**Complex Analysis**
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**Module - 1**
**The Arithmetic, Geometric and Topological Properties of the Complex Number**
**Lecture - 3**
**Topology of the Complex Plane**

Hello viewers, in this session, we will learn about the topological properties of the complex plane. So, by this, what I mean is that we have seen how what complex numbers are. So, these topological properties tell you how, roughly speaking, these complex numbers are knit together to produce a sort of continuum. So, let us first look at some important subsets of complex numbers.

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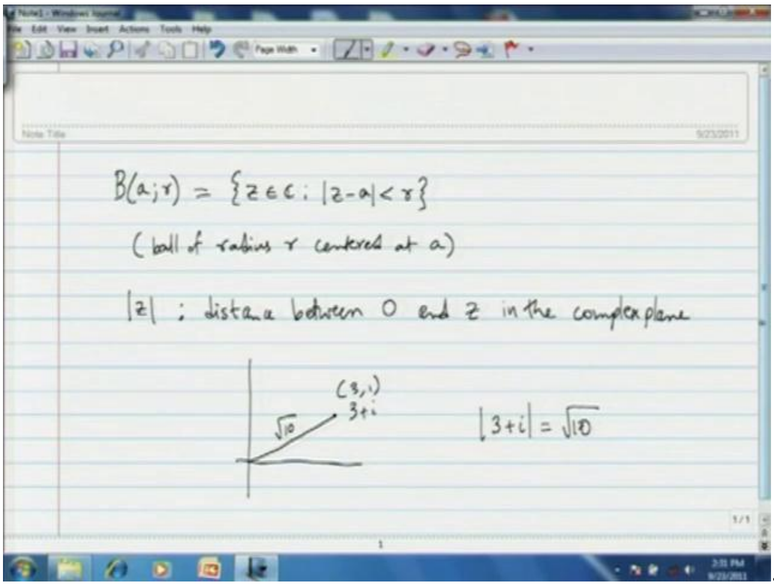


So, I will list some important subsets. So, here is B a r, which stands for the set of all complex numbers, such that the absolute value of Z minus a is strictly less than r. So, b there stands for the ball of radius r, read this as a ball of radius r centered at a. So, the geometric intuition behind this wording is that the absolute value, we know that the absolute value or the modulus of a complex number, gives you the distance between 0 and the number Z in the complex plane. We have seen this interpretation when we have studied the modulus of a complex number. So for example, if I take the number 3 plus i, 3 plus i; so it is represented by the point 3 comma1 on the complex plain and the distance from the origin of that point the Euclidian distance of that point from the origin is square root of10, which essentially is the modulus of the number of the complex number 3 plus i.

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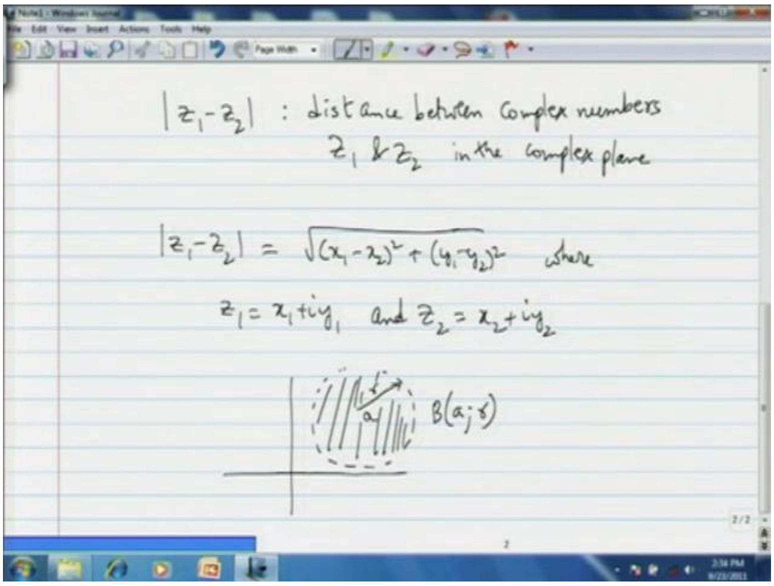


So, the modulus of the complex number Z 1 minus Z 2 in this way denotes the distance between complex numbers Z 1 and Z 2 in the complex plane. So, algebraically the modulus of Z 1 and Z 1 minus Z 2 looks like square root of x 1 minus x 2 whole squared plus y 1 minus y 2 whole squared, where Z 1 is x 1 plus iy 1 and Z 2 is x 2 plus iy 2. So, it indeed indicates the distance between points x 1 comma y 1 and x 2 comma y 2. So, in this set example or in this set that I was talking about B a r is basically the set of all complex numbers, which are at most are away from the point a.

So, this condition here absolute Z minus a or rather modulus Z minus a is restrictively less than r indicates that the complex number Z is at most at distance r away from the complex number a. So hence, pictorially B a r looks like the following. So, suppose this is your a, complex number a and r is some distance, some positive number, then it is the set of all numbers, which are at most r away from the from this complex number a. So, the shaded region is the required region. So, this is B of a r.

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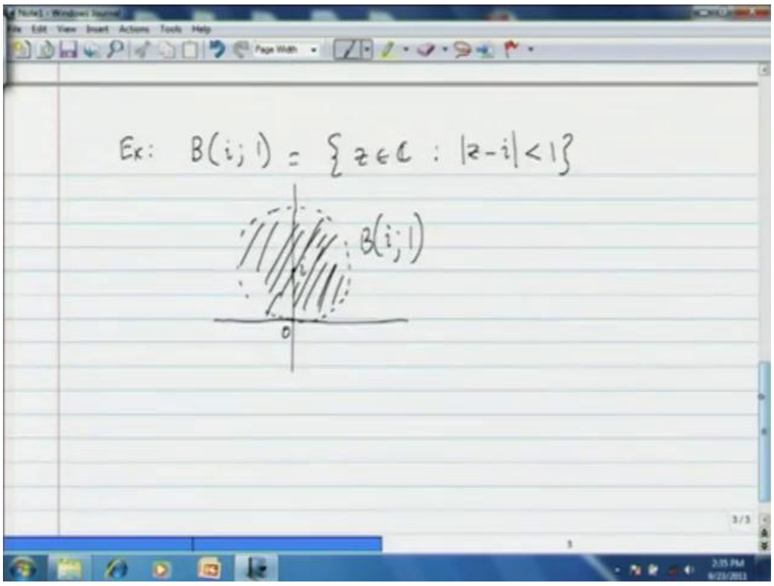


So, here is an example: B of i comma 1 is the set of all complex numbers, such that the modulus of Z minus i is strictly less than 1. So, on the complex plane we can picture this. So, here is your i, complex number i and let us say this is, well the origin is at a distance 1. So, a circle of, we will draw a circle of radius1 and so the shaded region is the required region. This is your B i 1.

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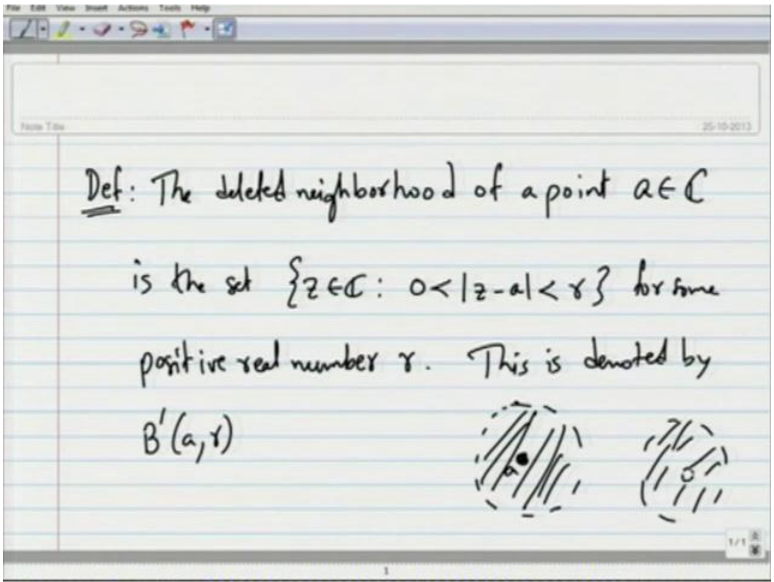


Here, we define the deleted neighborhood of a point in the complex plane. The deleted neighborhood of a point a in the complex plane is the set of all complex numbers, such that 0 is strictly less than the modulus of Z minus a, is strictly less than r, for some positive real number r. This is denoted by B prime a r. So, pictorially what we are doing is, we are considering an r ball around the complex number a, it is an open ball and we remove the point a itself. So, what we have is a deleted neighborhood B prime a r.

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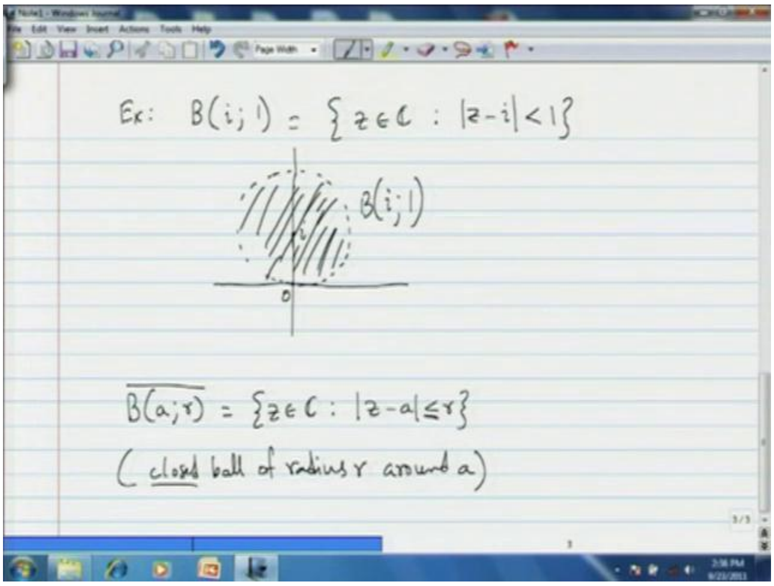


So, there is another important subset of this kind that we will study so it is indicated by B a r with a bar over it to indicate the set of complex numbers, which are at a distance less than are equal to r away from the complex number a and this is sometimes called the closed ball of radius r around a. So, we will shortly see this, a word closed formally in few movements and notice the difference between B a r and B a r bar. We have now included points, which are at a distance r away from a as well.

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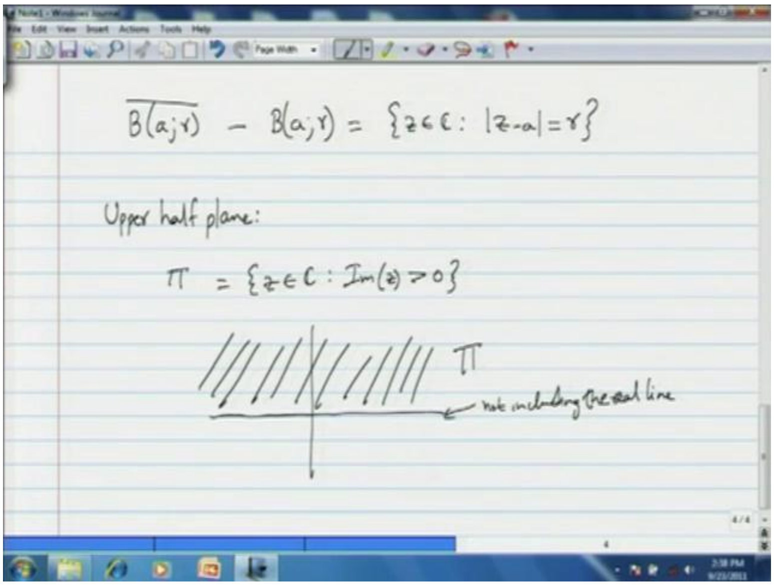


So, the set difference between the B a r and B a r bar is the set of all complex numbers, which are exactly are away from a, which essentially a circle of radius r around a. A comment here, this ball around the complex number a can be defined for any complex number and for any positive radius r.

Next we will see the upper half plane. This is at another important example of subsets of the complex plane. So, this is the set Pi, it is denoted by Pi. This is the set of all complex numbers, such that the imaginary part of Z is strictly greater than 0. So, on the complex plane the picture is as follows this set is all such points whose imaginary part is strictly greater than 0. So, this is not including the real line. This is pi, this is the picture of Pi. This is yet another subset of C that we will keep in mind.

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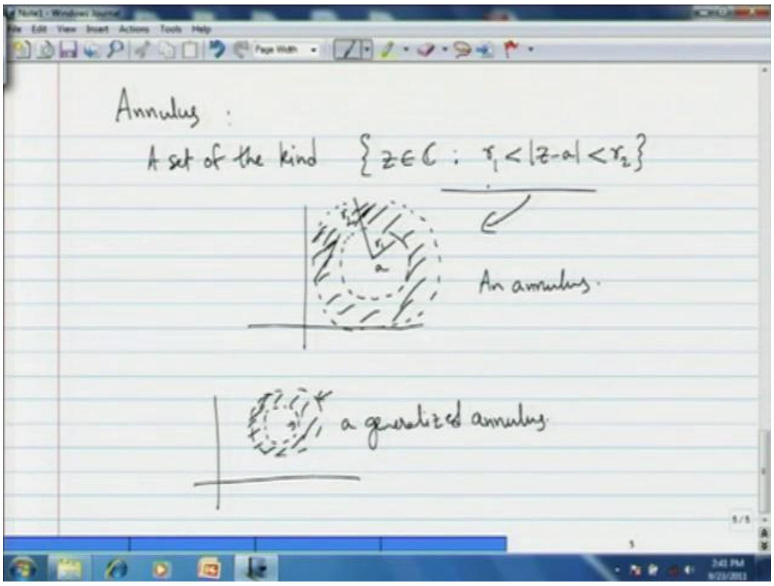


We will often talk about the annulus. So, an annulus is a set of the following kind, it is a set of the kind set of all Z belongs to complex numbers, such that r1 is strictly less than Z minus a is strictly than r 2. So, this essentially is the set of all points centered at or rather it is set of all points which are at least a distance r1 away and at most a distance r 2 away from certain point, certain fixed point a. So, here is a picture of it. So, if you take point a and fix a certain distance r 1, this is your r 1 and fix another distance r 2 and draw a circle of radius r 2 centered at a, then your set here the set here is essentially the shaded region and such a region is called an annulus.

So, there are three parameters for an annulus the center of the annulus, the inner radius and the outer radius. In a more generalized sense we will allow these circles to have different centers, sometimes. So, region of this shape will also be called an annulus, sometimes. So, here the center of the outer circle and that of the inner circle are different, so the enclosed region is generalized annulus. So, these are some of the names of sets, kinds of sets that the viewer should be aware of.

So, now we will talk about the topology of the complex plane. These properties are very essential and have a direct impact on the analysis of the complex numbers. So, the viewer who is familiar with real analysis will recall the importance of the open intervals in a studying calculus or functions of 1 real variable. So, likewise, the so called open sets, which we are going to define in a movement, play an equivalent role in complex analysis. So, essentially an open set is a set where around each point in the set there is some room to manure, so 1 can sit at that point in the set a at a fixed point in the set and then see that he or she is surrounded by some points, completely laying in the set. So, in a more concrete sense an open set is as follows.

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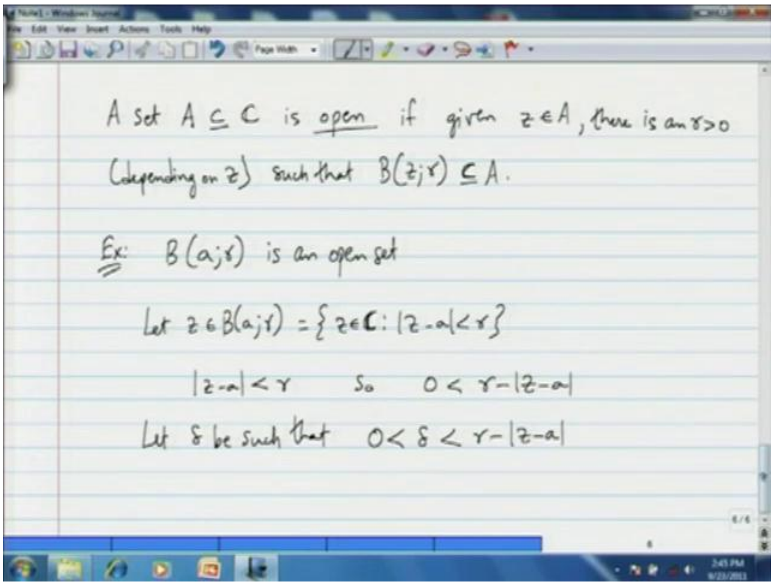


So, a set A contained in C is open if given Z belongs to A, there is an r positive depending on, possibly depending on Z, such that a born of radius r centered at Z is completely contained in the set A. So, this is concretely is an open set in the complex plane.

So example, well our first example is that B a r is itself an open set, is an open set. So, why is B a r open? So, let us see why this set is open. Let Z belong to B a r; recall this is the set of all complex numbers, which are at most r away from the point a from the complex number a. So, if we pick Z belongs to such a set, then we know that the absolute value of Z minus a is strictly less than r, so that, so 0 is strictly less than r minus absolute Z minus a. Since r minus the modulus of Z minus a is strictly positive, we can pick a number between this number and the 0. So, let delta be such that 0 less than delta less than r minus the modulus of Z minus a, and so what we are doing is the following.

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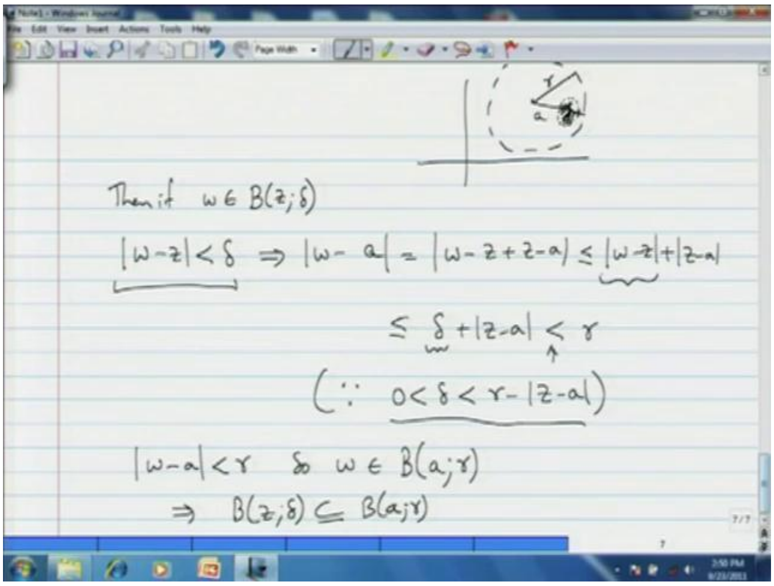


Here is your complex number a, and here is the number r and here is the circle of radius r around a. So, if you pick any Z which is at most r away from a, then the point is that you can now choose on the line joining a and Z, there is still some distance to the to the circle of radius r around a. So, you can choose some positive distance, which is slightly shy of the circle. So, here is this little piece is your delta and here is a small circular piece or a disk which is of radius delta around the point Z, which is completely lying in B a r.

So, this is a pictorial view and I am going to demonstrate that analytically. Choose a delta, such that delta lies between 0 and r minus absolute value or rather modulus of Z minus a and then, if a number w belongs to B Z delta, so this is a ball of a radius delta around the point Z then, the modulus of w minus Z is going to be strictly less than delta, by the definition of the ball. And then this will imply that w minus Z plus Z minus or rather, w minus a is equal to w minus Z plus Z minus a, is less than or equal to w minus Z plus the modulus of Z minus a, by the triangle inequality, which in turn is less than or equal to delta plus the modulus of Z minus a, which now is strictly less than r. Because delta was chosen to be between r minus the modulus of Z minus a and 0.

So, this portion that the modulus of w minus Z is less than delta is coming from the fact that w belongs to the ball of radius delta centered at Z. And then, this inequality here follows from this fact the choice of the delta. So, that shows that a the modulus of w minus a is strictly less than r, so such a w belongs to your B a r. So, an arbitrary w in B Z delta, ball of radius delta around Z is contained in or is an element of the ball of radius r around a. So, this implies that B Z delta is completely contained in B a r.

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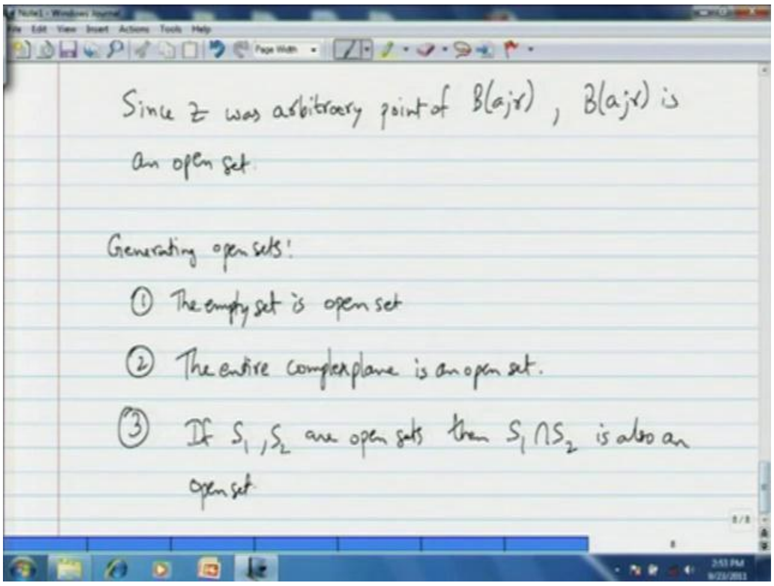


That shows that, since Z is arbitrary, Z was arbitrary point of B a r, B a r is an open set. If you choose any Z belonging to B a r, then you can come up with a delta such that B Z delta is contained in B a r. So that is takes to the definition of an open set. That confirms to the definition and so such sets are open sets.

Next, we want to see how to produce open sets. So, here is a list of ways to produce a open sets, generating open sets. One-the empty set is open. That is because the condition for openness is trivially true, there is no element to check condition upon, the empty set is trivially open. And then, the entire complex plane is open, the entire is an open set. In the first point I mean an open set, this is an open set and the entire complex plane is an open set. That is because of course if you pick any point in the complex plane, you can find a large enough disk around that point in the complex plane. So that is easy. So, the entire complex plane itself is open. And the third way to generate open sets is as follows if S 1 is an open set, S 1 comma S 2 are open sets, then S 1 intersection S 2 is also an open set.

So, why is this true? Well, if you take a point in the intersection or if you take a complex number, which is in the intersection of 2 open set, S 1 and S 2, then there is a ball of radius, ball of radius r around that point, which is contained in the intersection; where r here is the minimum of the radii of the balls, which are present in S 1 and S 2.

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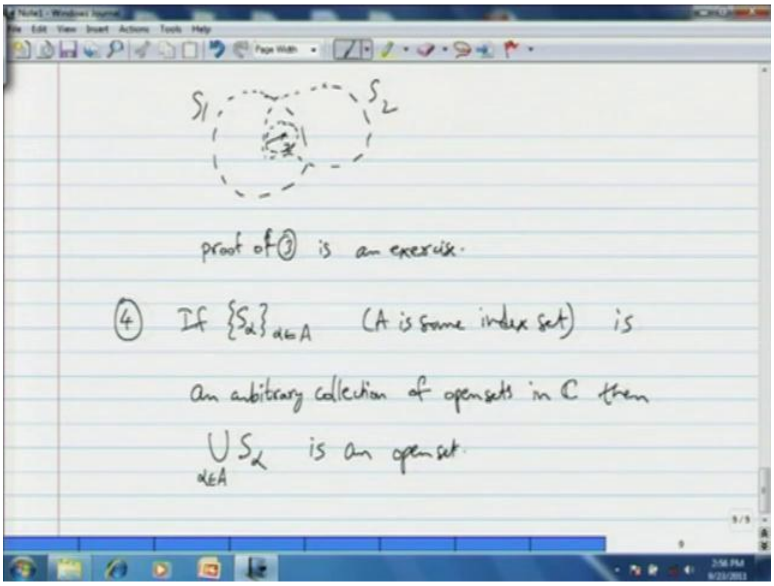


So, what I mean by that is as follows. So, here is an open set. Here, roughly speaking, here is an open set and then here is let say another open set. So, here I am giving a picture proof of this fact. So, if you pick Z in here, then you can choose a ball of large enough radius, which lies in the intersection. So, the radius of this ball can be chosen so that it is the minimum of the radius, radii of the balls around Z in the set S1 and the ball in S 2 around Z. So, a proof of this fact is left to the reader or to the viewer, rather. So, proof of 3 is a good excises, is an excises. Likewise if you have an arbitrary collection of open sets their union is also open.

So if S alpha such that alpha belongs to some index set A, A is some index set, is an arbitrary collection of open sets in the complex plane, then the union of these S alphas, alpha belonging to A is an open set. Now, the index set need not be a finite set or even countable. So, you can take an arbitrary collection of open sets and it is union will be an open set.

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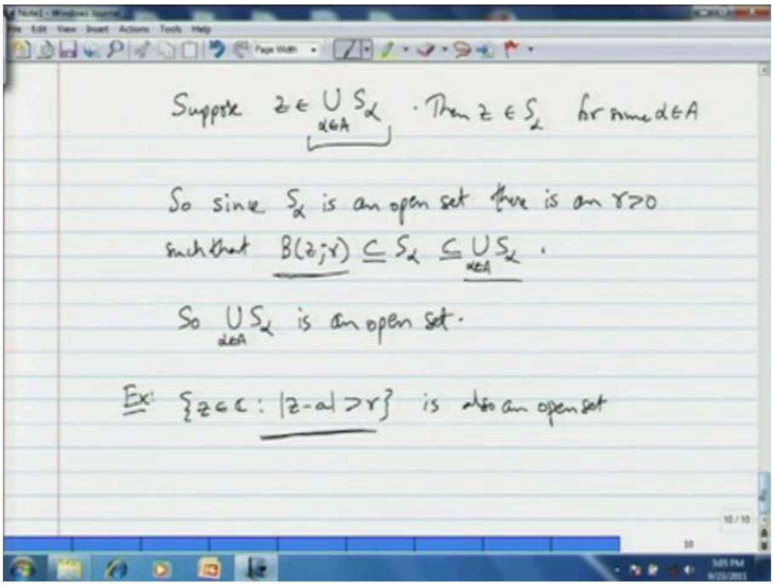


So I will present a proof of this fact. Suppose that Z is a complex number, which is in such a union, then what I have to exhibit is that there is a number r, a positive a real number r, such that a ball of radius r around Z is contained in this arbitrary union. So, since Z is in the union, then Z belongs to S alpha for some alpha belongs to A. After all this totality is the union of such as alphas. So, if Z is an element in there, then it should have come from some set S alpha. So, since S alpha is an open set there is an r positive such that B Z r is contained in S alpha that is by the definition of an open set. So, and then this S alpha of course is contained in the union of these S alphas. Since Z is an arbitrary complex number which is in the union, and we have shown that B Z r for such a choice of r is contained in the union, so we declare that this union is an open set. So, that is an easy proof.

Next, what we want to see is more open sets. So, due to these properties 1, 2, 3, 4 are these ways of generating open sets. So, if you take, so here is another example of an open set. So, similar to the argument made to show that B a r, the ball of radius r around a is open. We can argue to show that the following set of all complex numbers, such that the modulus of Z minus a strictly greater than r is also an open set. So essentially these are points which are strictly greater than r away from a. So, you can once again apply a triangle in equality to produce a delta ball around any point in the set to stay away from a, at least r away from a. So, I will let the viewer complete this excises. Once again, this the proof of this example is an exercise.

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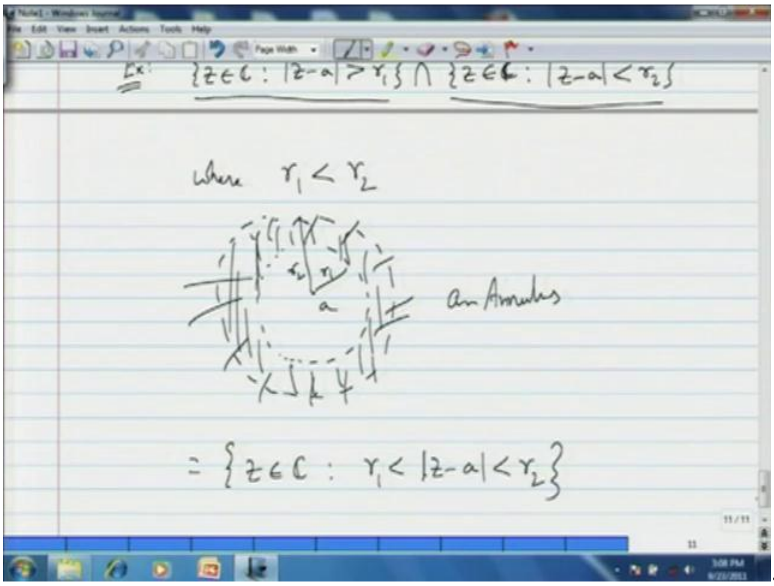


So, using this example and the earlier example I want to say that the following is also an example of an open set. So when you consider the set of all complex numbers, which are r 1 away, which are greater than r 1 away from a complex number a and then intersect it with the set of all complex numbers such that the modulus of Z minus a is strictly less than r 2, where r 1 is some chosen number strictly less than r 2, then what you get is the following. So, here is a and the first set is the set of all numbers which are outside of a circle of radius r 1, that is your first set here, and the second set here is the set of all points which are at most a distance r 2, notice r 1 is less than r 2. So, these are points, which are at most a distance r 2 away from a. so, let me use vertical lines.

So, the intersection of these two sets that is pictorially is an annulus and this can be written as set of all complex numbers which have a modulus like that between r 1 and r 2. So an annulus is essentially, of this kind is essentially an open set, is an open set. Why this is an open set? Well, it is an intersection of 2 open set, and that is a way to generate open sets. So, this is an open set.

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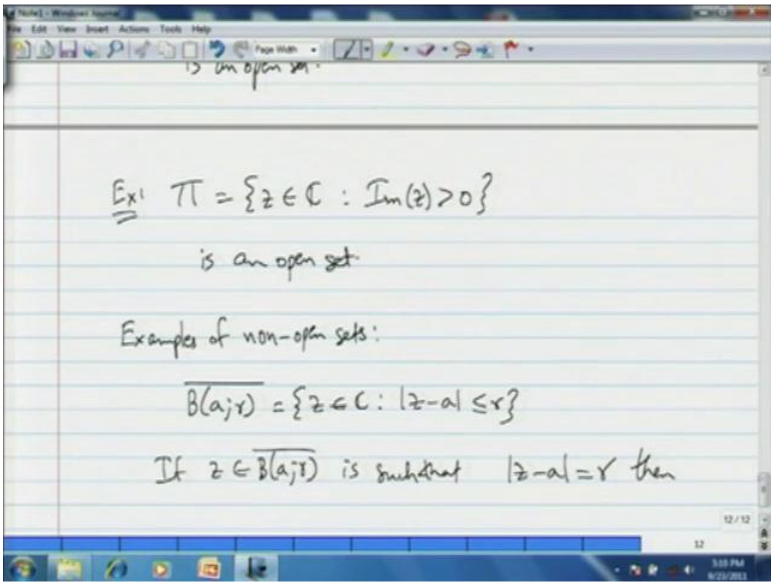


Next, we can also produce, well I wanted say that the upper half plane, set of all complex numbers such that the imaginary part of Z strictly greater than 0 is an open set. So, once again, the argument to prove that this is an open set is fairly easy and there viewer is encouraged to look at how to prove such a fact. Essentially the idea is that the imaginary parts of points in this set are strictly greater than the 0. So, if you pick any point, which is in the set, then you can find points around it whose imaginary part is still strictly greater than 0. So, the proof of this example is also a good exercise. So next, I want to give examples of sets which are not open. So, we have seen a close disk of closed ball of radius r around a complex number a.

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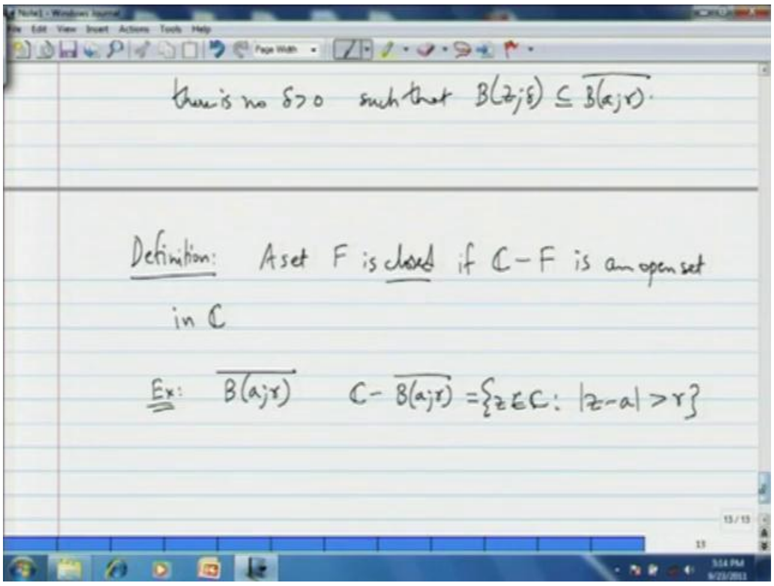


So, this is the set and I claim that this is not an open set, because if is Z belongs to B a r bar, is such that absolute or the modulus of Z minus a is equal to r, then there is no BZ delta contained in. so, I want to say that there is no delta positive such that B Z delta is contained in B a r bar. It is because if you consider any point which is exactly r away from the point a then, now if you draw any disk of radius delta around it, any ball of radius delta around it, there are points which are outside, lying outside this ball B a r, because there are always points, which are at a distance slightly greater than r in that b Z delta, for any choice of positive delta. So, this set is not an open set; this set B a r bar is not an open set.

So next, I want to define what is called as a closed set. So, before we move on I want to remark once again that these open sets essentially give you a room to manure, the definitions says that there is certain positive distance around each point in the set, so that you can move around in that ball. So, there is some elbow room for each of these points in the set and these open sets play the role of open intervals for real analysis. So, the open sets in complex analysis are in the complex plane are what open intervals are to real line.

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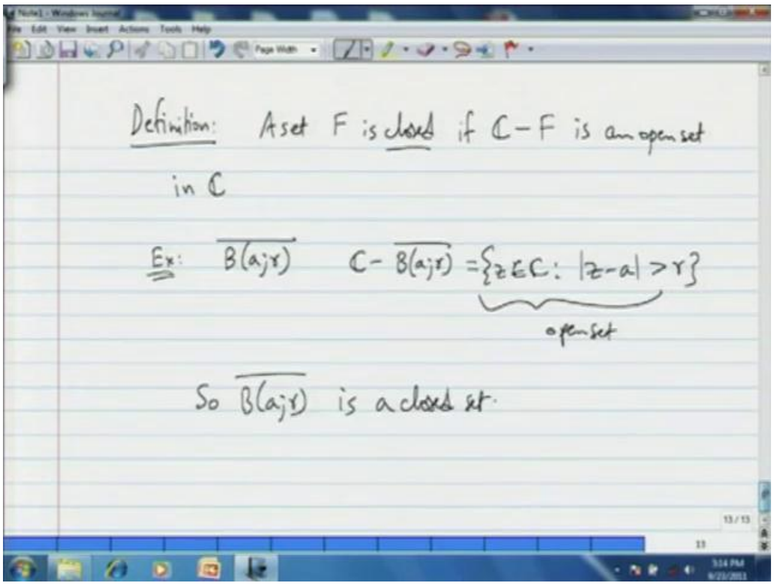


So next, let us define a closed set. So, a set S or let us say F is closed if C minus F is an open set in C. So, if the complement of a set is an open set in C, then such a set is called a closed set. So, for example, if you look at B a r bar once again, this is essentially or let say the complement of this in the complex plane is essentially the set of all points in the complex plane such that the modulus of Z minus a is now strictly greater than r, which we said is an open set. So, this is an open set. So, B a r bar is a closed set. And we will see more examples as we go along.

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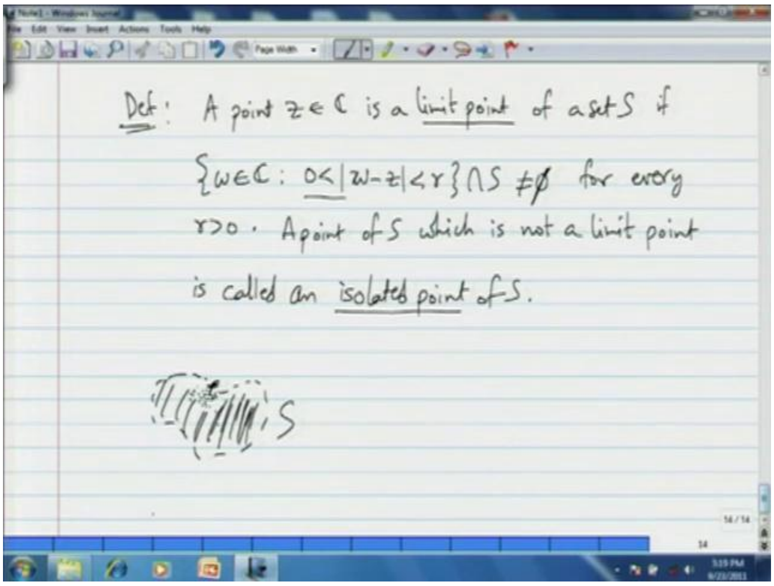

Next, I want to define a limit point. A point Z in the complex plane is a limit point of the of a set S, if the intersection of the set of all the complex numbers, which are, or let me say, a set of all complex numbers I have used Z, so a set of all complex numbers, which are at most r away in the section this S is non empty for every r positive. So, let me complete the definition; a point of S, which is not a limit point is called an isolated point of S. So, a limit point is essentially a point such that the there are points in the set S which come closer and closer to that point.

So this definition tells you that a pictorially this is what is happening. So here is a point and there is a set and then, here is a point and there are points. So here is a set S and then there are points, which are very close. So, the set S is what is contained inside this kind of curve that I have drawn. So, there are points inside the set, which are coming closer and closer to this cross mark point. So, such a point is called a limit point (( )). So the condition that 0 is strictly less than the modulus of w minus Z is removing the trivial case that any point of the set just becomes a limit point. Because if that condition were not to be there, then the modulus of Z minus Z for any Z belongs to S is equal to 0 and so that intersection S will always been non empty. So, this is the empty set, that I apologize, this definition this is not 0 but, the empty set. Please make a note of that.

So, this condition will be trivially met and we do not want to make every point of the set a limit point. The idea of the limit point is that there are really points in the set, which are coming closer and closer genuinely coming closer and closer to the point of interest, which we want to call as a limit point. So, that is a limit point and if a point of the set is such that it is not a limit point then that will be called an isolated point. So, I will give examples.

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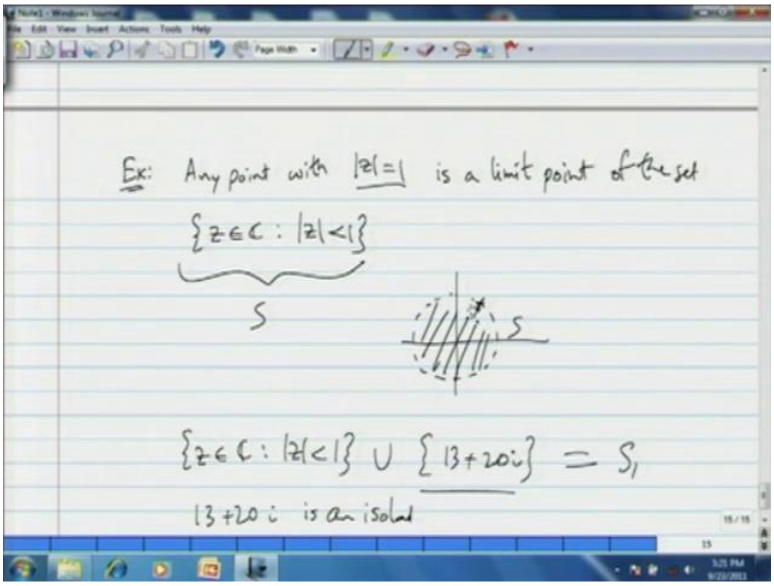


Any point with the modulus of Z equals 1 is a limit point of the set of all Z belongs to C such that absolute Z strictly less than 1. So, here I have picked the standard ball of radius 1 around the origin, so that is your set S, that is your set S and if you pick any point with absolute or with a modulus of Z equals to 1, then you can get arbitrarily close to that point using points in the set S. So that is, so any such point is a limit point. If you take the set Z belongs to C such that absolute Z less than 1 union a far away point, let say 13 plus 20i, then 13 plus 20i, let me call this set as S1, is an isolated point of S1. So, we will consider yet another example.

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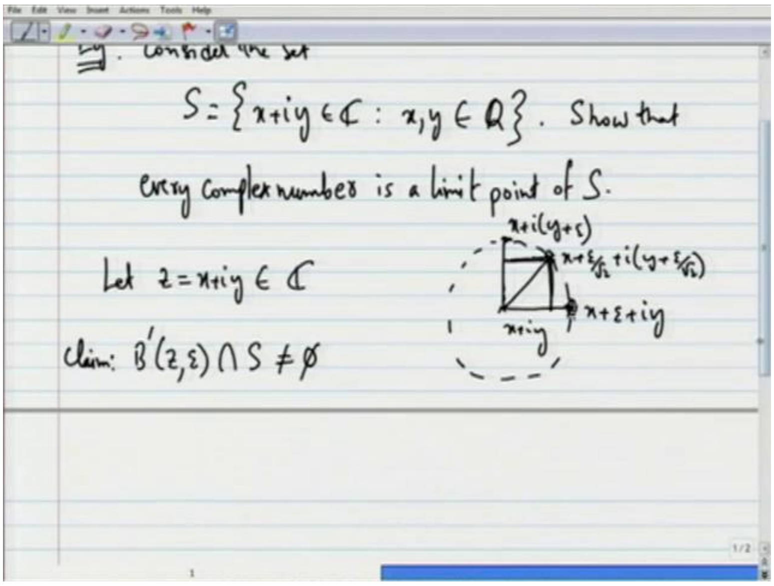


So here is yet another example of limit points. So, consider the set S equals set of all x plus i y in the complex plane, x y are real numbers, such that x comma y are actually a rational numbers. So these are complex numbers with rational coordinates. What we want to show that every complex number is actually a limit point of the set. So, it is easy to visualize this that q sort of all the points with the rational coordinates, they are actually dense in the complex plane. So, here is the proof of this fact. So every complex number is actually a limit point of the set. So here is the proof of this fact. Let you pick a complex number Z equals x plus iy.

Let x plus iy be an arbitrary, so here is the picture you have x plus i y and then you take any epsilon ball around it and then, this is a horizontal line; the coordinates of this missing point, which is on the boundary is x plus epsilon plus i y and the coordinates of this point, which is on the vertical line is x plus i times y plus epsilon. And then the coordinates of a point on the 45 degree line to the horizontal, the coordinates of this missing point which is on the boundary is actually x epsilon by root 2 plus i times y plus epsilon by root 2.

So, the reason I am considering this forty five degree line is that notice that if I take this square of a side epsilon by root 2, then every point in the interior of the square is completely contained in this epsilon ball. Now, what we are going to see is that the deleted neighborhood of this Z intersection the set S is actually a non empty that will make Z limit point of this set S. So, this is true for any epsilon and then that will make this Z a limit point of the set S.

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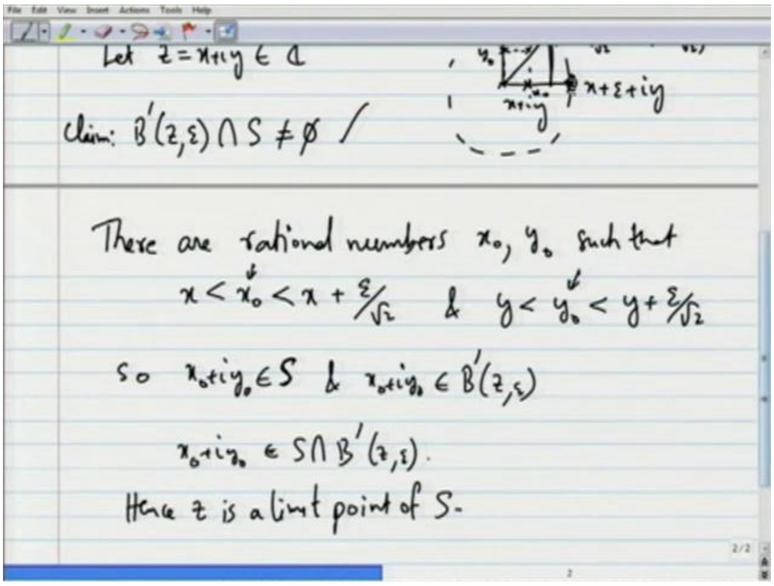


So this follows from the fact that rational numbers are dens and real numbers. So, there are real numbers or more particularly there are rational numbers x naught comma y naught such that x is strictly less than x naught, is strictly less than x plus epsilon by root 2 and y is strictly less than y naught, is strictly less than y plus epsilon by root 2. So, I already mentioned why I am considering x plus epsilon by root 2 and y plus epsilon by root 2.

So this x naught y naught lies somewhere in this picture. They will lie somewhere like that. This is y naught, this is x naught. Let us say the x naught plus i y naught is in the inside of this square, it is in the interior of that square region. So that, x naught plus i y naught is definitely a point in the set S. It is definitely a point in the set S because it is within the epsilon neighborhood; not only that and x naught plus i y naught is also in the B prime of Z epsilon. So x naught plus i y naught, sorry is in the set S because it has rational coordinates x naught and y naught are rational numbers and x naught plus i y naught is in the deleted neighborhood because it is inside the square.

So, in total you have that x naught plus i y naught belongs to S intersection B prime of Z epsilon, so the claim that this is non empty is true, that makes. So, hence Z the arbitrary complex number Z is a limit point of S, so that proves that every complex number is a limit point of this set S.