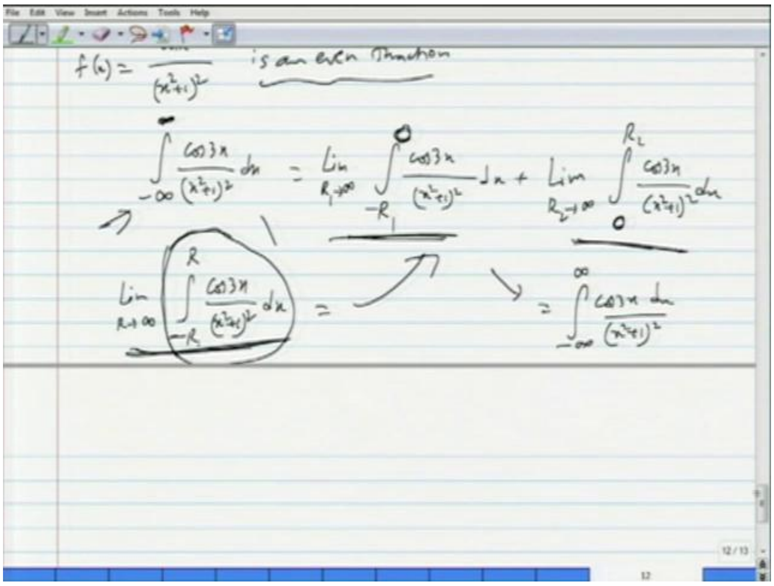

So, I am free to choose C, so I will choose C to be 0. So, that is the definition of this as we know from the analysis, but what is true is that when the function is even one can use these expressions to show that the limit as R goes to infinity of minus R to R of cosine three x by x squared plus 1 whole squared d x a. If we can show that this limit exists then this is equal to you know this expression on the right hand side, which is actually you know by definition integration from minus infinity to infinity of cosine 3 x d x by x squared plus one whole squared, okay?

So, if we can show that this limit exists and then it has to equal these two when the function is even it has to equal the sum of these two, when the function is even. So, we can calculate this particular given definite improper integral, so we will take this root and try to first take the contour in appropriate function and in appropriate contour to evaluate this particular integral minus R to R cosine 3 x by x squared plus 1 whole squared d x.

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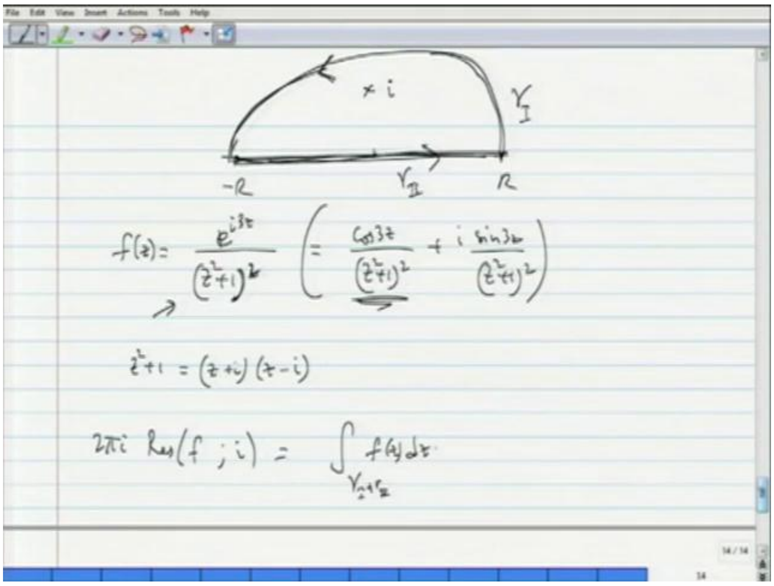

So, that minus R to R suggests that perhaps I have to stick to the real line from minus R to R, but what is the appropriate function. Well let us take f of z equals 1 by z squared plus 1 whole squared owing to the denominator, okay? Then this function or I should say e power three i z by that, so this is going to give me well cosine 3 z by z squared plus 1 whole squared plus i sine 3 z by z squared plus 1 whole squared. So, hopefully if I am able to say something about the integration of this from the real line from minus R to R, I can hopefully say something about its real part.

So, with with lot of that hope we can we will consider this f of z this function f of z this function e power i 3 z is entire, so there is no problem with the numerator, but in the denominator, notice that z squared plus 1 is z plus i times z minus i. So, there is a pole of order 2 at z equals i or minus i, okay? So, if we consider the upper half plane, then there is only one pole of order two namely i, so if we consider a semicircle like that and then this line on the real line from minus R to R so we take a orientation on this contour. It is a closed simple closed contour, so then the residue by the residue theorem, 2 pi i times the residue of f at the point i is going to give us, the integration on the considered contour gamma. Let me call this contour gamma, gamma 1 and gamma 2 1, again so integral over 1 1 plus gamma 2 of f of z d z. We know that limit as R goes to infinity of gamma 2 is what we are interested in.

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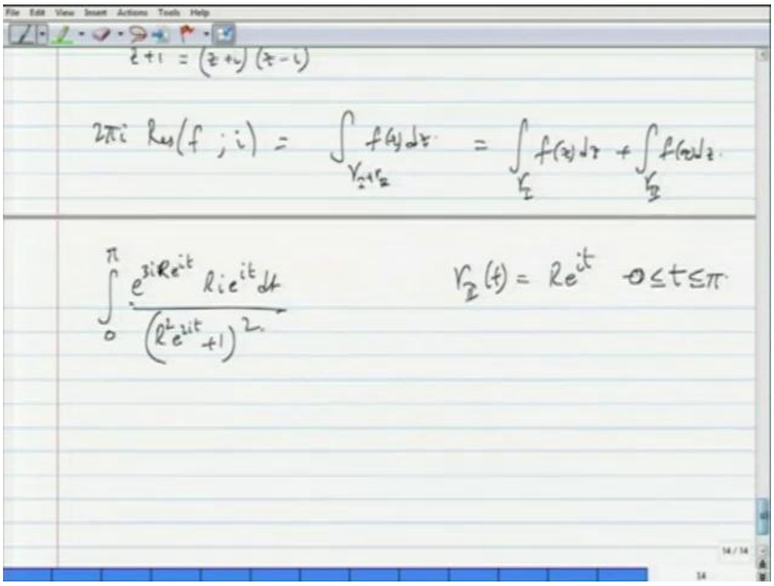

So, this is integration over gamma 1 f of z d z plus integration over gamma 2 of f of z d z. Let me first parameterize gamma 2 gamma 2 can be parameterized as gamma 2 of t is R e power i t t goes from 0 to pi, okay? So, this is 0 to pi, z is R e power i t, so I have i e power 3 i R R power i t and then the differentiation is R i e power i t d t divided by z squared plus 1 z squared is R squared e power 2 i t plus 1 whole squared.

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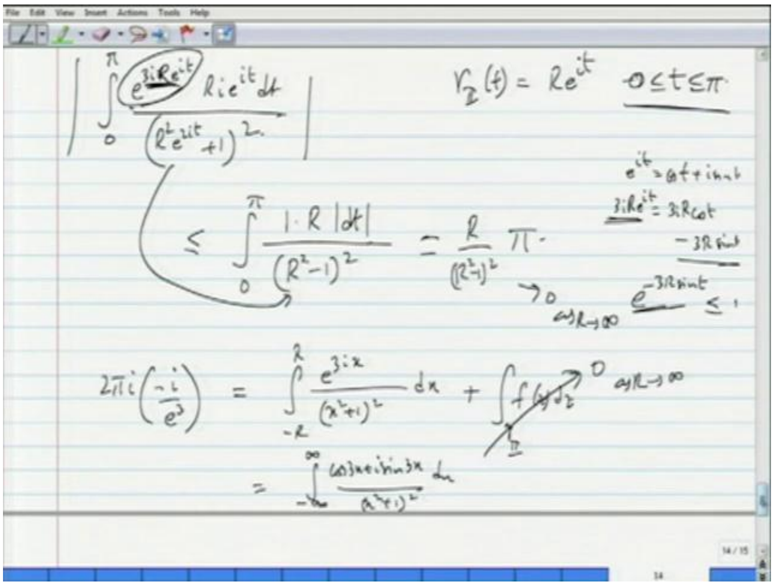

So, like we saw in the earlier problem, when we consider this in modulus this is less than or equal to the integration from 0 to pi like we saw in the earlier problem the denominator is less than greater than or equal to R squared minus 1 in whole squared. So, that can be done, so this part is okay. And the numerator we have, e power 3 i R e power i t, so if I write e power i t as cosine t plus i sine t, then 3 i R e power i t is going to be 3 i R cosine t minus 3 R sine t.

So, the real part of this complex number is 3 minus 3 R sine t and as t ranges from 0 to pi sign is positive. So, e raised to minus 3 R sine t, which is going to be the modulus of this complex number, the modulus of this complex number is going to be the e raise to the real part of the complex number in the exponent, so this is the modulus of that complex number. So, this is going to be less than or equal to 1. So, this is 1 times R times modulus of i e power of it is 1 modulus of d t. So, this is less than or equal to, well less than, this is equal to R by R squared minus 1 whole squared times certain (( )) its pi, the length of that curve and this goes to 0, as R goes infinity.

So, that is what we are interested in. So, then we have 2 pi i times so that the calculation of residue of f at the point I which is a singularity of f inside this contour, is easy. So, that is an exercise that one can see is minus i by e cube. So, using this equation here 2 pi i times the residue of f at i which 2 pi i times the residue of f at i, which is 2 pi i by eq. So, using this equation here 2 pi times the residue of f at i, which is 2 pi i by e q is equal to integration of gamma 1that is what we want. It is the integration of minus R to R of cosine sorry, e power 3 by x by x squared plus 1 whole squared d x plus this integral over gamma 2 of, I will just f of z d z.

Now, we are we are interested in limit at R goes infinity and we showed that limit of R goes to infinity, so f of z itself is 0. So, this 0 as R goes to infinity. This is the integral limit as R goes to infinity of this gives us integration infinity to minus infinity of cosine 3 x plus i sine 3 x d x divided by x squared plus 1 whole square. So, on the right hand, on the left side we have 2 pi by e q, which is a real number. So, real and imaginary parts of this indefinite integral converge because there are no imaginary parts on the left hand side. So, indeed the real part of this definite integral, which is minus infinity to infinity of cosine 3 x p x by x square plus 1 whole square is equal to 2 pi by e q. That is how we evaluate this identity.

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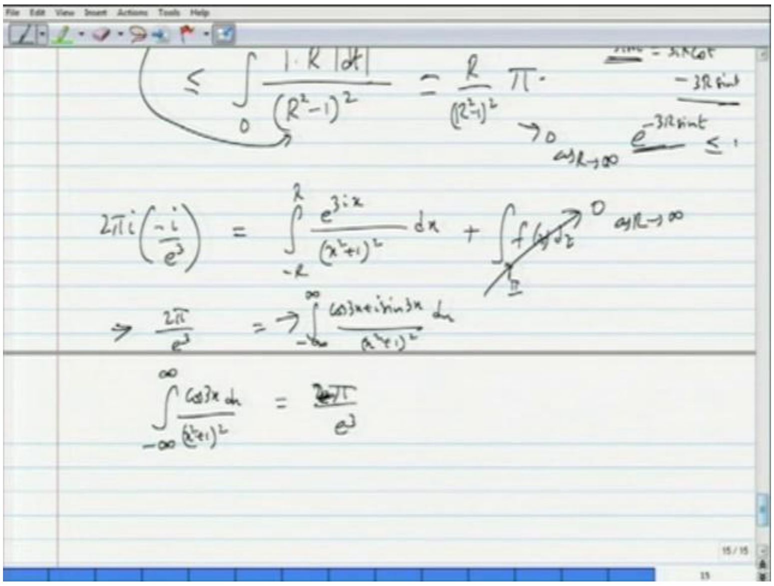

So, as one can see, the challenge in the problem lies in choosing an appropriate function and an appropriate contour for that function in order evaluate a definite integral. One can solve the more examples of these all, and like I mentioned at the beginning of this problem, one can look into the textbook for more example of this all, and evaluate definite integrals using Caushy's residue theorem. I will pause, stop.