Complex Analysis
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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 5
Residue Theorem and Applications

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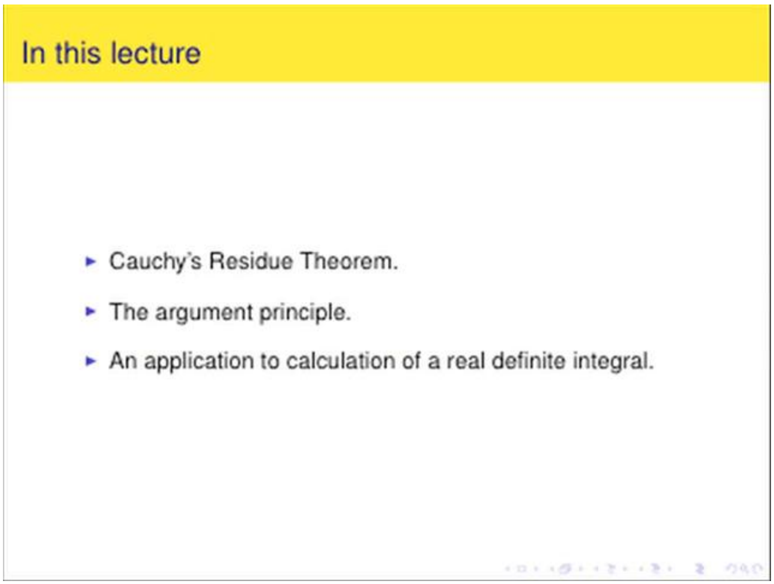

Hello viewers, in this session, we will see Cauchy's residue theorem and applications of Cauchy's residue theorem. So, we will conclude this course with the applications of Cauchy's residue theorem. So, we have seen that the coefficients in the Laurent series expansion are unique.

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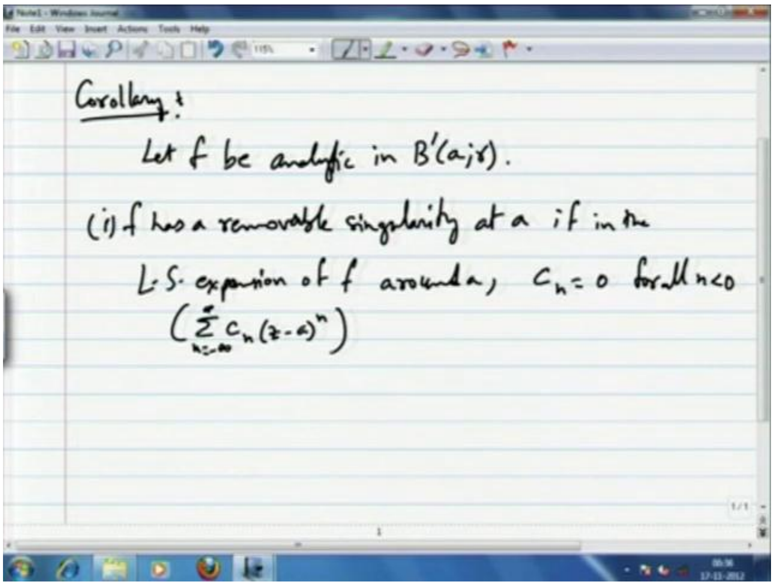

And then we have the following corollary to Laurent theorem. So, let f be analytic in b prime a, r and so it will have a Laurent series expansion in b prime a, r. So, the behavior of f can be predicted around a f has 1, f has a removable singularity, at a if in the Laurent series expansion of f around a, c n is equal to 0 for all n less than 0. So, I am assuming the Laurent series expansion as the form sigma c n z minus a power n, n from minus infinity to infinity. So, if the coefficients with n negative are all 0, then the nature of singularity at a is a removable which means, f can be made to be an analytic function in all of b a, r.

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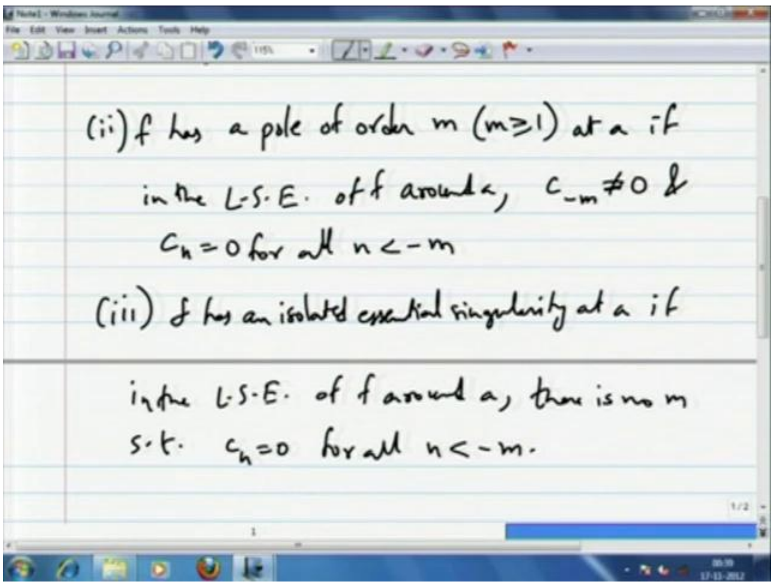

So, f has likewise f has a pole of order m, m greater then are equal to 1 at a if in the L S expansion, I will just L S E for Laurent series expansion of f around a, c minus m is non zero. This particular m is non zero and c n is 0 for all n less than are equal to sorry, n strictly less than minus m.

So, we have seen such a from earlier so, I will go back to the following that we have proved few or couple of sessions, I go. So, if f had a pole of order m we showed that f of z has this form and we called this part singular part. So, you see that this c naught is the c minus m under discussion currently. And then everything I mean the coefficients of z minus a power minus m minus one onwards, on the negative side are all 0. So, this is exactly what we are stating now, we are saying that if f has a I mean f has a pole of order m at a, if in the Laurent series expansion c minus minus n is 0 for all minus n less than minus m.

So, that is two and then three it has an isolated. So, f has an isolated essential singularity at a if in the L S E of f around a, there is no m such that c n is equal to 0 for all n less than minus m so, or you short it is the opposite of 2. So, it is the opposite in sense that you have infinitely many non-zero c n’s occurring on the negative side, or in the negative integers n so, that is an isolated essential singularity. So, that is just a restatement and it follows from the restatement of what we have already seen, and it follows from Laurent series are Laurent’s theorem, so it is a corollary.

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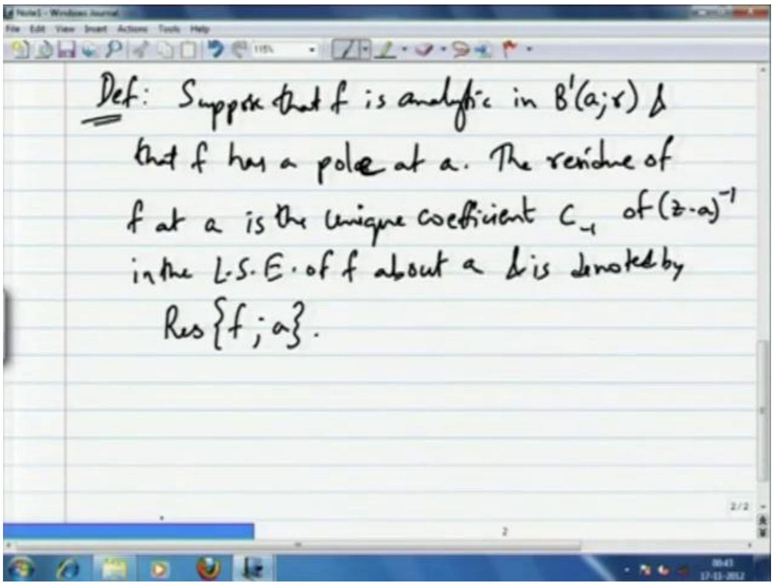

And we will make a definition as follows. So, suppose that f is analytic in B prime a, r in the deleted neighborhood of a and that f has a pole at a, then the residue of f at a is the unique coefficient c minus 1 of z minus a power minus 1 in the Laurent series expansion of f about a and is denoted by R E S residue of f at a. So, we can actually make this definition even for an essential singularity, but we will confine ourselves to poles of f at a that is the residue of f at a.

So, going back to this form once again the c m minus 1 will be called the residue of f at a. So, look at this form of f and then that is the residue. So, we proved I mean using the Laurent's theorem, we proved that such a, c, m minus 1 has a definite form and we showed that is unique. So, we will call that the residue of f at a.

What is the use of that residue once again I will go back to that session, where we saw the following lemma. We said that that c m minus 1 becomes important when we try to integrate f around gamma, a simple closed curve gamma. So, where this singularity a lies inside of that gamma so, then the coefficient of 1 by z minus a is the only term that survives when you integrate f over gamma. So, that is the content of the following theorem in this restricted sense.

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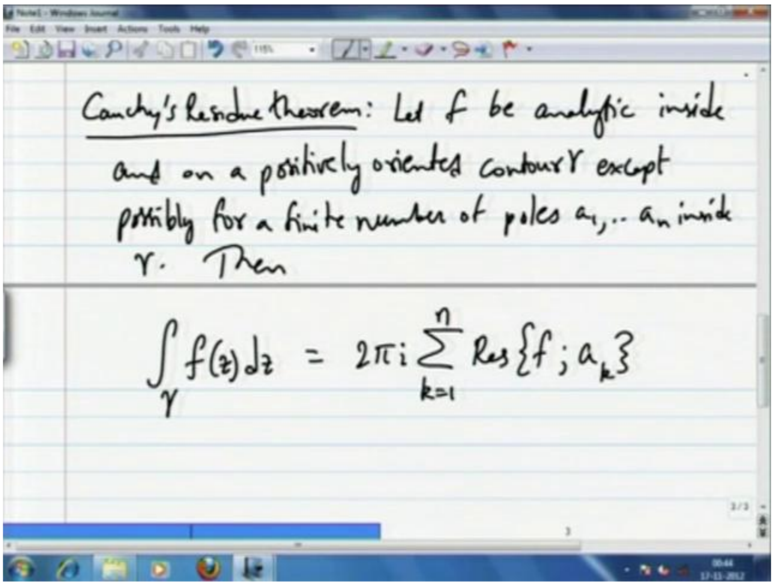

So, we call this theorem the Cauchy's residue theorem or a version of it really confined only to poles of f. So, let f be analytic inside and on a positively oriented contour gamma accept possibly, for a finite number of poles a 1 through a n inside gamma. So, there we will not allow the poles to lie on gamma inside gamma.

So, then the integration over gamma of f of z d z is going to give you, 2 pi i times the sum of residues k equals 1 through n of the residues of f at the points, a k at the singularities a k, or in this case poles. So, I will pause here to mention that the Cauchy's residue theorem holds in general, even if these singularities are essential, but I have stated here the Cauchy's residue theorem only for poles. And we will prove a Cauchy's residue theorem in this restricted sense and see its applications.

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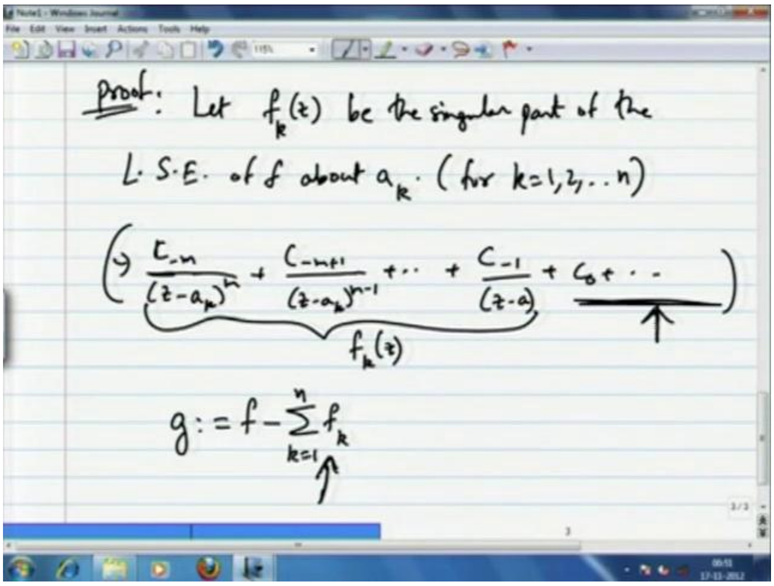

So, here is a proof of a Cauchy's residue theorem. The proof involves the technique we have used to prove this lemma, let me go back to the previous session once again. So, it essentially involves these techniques of expressing f in this manner. So, this is the Laurent series expansion of f in the neighborhood of a and what was important was that this f 1 of z was or has a removable singularity at a.

At most a removable singularity at a and it can be removed so that is the idea. So, what we will do is we will use the Laurent series expansion of f, around each of the singularity and then try to come up with an analytic function on all this inside of gamma. So, here is the technique let f k of z be the singular part of the Laurent series expansion of f about a k. So, recall the singular part is of the form 1 by or sorry c minus k by z minus I apologize I should not take k.

So, it depends on the order of a pole at the point a k. So, let us for the time being assumed its some m so, then it looks like c minus m z minus a k power m etcetera plus c minus m minus 1 minus m plus 1, rather z minus a k power m, m minus 1 etcetera plus the c minus 1 by z minus a plus etcetera c naught plus etcetera. So, this is the Laurent series expansion about a k. We are assuming I mean if a k has a, I mean a k is the pole of f of order m at a k. So, this is the singular part and this we are calling as f k of z for each a k we will do this. So, in a neighborhood of a k this expansion is valid and we will take this singular part f k of z.

And then now notice that f k of z is a function which has a singularity at a k, which has ah a pole at a k it is a function in it is own right, which has a which is a pole at a k of order m and it has no other singularity in the whole of the complex plane. In particular f k is a function which is defined on and inside gamma and except for the singularity at the point a k. So, let f k be the singular part of the Laurent series expansion of f about a k.

So, what we can do is for so this is for k equals 1, 2 so on until n. so then what we can do is we will construct a new function g equals f of z. So, g equals f minus sigma k equals one through n of f k of well f k. So like I mentioned each of these has singularities at respective a k's and they are analytic otherwise so on. All of the complex planes except the point a k these, all these functions are analytic. So, f minus this singular part in a neighborhood of a k will look like that and then and so.

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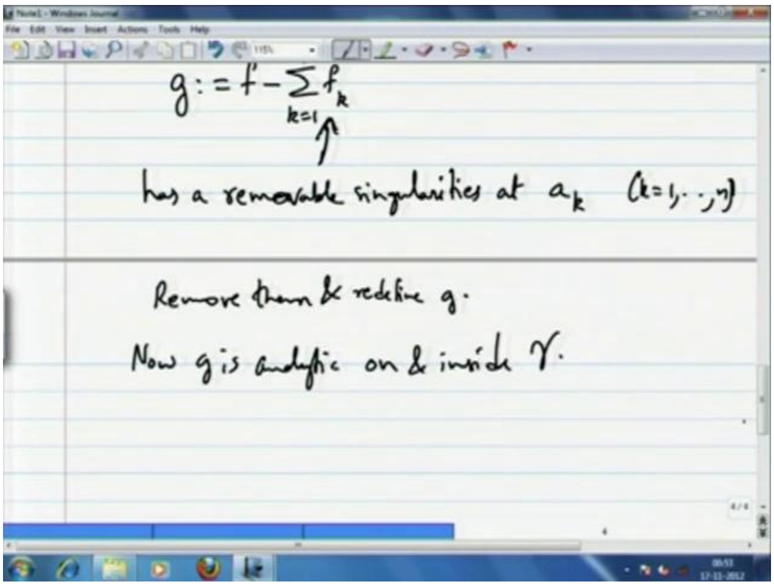

This has a removable singularity, has removable singularities at a k at each of the a k, k equals 1 through n. So, you can remove them we know how to redefine g so, remove them we will we know how to remove a removable singularities, we will redefine g of a k for example, to be the limit as z goes to a k of g of a k of g of z. So, we will remove them and redefine g and redefine g. So, we will exchange g with that new g the redefinition of g and then so. Now, g is analytic after this redefinition analytic on and inside gamma, which means it is analytic on an open set on and inside containing gamma and the inside of gamma.

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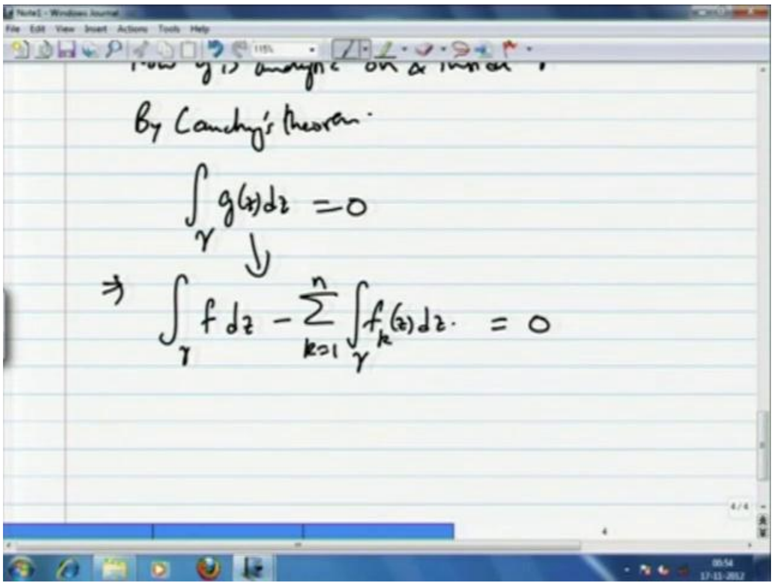

So, by Cauchy's theorem we know that by Cauchy's theorem. Now, we know that the integration over gamma of g of z d z is 0. What that implies is that the integration over gamma of well g is f minus sigma f k. So, integration over gamma of f d z minus sigma k equals 1 through n. So, I am actually exchanging the integration and the summation because this is the finite sum, we can definitely exchange the integration and the summation here, integration or gamma of f k of z d z that is equal to 0 this is basically your g.

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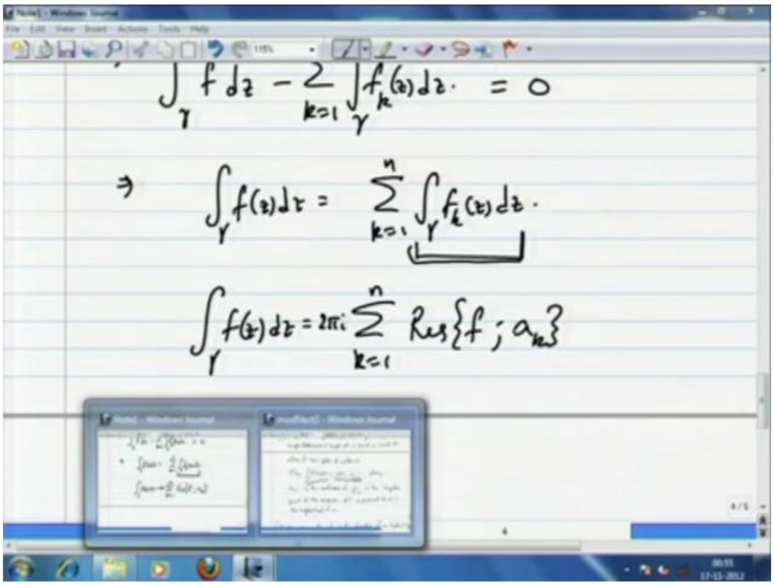

And then so, this implies that the integration over gamma of f of z d z is equal to sigma k equals 1 through n of integration over gamma f k of z d z, but we know something about this integration, we have already shown earlier this lemma. So, this lemma we have proved that this is nothing, but 2 pi i times that coefficient of z minus a power minus 1, which is c minus 1 in the current context. So, this is equal to sigma k equals 1 through n we define c minus 1 to be the residue of f at a k.

So, the integration over gamma of f of z d z is hence, equal to the sum of the residues of f at a k I apologize, I need a 2 pi i it is 2 pi i times that is a what of f. So, it is 2 pi i times the coefficient of z minus a power minus 1. So, I have 2 pi i times that so that is the proof of this restricted version of Cauchy’s theorem. Cauchy’s residue theorem and we will today see some applications of Cauchy’s residue theorem.

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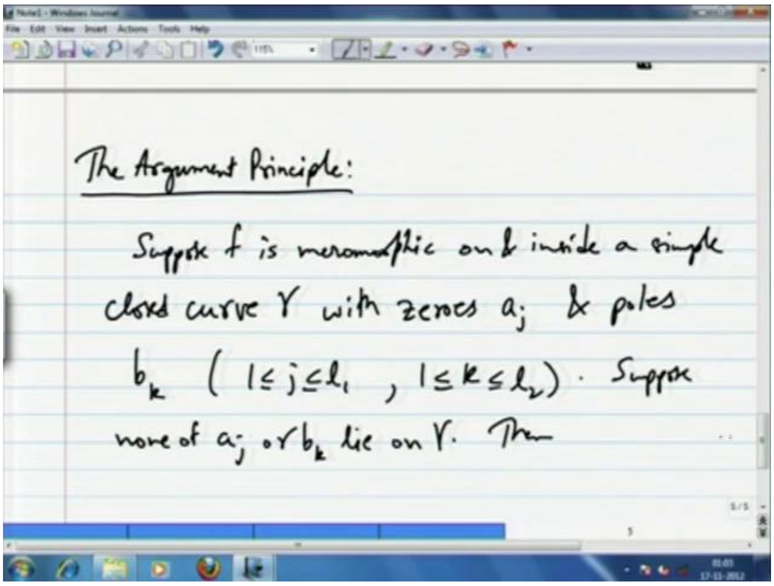

The first application to Cauchy's residue theorem is the argument principle so, the argument principle. So, we have already seen the counting zeros theorem and this argument principle is a version of it, is as a modified version of it. So, here is the statement. So, suppose f is a meromorphic on and inside simple closed curve gamma with zeros a j and poles b k, where j runs over some index and k runs over some finite index. So, let say one less than are equal to j less than are equal to l 1 and 1 less than are equal to k less than are equal to l 2, the l 1 and l 2 are unimportant except that there are finite number of poles and zeros inside of gamma. Suppose none of a j or b k lie on gamma.

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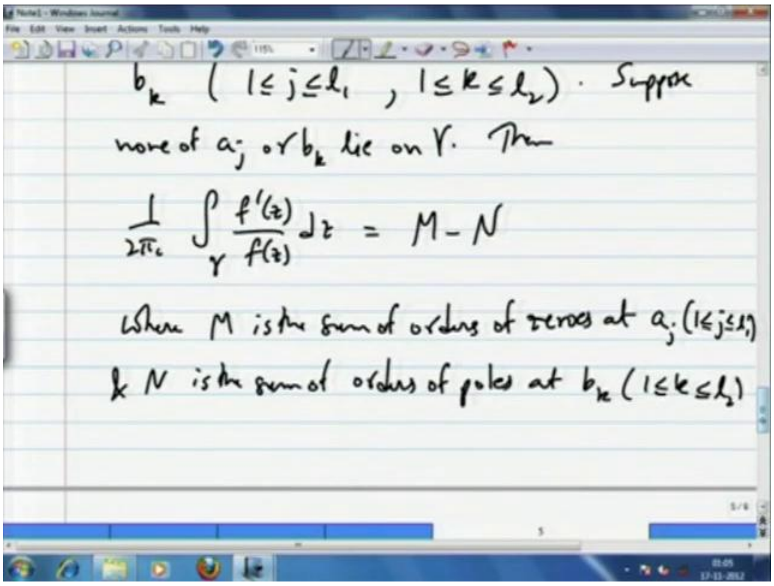

So, with this assumption then the integration 1 by 2 pi i times the integration over gamma of f prime of z by f of z d z is going to give you capital M minus capital N. Where capital M is the sum of orders of zeros at a k at a j, j form 1 through l 1 and N is the sum of orders of poles at b k, k runs from 1 through l 2. So, you add up all the orders at each of the pole b k that is capital N, and add the some add the orders of zero set each of the a j and that will be your capital M.

So, we have already seen the counting zeros theorem, we have uncounted this integral and there we showed that 1 by 2 pi i times integration over gamma of f prime by f d z is equal to capital M, if f is analytic on and inside gamma. So, here it is a modified version if we have poles in addition to zeros. So, we are going to show that this integral will gives us M minus N.

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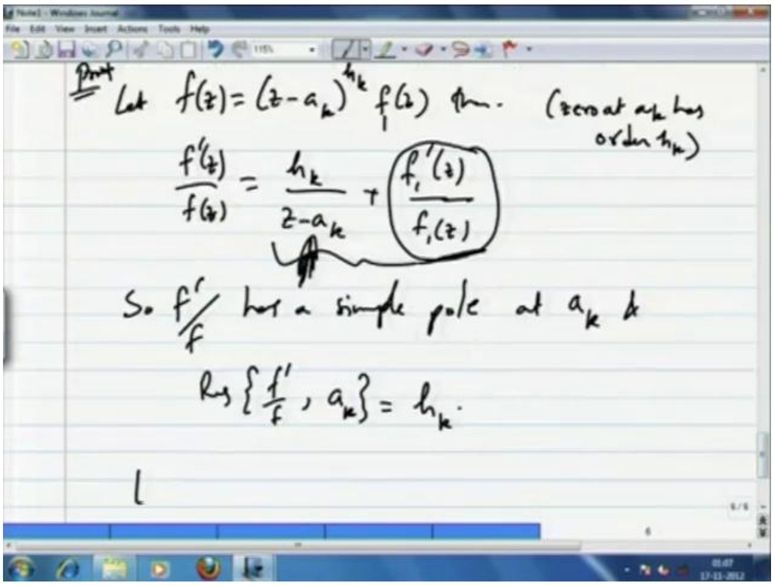

So, proof it is given that f of z has a 0 at a j. So, f of z let us suppose is equal to z minus a power h or let me call that a 1 power h or a k power h in general. So, f of z is equal to let f of z equal z minus a k power h k times some f 1 of z. So, in a neighborhood of a k we can definitely write f of z in the following fashion then, we know that f prime over f we did this calculation earlier. So, this gives us h k by z minus a k plus f 1 prime of z divided by f 1 of z. So, here we are assuming that 0 at a k has order h k.

So, what that means is f 1 of a k is not 0 so f prime by f has a simple pole at a k and the residue of f prime by f at a k is going to be h k that is the residue. So, I mean this is this function is going to be analytic. So, this is this has it is Taylor series expansion in the neighborhood of a k. So, this is the only singular part so that gives us h k that is gives us that the residue is h k. Likewise, if that works for any k 1 through l 1.

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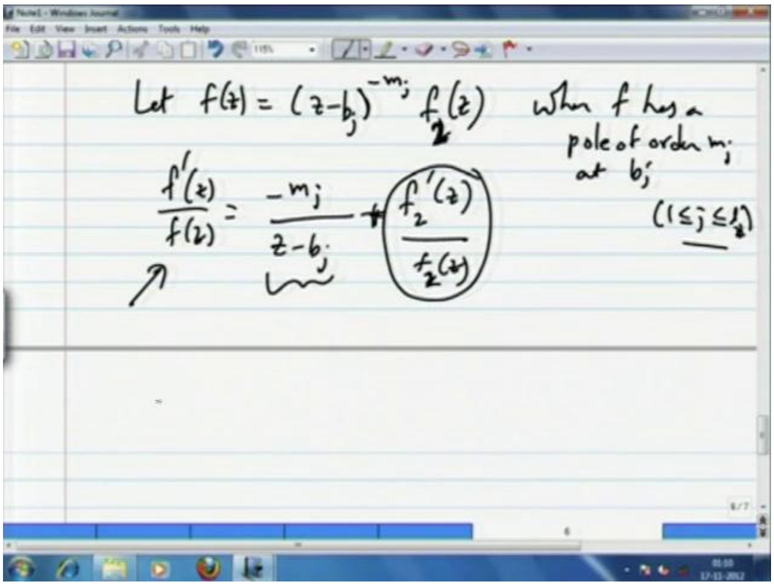

And then let f of z is neighborhood of pole will do the following let f of z equal z minus b j power minus m j times f 2 of z, where f has pole of order m j at b j I should have use b k's and a j does not matter so it is one and the same. So, here we will let k run through 1 from 1 through l 1 and j run from 1 through l 2. So, this f 2 need not be same for all of these j's.

So, this f 2 is different for each of this b j's, but nevertheless what we have is f of z is of this form and this gives that f prime by f, f prime of z by f of z simple calculation shows that, this is minus m j divided by z minus b j plus f 2 prime of z divided by f 2 of z. And this expression is analytic in a neighborhood of b j. So, it has it is own Taylor series expansion. What that means, is this is the only singular part of the function f prime over f.

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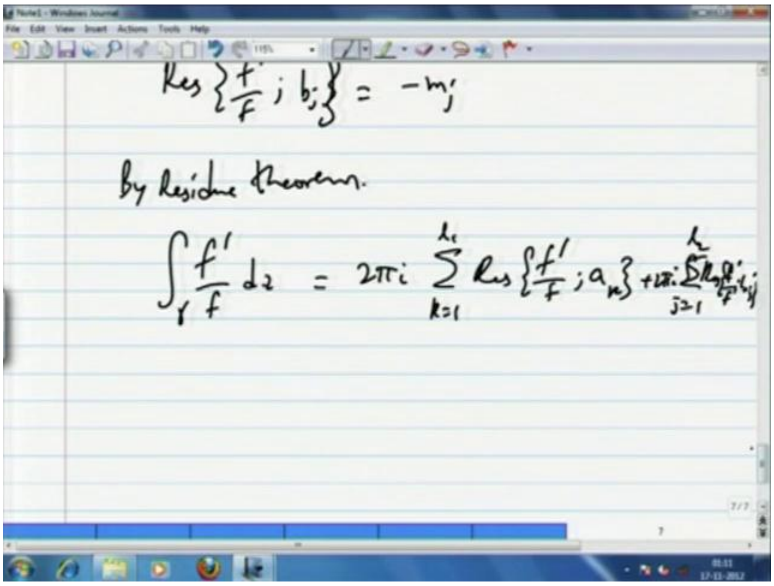

So, that gives us that the residue of f prime over f at that point b j is going to be minus m j, so minus m j so that is it is so it is the negative of the order of the pole at b j. So, by residue theorem then, we know that the integration over gamma of f prime over f d z is going to give us this 2 pi i times, the sum of residues of f prime over f at a k, k runs from one through l 1. And then plus 2 pi i times the residues sigma residue of f prime over f at b j, j runs from 1 through l 2 so that gives us, this is 2 pi i times the sum of orders of zeros capital M and the sum of orders of poles is capital N. So, that gives us M minus N which is what we want. So, it is the proof of the argument principle.

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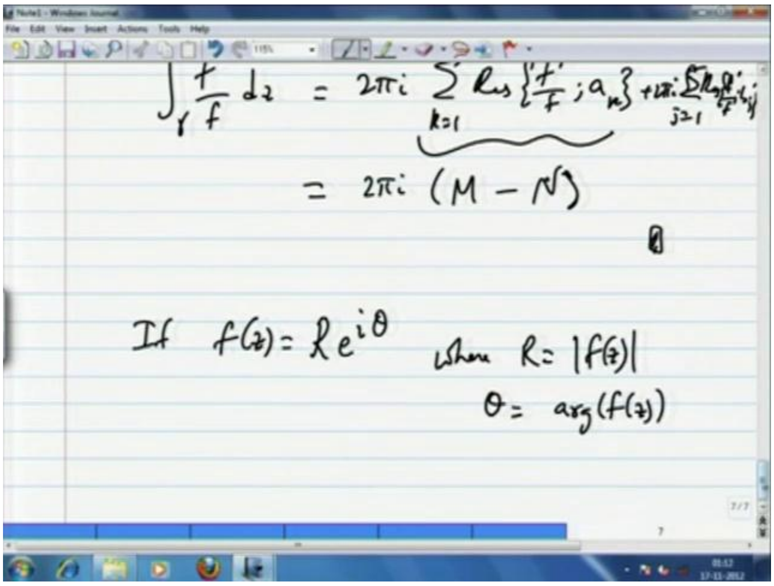

Now, an inside into why this is called the argument principle. So, if f z is written as R e power i theta, where R of course, is the modulus of f of z and then theta is the argument this f of z.

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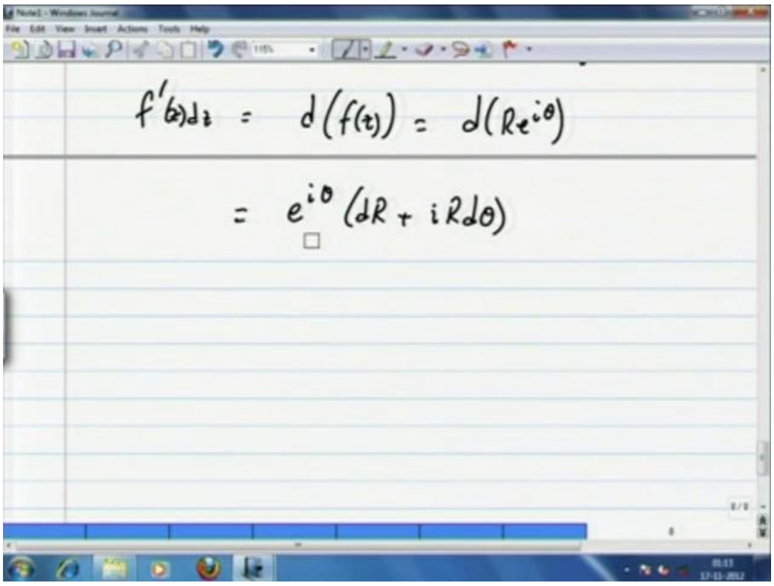

So, f prime, what is f prime of z d z so, that is thought of as d of f of z right and this is d of R e power i theta. Now, because f is R e power i theta. So, this is e power i theta times d R plus i times R d theta. So, once differentiating R i get d R power i theta once differentiating e power i theta i get i times R e power i theta d theta. So, I can extract an e power i theta and then I get this form.

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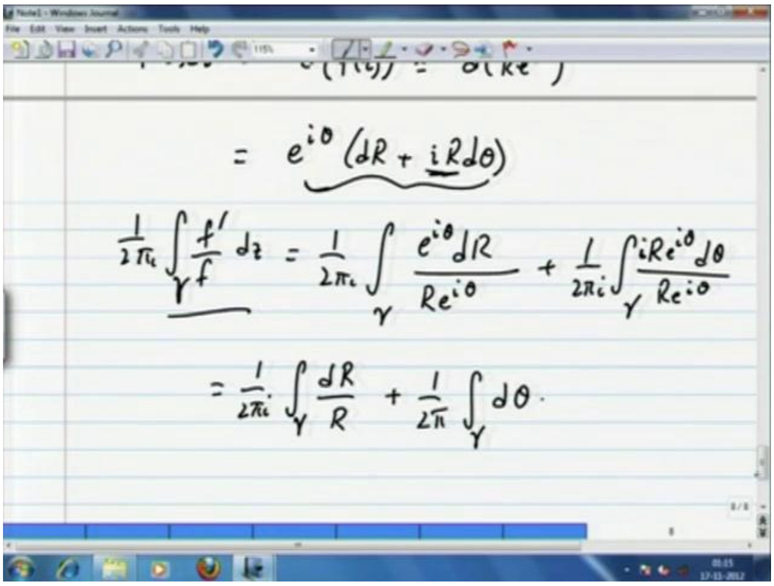

So, 1 by 2 pi i times the integration over this gamma of f prime over f d z is really 1 by 2 pi i times integration over. So, f prime d z I am using this form for f prime d z so, f prime d z will give me e power i theta d R divided by f of z, which is R e power i theta over gamma plus 1 by 2 pi i times integration over gamma of R e power i theta d theta divided by R e power i theta.

So, I am just using this and separating this integral into 2 pieces. So, I guess I have a i here sorry, this is i R. So, I get i R e power i theta at d theta. So, then this gives me after some cancelations, this gives me 1 by 2 pi i times integration over gamma of d R by R plus 1 by 2 pi i. Well i cancels i so, I have 1 by 2 pi times integration over gamma of d theta.

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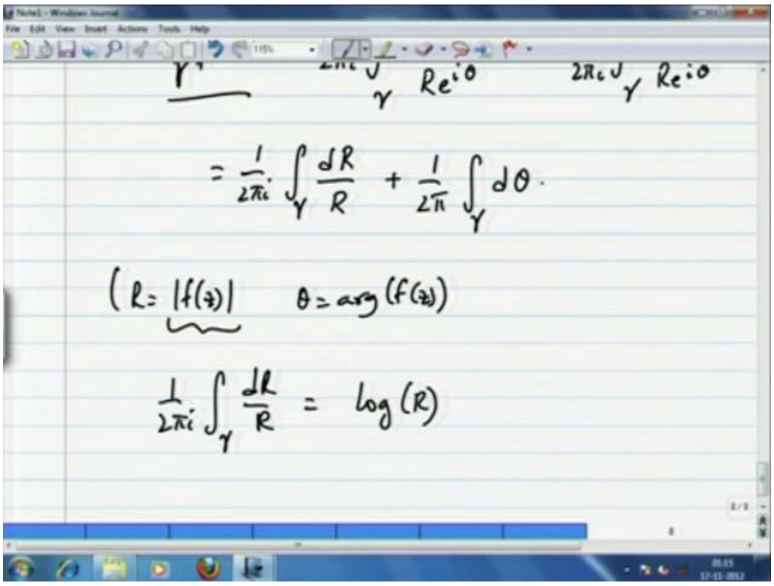

Recall R is the modulus of f of z and then theta is the argument of f of z, and this gives us 1 by 2 pi i times integration over gamma of d R by R, this is nothing but the logarithm of R as it changes. The logarithm of modulus of f of z as z varies over this gamma.

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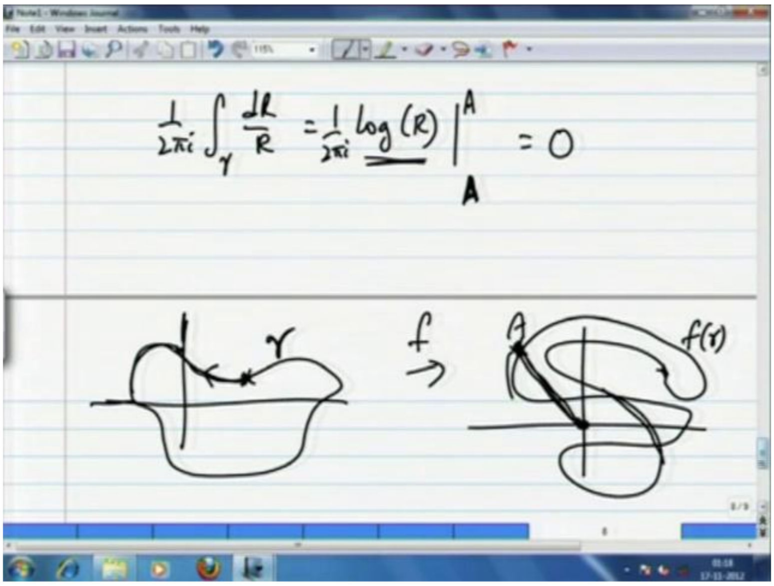

So, you have here is the picture so, you have gamma some simple closed curve somewhere here, and then your modulus of f of z well firstly, this gamma is taken by f to some closed pair like that, impossibly with self intersections does not matter. What is important is no 0 of f lies on gamma. What that means is f of z is not 0. So, 0 is not in the image this curve, which is the image of gamma under f does not pass through 0. So, no 0 of f lies on gamma so, this is your f of gamma.

So, the modulus of f of z on gamma so when it starts, it starts with some point this gamma is oriented in some fashion. So, when you start here possibly you start here, let say and as gamma is traversed, this curve is traversed this whole curve is traversed and then when you reach back this point, you reach back this point and then modulus of f of z whatever, this is this is the modulus of f of z this is length of this segment.

So, you come back to that point and this is log R between that point and that point. So, if I call this a so pardon my sloppiness this is between a and a. So, since the log of since the modulus of f of z returns to the same point this gives us a 0. So, this is a 0 1 by 2 pi i times this well, but that is the integration of d R by R over gamma is 0.

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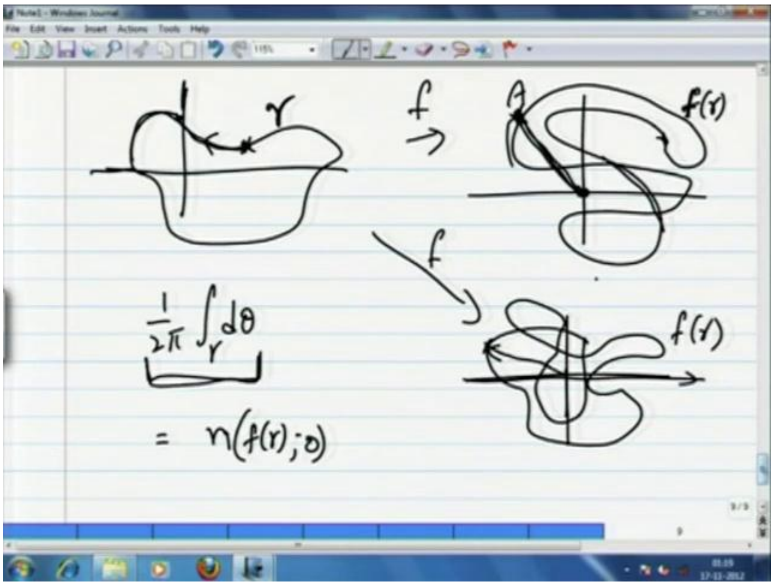

But we cannot say the same about the other integral, which is left out 1 by 2 pi times integration over gamma of d theta. This picture might given impression that I assume that f of gamma does not surround 0, but that is unnecessary for the above proof. So, f of gamma could look like that as well does not matter so, does not matter. What I said above holds true. However, f of gamma looks like so in the case of 1 by 2 pi times integration over gamma of a d theta. So, here if you start at a certain point like this so then you keep track of you will sort of keep track of how, the argument is changing as you run along gamma in that domain.

So, f of gamma will be tracing this curve so, you will keep track of how the argument is changing sort of when you rotate around 0 roughly speaking once, you will pick an argument which is 2 pi. So, it depends on the index of f of gamma around 0. So, this is actually the index of so, we define the index anyway so at that this that index of f of gamma around 0. So, this picks up this number picks up change in the argument, as gamma is traversed, the change in the argument of f of gamma.

So, this integral in a sense picks up that difference in the argument, that difference in the argument which is the, which is the index of f of gamma around 0 and hence this is called the argument principle. So, this theorem has that name because of this phenomenon so, that is the first application of Cauchy's residue theorem. So, as the next application we will consider applying Cauchy's residue theorem, to evaluate some definite integrals definite real integrals. So, these are improper definite integrals.

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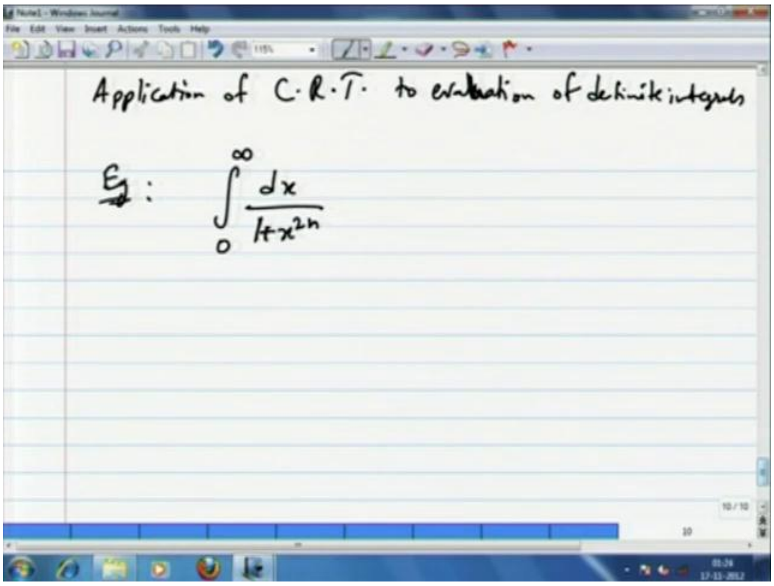

So, this is a application of Cauchy's residue theorem to evaluation of definite integrals. So, let me start with the following example so, suppose I want to evaluate 0 to infinity d x by 1 plus x power 2 n. So, when studying one variable calculus or of real variable so, evaluating this integral might be very difficult, but using complex analysis we can actually or by using the Cauchy's residue theorem, in particular we can actually evaluate this integral much easily.

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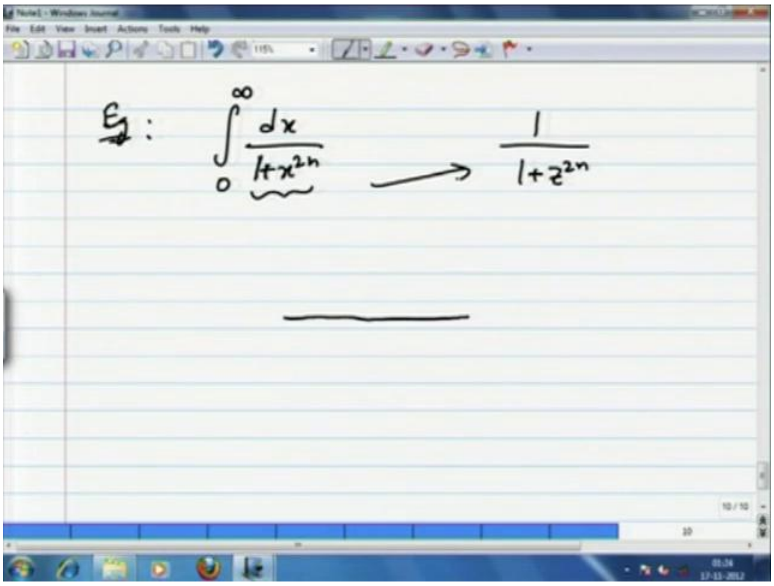

So, here is the strategy what we will do is the key is to pick the right kind of contour on which to integrate the function 1 plus z power 2 n. So, inspired by this function 1 plus x power 2 n, we pick up the function 1 plus z power 2 n.

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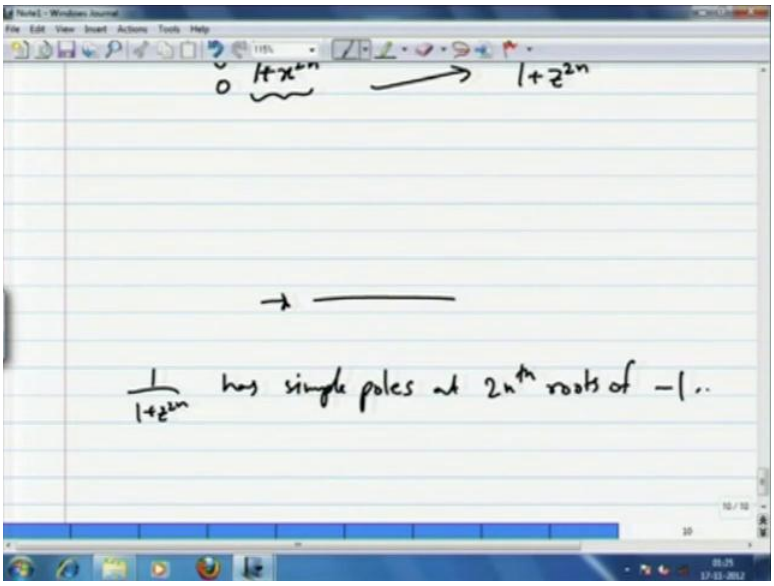

1 by 1 plus z power 2 n rather has simple poles at 2 nth roots of minus 1.

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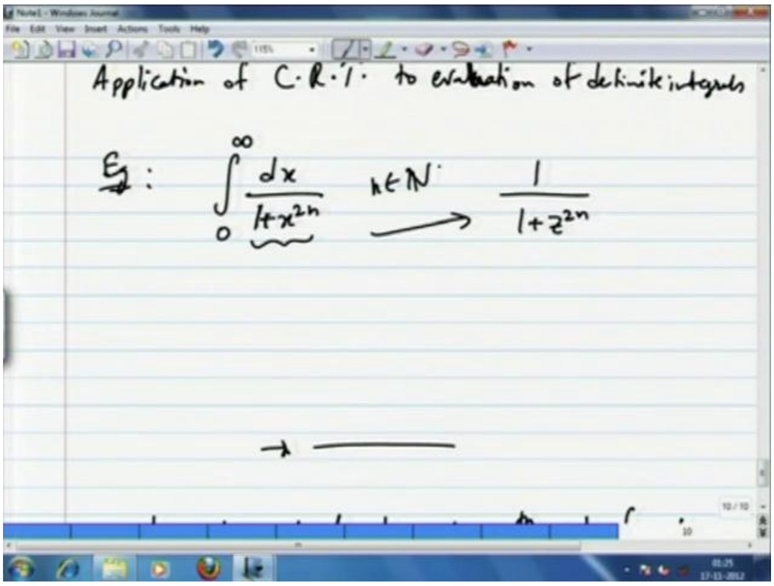

So, I should have mentioned that n belongs to n, n can be any natural number. So, it has I mean it has simple poles at 2 nth roots of minus 1 and n of them occur in the upper half plane and n of them occur in the lower half plane. So, excluding the simple case z is equal to or n is equal to 1, we will consider n greater than 1. So, in that case n of them lie in the upper half plane and none of them lie in the lower half plane.

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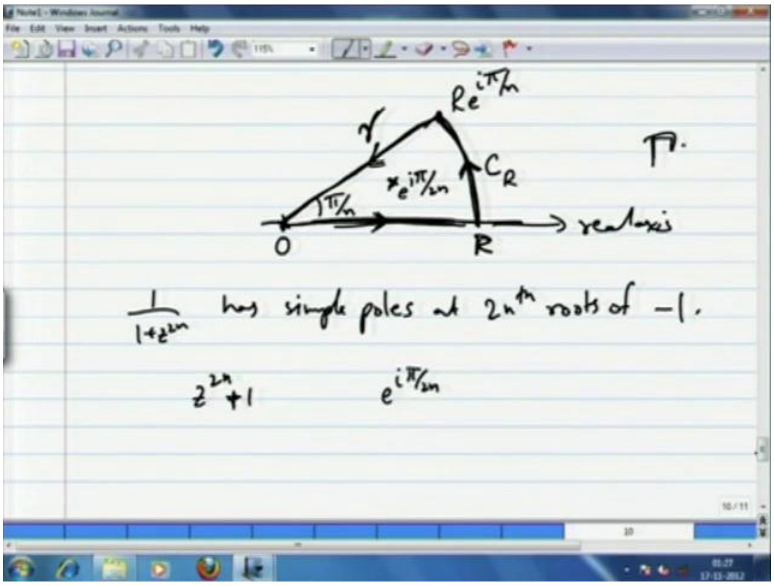

What we will do is we will pick segment sector of a circle like that, which contains I mean in the in the complex plane will pick a sector like that, here is the real access. And here is a sector of angle pi by n. So, we know that this contains 1 root of z power 2 n plus 1 namely e power i pi by 2 n. So, e power i pi by 2 n is a is 1 root of z power 2 n plus 1, 1, 0 of that function and hence it is a pole of that and it is a simple pole, and here is a sector containing that.

So, this is the real axis and this is 0 and this is the point R. A variable R on the real access and so this point will be R e raise to i pi by n and we will pick a contour here, with the contour we will pick is this is the join of these 3 smooth parts. So, we start from 0 go to R and then traverse this circle and then comeback along this path. We will call that gamma so this we will call gamma, this we will call C r for a portion of the circle of radius R, around 0 and we will call the whole curve as capital gamma, whole path as capital gamma.

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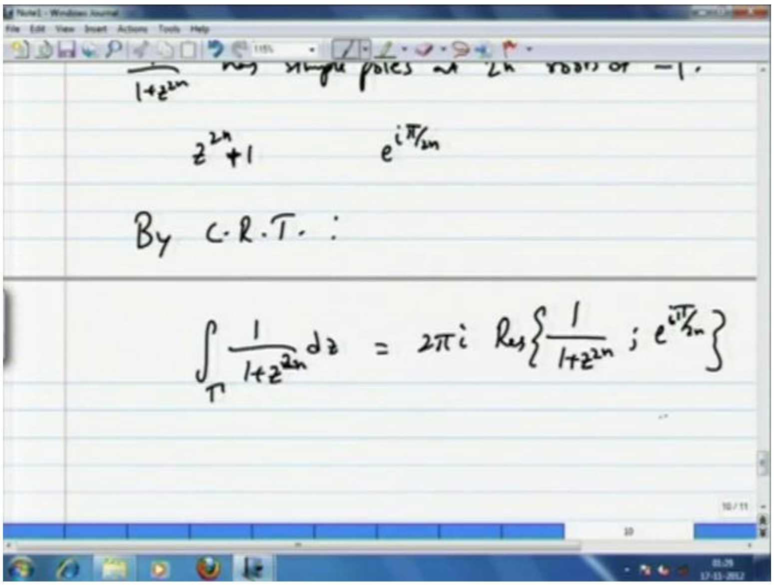

So, by Cauchy’s residue theorem, what we know is that the integration on this whole path gamma of 1 by 1 plus z power 2 n d z is going to give us 2 pi i, finds the residue of the function 1 by 1 plus z power 2 n at the point e power i pi by 2 n, because this function 1 by 1 plus z power 2 n is analytic on, and inside this gamma except at the simple pole e power i pi by 2 and n. So, it gives us this thing so, let us look at how to manipulate the left hand side to get the required integral.

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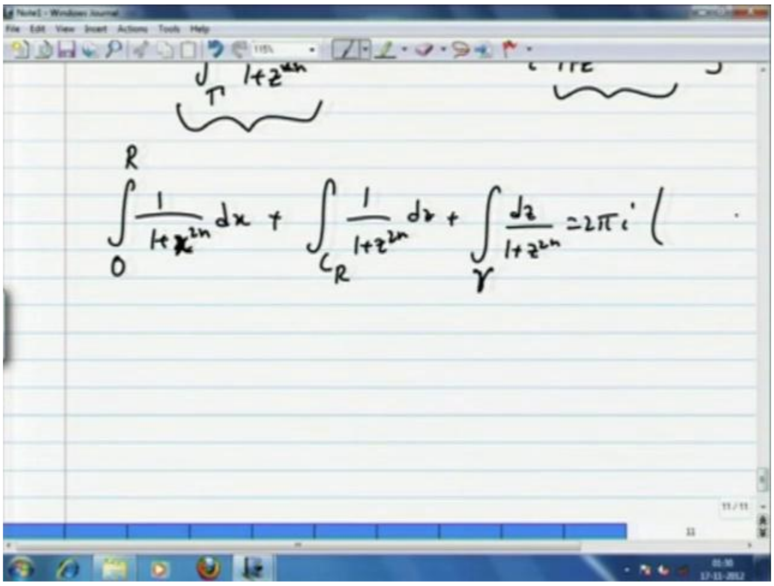

So, we can split the left hand side into 3 integrals depending on the 3 paths. So, the first path is 0 to R so that is the side, I did not call any name. So, it is a real integral so it is from 0 to R of 1 by 1 plus x power 2 n d x. Since, it is a real number I can just use x and then the second portion is the circle of radius capital R so on. The circle of on this portion of the circle of radius capital R, I have 1 by 1 plus z power 2 n d z and then I have the third portion, which is d z by 1 plus z power 2 n on the path gamma. So, this is equal to 2 pi i times, well what is the residue of 1 plus 1 by 1 by z power 2 n at e power i pi by n. So, let us calculate that here in parenthesis well we know that, that is a simple pole.

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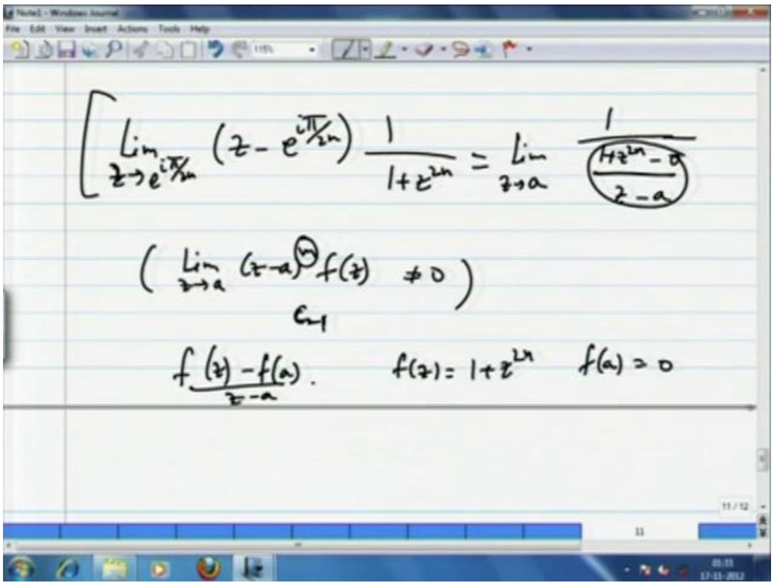

So, the limit z goes to e power i pi by 2 n of the residue of this function at e power R i pi by 2 n is going to be the limit of that times z minus e power i pi by 2 n times, this function 1 by 1 plus z power 2 n. Since, we know that we have a simple pole for this function at e power i pi by 2 n, this is the residue and this can be calculated to be well this is the limit as z goes to e power.

So, let me just call that point a for simplicity this is 1 by 1 plus z power 2 n and 1 plus z power 2 n is anyway 0 at e power i pi by 2 and divided by z minus a. So, this looks like f of z minus f of a by z minus a. Here f of z is 1 plus z power 2 n, and we know that this has a 0 at e power i pi by 2 n. So, f of a is 0 and then divide by z minus a and so we know that the limit as z goes to a of this is f prime of z f prime of z at a sorry f prime of a.

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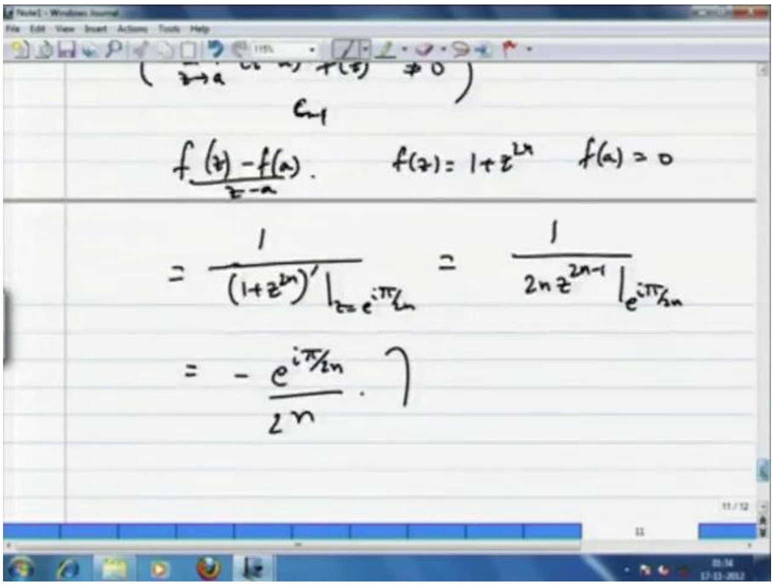

So, this is going to be the derivative of 1 plus z power 2 n evaluated at the point z equals e power i pi by 2 n. So, that gives us 1 by 2 n z power 2 n minus 1 evaluated at e power i pi by 2 n, which gives us minus e power i pi by 2 n divided by n. So, this is divided by 2 n actually and so, the right hand side so this is all in parenthesis.

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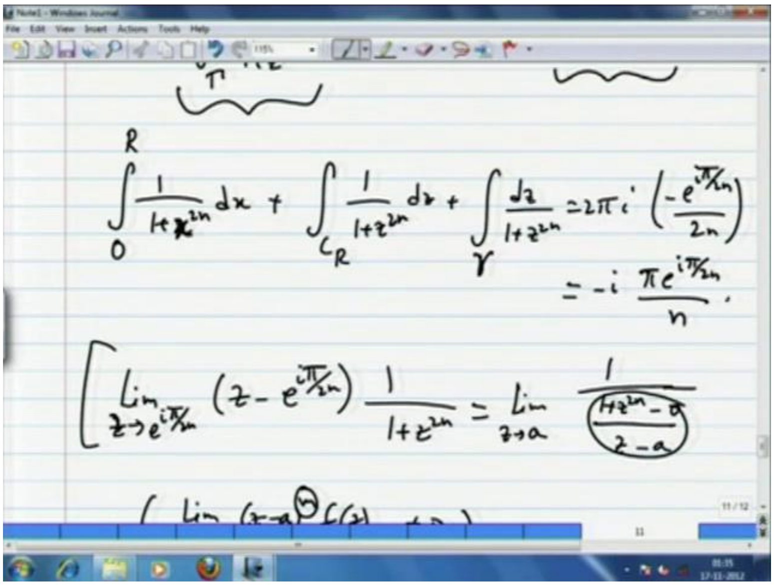

So, the right hand side here is 2 pi i times, this residue which is minus e power i pi by 2 n divided by 2 n. So, this gives us this is equal to when I multiply 2 pi i to it, I get minus i times pi e power i pi by 2 n divided by n.

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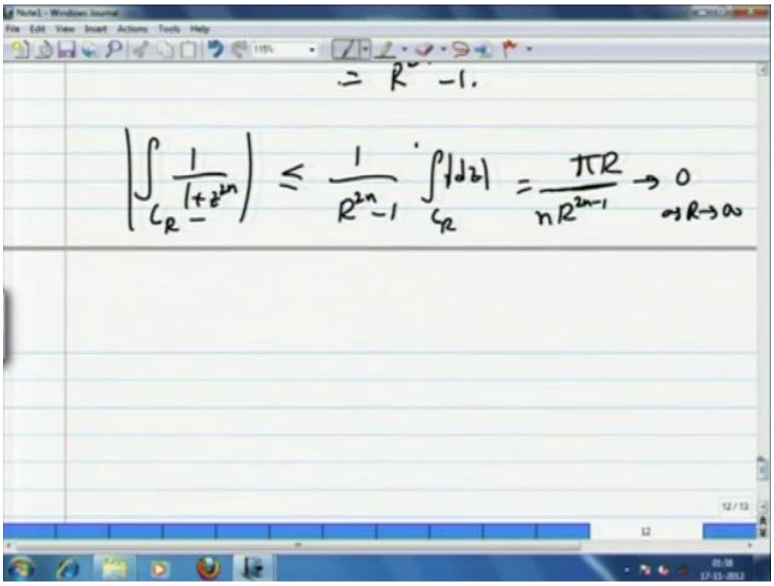

So, the left hand side if e if we look at the left hand side, one of the integral is on the circle of radius R are on the portion of circle of radius R of 1 by 1 plus z power 2 n. So, on c r, d z so on c r the modulus of 1 plus z power 2 n notice is greater than by the triangle inequality, this is greater than or equal to the modulus of z power 2 n minus 1. Well actually in absolute value bit for large enough for large mod z mod z is greater than 1.

So, this modulus will be greater than 1. So, the absolute value of this will be equal to this by the triangle inequality of one kind. And so this is and the modulus of z on c r is simply R. So, this is greater than are equal to r power and this is equal to R power 2 n minus 1. So, this integral on c r 1 by 1 plus z power 2 n we will estimate this, this on the c r is less than or equal to 1 by R power 2 n minus 1 that is the denominator and then I have d z.

So, the integration of so the modulus of d z on c r, c r is a portion of a circle of angle pi by n. So, this gives me pi R by n times R power 2 n minus 1 and the point here is that when I let, R go to infinity. So, if I let R tend to infinity so, I will make this peace bigger and bigger, the sector of the circle bigger and bigger. So, this R tends to infinity. We are interested in R tends infinity. So, this tends to 0 as R tends to infinity.

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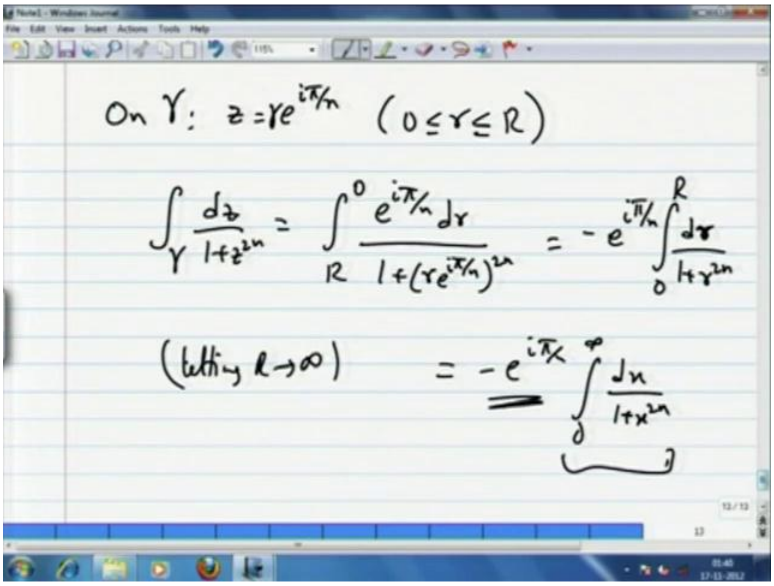

So, on gamma r or on gamma we have on gamma, we have z has the form e power r e power i pi by n. So, we are trying to parameterize this piece now. So, on this piece, what we have is gamma of T, so this can be parameterize like little r e power i pi by n, the angle is fixed, the argument is fixed; here r varies from 0 to capital R or in or the direction of this is, such that r you know little r starts from capital R and goes until 0; so we have this is equal to, on gamma, we have z is that. And then so the integration on gamma of d z by 1 plus z power 2 n is integration from r to 0 of e power i pi by n d r divided by 1 plus r e power i e pi by n raise to 2 n and the denominator can be simplified to be well, will extract a minus e power i pi by n, and then change the limits of integration from 0 to r and then I have d r divided by 1 plus r raise to 2 n. So, the denominator can be simplified and I have that and now if now by letting r tend to infinity, what we have is I mean I can substitute x instead of r.

So, I get this is equal to by letting r tend to infinity, this is equal to minus e power i pi by n integration from 0 to infinity of d x by 1 plus x power 2 n. So, which a coefficient times, what we want and for the remaining peace 0 to r 1 plus x power 2 n d x limit as r goes to infinity of this is what we want.

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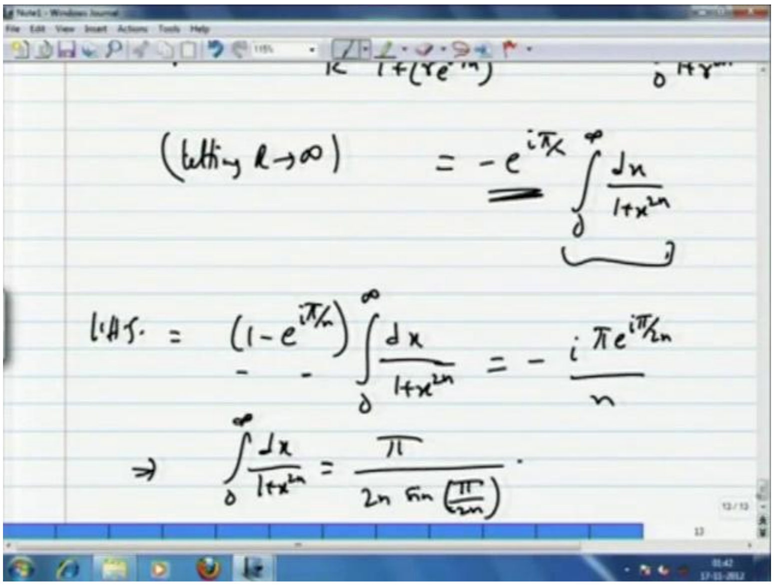

So, in summary, the LHS is equal to 1 minus e power i pi by n times, what we want 0 to infinity d x by 1 plus x power 2. So, by choosing an appropriate integral appropriate function, we are able to convert the definite integral into, we are able to express the definite integral into some complex integral, and then we are able to use the residue theorem to get its value, and then this is equal to minus i pi e power i pi by 2 n divided by n which is our calculation earlier. So, this is the residue so this is our calculation here, so this is so this the RHS.

So, this implies the integration from 0 to infinity d x by 1 plus x power 2 n is equal to well, upon simplification I will divide the right hand side by this factor. I will end up with pi divided by 2 n times sin pi by 2 n so the Cauchy’s residue theorem can be use to evaluate some real integrals. So, the viewer is advised to practice more excises of this kind from that references or text books. So, with this example, I will conclude this course here.