Complex Analysis
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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 4
Laurent's Theorem

Hello viewers, in this session, we will see Laurent's series expansion of functions in a neighbourhood of singularity particularly polls and essential singularities. And we will also see the Cauchy's residue theorem, so firstly to motivate this discussion, let us look at the expansion of 1 by 1 minus z the, the, the Taylor's series expansion of 1 by 1 minus z which is around 0.

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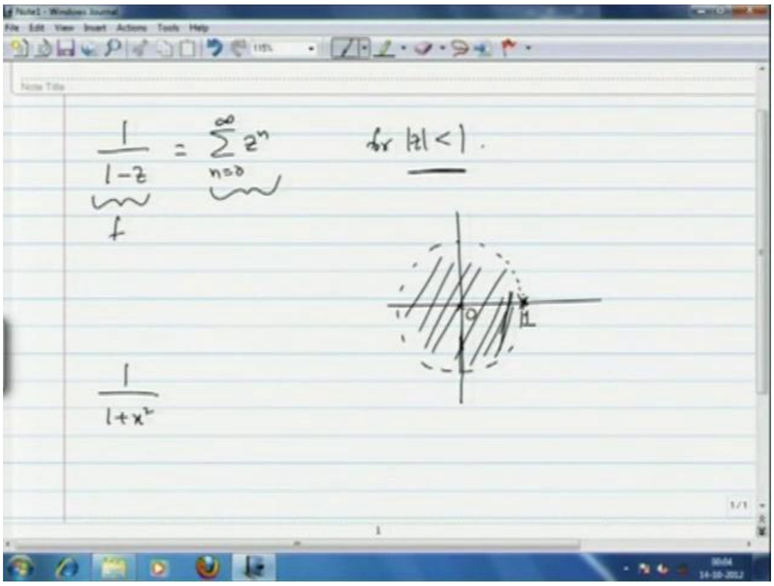

This is equal to sigma z power n, n equals 0 through infinity and then this is valid for modulus of z strictly less than 1. And now after having studied singularities, we know that this, this power series expression of this analytic function, function analytic at 0 is only valid on a ball of radius open ball of radius 1 because this function f has a singularity has a simple pole at z equals 1. So, that is the scenario what is happening is that here is 0 and then function has a simple pole at z equals 1, simple or not that is not the point, but it has a pole at 1.

Then, so the Taylor's series expansion around 0 is valid in a ball of radius 1 open ball of radius 1. That ball can have maximum radius 1, because at one there is a certain resistance there is a poll, so the Taylor's series expansion cannot go beyond 1. So, that is the picture, okay? So, also when I, when we studied Taylor's series we sort of looked at real function f of x equals 1 plus x square and I and I also give this example to say, it is not clear why in the real setting, this function 1 by 1 plus x square has a Taylor's series expansion around 0 which has radius of convergence only 1?

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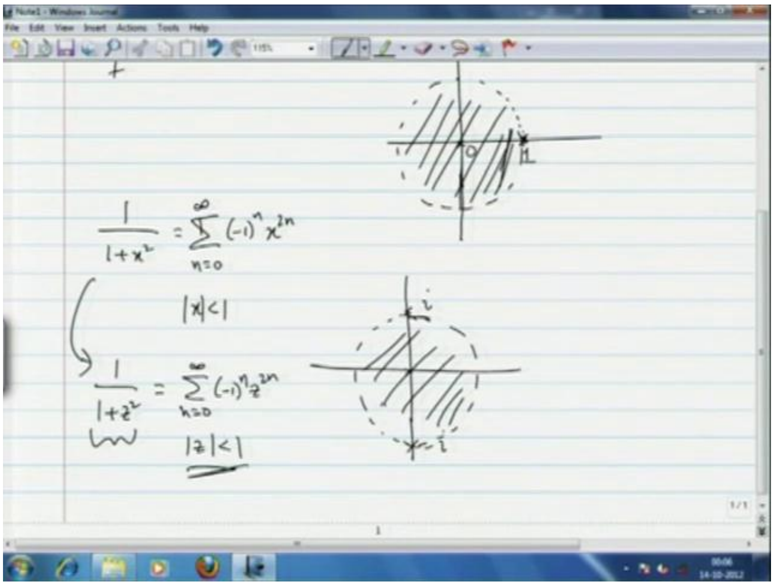

So, in the case of real numbers it is not immediately clear why that happens? So, this that Taylor's expansion by the by is sigma n equals 0 through infinity minus 1 power n x power 2 n and this is valid for modulus of x less than 1. So, the situation becomes clear when we gradate to complex function 1 by 1 plus z square. So, let us consider the corresponding complex function 1 by 1 plus z square and it has a very similar Taylor's series expansion. It is n equals 0 through infinity minus 1 power n z power 2 n around z equals 0 and this is valid for modulus of z strictly less than 1.

In the case of complex functions we know, why the modulus has to be less than 1 because at z equals plus or minus i, this function has a singularity minus i or plus i the denominator is, is 0. So, this function is undefined you have a ball of radius maximum 1 which is clear of these singularities of this function. So, the Taylor's series expansion stops at you know one the radius of convergence is 1. Earlier we consider the function 1 by 1 minus z, which is equal to 0 through infinity z power n for modulus of z less than 1, alright?

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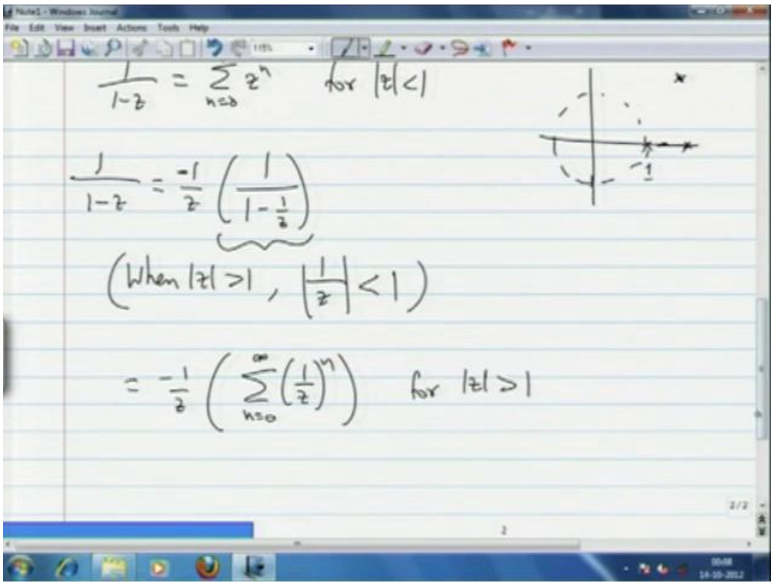

There is a poll at 1 and so this expansion has to stop there, but what we can do is of course, it is analytical elsewhere. So, it is analytic everywhere out you know other than at the 0.1 on the complex plane. So can we do something else to write 1 by 1 minus z as a certain series outside the disk of radius 1. So, in an open set, which is outside the closed disk of radius 1. So, yes we can do that by small manipulation, we write 1 by 1 minus z equals 1 by z or minus 1 by z times 1 by 1 minus 1 by z.

So, when modulus of z is greater than 1, so outside the closed disk of radius 1 modulus of z is greater than 1. When this happens, 1 by z modulus strictly less than 1. So, we can expand this factor 1 by 1 minus 1 by z as a power series, okay? So, what we can do is we can write that as minus 1 by z times sigma n equals 0 through infinity I by by z power n. So, that is the geometric series and this is valid for modulus of z greater than 1.

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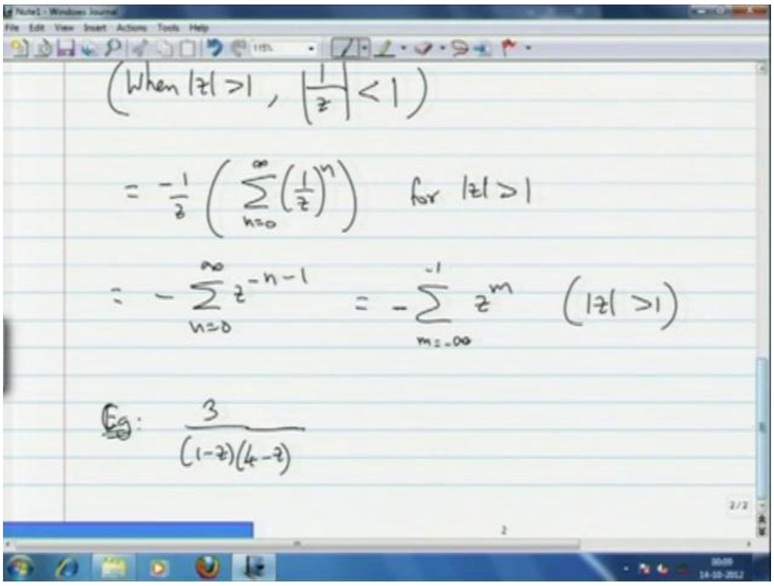

So, this is equal to minus sigma n equals 0 through infinity z power minus n. Then there is a another factor then that gives me a minus 1, so I will rewrite this as sigma minus m goes from minus infinity to minus 1 z power n. By re indexing it I can write this as m goes from minus infinity to minus 1. What that means is I am allowing the index to start at minus 1 and go to negative integers until forever. So, but we know that this series converges and this is for modulus of z greater than 1. So, then we have expanded 1 by 1 minus z outside the disk of radius 1, closed disk of radius 1. We will use this to do the following to consider this function, we will consider an example, 3 by 1 minus z times by 4 minus z.

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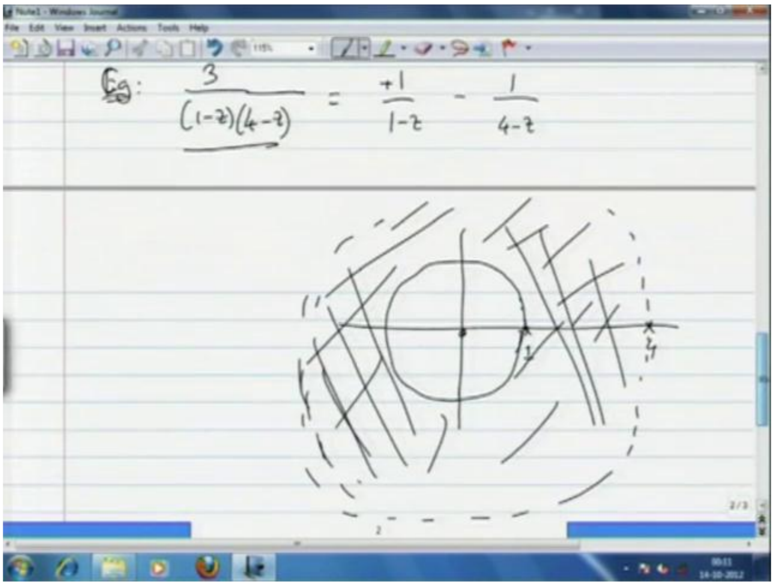

This function has two singularities; one is at 1 and another is at 4. So, there is a singularity at 1 and it has a singularity at 4. So we, we just did something to expand one by 1 minus z outside the closed disk of radius 1. So, we have an expansion for 1 by 1 minus z outside of this disk. Then we can do, we can also do that, we can expand 1 by 4 minus z inside a. So, I will do the hashing the other way inside a disk of radius 4 around 0. So, so we will see that we have an expansion of this function in an annular region between a a disk of radius 1 and disk of radius 4. So, let me write this as partial fractions, we can write this as a minus 1 by 1 minus z minus 1 by 4 minus z or I apologise it should be plus 1 by 1 1 minus z.

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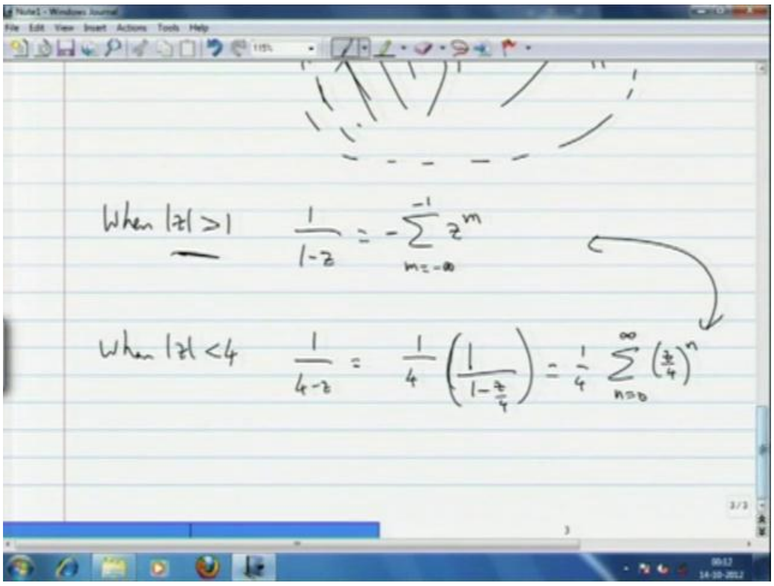

I can expand 1 by 1 minus z. So, when when modulus of z is greater than 1. I know that 1 by 1 minus z is equal to sigma from above minus of m equals minus infinity into minus 1 z power m. This is converges to 1 by 1 minus z when modulus of z is greater than 1. Likewise when modulus of z is strictly less than 4, this other part 1 by 4 minus z gives you 1 by 4 minus z can be written as 1 by 4 times 1 by 1 minus z by 4, which is equal to 1 minus 4 by sigma n equals 0 through infinity, z by 4 power n, okay?

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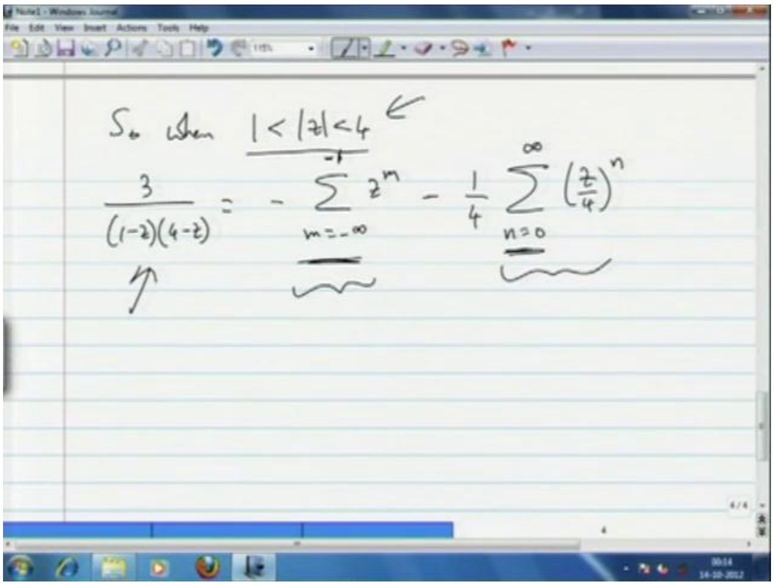

So, put together these two expansions are valid on a common region, namely when, so when 1 less than modulus of z is strictly less than 4, we have this function 3 by 1 minus z times 4 minus z can be written as minus of sigma m equals minus infinity through minus 1 of z power m minus 1 by 4 n equals 0 through infinity z by 4 raise to n. So, you see that in the common region, so the indices one of the indices is running through the negative integers from minus one, through ever and one of the indices is running in the positive direction through the non negative integers, n equals 0 through infinity, okay?

So, we have a sort of series which is double ended we are going to call these as double ended and each of them converges in common region individually. Hence, we say that when we put them together like this, they converge to this function on the on the annular region between 1 and 4.

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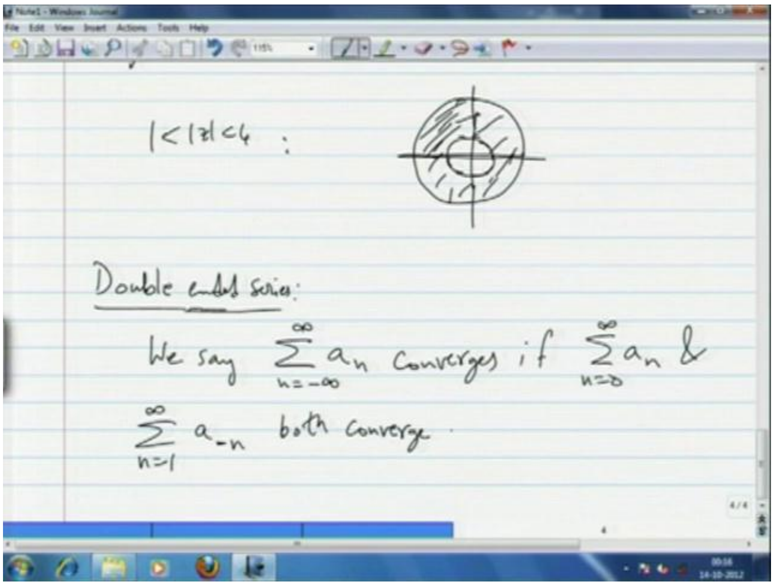

So, write I mean 1 less than mod z less than 4 is an annular region, so it is the region between disk of radius 1 and a disk of radius 4. So, it is all this region not including the boundaries, okay? This motivates us to expand certain functions in annular regions and we will introduce what is called as Laurent's series of functions in annular regions. So, first I will start with double ended series. We say sigma n equals minus infinity through infinity a n converges if sigma n equals 0 through infinity a n and sigma n equals 1 through infinity of a minus n. What that means is the the subscript of a runs through negative integers both converge.

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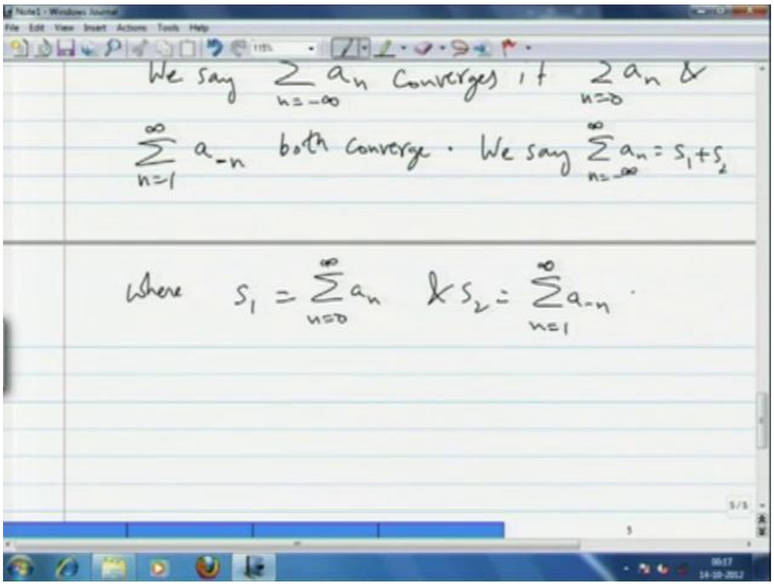

So, if both of them converge we say that sigma n equals minus infinity through infinity n converges and in this case we say, what does it converge to well n equals minus infinity through infinity a n is equal to S 1 plus S 2, where S 1 is n equals 0 through infinity a n and S 2 is other quantity n equals 1 through infinity minus n, which we agreed already converge, which we supposed already converged, okay?

So, if we call the convergent quantities S 1 and S 2, then the original quantity n equals minus infinity through infinity is S 1 plus S 2. We say that is S 1 plus S 2 and with this convention what we have is the Laurent's theorem, which talks about expanding functions in an annular region as series, as double ended series.

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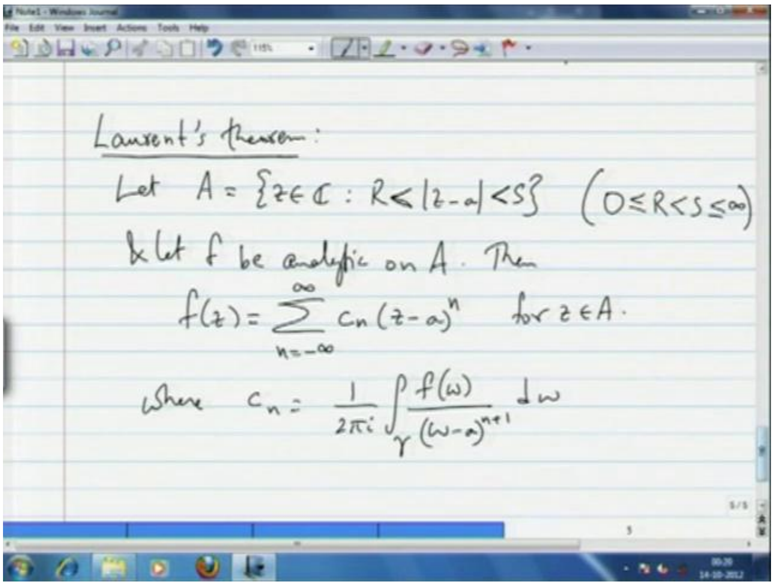

So, we look at the annular region A equals z belongs to C such that R less than modulus of z less than a strictly less than S, okay? So, let let A be this annular region, what we will allow is, we will allow R 2 equal 0, but it is strictly less than S and S can be as large as it wants it could be infinity. So, we will allow the radius R of I mean the lower radius of this annular region to to actually collapse to a point. Then we will allow S to be as large as infinity.

So, but here we have a strict in quality, so here or here we have a strict in quality. Then and let f be an analytic on A, so f is a function complex function which is analytic on A then f of z can be written as sigma n equals minus infinity through infinity C n z minus a power n. So, you can expand f as a double ended series z belonging to this annular region, where these coefficients are not alien. We can select coefficient C n to be 1 by 2 pi i integration around gamma of f of w by w minus a power n plus 1 d w, where that a is the centre of that annulus annular region and where gamma and where gamma is a circle.

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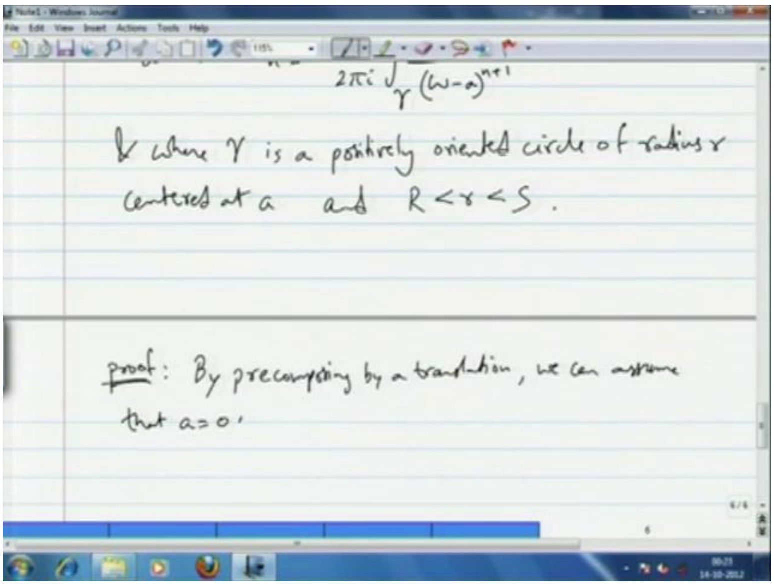

So, its trace is circle, is a positively oriented circle of radius R centred at a and of course, the r lies between R and S. So, that gamma is contained in the annular region and this is a, so what that means is the the trace of gamma is is a circle and gamma is oriented in the counter clockwise direction on that circle and it travels once around that circle. So, that is your gamma and when gamma is such you can select these C n's to be 1 by 2 pi i of that integral, okay?

So, it is interesting the coefficients look very similar to the coefficients of Taylor's theorem. Expect that you now have a double-ended series, when you have a annular region. So, that is Laurent's theorem, so here is the proof of Laurent's theorem, what we will do is we will do is recomposing by a translation. We we can assume that a is equal to 0. So, we are moving everything, we are moving all of this region to be centred, this annular region to be centred at 0. So, the function f sort of changes, but that is okay with us, okay?

So, we will assume that a is equal to 0 and if you want to modify in another fashion you can actually assume a equals 0. Work it out and then transfer these series to you know a series centred at a by classifying you know the functions to be of two types, you know series around 0 and series around a.

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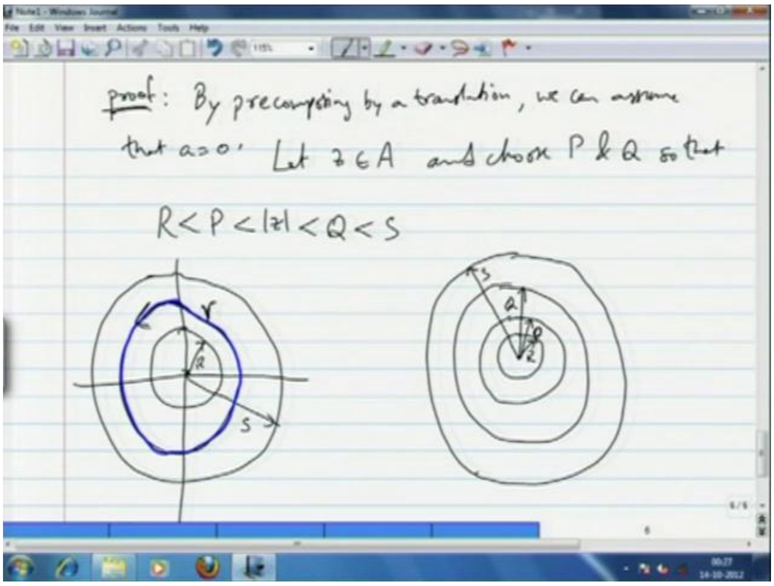

Anyway, so you can do this and choose let let first fix let z belongs to a. You all fix one point on the annulus and choose and choose P and Q, so that R strictly less than P is strictly less than modulus of z is strictly less than Q is strictly less than S ok. So, I will show you a picture a picture is more helpful here, so here is your annular region. Once again could be 0 and S could be infinity. Let us suppose that these are the coordinate axis that is the annular region.

So, this is radius R and that is of radius that is the circle of radius S and the region in between is what you want and gamma is a contour like that, oriented in the positive sense, that is gamma, colour that is your gamma, okay? So, now your choice of P and Q are such that I will not draw the coordinate axis, so that there is less clutter. So, here is your R circle of radius R, here is the circle of radius P around origin, here is the circle of radius Q and here is the circle of radius S. So, you have R this is Q sorry P and that distance is Q P and finally, that distance is S.

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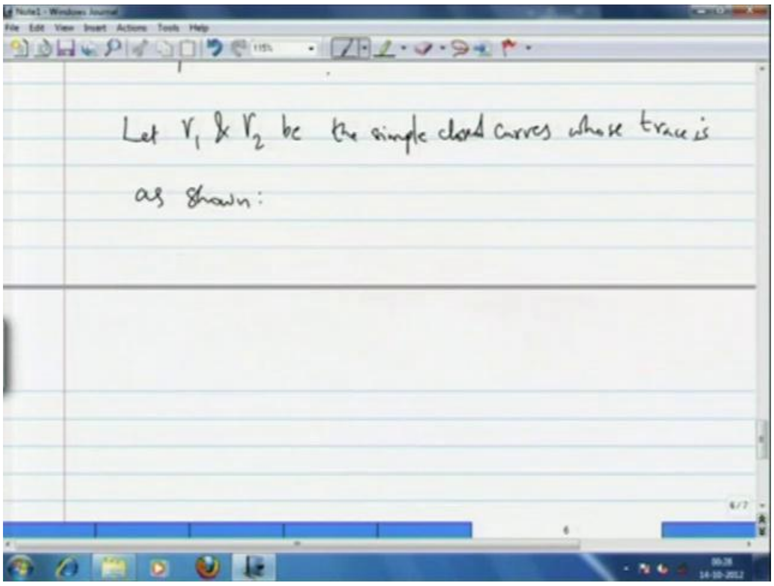

Let gamma 1 and gamma 2 be the simple closed curves, whose trace is I will draw another picture is as shown, whose trace and orientation as shown, I need another picture. So, here is now I will eliminate the curve of sorry the circle of radius R and S, so here I am only going to draw circles of radius P and Q around origin.

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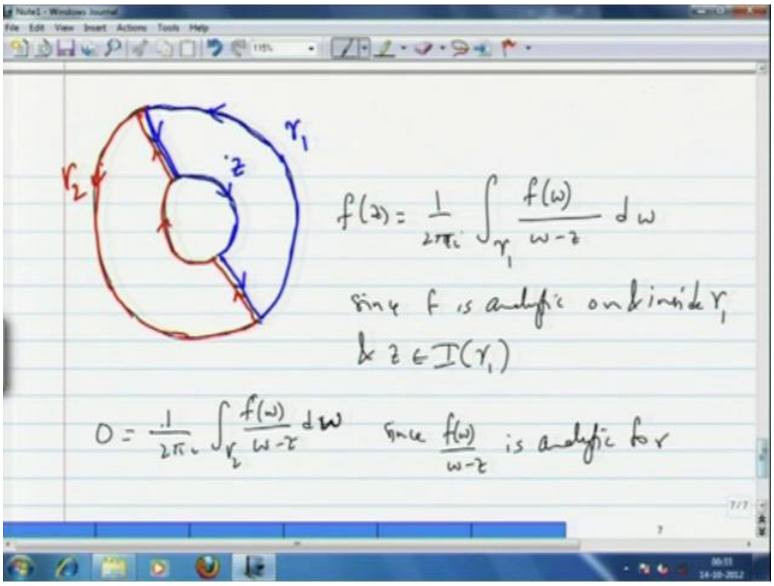

So, here is your P and then here is your circle of radius Q and your gamma 1, I will draw it in blue. It is the contour which starts here you could assume or anywhere here, firstly z is a point here. Then it is any... Notice your z the modulus of z lies between P and Q,so z is inside here. So, gamma 1 goes in this direction say and travels in this direction. Then goes around in that direction half way and it goes this way. Then it it follows the half of the circle of radius Q. So, that is your gamma 1 and gamma 2 is a contour which starts somewhere here you can assume.

So, it goes around on this circle of radius Q and it goes in the opposite direction on this. I am drawing it a little to the side because I do not want to clatter that line, but especially it goes in the opposite direction along the same line and it goes around in in this direction around a circle of radius Q and it only covers half of the circle and goes back that way. So, it traces the same line once again, the same common and it touches that point and then goes back like that, okay?

So, that is your contour gamma 2, that is the trace and orientation of the contour gamma 2. Now, the point is the integration along these line segments that you see here, which those lines segments, which pass through the annular region between P and Q circles of radius P and Q the integration along that cancels. And then when you integrate along gamma 1 plus gamma 2, what is going to remain is only integration along along the outer circle and along the inner circle in the opposite direction.

So, that is the idea. So, so f of z now is equal to 1 by 2 pi i times integration over gamma 1 of f of w by w minus z d w, because gamma 1 contains z in its interior and f is analytic on and inside of gamma 1, okay? So, since since gamma f is analytic on and inside gamma 1 and z belongs to interior of gamma 1, so that is the tray. Also 1 by 2 pi i times integral over gamma 2 f of w by w minus z d z, what is that see d w sorry, what is that? That is equal to 0 because z lies in the exterior of this curve gamma 2, since f by f of w by w minus z is analytic for w belongs to inside of gamma 2 and w belongs to gamma 2 star.

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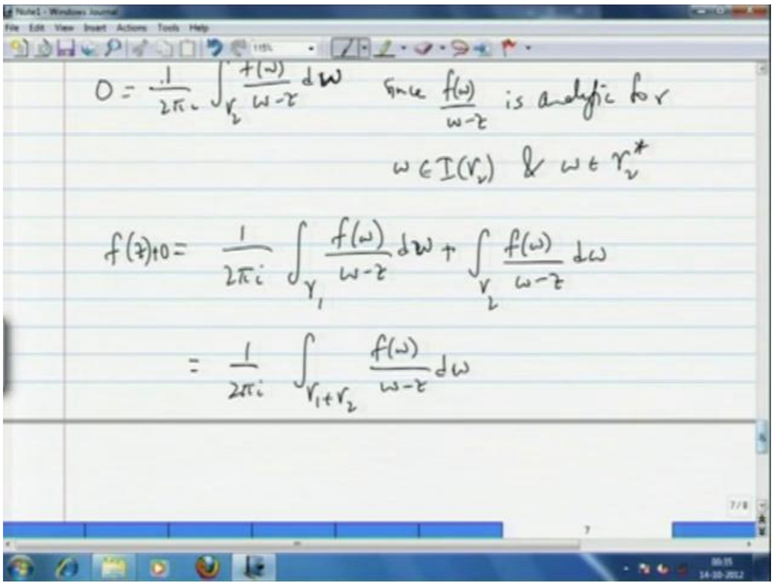

Whether I should say f of w by w minus z is analytic on and inside inside the contour gamma 2, but when I say f of w by w minus z there are two variables, so I should exactly specify which variable I am talking about. So, I have, I have been clear I am saying that with respect to w this function f of w by w minus z is analytic on and inside gamma 2 and so this integration is equal to zero, okay? By combining these two things what we can say is that f of z plus 0, which is f of z is equal to 1 by 2 pi i times integration over gamma 1, gamma 1 f of w by w minus z d z plus integration over...

I will rather take the minus of this, minus so f of z minus 0 technically, is minus integration over gamma 2 f of w by oh sorry, I need a plus, I apologise, I need a plus. f of w by w minus z d w I need a plus. So, what that transfers to is that 1 by 2 pi i integration over gamma 1 plus gamma 2 f of w by w minus z d w. This is a d w d w etcetera and gamma 1 plus gamma 2. Like I explain gamma 1 plus gamma 2 will give you integration on the circle of radius Q in the positive direction and integration on the circle of radius P in the negatively negative direction, okay?

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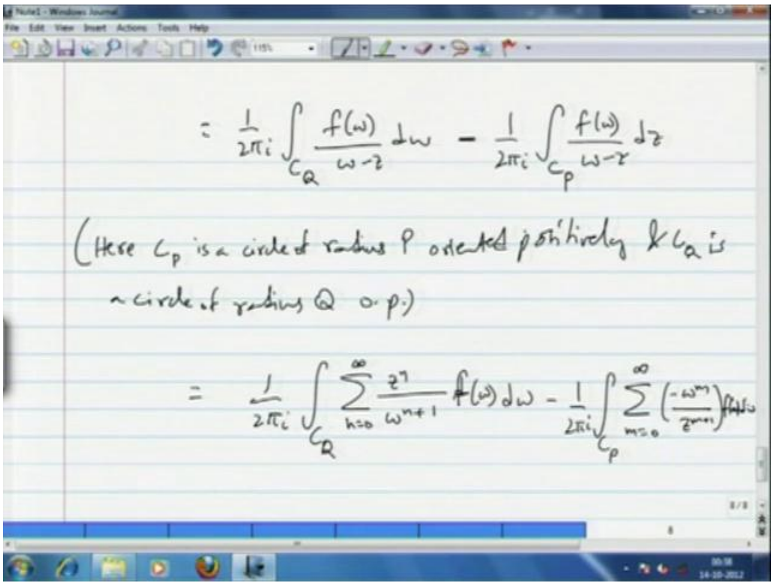

So, this is equal to 1 by 2 pi i integration on I will say C Q, I will write what this is C Q f of w by w minus z d w minus 1 by 2 pi i integration on C P f of w by w minus z d z. Here C P is a circle of radius P oriented positively short and C Q likewise is circle of radius Q oriented positively. This is they go around once only once on those circles, those are the contours and that is equal to 1 by 2 pi i first integration on C Q of sigma n equals 0 through infinity z power n by w power n plus 1 f of w d w minus 1 by 2 pi i integration of on C P sigma m equals 0 through infinity of minus w power m by z power m plus one f of w d w I am going to explain.

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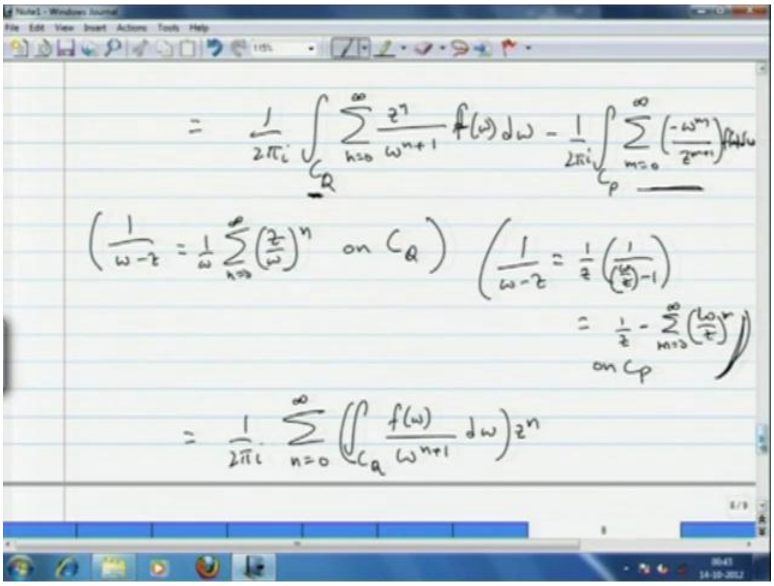

So, what I am doing here is that I am taking 1 minus w minus z and notice that z is in between the circles of radius Q and P. So, what that means is that or you can look at this inequality sorry, you can look at this inequality modulus of z is in between P and Q. So, w when it ranges on C Q the modulus of on, is equal to sigma h z by 1 by w times z by w power n n equals 0 through infinity on on C Q, okay? So, the modulus of z by w is strictly less than 1 when w is on C Q. So, 1 by w minus z can be expanded as geometric series by pulling out 1 by w and you have 1 by 1 minus z by w which you can expand as geometric series.

Likewise for the second term what I am using is when you have w on C p w by z is going to be less than 1 in modulus. w by z can I mean, can then be used you can use geometric series to to expand 1 by w minus z, you can do that trick we did that beginning of this session, for 1 by one minus z. What you can do is, you can pull out a 1 by z times 1 by w by z time minus 1 and that can be expanded as geometric series. So, you get 1 by z times minus of sigma n m equals 0 through infinity of 1 of w by z power n, hence the second expression as the series. So, this is I should say on C P because the modulus I mean P is strictly less than the modulus of z, you can do this, for w belongs to series.

So, that is you know that is how we introduce these series into the integrands. How we want a lemma which exchanges the order of summation and integration. So, that we will recall certain lemma we proved before Taylor’s theorem, okay? So, it is an exercise for the viewer, it is an easy exercise to show that the order of integration and summation can be exchanged. So, please try that and when you do that exercise this is going to be equal to sigma n equals 0 through infinity integration over C Q of f of w by w power n plus 1 d w times z power n. Maybe I I want to put the 2 pi i inside as well. So, I will write n equals 0 through infinity sigma n equals 0 through infinity 1 by 2 pi i and then plus sigma m equals 0 through infinity 1 by 2 pi i times integration over C P w power m f of w d w.

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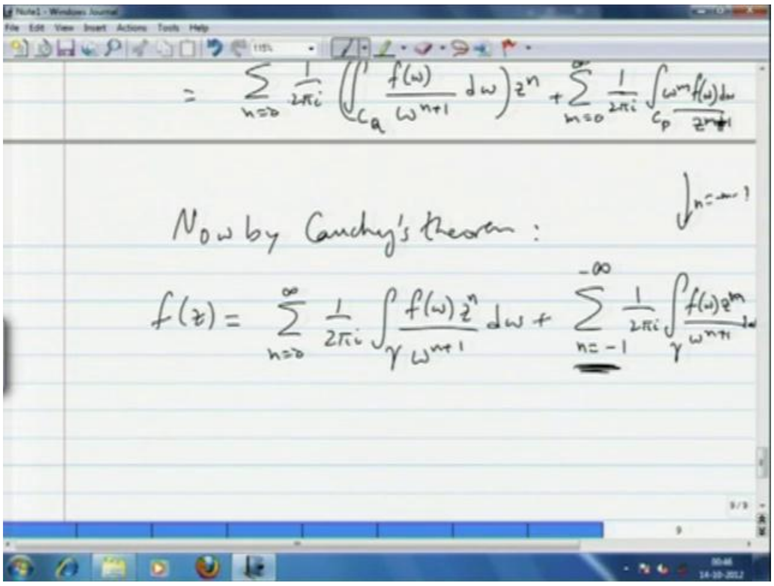

Then there is 1 by z power m plus 1 1 by z power. So, I will write simply divided by z power m plus 1 and so now, by certain version of Cauchy's theorem integration over C Q can be replaced by integration over gamma. Now, by Cauchy's theorem is integration f of z is equal to, so this is all equal to f of z. So, this f of z equal to n equals 0 through infinity 1 by 2 pi i integration over gamma.

So, I am replacing C Q with gamma f of w by w power n plus 1 d w plus integration from n equals minus 1 through minus infinity or minus infinity through minus 1, whichever way 1 by 2 pi i integration over gamma of f of w d w f of w, I will say z power m divided by w power n plus 1 d w. Missing a z power m, I apologise, I should have a z power m here this is z power m, okay? So, putting these together, so I have exchanged the index here, I have converted m to n taking substituting n equals minus m minus 1. So, I am taking m equals minus m minus 1, so putting these together we have what we want.

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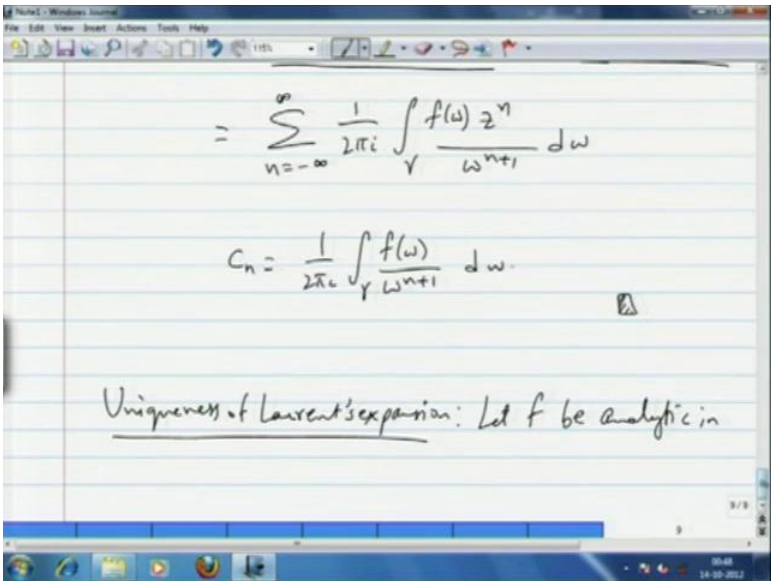

This is equal to sigma n equals minus infinity through infinity through firstly both of these converge. Of course, and then this is equal to n equals minus infinity through infinity of 1 by 2 pi i integration over gamma of f of w times z power n by w power n plus 1 both these. Now, look the same d w, so your C n is indeed 1 by 2 pi i times integration over gamma of f of w by w power n plus 1. So, that completes the proof of the Laurent's theorem.

So, we can expand f of z as series in an annular region where n is analytic. The important step, that we did not, we left as an exercise to the viewer is the step here, that this the series and the summation order of summation and the integration can be exchanged. So, please complete that exercise and when you complete that exercise you have the proof of this theorem. Now, what we are going to do is we are also going to show that this series expansion is unique, so uniqueness of Laurent's expansion.

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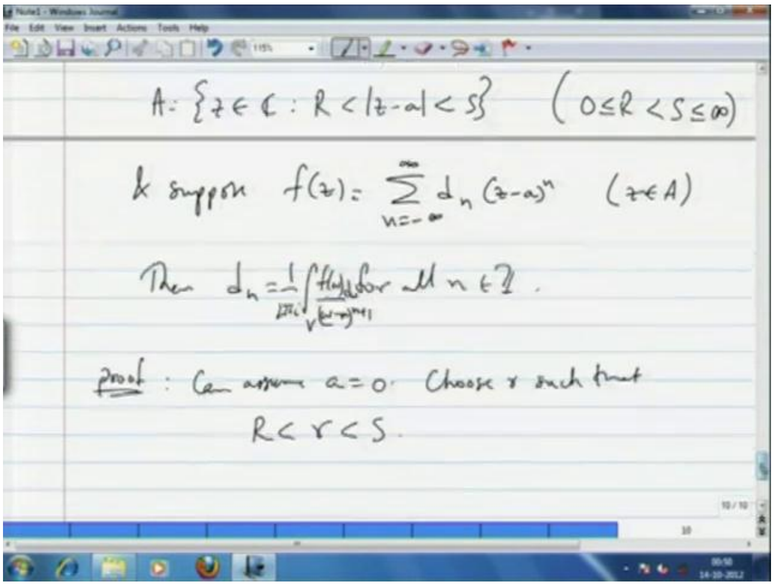

Let f be analytic in a equals z belongs to C the same setup R strictly less than mod z minus a strictly less than S. You will allow R to be 0 and you will also allow S to be infinity. Suppose that f of z is equal to sigma n equals minus infinity through infinity d n z minus a power n. Suppose you are able to expand f of z as a double ended series in some other format with other coefficients d n z belongs to a. Then it has to be that d n is equal to C n coming from above for all n belongs to z. I can directly say this as d n is equal to 1 by 2 pi i integral gamma f of w by w minus a raise to n plus 1 d w, okay?

So, what that is saying is that if you are able to expand f as a double ended series, then the coefficients have to be form that Laurent's theorem specifies and the proof is as follows. You can assume that, can assume a is equal to 0 choose r such that r lies between capital R and capital S.

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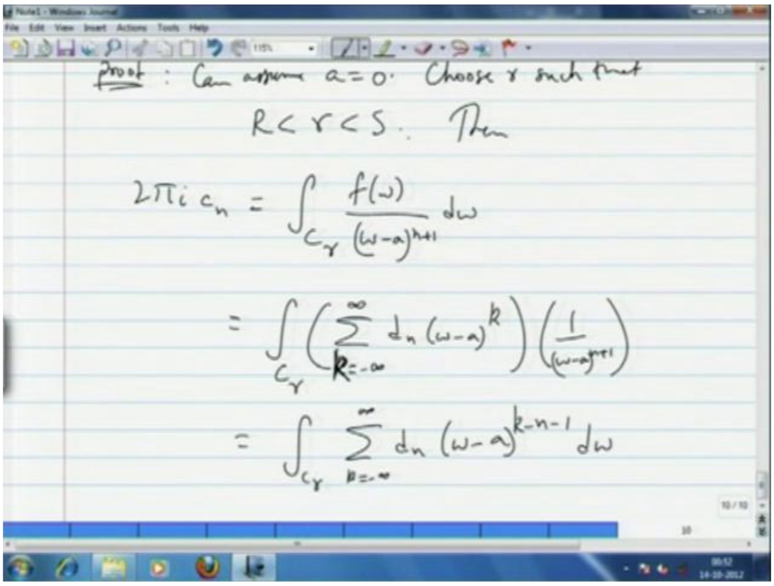

Then, 2 pi i times C n the n eth coefficient from the theorem, this C n coming from previous theorem, is equal to integration over C r f of w by w minus a power n plus 1 d w that is that is the coefficient here, okay? So, and then that is equal to integration over C r C r is a circle of radius r. Here once again oriented in the positive sense. So, I am not writing that here this is equal to sigma R equals minus infinity through infinity of d n w minus a. So, this is r k equals minus infinity through infinity of d n w minus a power k because, I can write f like that and then times 1 by w minus a power n plus 1, okay?

So, f we assume can be written as d n times z minus a power n for z belongs to a. Here the w belongs to is on the contour C r which means it belongs to a, so you can write f of w in that fashion and then that is equal to integral over C r sigma k equals minus infinity through infinity of d n times w minus a power k minus n minus 1. So, I cancel the factors of w minus a to get that. I just changed the exponent, so this is equal to integration over C r sigma k equals 0 through infinity d n w minus a power k minus n minus 1 d w plus integral over C r sigma m equals 1 through 1 through infinity d minus n.

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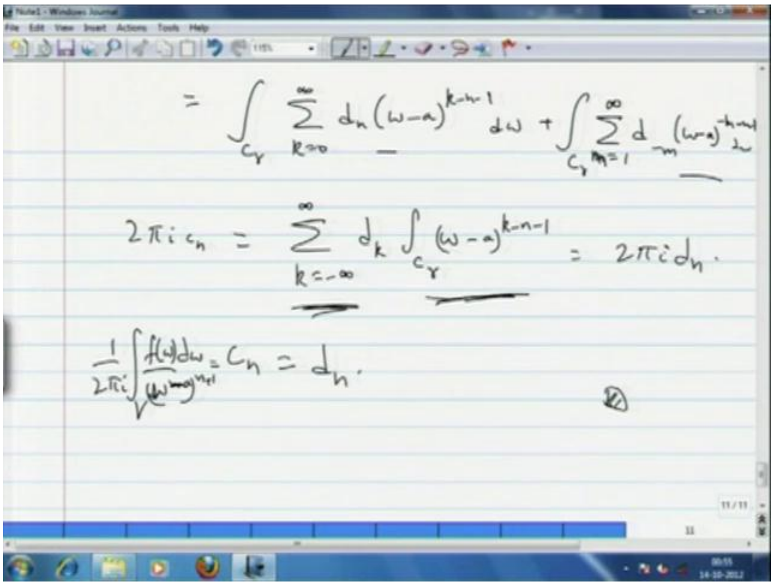

So, I am splitting the integral on two series w minus a power minus m minus n minus 1 d w. Then this is 2 pi i times C n now is equal to, once again I will invoke that theorem which exchanges the order of summation and integration. So, that gives me k equals minus infinity through infinity of d k times integration over C r of w w minus a power k minus n minus 1. So, I actually never used the fact that I can assume that a is equal to 0, okay?

So, actually you can eliminate that, I have been doing this round a itself. So, this boils down to the fundamental integral and we know that this integral. So, I am summing these two things, putting these two things together. We know that the fundamental integral is 0 except when the exponent is a minus 1, so this gives us 2 pi i. The summation all the terms in this summation are 0 except when this exponent k minus n minus 1 equal to minus 1, so which means this is when k equals n, so you are left with just d n, sorry two pi, I I am sorry.

So, that tells you that C n which is coming from the previous theorem, which is 1 by 2 pi i all that integral over gamma f of w by w power n w minus a power n n plus 1 t w, that is your C n is equal to d n. So, that proves the uniqueness part, part of Laurent's expansion. So, there is a unique expansion as double ended series of f where the coefficients are of this form or of you know this, this form which is 1 by 2 pi i times integration over gamma f of w by w minus a power n plus 1 d w. So, I will stop here.