Mathematical Complex Analysis
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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 3
Essential Singularity & Introduction to Laurent Series

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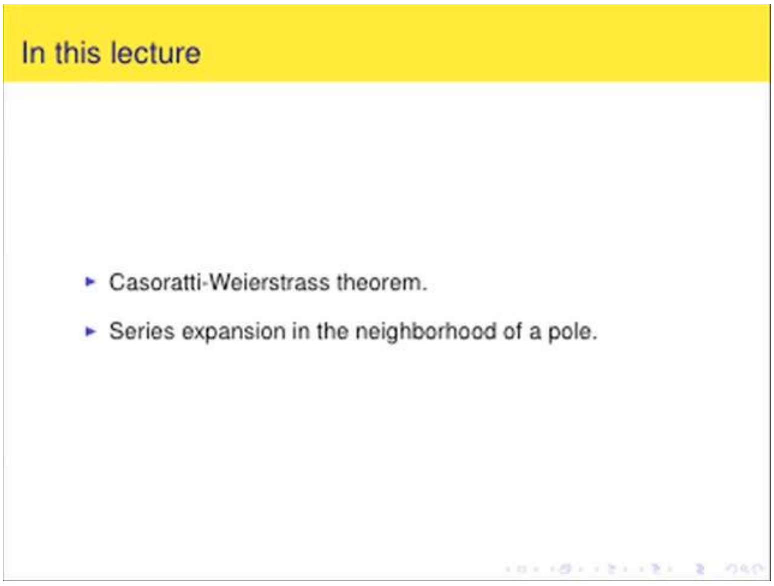

Hello viewers, in the last session we have define an essential singularity, and we we defined it to be the behavior of a function, which is analytic in a neighborhood, such that its limit as z goes to that singularity does not exist. Due to the reason that either there are two sequences in the neighborhood in any neighborhood of a point, which go to, which take f to different limit or or one of them goes off to infinity and the other goes to some finite limit, okay?

And we also claimed that we will show that more, a more complex behavior is exhibited by a function in the neighborhood of an essential singularity. So, in this direction, we will see the Casorati Weierstrass theorem, which says that the the function f assumes or comes arbitrarily closed to every complex number in a neighborhood, in every neighborhood of an essential singularity. So, that is the statement.

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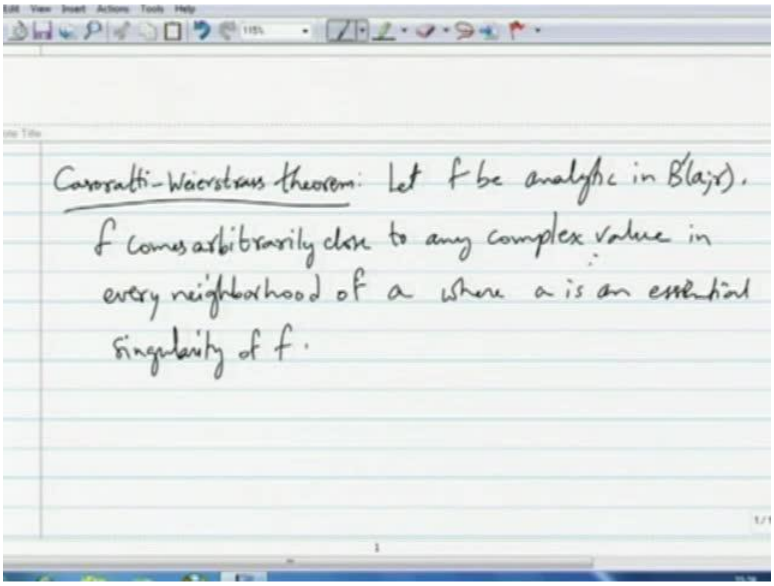

So let me write that down Casorati Weierstrass theorem. Let f be analytic in B prime a r that is an deleted neighborhood of a and f comes arbitrarily closed to any complex number. So, emphasis on any any complex value in every neighborhood of a, where a is an essential singularity of f. So, if a, if a is an essential singularity of f, then f comes arbitrarily closed to any complex value in every neighborhood. So, it is a very strong kind of description of this behavior and we will see a proof of this fact. So, let me remind you how we define this. So, emphasis on any, any complex value in every neighborhood of a, where a is an essential singularity of f.

So, if a, if a is an essential singularity of f, then f comes arbitrary closed to any complex value in every neighborhood. So, it is a very strong kind of description of this behavior and we will see a proof of this fact. So, let me remind you how we defined this is essential singularity here is the slide from the last lecture. So, conditions one and two neither of them should holds, which means there is no alpha for which one holds or is there alpha for which this other second conditions holds, okay? So, that is, that is an essential singularity and we are going to use that.

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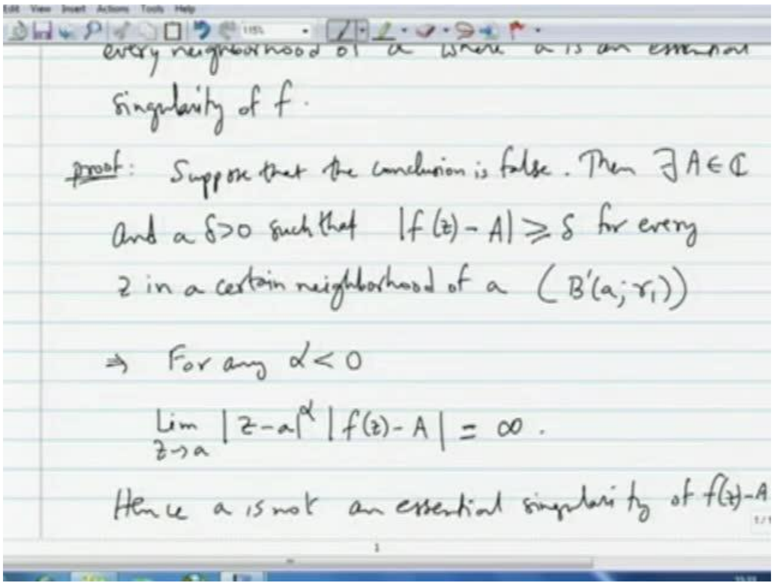

So, suppose, suppose that the conclusion is false. i f does not come arbitrarily close to any complex value in every neighborhood. What that means is there is a complex number capital A to which f does not come arbitrarily closed. So, there is a capital A ended delta. So, then their exist A capital belongs to C and A delta positive such that the modules of f of z minus A is greater than or equal to delta for every z in some, in a neighborhood of A; so for every z in a certain neighborhood of A.

So, that has to happen, because so what that means is it does not come arbitrary close to A, the complex value A. So, then this implies that for any alpha less than 0 limit z goes to A modules of z minus A power alpha times modules of f of z minus A. So, we are using the condition to on well before I use the condition, let me say that this is this has to be infinity, that is because z minus A power alpha when alpha is negative, tends to infinity as z tends to A. Your f of z minus A is bounded away from 0, it is a positive number its greater than or equal to delta in modules, in modules.

So, you have ah non-zero number in the numerator divided by divided by z minus A power alpha in modules, and then so, so that as so go to infinity. z minus A power alpha modules go to infinity, so this is infinity, that is easy. Hence, A is not an essential singularity by the definition of essential singularity, hence is, hence is not an essential singularity of the function f of z minus A, right? Because condition two is satisfied, let me go back to that condition two, condition two is satisfied for f of z minus A not for f of z exactly, but f of z minus A. So, we will see that this leads to contradiction.

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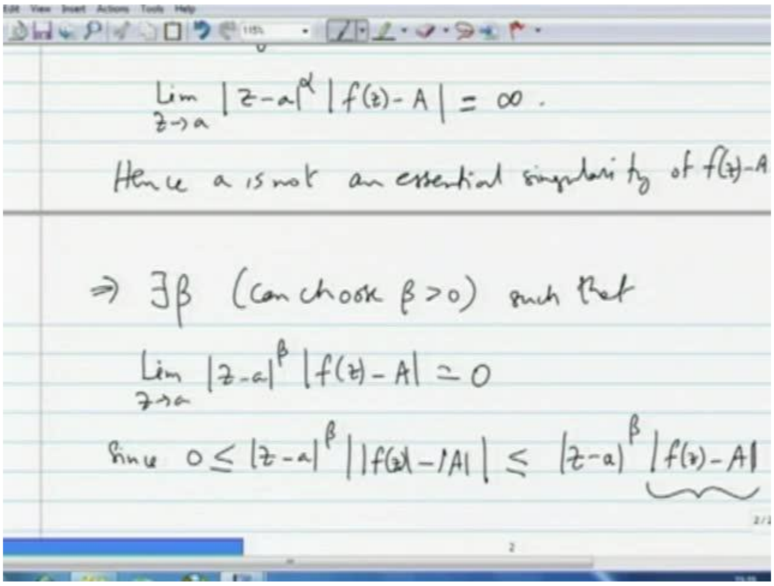

So, so then there is implies there exist a beta, can choose beta to be a positive, because we have seen that there is a integral divide point. If if condition one holds then two holds and if two holds one holds, I am talking of the conditions of that I have just shown and there is a integral divide point where condition one will hold to the right and condition two will hold to the left, on the real line, okay? So there exists a beta, can choose beta to be positive, because I mean that integral divide beyond that integral divide you can choose a positive number.

So, can choose a beta positive such that, the limit as z goes to A modules of z minus A power beta times modules of f of z minus A is equal to 0 because that second conditions holds the first condition also holds, okay? So, since 0 less than or equal to modules of z minus A power beta times modules of modules of f of z minus modules of A is less than or equal to modules of z minus A power beta modules of f of z minus A. So, I am using the triangle inequality on this stuff here f of z minus A modules that is greater than or equal to this thing here. So, I am using triangle inequality in one form.

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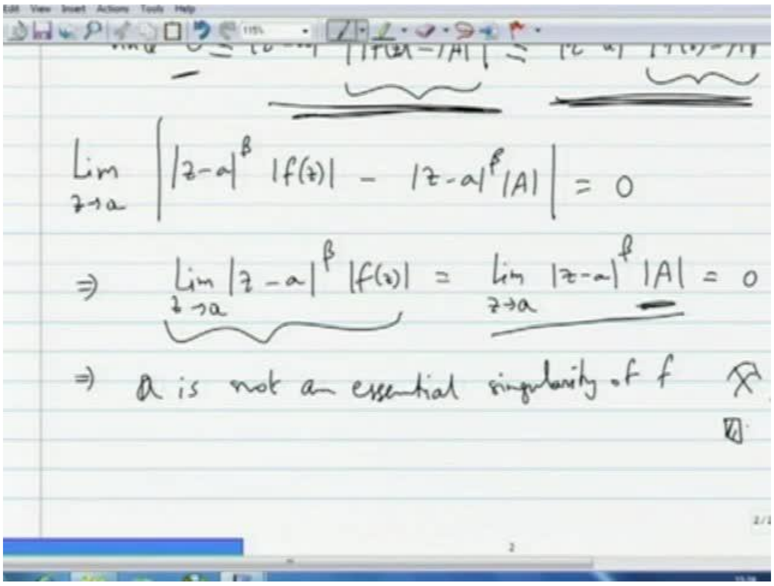

Since, that it is true the limit as by the Sandwich theorem, the limit of the the quantity here goes to 0 and this is a 0 anyway, so the limit as z goes to 0 of the modules of z minus A power beta, the modules of this times a modules of f of z minus modules of z minus A power beta modules of A. So, I am multiplying modules of z minus A beta into the modules and then the modules of this A is equal to 0. I am taking the quantity in the middle and multiplying that out, okay?

So, so then this implies that that the limit as z goes to 0 of a certain quantity complex quantity or or real quantity sorry, is equal to 0 so the quantity has to be 0 in the limit the limit as z goes to 0 of modules of z minus a power beta modules of f of z is equal to limit as z goes to sorry, this is A I apologize, this is limit as z goes to A modules of z minus A power beta modules of A. And beta is positive and we know that A is a fixed complex number, which we assumed f does not approach closely in a neighborhood of a little a, okay?

So a a is a fixed complex number. So, this limit we know is 0, right? Because beta is a positive quantity z minus A goes to 0 as z goes to A. So, this is 0, so what this implies is f. So, this now in the limit is equal to 0, what that says is that f does not have A is not an essential singularity of f, because condition two held for a certain beta A is not an essential sorry, the condition one held for certain beta not an essential singularity of f. That is a contradiction to our assumption that A is an essential singularity. So, that cannot happen, so f comes arbitrarily closed every complex number in the neighborhood of an assumption singularity. So, it is the Casorati Weierstrass theorem.

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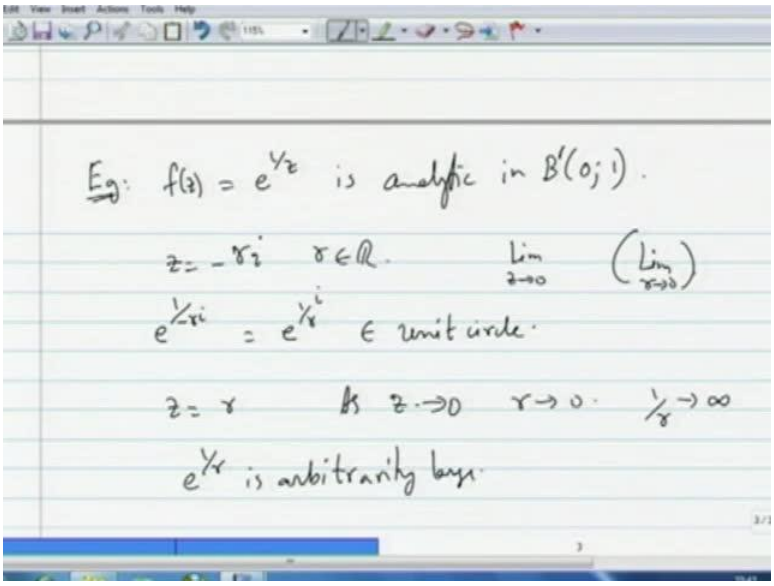

We will look at some examples of an essential singularity e power 1 by z. For example, has an essential singularity at z equals 0 of course, it is not defined at z equals 0. So, this is analytic in B prime of 0 let us say 1 in in the deleted neighborhood of any, in a deleted bowl of radius 1 centered at 0. You notice that when you take z equals minus minus r i for some real number r, r belongs to r. Then what you get is limit z goes to 0, will mean that limit r goes to 0, limit z goes to 0 of this quantity.

What will that be? Well firstly what is e evaluated at such a z minus r i will give you e power 1 by r times i. This quantity whatever this is, the 1 by r is a real number, so this belongs to unit circle. i it has modules 1. So, it is, it stays in the bounded complex plain, that is the point. Whereas if you take z is equal to a real number r, then as z goes to 0 as z goes to 0 r goes to 0 and e power 1 by r this is, I mean 1 by r is arbitrary large. So, e power 1 by r is also it is a, it is a real number and this is arbitrarily large.

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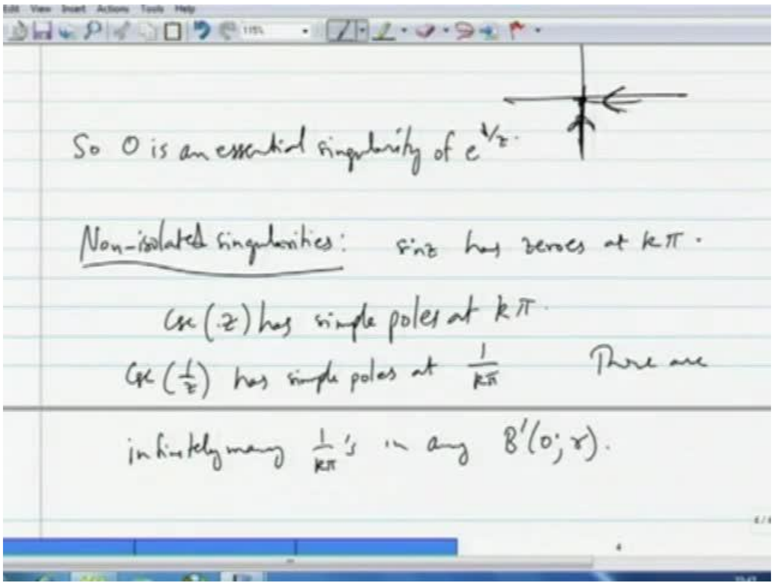

So, where as when you approach along that, along the, along one direction to 0 namely from the negative imaginary axis to 0 your e power 1 by z stays bounded it stays in the bounded circle. Whereas when you approach 0 from the positive real axis, this way you see that e power 1 by r is unbounded, okay? So, at least you can conclude that limit as z goes to 0 of e power 1 by z does not exist and that will immediately give us the fact that it is an essential singularity because that condition one or two, which we gave earlier would not hold. And so it is an essential singularity.

By Casorati Weierstrass theorem, we know more we know that e power 1 by z now comes arbitrarily closed to every complex number in every neighborhood of 0. So, that is an example of essential singularity. So, 0 is essential singularity of e power 1 by z. So, that is the discussion of of poles and singularities and of course, we also discussed removable singularities, which are the only three kinds of isolated singularities. We have seen the the dual dual behavior of zeroes and poles and how they are tightly knit? We also gave a Lemma in that connection and that sort of wraps of power discussion about isolated singularities, okay?

So, the viewer should also be aware that there are non isolated singularities a function a function can have non-isolated singularities. So, non isolated singularities, I will give an example, we will not discuss them at length, but they can be such sine z has zeroes at k pi. So, if I consider cosecant 1 by z. So, firstly so cosecant z has singularities or again actually we saw that it has simple poles at k pi k naught equal to 0 at k pi in indeed at k pi. So, cosecant 1 by z when you consider cosecant 1 by z this this has simple poles at 1 by k pi, there are infinitely many many 1 by k pi's in any B prime of 0. That is what I want to say. So, essentially 0 is definitely singularity of cosecant 1 by z and and in every neighborhood you have infinitely many singularities of cosecant, okay? So, 0 is a limit point of of this singularities. So, it is not an isolated singularity.

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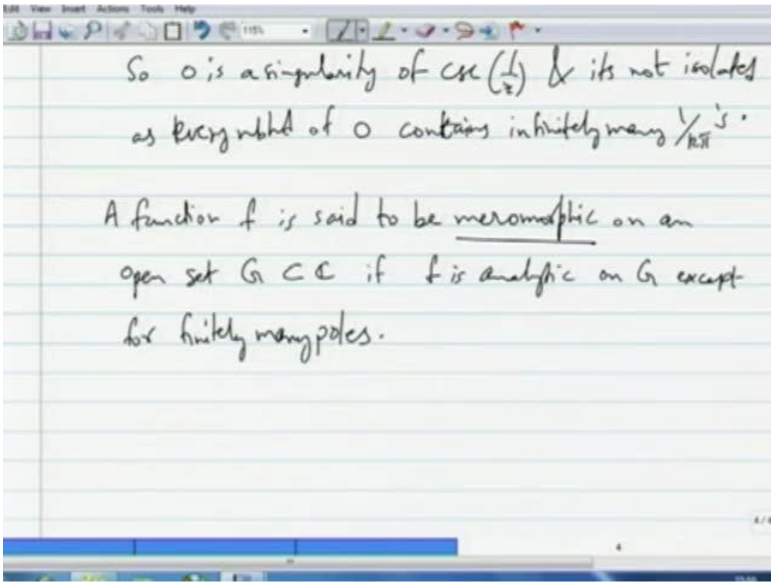

So, zero is a singularity of cosecant 1 by z and it is not isolated as there are as every neighborhood of 0 contains infinitely many 1 by k pi s, which are all again singularities of cosecant 1 by z. So, there are non isolated singularities as well, but will confine ourselves to isolated singularities and the can either be poles or essential singularities and the removable singularities are a peculiar. They they sort of can be forgotten in the sense that you can always extend the function to be analytic at that point itself.

So, it is it is sort of fake singularity there is a lake of information about the function f at that point and when you have that information the function can be made to analytic. So, that is the discussion about singularities and the following terminologies often used consequent function f is set to be meromorphic on an open set G contained in C if f is analytic on G except for finitely many poles, okay?

So, if an open set f contains finitely many poles, then you call f to be a meromorphic. So, that is that is a terminology, which is often used and now, what I am going to do is, consider a different discussion, okay?

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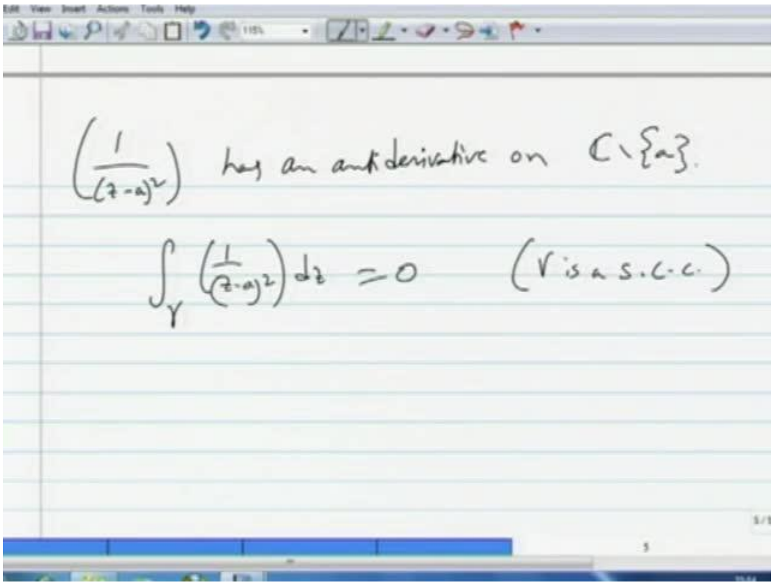

Since, we know that 1 by z minus a square is has an anti-derivative on… Let us say C minus a, we know that the integration of for gamma 1 by z minus A square we saw this d z is equal to 0, whether or not a is in on the inside of gamma, I should say that gamma is a simple closed curved, okay? So, so that gives independence of path in the sense, that path when you integrate 1 by z minus a square in C minus a, so here is a point a let us suppose.

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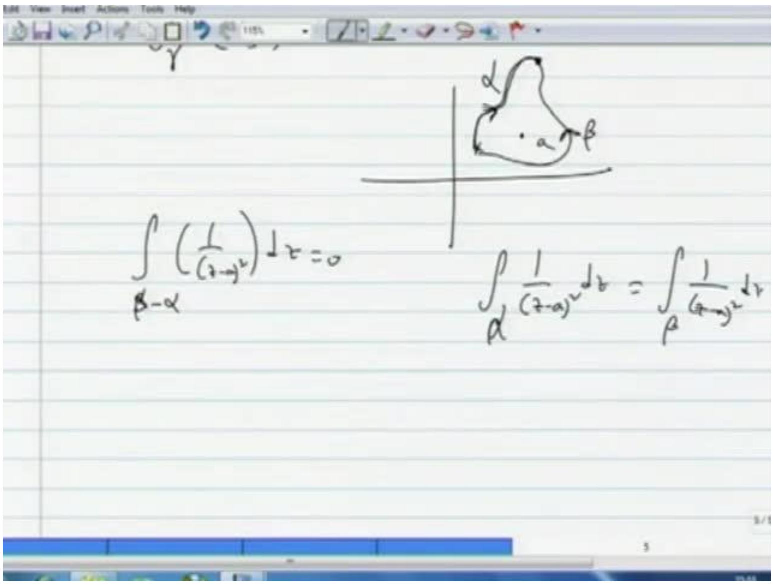

So, it does not matter whether you integrate 1 by z minus a in this fashion. Here is a curve gamma and then I mean now, this is a different gamma. So, I should probably say this is alpha or whether you integrate 1 by z minus a square around some other path beta oriented path beta. So, because the integration around alpha or beta minus alpha or alpha minus beta of 1 by z minus a square d z is equal to 0, the integration on alpha of 1 by z minus a square d z is equal to the integration around beta of 1 by z minus a square d z.

So, we saw that when I mean that that happens because 1 by z minus a square has an anti derivative in C minus a, okay? So, although we know that 1 by z minus a square is not actually analytic inside of inside of this curve gamma of course, at a it has a double pole. Now, the question is, when does a function have an anti derivative, or when is a integral of a function f of z t z is equal to 0?

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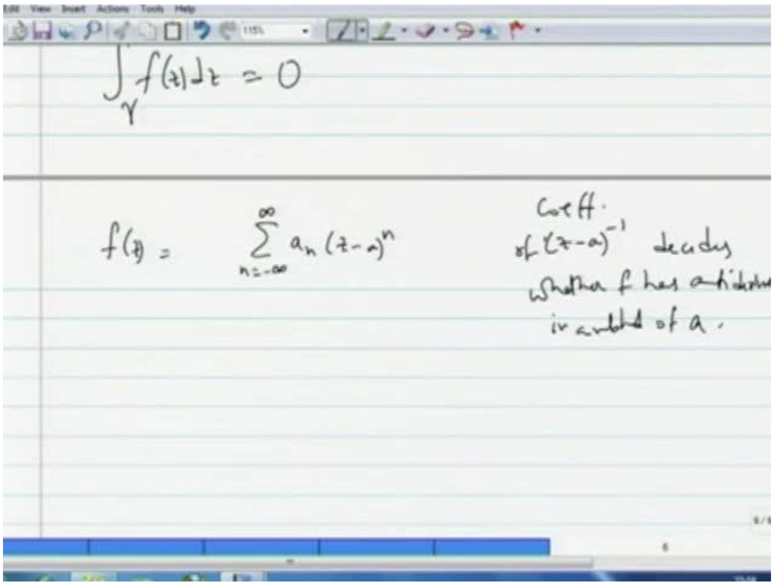

Whether or not s is analytic inside inside of gamma? So, we will only allow f to have isolated singularities, in case it skips to be analytic inside of gamma. Okay? So we will avoid non isolated singularities for this discussion. So, now we will see that a function f, will, will happen to have anti derivative, when when we write f as a certain power series as a certain double series actually sorry. So, we will we will now write f locally as n goes from minus infinity to infinity will see what all this means of a n times z minus a power n in a neighborhood of a whether or not f is defined at a, okay/

We will see that the co-efficient of z minus a power minus 1 the co-efficient of this decides, whether f has anti derivative in a neighborhood of neighborhood of a. So, that that we will see and this kind of a series is a double series. So, we have to introduce what that means and then, and then the motivation for this is clear.

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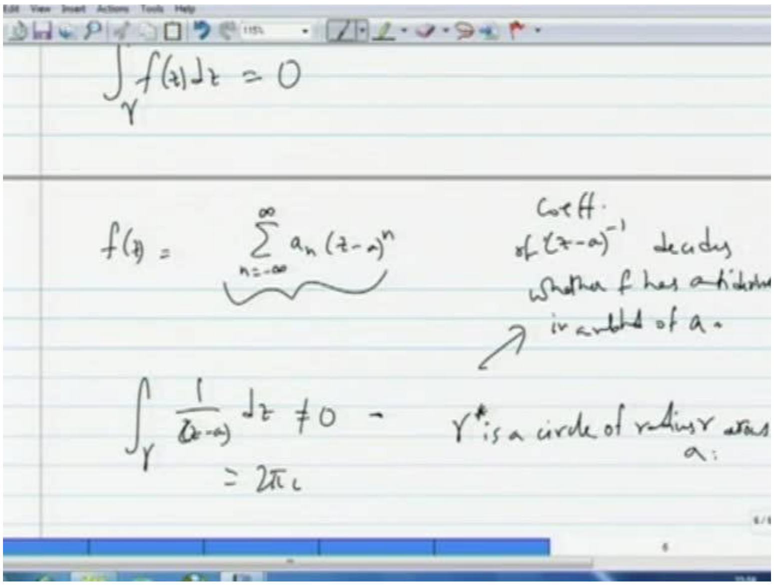

We know that 1 by z minus a does not have a single valued function 2, which it is a derivative in a neighborhood of a. After all we know that the integral of this is a not a 0, it is actually equal to 2 pi I, that is the fundamental integral. When gamma is a circle of radius r gamma star is a circle of a radius r around a, which goes I mean when parameterized it goes around once, okay?

So, this is a in the positive direction, so this is 2 pi i so it is non zero is a point. So, one by z minus a is not that derivative of a single value function in any neighborhood of a. So, they there is a resistance their and hence, and hence 1 by z minus a fails this condition and, and we see that when we can write f of z locally like this. Near a singularity a, it is the co-efficient of this 1 by z minus a really, which matters and, and if that co-efficient is 0, we will see that f is the derivative of a single value analytic function in the neighborhood of a or another words it has an anti derivative, okay?

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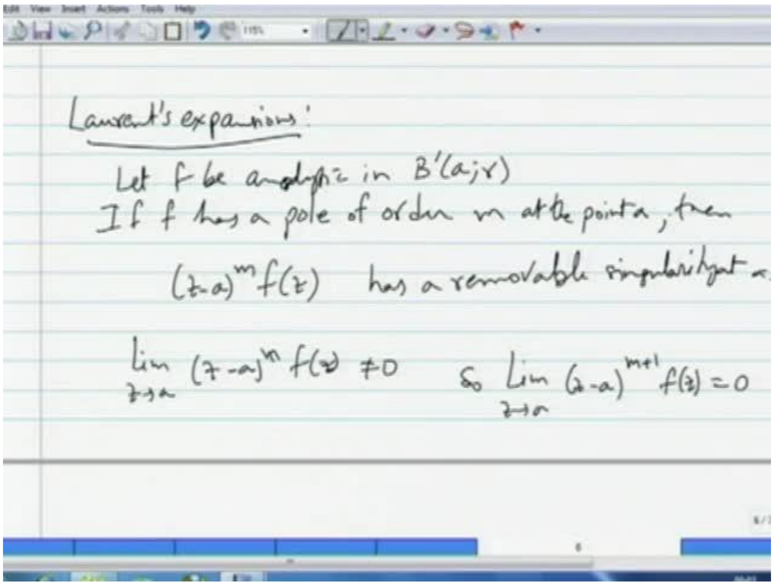

So, for that we will need some machinery, so in particular we are going to need the Laurent's expansions, so in a annulus. So, first let us examine the case of a pole. So, if f has a pole of order m at a at point a and is, so I should have said this let f be analytic in B prime a r. So, its analytic in a deleted neighborhood of a and if f has a pole of order m at a at the point a, at the point a, what you can do is, firstly we know that then z minus a power m f of z.

This has a removable singularity at the point a, that is because we know that when, when you have a pole of order m at a, the limit as z goes to a of z minus a power m f of z is non zero quantity. It is a, it is a complex numbers, which is non zero, that is when you say that it has a pole of order m, okay? So, if you jack up the power of z minus a, a bit. So, if you take m plus 1 of f of z that is equal to 0. So, that condition will tell you that z minus a times this function this z minus a power m f of z in the limit is equal to 0, which means you have a removable singularity for this function at z equals a, okay?

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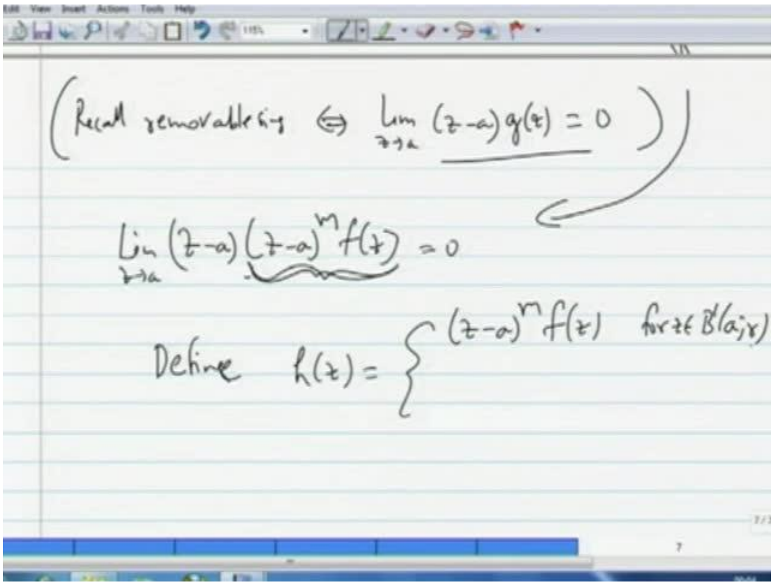

So, right that is a condition recall removable singularity. If and only if limit z goes to a z minus a g of z let me say is equal to 0, so in this case the function g under consideration is z minus a power m f of z. So, this is so this condition is telling you that, limit z goes to a z minus a times z minus a power m f of z. So, this is the function g, written here that is equal to 0. So, that has a removable singularity, this function has a removable singularity, So, now you redefine you define, new function h of z is equal to z minus a power m f of z for z belongs to B prime a r.

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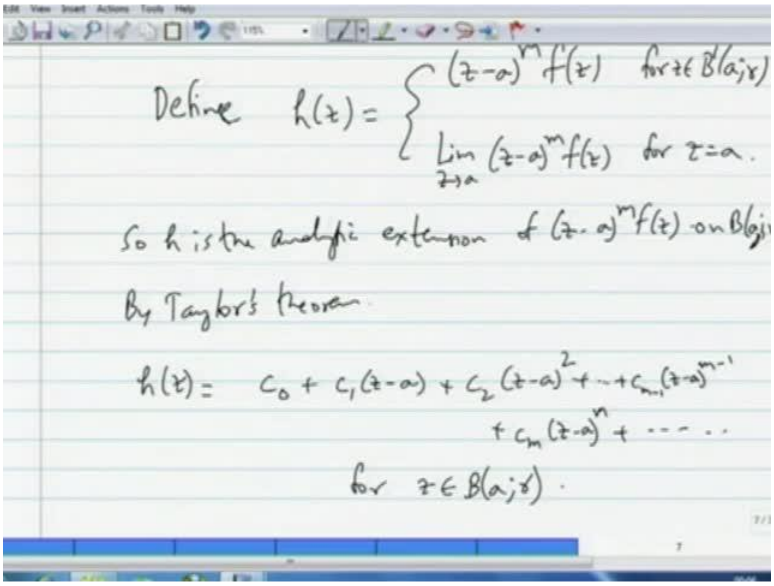

This is limit as z goes to a z minus a power m f of z for z equals to a, so you defined this two the this limit at that point, so that you can remove the singularities. So, h is the analytic extension, of z minus a power m f of z. Notice it is the analytic extension of z minus a power m into f of z, not of f itself, okay? So, it is an analytic extension of this function. So, on B a r. So, what what you can conclude is h as a local power series expansion at a because its analytic.

So, by Taylor's theorem you can write h as power series. So, h of z is equal to h of a plus, well I will suppress h etcetera. So, I will just write C naught plus C 1 times z minus a, we know what this co-efficient C naught C 1 are. They are in terms of the derivatives of h and then plus C two times z minus a square etcetera plus so on. What is important is we also have a C m minus 1 z minus a power m minus 1 plus C m z minus a power n plus, so on further. For z belongs to B a r.

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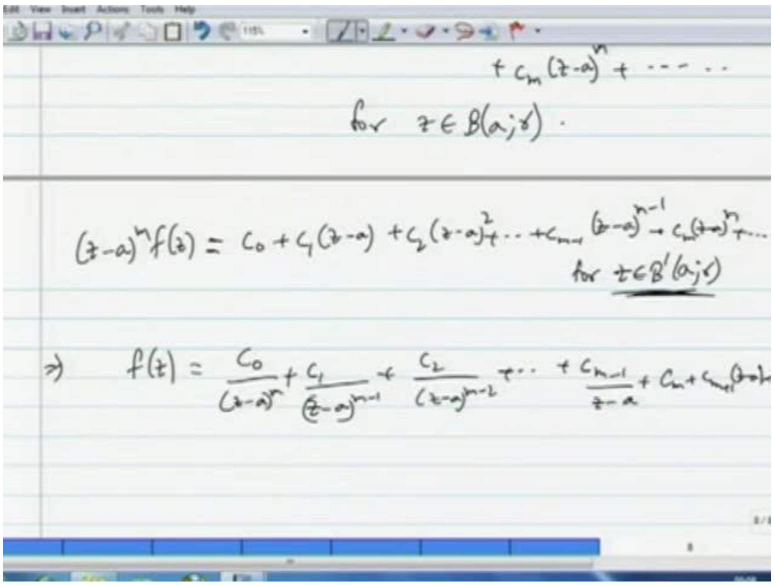

So, then let me substitute h of z is equal to z minus a power m f of z, where I will not allow z to be a. Except at the point a if I substitute z minus a power m f of z is equal to h of z what I get is C naught plus C 1 z minus a plus C 2 z minus a square plus so on Plus C m minus 1 z minus a power m minus 1 plus C m z minus a power n plus so on. For z now belongs to B prime of a r, which implies a f of z is…

So, I will divide by z minus a power m because z. Now, is not equal to a so, f of z is C naught by z minus a power n plus C 1 divided by z minus a power m minus 1 plus etcetera m minus 2 plus so on plus C m minus 1 by z minus a plus C m by sorry it is just C m plus C m plus 1 z minus a plus so on, okay? So, this portion looks like a power series. It has co-efficients C m plus C m minus C m plus 1 z minus a C m plus 2 z minus a square etcetera; okay?

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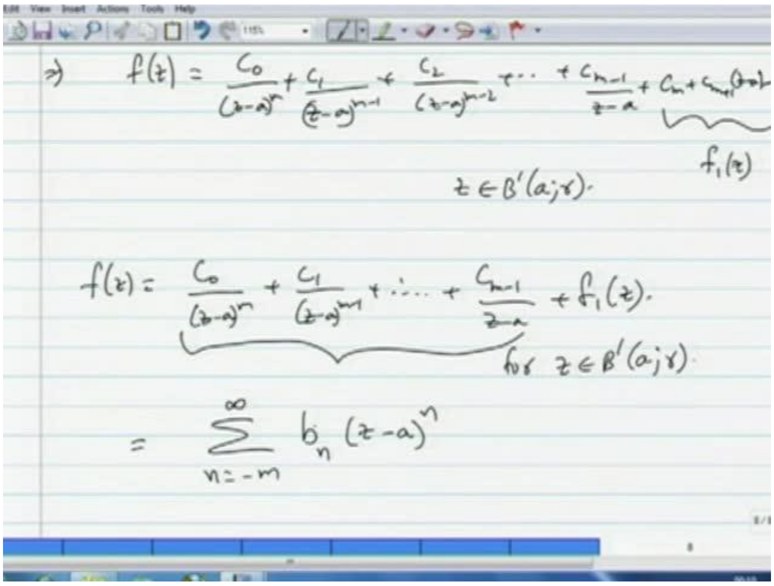

So, let me denote that portion by f 1 of z and by the way this true for z belongs to B prime of a r. So, let me denote that by f 1, so f of z is C naught by z minus a power m plus C 1 by z minus a power m minus 1 plus etcetera plus so on until C m minus 1 divided by z minus a plus f 1 of z f one of z is analytic on B a r actually, okay? But we will only use the fact that it is analytic on B prime of a r, there is no similarities of f 1 at a. So, this is true for z belongs to B prime of a r.

So, what we see is that all B prime of a r, f of z when it has a pole of order m, can be written in this peculiar form in in this form, where you have co-efficients to z minus a to the negative powers until negative m, where m is the order of the pole, okay? So, I mean you can say this is sigma n goes from minus m to infinity z minus a power n. So, m starts at minus m and goes on until infinity and then you have f of z can be represented in this form. So, this portion which is, which has negative powers of z is called that singular part of f, okay?

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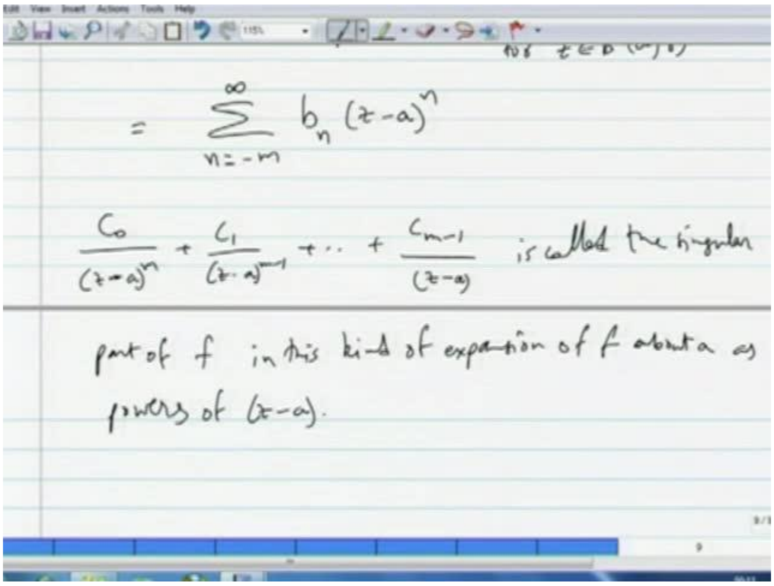

So, C naught by z minus a power m plus C 1 by z minus a power m minus 1 plus so on plus C m minus 1 by z minus a, is called the singular part of f, is called a singular part of a f. In this kind of expansion of f about a as powers of positive or negative powers of z minus a; so, we see that by the uniqueness of part of Taylors theorem. So, we saw that the power series representation of h of z is unique the co-efficient C naught C 1 etcetera, they are unique, they are actually certain derivatives of this h itself. So, for example, C 1 is a first derivative of h a t divided by one factorial etcetera. So, we saw the uniqueness of the co-efficients that uniqueness boils round to the uniqueness of representation of f in this fashion. And this kind of expansion for the pole is called the Laurent series expansion for a pole, okay?

So, if a function is analytic in a neighborhood of a and it has a pole let a, then this expansion is called Laurent’s expansion. We will see more general scenarios, but so this is true. And I want to motivate, I want to consider the motivation I have given earlier. So, once again, since the power series converge in the neighborhood of a, so I am taking about h here. Since, the power series converge in such a fashion, that if you want to find the derivative of h, you can find that derivative term by term of of these series and then take the submission, okay? So so also the integration of h can be done term by term. And so that boils round two integrating f term by term.

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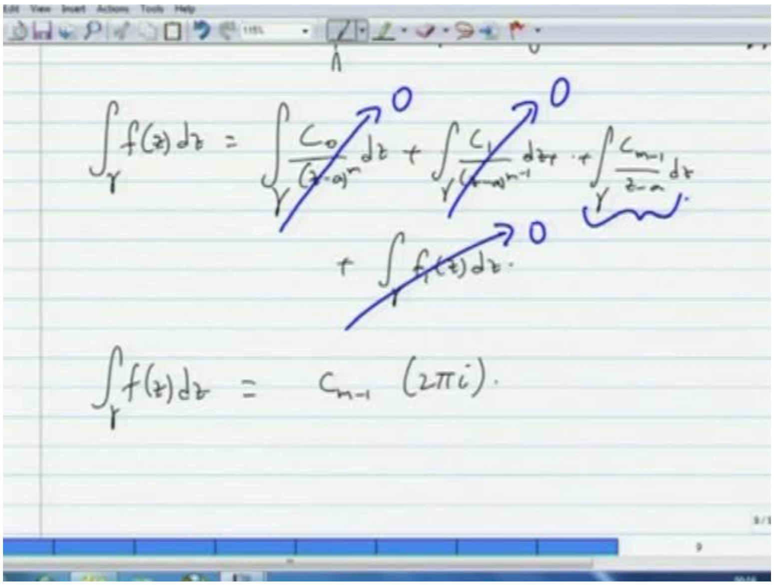

So, what I mean by that is if gamma is a simple closed curve oriented in a clockwise in the positive sense, counter clockwise direction, oriented positively. So, gamma is a simple closed curve, I should say in in B prime a r, such that such that a belongs to the inside of gamma. If a does not belong to the inside of gamma, then there is nothing much to say because the integration is 0. Then what happens is that integral along gamma of f of z d z, what is that going to be? That is going to be, well I just said that I can integrate all of this term by term.

Well if you consider f of f 1 to be I mean this 1, if you consider to be analytic function, so there are finitely many terms, okay? So let me integrate that that finitely many terms. The integration of this is going to be the integration of C zero divided by z minus a power m d z over gamma plus etcetera integral over gamma C 1 divided by z minus a power m minus 1 d z. So, on plus integral over gamma of C m minus 1 by z minus a d z plus integrate over gamma of f 1 over z d z, okay? So, f 1 is analytic on and inside of gamma, so we know that this integral is 0. So, we know that this integral is 0 and we know we have already done the fundamental integral the integration of 1 by z minus a power m is 0.

Except for when m is equal to 1 all this guys are 0 except for this part and we know the integral of this, when gamma is a simple closed curve oriented in the counter clockwise direction. So, the integration of that is 2 pi I, so this gamma f of z d z is equal to C m minus 1 time 2 pi, okay? So, it is actually the co-efficient of 1 by z minus a in this kind of a expansion of f of z, which we are going to call Laurent's expansion, which a the co-efficient 1 by z minus a tells you whether this integral is 0 or not? And when this integral over gamma of f of z d z is 0 for every gamma in the B prime of a r, then we know that f has a an anti derivative, f has an anti derivative. So, it all boils round to for at least the case of a pole, it all boils round to whether C n minus 1 is 0 or not, in order that f has anti derivative in B prime of a r. So, let us summarize this.

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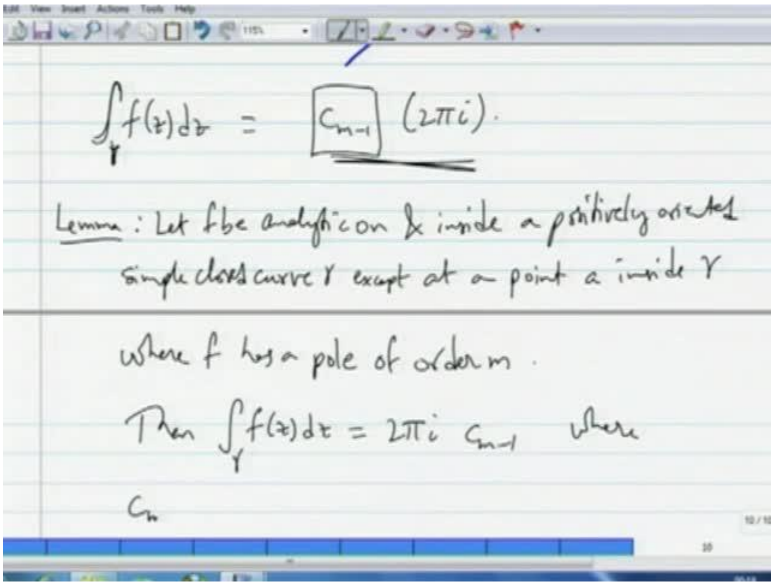

So, I will summarize what we will summarize what we have done as lemma. So, f be analytic on and inside a positively oriented simple closed curves gamma, except at a point a inside gamma, where f has a pole of order m.

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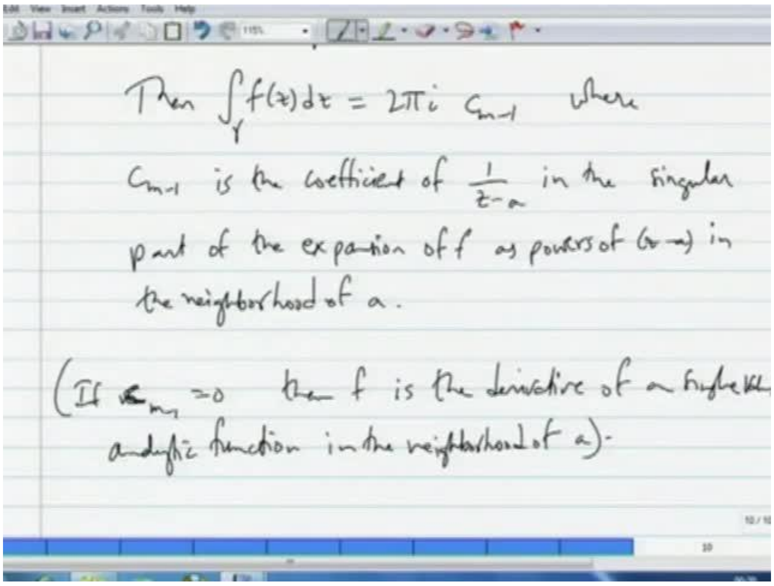

Then integral gamma f of z d z is equal to 2 pi times C m minus minus 1, where C m minus 1 is the co-efficient of 1 by z minus a in the singular part of the expansion of f as powers of z minus a in the neighborhood of a; so that is the summary and if C m minus 1 is equal to 0. So, not I mean I I can state it separately if C m minus 1 is equal to 0, then f has or f is the derivative of a single valued analytic function in the neighborhood.

So, it is it is identically equal to the derivative of a single valued analytic function in the neighborhood of a, at least in the neighborhood. So, far as a gamma is in that, I mean that neighborhood is contained inside of gamma in the statement of gamma. So, it has an anti derivative. So, we will continue to develop Laurent's theories of expansion and we will see the Cauchy's residue, theorem in the next session. I will stop here.