Complex Analysis
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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 2
Poles Classification of Isolated Singularities

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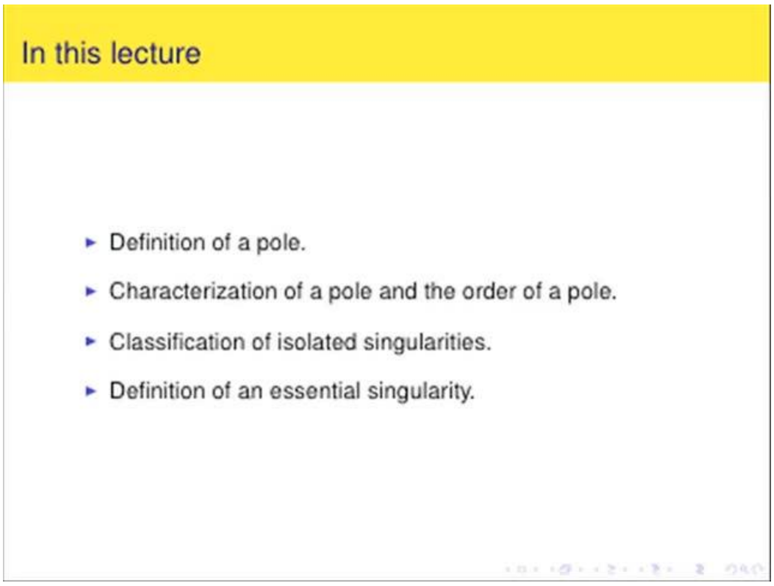

Hello viewers, in the last session, we learnt about removable singularities. So, couple of comments are in order, firstly that we can redefine the function. So, recall what a removable singularity is, it is such a singularity of a function of, an analytic function in the neighbourhood that there is an extension of that function to an analytic function at the singularity itself.

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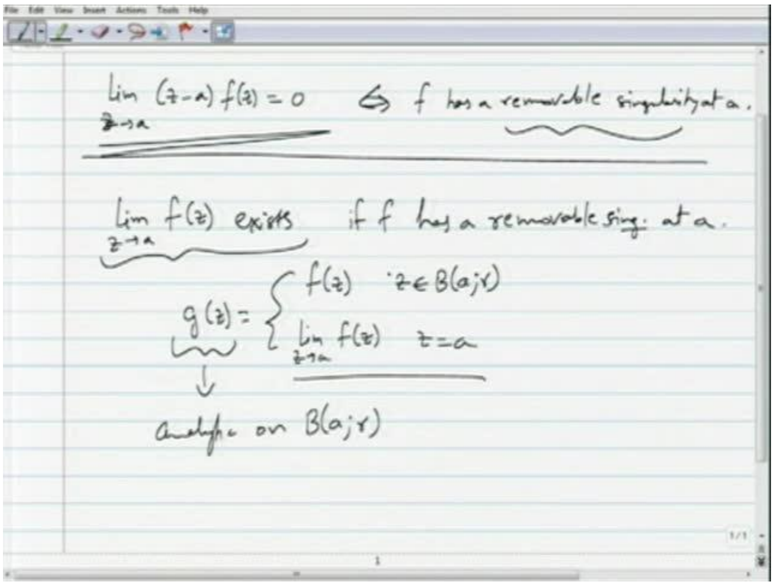

So, last time we showed a theorem that limit z goes to a. So, under the assumption that a is a removable singularity of a function f if limit z goes to a z minus a f of z is equal to 0, so this is if and only if f has a removable singularity at a, okay? So, recall what that means? It means that there is an analytic extension of f at the point a. So, often we will, we will whenever there is a removable singularity, we will say that we will remove the singularity at a for f. What that will mean is that, we will consider the extended function, So, kind of a singularity is a fake singularity because all that is lacking is information of the value of f at a. So, by the theorem that we proved last time, this is the theorem that we proved last time. I mean briefly stating, so what we have is that now in retrospect limit z goes to a f of z exists.

If f has a removable singularity at f removable singularity at a I will give a shortcut at a, a sing means singularity, okay? Then what we have is g of z is equal to f of z in ascertain for z belongs to b a r and it is equal to limit as z goes to a f of z if z is equal to a. So, that is your redefinition of f, so this g is now analytic on last time, we proved that this is analytic on B a r. So, we had not exactly prove this, but in retrospect this exists, if and only if this limit exists. So, in retrospect we can define g of z to be this at z equals z and so that g of z is going to be analytic on B a r, alright?

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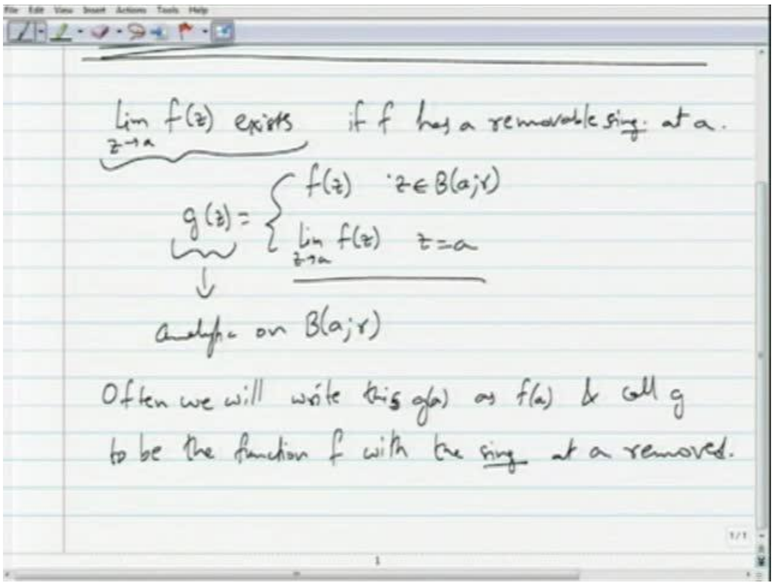

So, so often we will often, we will write this, this g as or I will say that often we will write g of a as f of a, f of a and call g to be the function f with the singularity at a removed. So, we will say that their singularity at a has been removed of f has been removed and the new function with f of as g of a is the extended function. So, this is a piece of notation if you will or language. So, next we will consider that other kind of singularities namely poles.

So, recall in the examples that I gave in the last session, there was a case where where the the the modulus of the function in a small neighbourhood around the singularity first standing to infinity as the the z the variable approaches the singularity. So, this is a condition we will use to make a definition of a pole.

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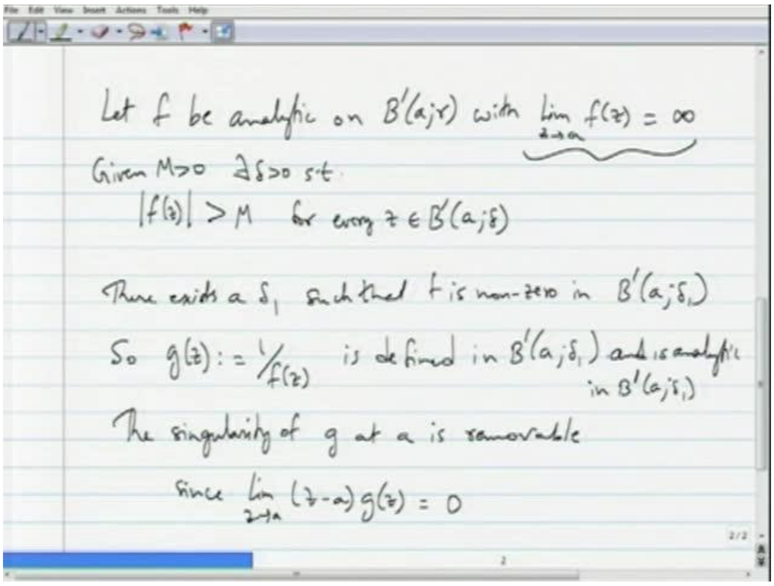

So let, let f be analytic on B prime a r. So, with limit z goes to a, so it is analytic in a punctured neighbourhood of a, but when you consider the limit as z goes to a of f of z that is infinity. So recall what that means? It means that the modulus of f is arbitrarily large in, in a small neighbourhood. And it is, it is large for every value of z in an arbitrary arbitrarily small neighbourhood. So, what this means recall, limits as limits tending to infinity.

What that means is that modulus of f of z is greater than, so given M positive there exists a delta positive such that modulus of f of z is strictly greater than M for every z belongs to B B prime of a delta. So, that is what this means. So, that is true for every M, so given any M positive we can do this. So, that is your limit of f tending to infinity, okay? So, by this very definition picking M to be M to be a 1. Let us say we can say that there is a delta such that modulus of f is not 0 in that delta neighbourhood of a. So, there exists a, there exists a delta 1, such that f if non-zero in B prime a delta 1. Simply picking M equals 1. For example, gives this delta 1, so g of z defined as 1 by f of z.

Let us notice the behaviour of g, is defined at least, is defined in B prime of a delta 1. Not only that g is defined not only in b prime of a delta 1, but at a we will notice that g has a removable singularity g defined this way as a removable singularity. So, g has the singularity of g is defined in this and I should also say and is analytic there in B prime a delta 1 because its 1 by an analytic function it is also analytic. So, the singularity of of g at a is removable, that is because our favourite condition limit z goes to a z minus a times g of z.

Let us notice what that is, that is that is equal to 0, because 1 by f of z is arbitrarily large in modulus and sorry, f of z is arbitrarily large in modulus. So 1 by f of z is very small in modulus and so is the modulus of z minus a. So, in modulus z minus a times 1 by f of z tends to 0. So, in modulus when some quantity tends to 0, you can conclude that the complex quantity itself tends to 0. So, that is, that is why limit z goes to a z minus a, g of z is 0, so Also notice that since 1 by f of z itself tends to zero in modulus, limit of g as z goes to a is actually equal to 0. So, we can redefine g at we can remove the singularity of g at a and define g of a, to be 0. So, that is the value of the limit as z goes to a of g of z.

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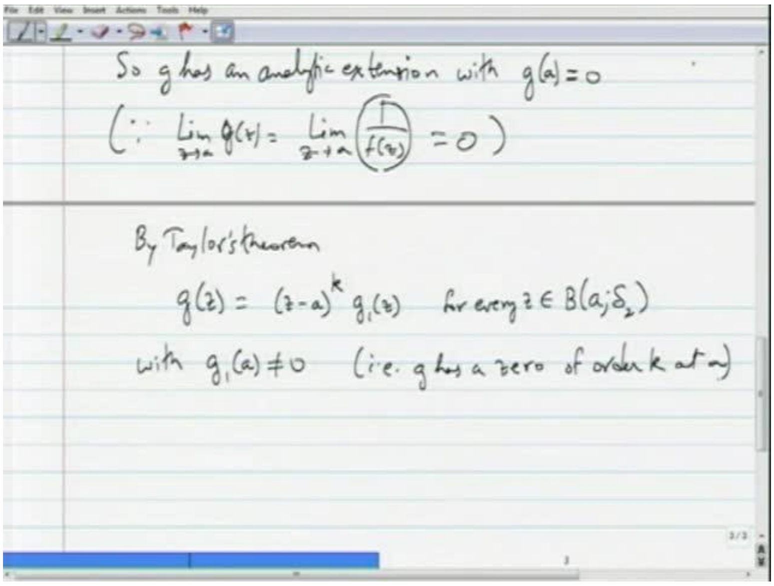

So, we can define, so g has an analytic extension with g of a is equal to 0. So, notice that I am not giving a new name to the analytic extension, I am not calling it some H. I am once again calling the analytic extension also as g, you know like I said in the remark above so g has an analytic extension with g of a is equal to 0. That is since limit z goes to a g of z is equal to i will write below sorry limit z goes a g of z g of z is equal to limit z goes to a 1 by f of z; and in modulus f of z is arbitrarily large in every I mean for a every point in neighbourhood.

So, this tends to 0 in modulus this quantity. So, this is equal to 0, so we redefine that to be that and g is like that. Now, the extended function g which are, which we are still calling it g has a 0 at a from the study of zeros of analytic function from earlier sessions. We know that it has to have that 0 of g at a has to have some order. So, by by Taylors's theorem, so by Taylor's theorems and its conclusions by Taylor's theorem, we can write g of z is equal to z minus a power k g 1 of z for every z belongs to B a. I will have to change this delta to some delta 2, because Taylor's theorem is a local theorem.

So, the Taylor's expansion for g, g is an analytic function now. Taylor's expansion for g around a is valid in some two neighbourhood of a and g has a 0 at a, so we can pull out a factor z minus a power k from g with, with g 1 of a naught equal to 0. i e g has a 0 of order k at a. This, this is something we know from earlier. So, g one of a is non-zero. So, what we can do is we can now substitute what g of z is, okay?

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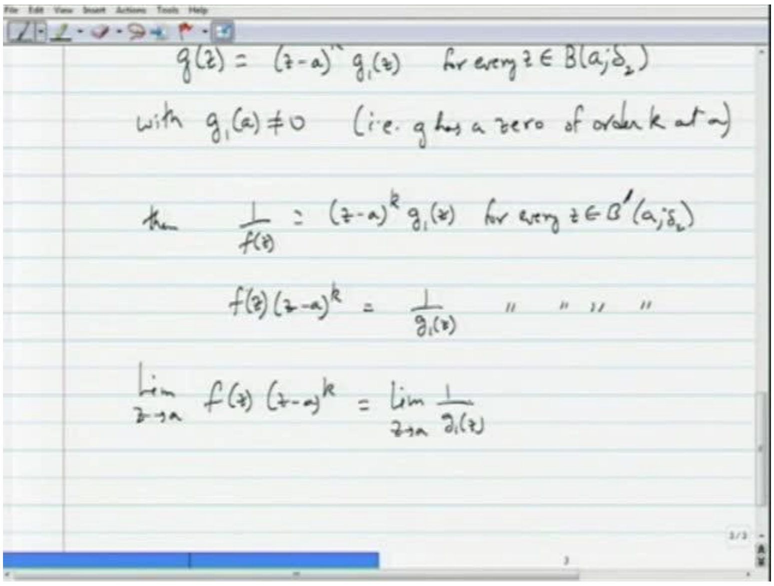

1 by f of z is equal to z minus a power k g 1 of z for every z belongs to notice I am deleting a itself because f has a singularity at a. So, in B prime of a delta 2 and so g 1 of z now or I will say f of z times z minus a power k is equal to 1 g 1 of z. Once again for every z belongs to b prime a delta 2, so when I take the limit as z goes to a I can take the limit of this quantity on the left hand side, that is equal to the limit as z goes to a 1 by g 1 of z.

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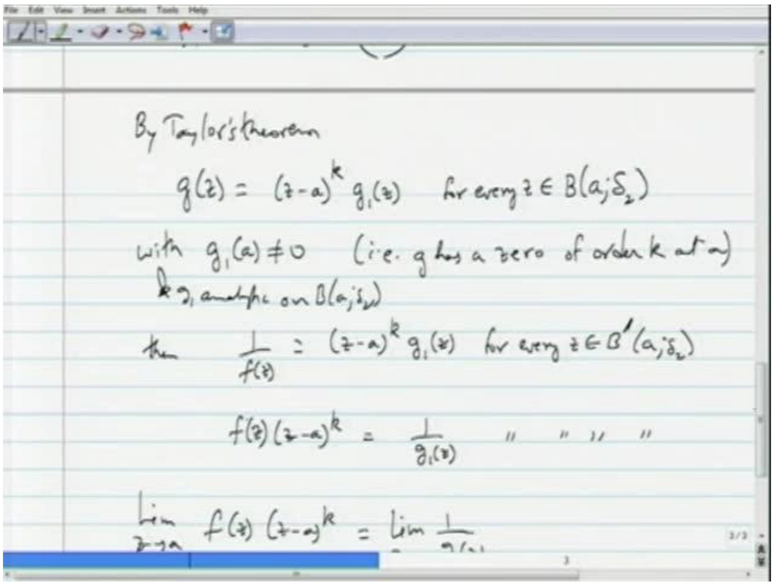

I probably should have remarked that with g 1 of a not equal to 0 and g 1 analytic on b a delta two. So I will need that because analyticity gives me continuity of g 1, so this is equal to 1 by g 1 of a and g 1 of a is non zero. So, 1 by g 1 of a is also a non-zero complex number.

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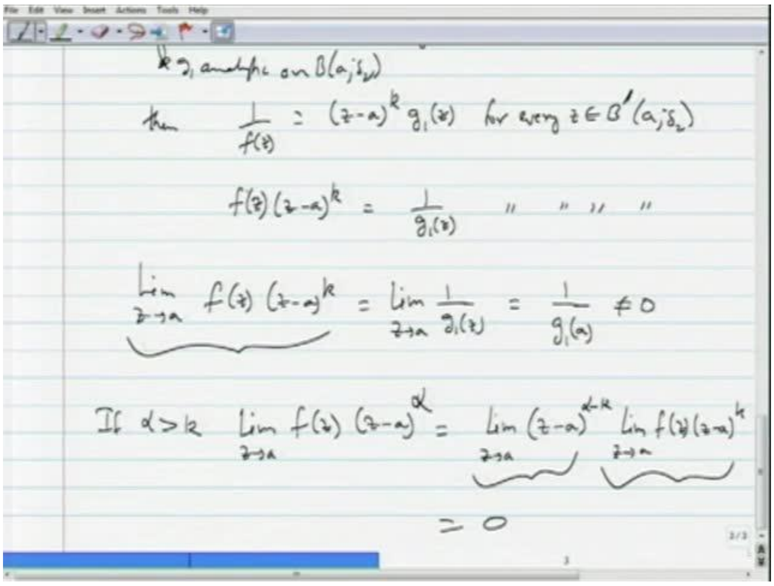

So, the limit as z goes to a of f of z times z minus a power k is a non zero quantity like this. So, that is a conclusion we draw and are noticed that if, if alpha is greater than k, then limit z goes to a f of z times z minus a power alpha is equal to limit z goes to a z minus a power alpha is greater than k. So, I can subtract a k and then limit z goes to a f of z times z minus a power k. So, this quantity is finite complex number and this quantity is 0. So, this is equal to 0 also if alpha is less than k.

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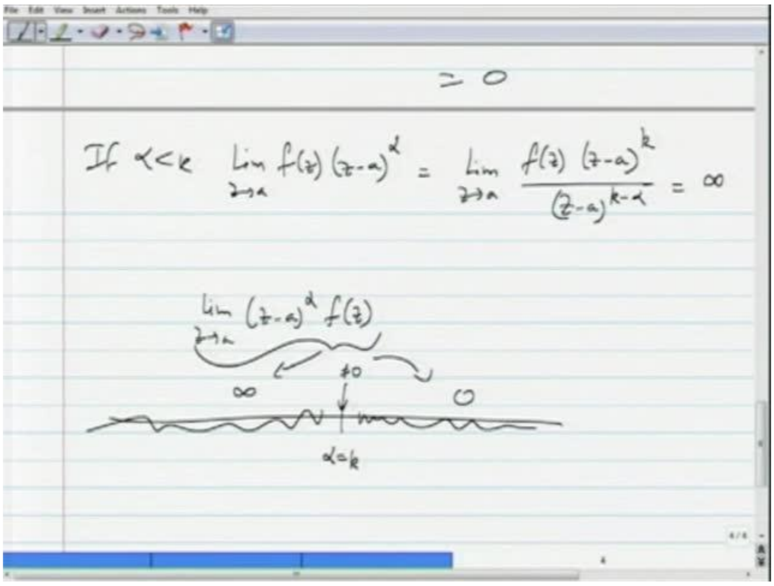

If alpha is strictly less than k limit z goes to z f of z times z minus a power alpha can be written as limit as z goes to a well you are f of z times z minus a power k divided by z minus a power k minus alpha, which is positive k minus alpha is positive. So, have a we have a term pending to a non-zero complex number in the numerator finite complex number. The denominator we know blows up in modulus it has a very large modulus as z goes to a this, this limit is equal to infinity.

So, we have a tie at k, so that is the situation f of z times z minus a power alpha is non zero at alpha equals k and for alpha less than k it is the limit is infinity. For alpha greater than k we have limit is 0, so this is the dividing point. So, I am looking at the limit as z goes to a of z minus a power alpha f of z for an integer alpha as alpha varies alpha is equal to k is the dividing point at this point this limit is this limit is non zero, some finite complex number. For every alpha greater than k this is this is 0. This limit is 0 and then for every alpha less than k this is infinity this limit is infinity. So, there is a neat divide like that for this limit.

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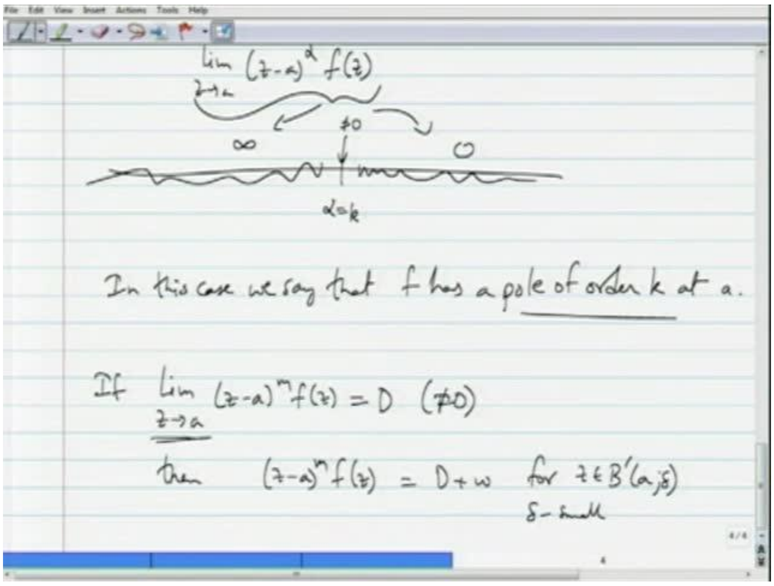

So, what we will say is that say in this case we say we say that f has a pole of order k at a. So, let me allow me to go back. So, if 1 by f of if 1 by f of z has, I mean it has a removable singularity at a. If 1 by f of z with its extension has a 0 of order k at a, then we say that f has a pole of order k at a. So, that is, that is pole of order k. So, we have a order for a pole not only that if the limit as z goes to a of z minus a power M. If the, the, the condition goes other way around power M of f of z is non-zero.

If if this happens, then f will have a pole of order M at a, so it is an if and only if condition if this limit exists and is non-zero, then f will have a pole of order k or M whatever that constant is at a. And if it has a pole of order k or M then that limit accordingly will be non-zero, okay? So if I am I am trying to show the other direction, I will give a Heuristic proof, if limit z goes to a z minus a power M f of z is non zero, let us call it D. It is some D not equal to 0.

What I will show is f as a pole at a of order M. So, if that happens then, then what happens to z minus a power M f of z in a neighbourhood in a, small neighbourhood of a. This is equal to d plus some w where is w is a, a complex number with very small modulus where for first I should write for, for z belongs to B prime a delta delta small. w has a small modulus, okay? So that is what limit means anyway, so the modulus of z minus a power M f of z is between d plus epsilon is, is between the modulus of D plus epsilon and the of D minus epsilon. So I can I can make this kind of statement, so given an epsilon I can do that. So, I can make this statement.

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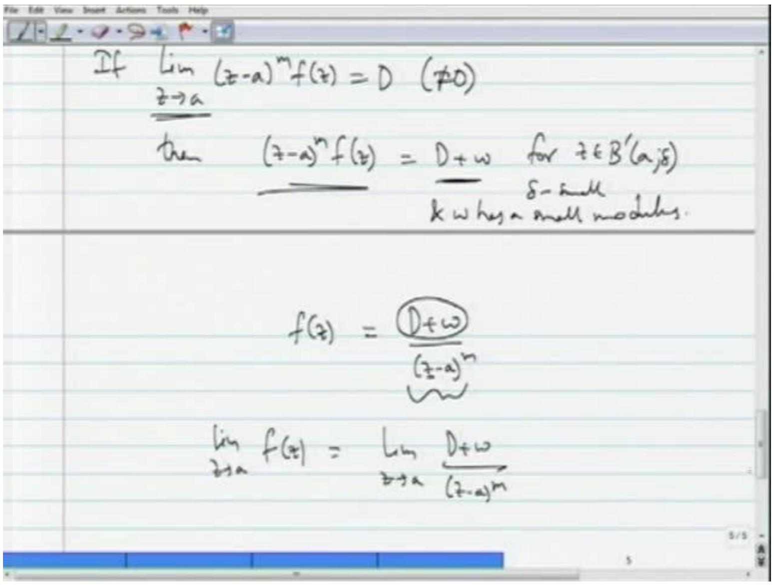

So approximately the modulus of the right hand side is closed to D, the modulus of d. So, in order to compensate the modulus of the left hand side is growing smaller and smaller as z approaches a, so the modulus of f has to compensate for the loss. So, the modulus of f has to go up so f of z is equal to d plus w by z minus a power M. So, this modulus stays closer to d its a finite quantity and this is becoming arbitrarily small, so as limit as a times z tends to a. So, the limit as z tends to a f of z has to be arbitrarily large. So, limit as z tends to a f of z is equal to limit as z tends to a d plus w. w is not a fixed quantity it also varies as that tends to a, but it stays close to 0, so z minus a power M w the the limit of w really is 0. So, then we will get this, so it is a heuristic proof. That is not really proof I mean we can workout using the limit definition, okay?

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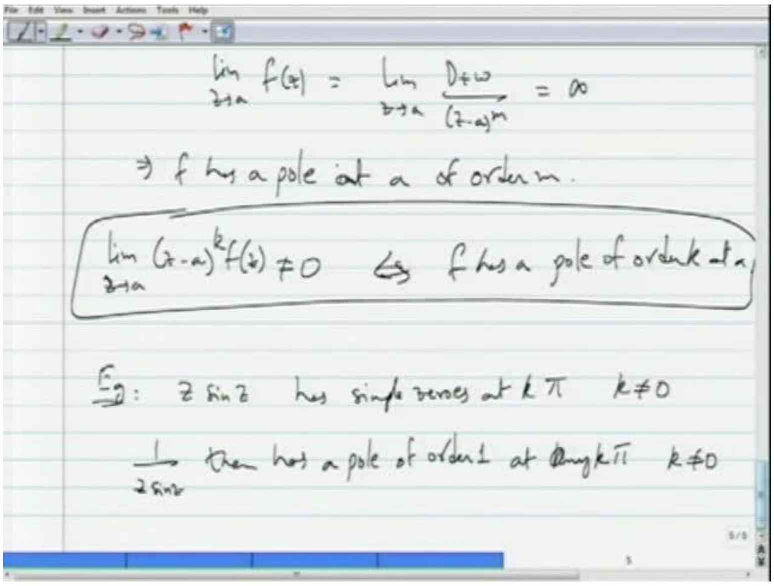

So this, this implies that f f has a pole at a of order M. So, in conclusion we have that limit z goes to a z minus a power M or k let me say k of f of z is non zero, if and only if f has a pole of order k at a. So, this is the conclusion so this is neat and useful, okay? Then here is an quick example, so let us consider z sine z the function f equals z sine z. So, we know that z sine z has a simple zeros at k pi k naught equal to 0. So, if we consider 1 by z sine z this has then has a pole of order 1.

So, I am going in the opposite direction here, I am looking at the function g and looking then at the function f. s So, what we will say is that say in this case we say, so 1 by z sin z has a pole of order 1 at any k pi at any k pi k not equal to 0. If k is equal to 0 you are looking at z equals 0. Then z is 0 and sine z is 0 as well. So, z sine z has a 0 of order 2 at at z equals 0.

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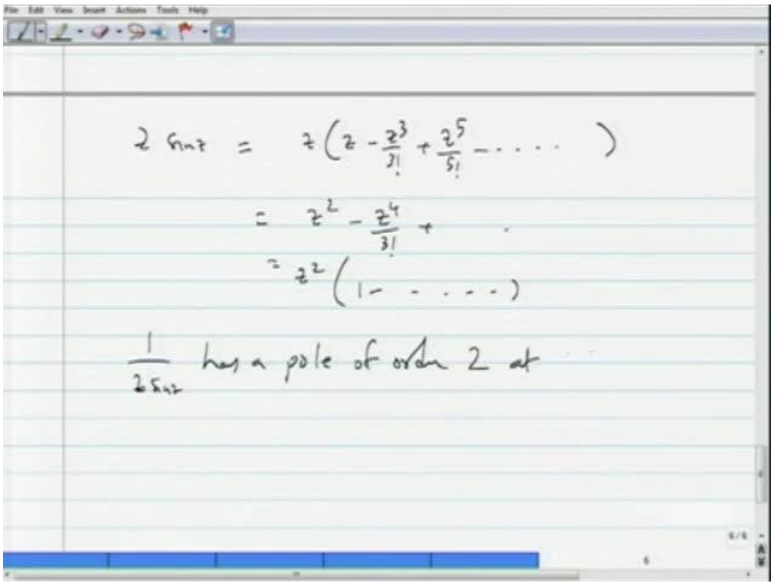

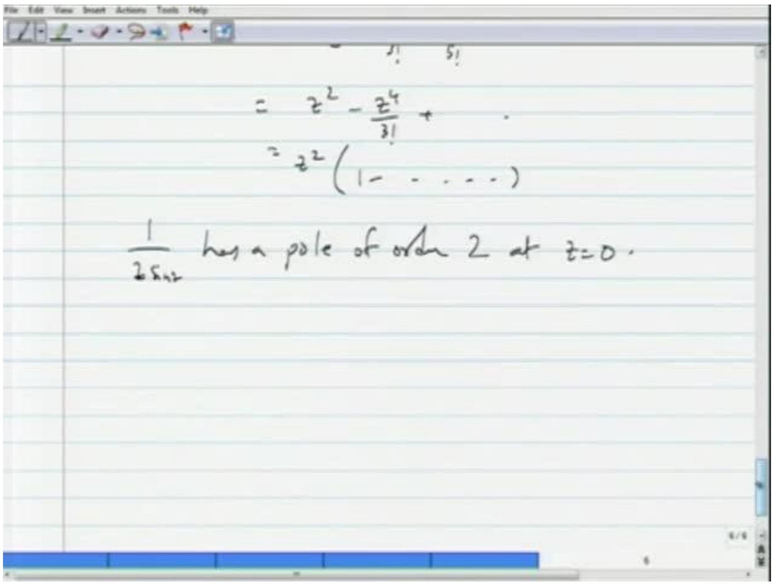

That can also be inferred from the Taylor's expansion of sine z, this is equal to z times z minus z cube by three factorial plus z power five by five factorial etcetera, that is the Taylor's expansion for sin z. This is equal to z squared minus z power 4 by 3 factorial etcetera, so you can factor out a z, I could have done that a layer 1 minus etcetera. So, z sine z has a a 0 of order 2 at at 0, so then 1 by z sin z has a pole of order 2 at...

So, it is a simple minded of a pole. So, poles and zeros work in mutually opposite ways, like we have seen. If f has a pole then, 1 by f has a 0 at at an at a point a. Then by that I mean it has a removable singularity and once you remove the singularity there is a pole there is a 0 of 1 by f. So, so what we can state is the following.

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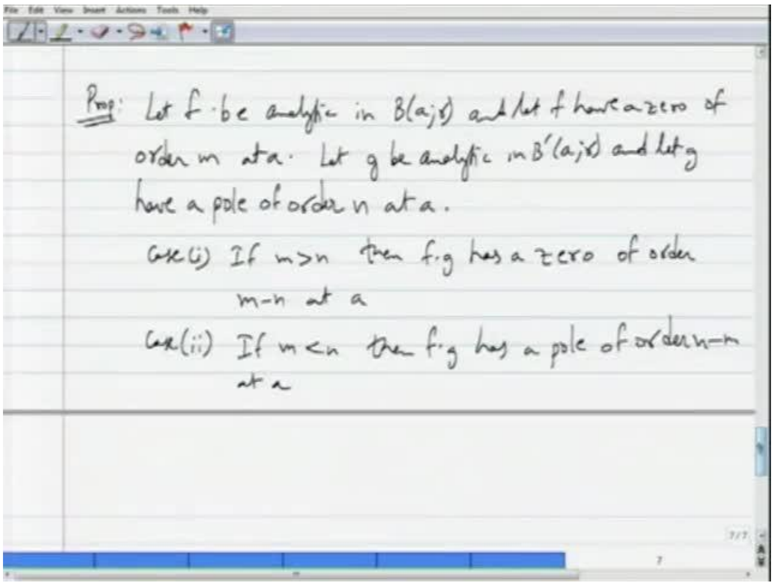

So here is a simple proposition, it is sort of the cancelling behaviour of zeros and poles. Let f be, let f and g be analytic. I will say that, f I need different domains. So, I will say differently. Let f be analytic in B a r and let f have a 0 of order m at a. Let g be analytic in B prime a r. Let g have a pole of order n at a. Case 1, if m is greater than n then f times g has a 0 of order m minus n at a. Case 2, if m is less than n then f times g has a pole of order m minus n at a. So, this sort of cancel each other, so if the poles order is more than pole dominates and then there is a pole of order m minus n there.

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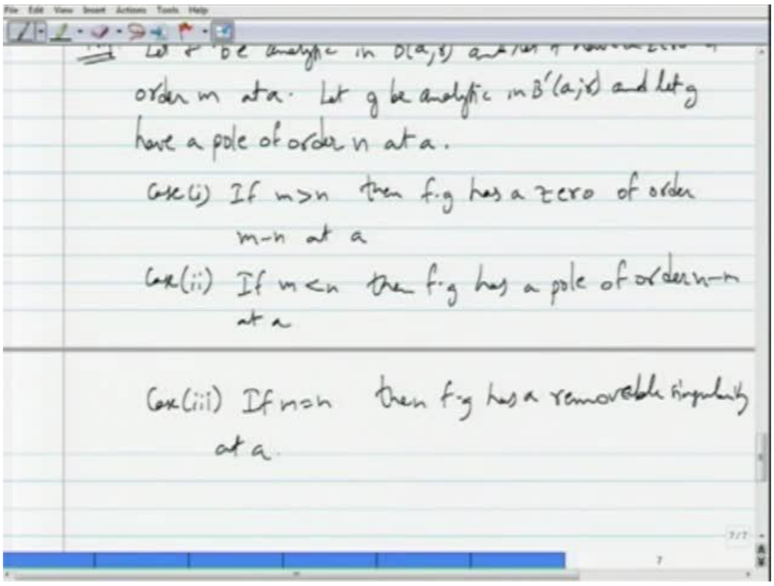

And then in the third case, if f m is equal to n, then f times g has a removable singularity. It can be removed removable singularity at a. So, these are the three cases, so this is how the pole and 0 cancel each other. So, that is a simple proposition and the proof is pretty straight forward using what we have already done. So, the viewer can provide the proof as an exercise just to revisit all the facts that we have we have to earlier. So, next I want to consider the other kind of singularities.

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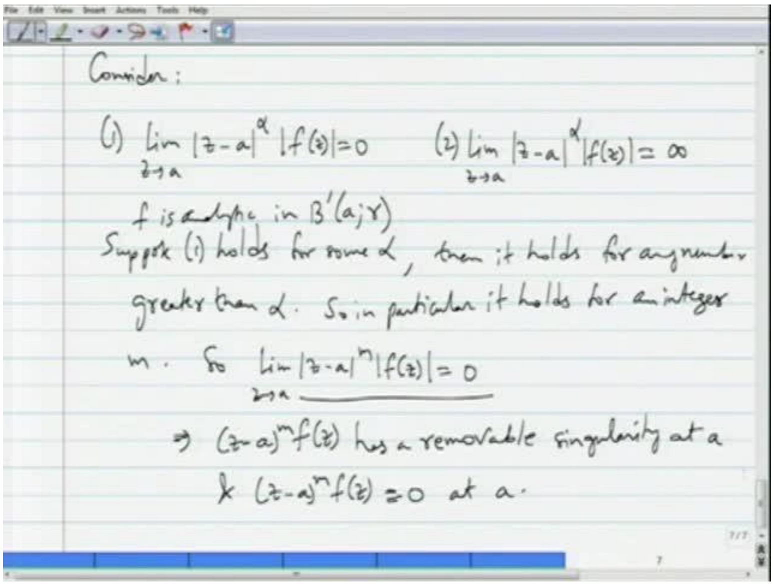

So, before I do that let us consider the following, to arrive at the other kind of singularities what other behaviour can f exhibit around a singularity? So, if f is analytic in B prime of a r, we have seen that f could actually be analytic in all of B a r it can be extended to be an analytic function at a, which is in the case, when a is removable singularity. Then f could have a pole at a in which case, well a by definition limit z goes to f of z is infinity. Then we have seen its behaviour. Now, we will consider what are the behaviour could it have at in the neighbourhood of a. Let us consider the following two equations first or limits first limit z goes to a modulus of z minus a power alpha times modulus of f of z is equal to 0.

Limit z goes to a modulus of z minus a power alpha modulus of f of z is equal to infinity. At least in the case of a pole we have seen that, there was a dividing point which was the order of the of the pole. So the value of alpha for which one held was any alpha less than k sorry, greater than k and the value of alpha for which this two held was anything less than k, where is the order of that pole. So, we will we will examine these two more closely. Let us first suppose that, suppose 1 holds for some alpha. So, given a function f suppose 1 holds for some alpha. We are working under the or supposition that f is analytic in B prime a r for some r positive, okay?

Suppose 1 holds for some alpha, then it holds because the limit is 0, so you can jack up the power alpha as much as you wish. The limit z goes to a of modulus of z minus a power alpha f of z in mod in modulus will be equal to 0, then it holds for any alpha greater than in any real number. Let us say greater than alpha, it is the first statement. So, in particular it holds for an integer m. So, and you pick an integer from greater numbers greater than alpha. So, for the time being I am assuming that 1 holds for some real number alpha.

So, then it holds for an integer greater than that, so limit, so limit z goes to a modulus of z minus a power m modulus of f of z is equal to 0, okay? So, this implies that we can, we can increase the power by one more, z minus a power m plus one f of z the limit of that as z goes a is also 0. So, this implies that z minus a power m f of z has a removable singularity. Singularity at a a and since the limit of this function itself is 0. As z goes to a, we saw how to redefine functions at removable singularity. We will take that limit of that function as z goes to a and, and so this and z minus a power m f of z vanishes i e is equal to 0 for considering the extended function is equal to 0 at a. So, that is how we redefine.

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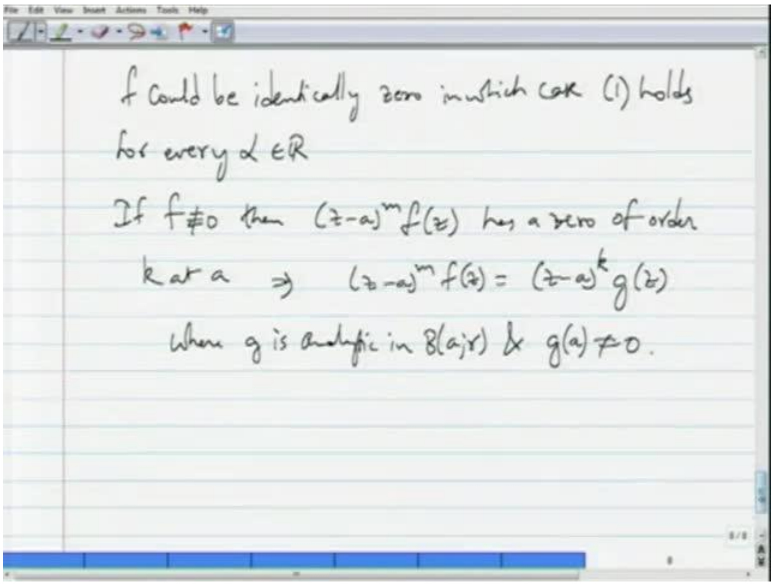

So, case one what could happen is f could be identically 0, if z minus a power m f of z is 0. One of the cases is that f could be identically 0, is the uninteresting case in which case one holds we considered two equations here. So, one holds for every alpha belongs to r, so limit z goes to we started by assuming that it holds for one particular alpha and what we have concluded is that z minus a power m f of z in modulus is equal to 0, in the limit. So, that will allow us to conclude in the case that f is identically 0. That alpha I mean for any alpha that one holds.

So, this is the uninteresting case, but what is more interesting is that if f is not identically 0, then z minus a power m f of z we saw that it is already 0. The analytic extension of this is 0 at a, so has a 0 order. Let us say k at a, this implies z minus a power m f of z is equal to z minus a power k times g of z, okay. g of where g is analytic in B a r and g of a is non zero in some neighbourhood of a g is analytic and g of a is non zero.

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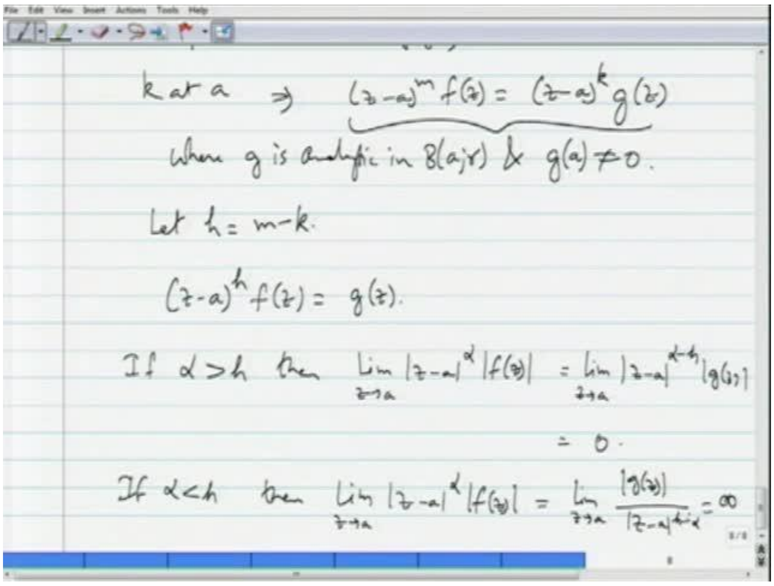

So, by subtracting let, let us, let h equal m minus k. So, we will make a, from this equation what we can do is we can write this as z minus a power h f of z is equal to g of z. Now, if if alpha is greater than h g is analytic, so if alpha is greater than h, then limit z goes to a modulus of z minus a power alpha modulus of f of z is the modulus of g of z is the limit as z goes to a modulus of z minus a raised to alpha minus h modulus of g of z. So, and that is equal to 0 because g is analytic at a g has some limit g of a and then modulus of z minus a alpha minus h raised to alpha minus h is 0, okay?

Likewise if alpha is strictly less than a, then limit z goes to a modulus of z minus a power alpha modulus of f of z is equal to limit z goes to a of modulus of g of z divided by modulus of z minus a power alpha minus h minus alpha. That is infinity that limit is infinity, so there is a divide h is integer.

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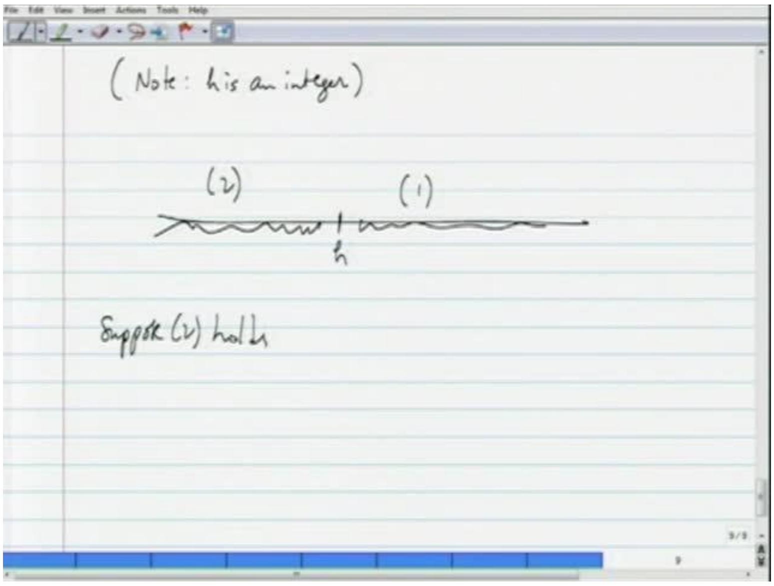

Notice h is note h is an integer, so in summary we are able to conclude the following. So, if we assume that one holds under the supposition that this, this limit one holds we are able to conclude that, there is a an integral point h on the real line. So, there is an integral point h on the real line such that for anything less than h 2 holds and then for anything greater than h 1 holds.

So, it is an integral divide here and then so it is a first conclusion and then likewise if if we assume that suppose two holds one can likewise show. Suppose, two holds, one can likewise show that there is an integral divide on the real line for which two holds for every integer less every number real number less than h and one holds for every real number greater than h.

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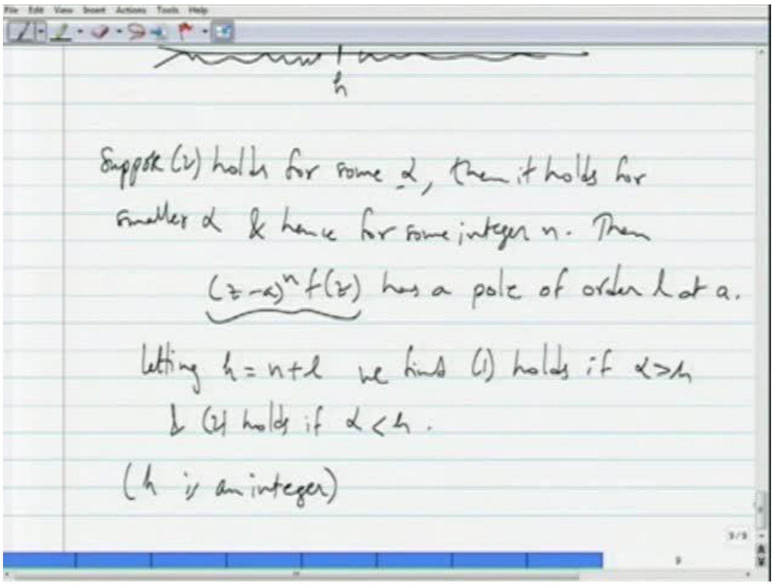

So, I will, I will show that, so suppose two holds for some alpha then it holds for smaller alpha. That is because you know if things tend to infinity and you are your decreasing the power of z minus a. So, then the modulus is I mean the modulus of f of z is competing with the modulus of z minus a power alpha and is any way tending to infinity. Now, if you reduce the power of the quantity which, which becomes smaller the competition is more fears in the sense that or competition is you know pretty much winning for f of z. So, it becomes even more in modulus, so this limit for alpha less than that particular or quantity less than that particular alpha, it still tends to infinity.

So, then it holds for smaller alpha that is easy and hence for some integer n less than alpha really. So, then once again we can do the same analysis then z minus a power n f of z similar analysis, so has a pole of order l at... So, that is because the modulus of z minus a power n times modulus of f of z in limit is equal to infinity. So, this function has a pole let us call the order to be l, so then what we know is that letting, letting h equals n plus l like we have done for the other case, we find one holds if alpha is greater than h. So, I am writing the conclusion, it is easy to argue like we have done here, like we have done all of here. So, one holds if alpha is greater than h and two holds if alpha is less than h. Once again here h is an integer notice h is defined to be n plus l and once again on the real line.

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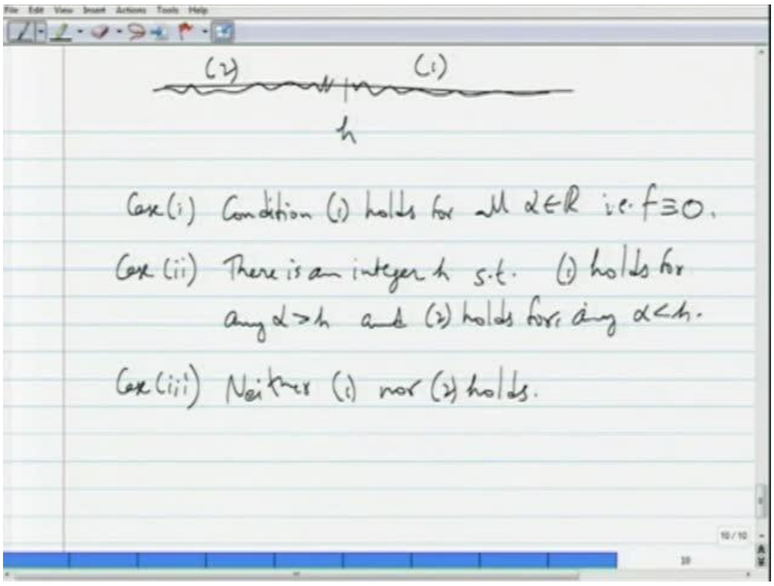

We have a divide at h for alpha less than h we have that two holds and alpha greater than h one holds. So, there are based on this analysis there are, if you assume that one holds then there is a integral divide on the real line where one holds for values greater than that integer and two holds for values less than that integer. Likewise if we assume two holds for some alpha the same kind of situation situation prevails.

So, one and two are mutually related the the the two kinds of limits that I stated earlier are related in this fashion. So, there are three cases based on this analysis case one condition one holds for all alpha belongs to r i e f is identically 0. That could be a case if f is analytic in deleted neighbourhood of r there is an integer. The second case is there is an integer that is with the suppositions. So, there is an integral divide so there is an integer h such that one holds for condition one holds for any alpha greater than h and condition two holds for alpha less than h for any alpha less than h. Then there is a case three, we saw that if one holds two should hold and if two holds one should hold unless there is case one where one holds for everything.

So, case three is that neither one or two holds in the neighbourhood of a, so what that is telling is that neither does f tend to infinity in modulus near a nor does nor does f have a 0. Of course, at a nor does f have a removable singularity. And f of course, we are assuming is not identically 0, which is the uninteresting case. So, in this kind of scenario, what, what really one can say is that f f if oscillating in its values in small neighbourhood surrounded. So, if you consider a small enough of neighbourhood they f could be possibly very large or possibly very small, but both the kind of behaviours should exist.

May be it, it tends to one limit and then there is a subsequence which we can consider in the neighbourhood, which tends to some limit and then there is another subsequence in the neighbourhood which tends to yet another limit, so that, that can be a possibility that is the case three, okay? So, neither one or two holds, which means, so that is the behaviour like I explained. But we can make a very strong statement in this direction of the behaviour of f in case three and that is the Casorati Weierstrass theorem. So, I will, we first call case three we will give case three a name.

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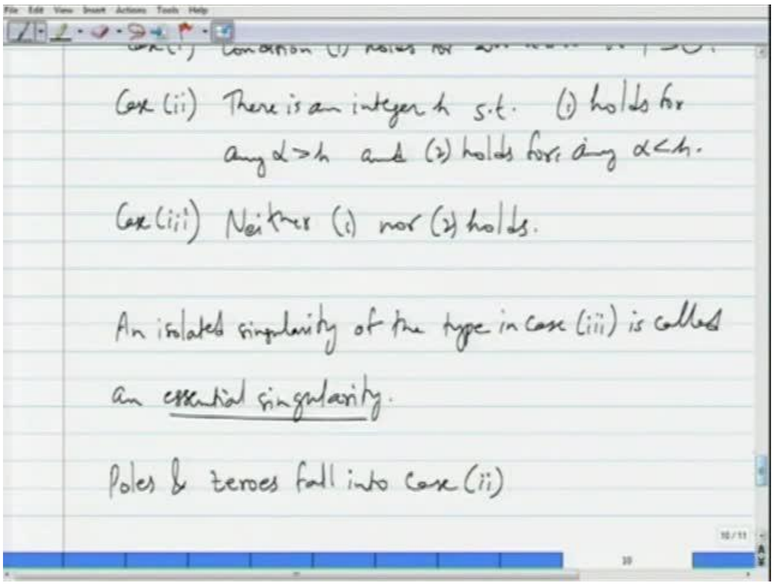

An isolated similarity of type three, of the type in case three is called an essential similarity. So, that is that is the definition of an essential similarity. Also, notice that case one is uninteresting in the sense that its well trivial case where f is identically 0 poles and poles and zeros fall into case two. What I mean by that is if f has a 0 or if f has a pole then, then case two holds even if it has a removable singularity. Then, then case two holds, so we saw that, so there is a integral divide in the case of poles the divide is at the integer, which is the order of the pole.

In case of zeros divide is at the 0 the order of the 0 at a in the case of removable singularity. Well it depends, it depends the limit as z goes to a of f of z. So, limit z goes to a f of z could be 0, which will, which will actually then give us a case where the order of the 0 will matter if the limit z goes to a f of z is non zero. Then the divide the integral divide is, is really at, at h equals 0. So, removable singularities and poles and zeros fall in case two. Okay?

Then everything else in some sense well everything else is case three if neither one or two holds, then then we have what is called an essential singularity. So, we will see more about the behaviour of a function which is analytic in a neighbourhood of a and it has a isolated essential singularity at a. Further so we will particularly state, the state and prove the Casorati Weierstrass theorem.