Complex Analysis

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Module - 6
Isolated Singularities and Residue Theorem
Lecture - 1
Removable Singularities

Hello viewers, in this session, we will learn about the singularities in particular isolated singularities of an, of an analytic function. So, firstly a point in the complex plane, where a function f is analytic is called a regular point. So, a singularity is such a point, where f is not defined, so of particular interest are singular points, which are surrounded by points, which are regular or in other words if there is a point a, such that in the neighborhood of a f is analytic. Then such a point is said to be a singularity, isolated singularity of f and we are interested in predicting the, the behavior of f. Then around this point are using the behavior of f around this point in order to classify such isolated similarities. So, I will first define a regular point and a singular point.

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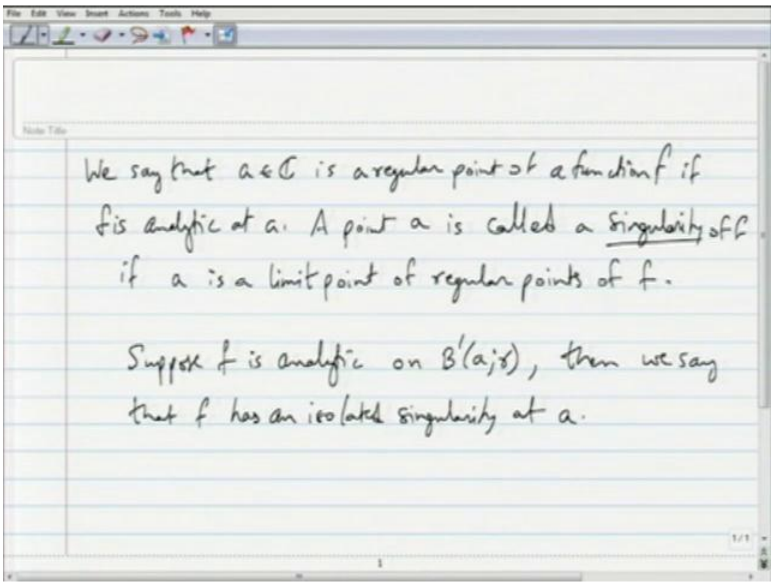

So, a regular point we say that a belongs to C is a regular point of a function f. If f is a analytic at a, and a point a is called a singularity of a of f rather called a singularity of f. If a is a limit point of regular points of f, so that is a singular. So, suppose f is analytic on B prime a r, then we say that f have an isolated singularity, at a ok. So, the first trivial case is the case, where f is actually analytic at a. So, it is a, it is analytic in B prime a r recall B prime a r is the is the deleted neighborhood of a.

So, we are removing a from the ball of radius r around a. If f is analytic in this de punctual neighborhood or deleted neighborhood it could happen that actually f is analytic at a, but we did not define it there, that is all. So, in that event we say that such a point is a removable similarity of f at a and there are other cases, but first this, this is I mean first this is the trivial case, which we will consider and then we will classify other kinds of behavior of f around a.

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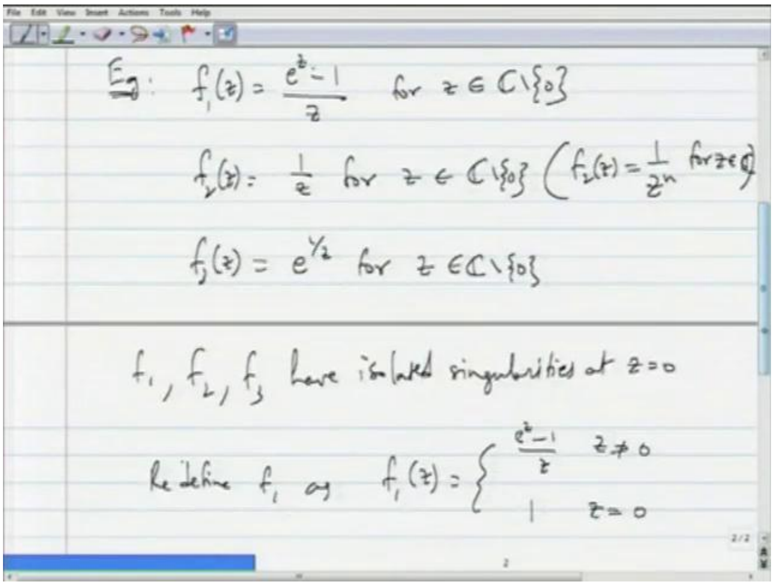

So, firstly ill start by giving some examples, so consider these functions f 1 of z define by e power z minus 1 divided by z for z belongs to C and f 2 of z. So, firstly I will I will complete these three examples and f 2 of z is 1 by z for z belongs to C. One can actually consider f 2 of z equals 1 by z power n. n is any integer for z belongs to C, okay? f 3 of z is equal to e power 1 by z for z belongs to C. I apologize C minus 0 at 0 at this function is not defined, so C minus 0 and even this is C minus 0.

All these are functions on C minus 0, so all these have singularities at the point 0. So, f 1, f 2, f 3 have isolated singularities at 0 at z equals 0. And we could I mean f 1 could be redefined to be, so these are three different kinds of isolated singularities as we will see. f 1 could be redefined as f 1 of z, so let us redefine f 1. As f 1 of z is whatever it is given to be e power z minus 1 by z for z naught equal to 0 and let us define it to be 1 for z equals

0. So, since we know that the limit as z goes to 0 of this quantity e power z minus 1 by z is actually equal to 1. So, that limit is 1, so if you redefine f 1 to be 1 at 0, then f 1 is actually analytic.

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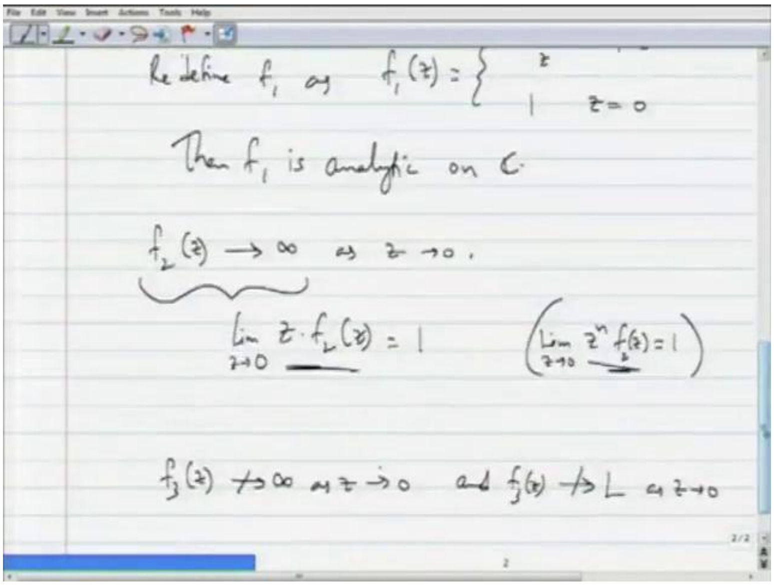

One can check f 1 is analytic con the whole of the complex plane. So, we have actually in effect remove the singularity at at 0 by by redefining f 1 at 0 itself to be 1, which is the limiting value of the definition of f 1 and a neighborhood. So, if the limit we will see, that if the limit of f of a function f in a deleted neighborhood, as z goes to that a isolated singularity exists, then, then f will be analytic. Of course, f is analytic in the deleted neighborhood, then f is analytic in the whole of the disk and that singularity can be removed.

So, such a kind of singularity will be suggestively called as a removable similarity. So, another way of saying this is the function, f if it can be extended to an analytic function, even at the point which is the singularity. Then, then a such a kind of singularity is removed, such a kind of isolated singularity is removable, okay? We, we will define that in a movement, so but f 1 here has such a kind of singularity. f 2 f 2 of z has another kind of singularity, f 2 of z notice that it tends to infinity as z tends to 0, whether you consider this definition or the that definition of f 2 on C minus 0, here also I should have select C minus 0.

So, which ever definition we consider, we know that as z tends to 0, the modulus of the denominator becomes larger and larger. So, f 2 tends to infinity. So, this kind of singularity, so this kind of singularity has a definite pattern, f 2 of z in modulus tends to infinity or f 2 of z tends to infinity as we call if and also. Notice that, z times f 2 of z if you consider the first definition limit as z goes to a 0 of z times f 2 of z is 1 are if you consider the other definition in the parenthesis of f 2. Then limit as z goes to 0 of z power n times f of z f 2 of z is equal to 1. In either case the limit of the appropriate power of z minus 0, which is z times f 2 is a non zero quantity and that actually characterizes the the the kind of singularity of f 2 and 0.

So, f 1 is actually analytic of 2 is said to have a pole at the at the singularity 0. The singularity zero of f 2 is said to be a pole, okay? Then we will see something we will define something called the order of the pole that integer n here or one in this case we will we will call that as the order of the pole. We will define that more concretely and that is another kind of singularity. Finally, you notice has different kind of singularity f 3 of z does not tend to infinity as z tends to 0 and sorry as z tends to 0 and f 3 of z does not approach a limit.

f 3 that does not approach a finite, limit n as z approaches 0. There is no finite limit for a f 3 nor does it uniformly go off to infinity, so it, it is sort of jumping back and forth. Well, see a more concrete pattern to this jumping and then this kind of singularity is the third kind we will call this an essential singularity. We will see that the the Casorati Weierstrass theorem for these kind of singularities, okay? In a sense these are the only three kinds of behavior, we will also show that these are the only three kinds of behavior exhibited by a function, which is analytic in a deleted neighborhood of a point a, okay? So, let us proceed to classify the isolated singularities of a function. So, we will start with the removal similarity like I have said.

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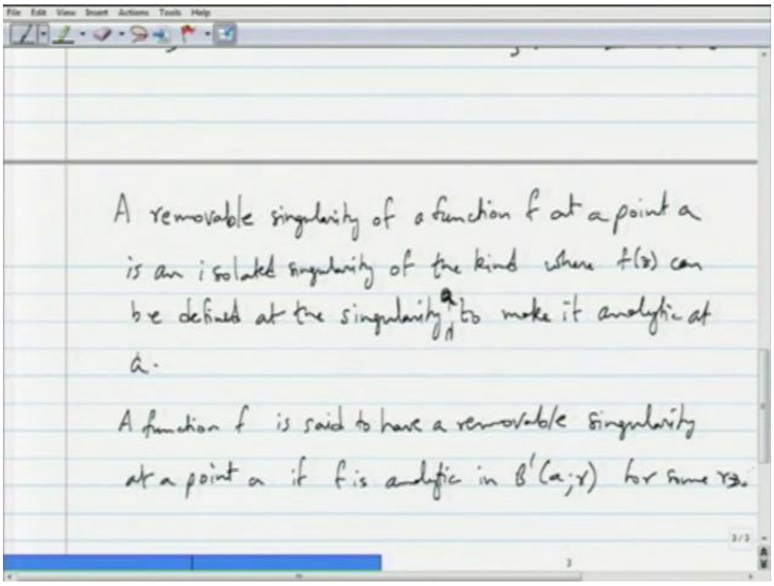

So, I will first define a removal similarity. If function f is said to be, is said to have a removable similarity at a point a, if f is analytic in B prime a r for some r positive.

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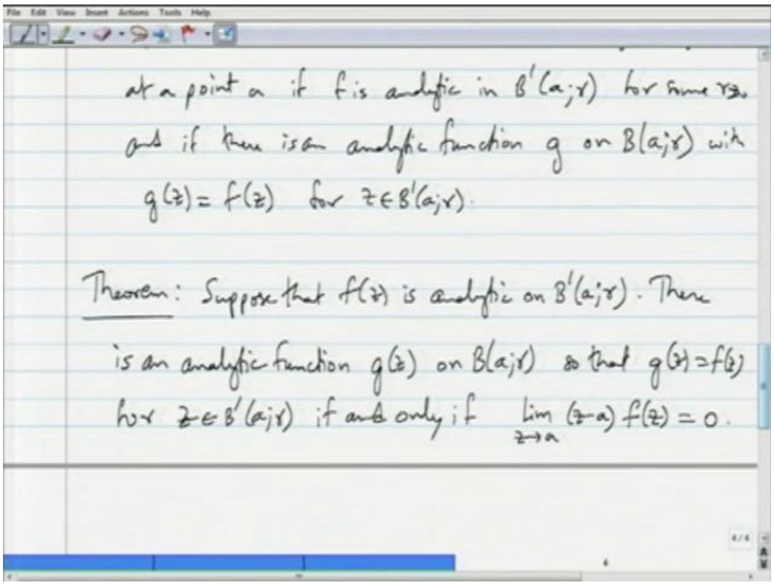

And if there is a function there is an analytic function g on the whole of B a r which means it it is already defined on at the point a itself with with g of z is equal to f of z for z belongs to B prime a r. So, g is an analytic function on B a r which agrees with f of z at all points in B prime a r. So, then f is said to have removable similarity, we will give a criteria to to identify removable similarity for a function f. So, here is the criteria, so here is the theorem, which states the criteria.

So, suppose that f of z is analytic on B prime of a r where r is some positive quantity, there is an analytic function g of z on B a r. So, that g of z is equal to f of z for for z belongs to B prime a r if and only if limit as z goes to a z minus a times f of z is equal to 0. So, if this condition is satisfied that the limit as z goes to a of z minus a f of z is 0, then then there is an analytic extension of f to the whole open disk B a r. If f is already analytic on B prime of a r. So, and it goes the other way round. Of course, if g matches with f on B prime of a r and g is analytic then limit z goes to a z minus a times f of z will be 0 by continuity of g. So, so that direction is easy, but the other direction requires some work, okay?

So, in retrospect this is equivalent to saying that there is an analytic extension of f on to the disk B a r, if and only if the limit z goes to a f of z is defined, if the limit exists. So, I am saying in retrospect in retrospect of what we are going to do, so if and if if we prove this theorem after we prove this theorem, we will see that that will imply that the statement that I have said. That limit z goes to f of z if it exists, then f can be redefined at a in order to make it analytic on the whole disk B a r. So, that is the theorem and in order to prove this theorem we will first see couple of lemmas, okay?

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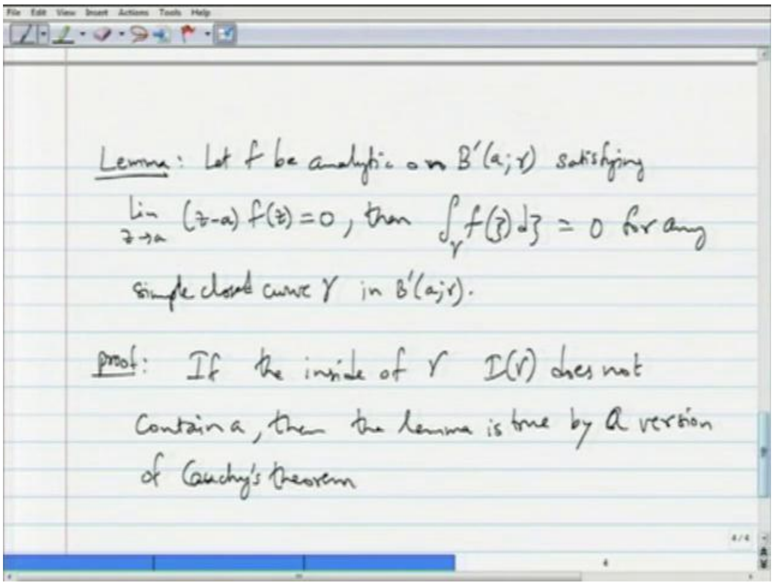

So, lemma before we prove this we will see this lemmas. Let f be analytic on B a prime r or B prime a r satisfying limit z goes to a z minus a f of z is equal to 0. Then integration over gamma f of zeta d zeta is equal to 0 for any simple closed curve gamma in B prime of a r. So, gamma should lie in the deleted d disk B a r B prime a r. Then integration over gamma of f is, so it is a modification to Cauchy's theorem.

So, what we are saying is that we can have a point a, at which we have a condition limit z goes to a z minus a f of z is 0. With these kind of exceptional points the Cauchy's theorem still holds that the integration around any simple closed curve of is 0, if f is analytic in B prime here. So, the proof is simple, so if the inside of gamma i of gamma recall we have define what the inside of a contour gamma is does not contain a, then the lemma is true automatically by version of Cauchy's theorem, that we proved already because then your inside of gamma is completely contained in the domain of analyticity of f. So, then this lemma is true. So, if a belongs to inside of gamma, then we ha we need some modification.

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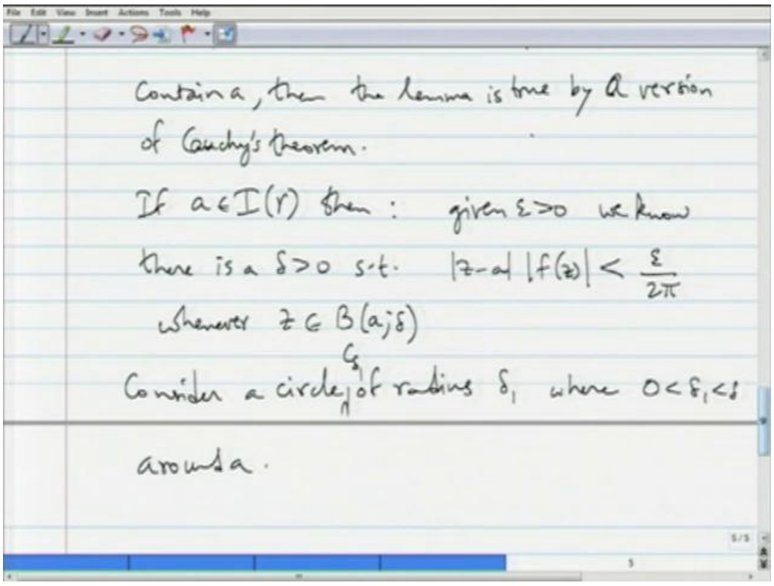

a is inside of gamma, then given epsilon greater than 0, we know there is a delta positive because the limit the the the the said limit exists of z minus a f of z and it is equal to 0, given epsilon greater than 0 there is a delta greater than 0, such that modulus of z minus a times modulus of f of z is strictly less than epsilon by 2 pi. We need this at just print factor. So, this is true when ever z belongs to a ball of radius delta around a. So, then now consider now consider a circle of radius delta 1 with where delta 1 is strictly less than delta delta 1 is positive strictly less than delta. So, I will call this circle C delta 1, consider a circle C delta 1 of radius delta 1 around a. So, the center of the circle is a and then integral over gamma f of zeta d zeta is equal to the integration around this circle. Now, C delta 1 of f of zeta d zeta.

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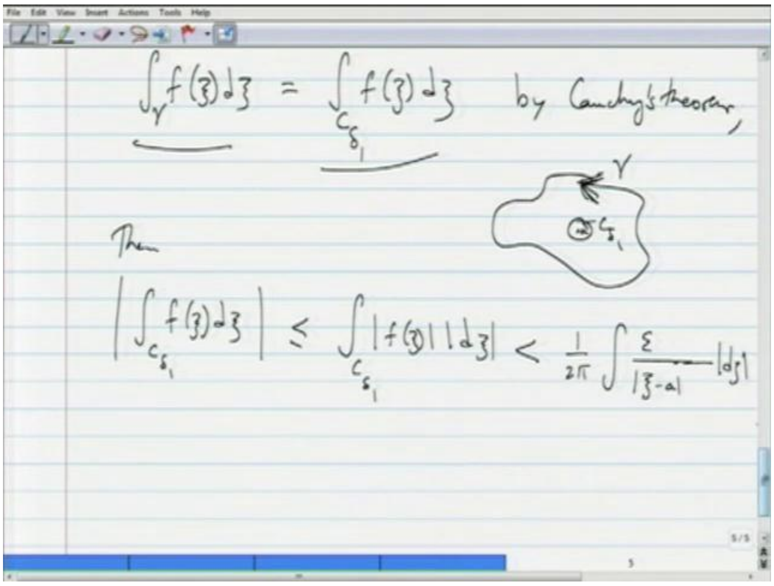

This is because, now if you have this contour gamma oriented in the positive direction, if we take a circle of radius delta 1 around a then, we know by one version of Cauchy's theorem, that the integration over gamma of f of zeta d zeta is going to equal integration over the contour C delta 1 f of zeta d zeta. So, these are one and the same by version of Cauchy's theorem. So, this is by Cauchy's theorem for simple closed curves, so then then the modulus of of this integration f of zeta d zeta is less than are equal to the integration over C delta 1 of the modulus of f of zeta times modulus of d zeta is strictly less than well 1 by 2 pi times integration of epsilon.

Because modulus of f of zeta is less than epsilon by sorry, mod modulus of z minus a f of z is less than epsilon by 2 pi. I have modulus of f of z less than epsilon by modulus of z minus a in this case zeta minus a times modulus of d z d zeta, okay? So, notice this is true for every z in B a delta and this contour C delta one lies in completely inside this B a delta. So, for all points on the contour C delta 1 this this inequality holds this inequality. This estimate in terms of epsilon holds and so and so we can we can say that the modulus of f of z is less than or equal to we can say less than are equal to 1 by 2 pi epsilon by modulus zeta minus a m mod d zeta a n.

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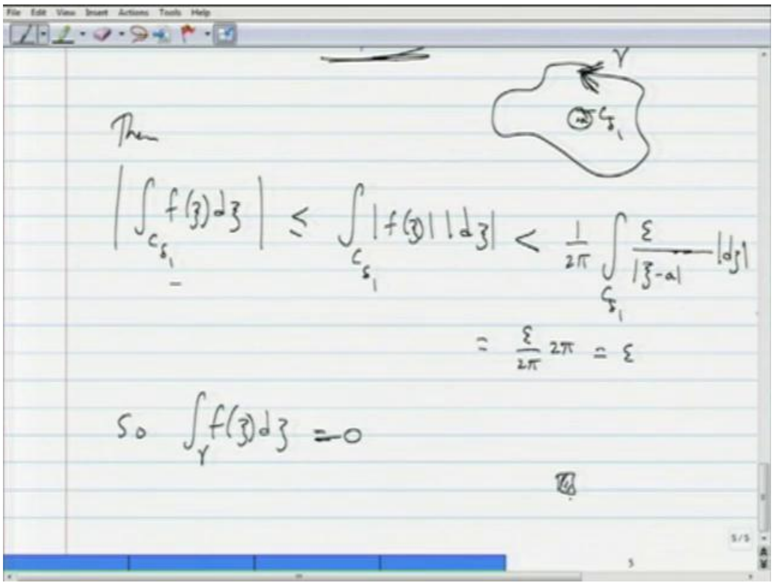

Then divide by 2 pi this is on C delta 1, so this is equal to epsilon by 2 pi times 2 pi. So, this integral 1 by mod z minus zeta minus a mod d zeta is 2 pi. So, that cancels and this is equal to epsilon. So, this is less than epsilon, so this was a strict inequality sorry. So, this is a strict inequality here, so I get epsilon. This is strictly less than epsilon, so, this so in summary this integral is equal to this integral and this is arbitrarily small. So, so integration over gamma f of zeta d zeta is equal to 0.

So, notice that we have we have we have proved this by using a technique similar to Cauchy's theorem, but we are using Cauchy's theorem itself once again. What we are doing is, we are considering this contour gamma which contains a in its interior. We are sort of contracting this this contour to a very small circle around a and then we are estimating the value of the function f on that circle around a itself. So, that is a teahnique very similar to 1 and Cauchy's theorem, but this condition, this condition that the limit z goes to a z minus a f of z is equal to 0, helps us give this estimate.

Then we can say that the integral of f around that circle is 0, so the around the the integration over gamma itself is of f of z is… So, that is the proof of this lemma there are only two cases that a is in the inside and a is not in the inside of gamma. So, in either case I have showed that the integral is 0, so that proves this lemma and we need one more lemma before we can prove the theorem.

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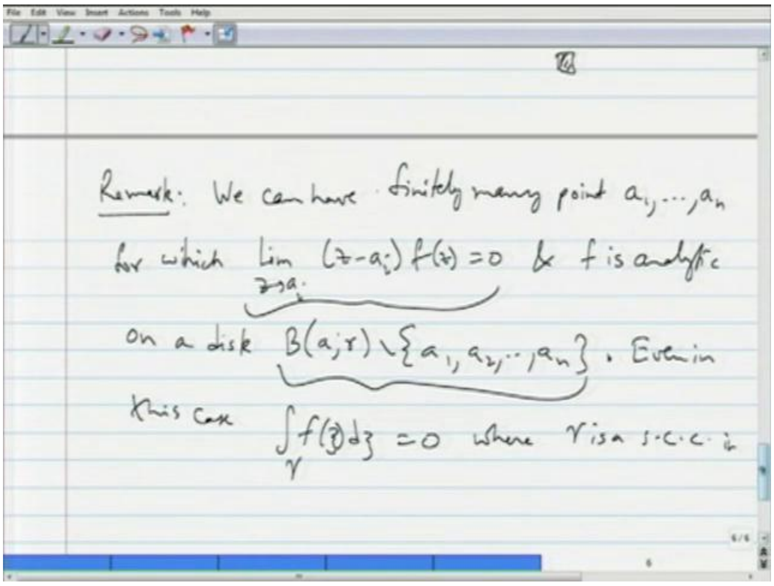

But first notice that, note so the here is the remark on this lemma we just proved we can have more than one points, we can have finitly many points a one through a n for which limit z goes to a i z minus a i times f of z is equal to 0. And and of course, with assumption that f is analytic on well on a disk B a r minus the points a 1 a 2, so on till a n. Even in this general scenario, we can show that we can show that using the very same thing, we can, we can contact these disks a 1 through a n, which are finitely many points. We can contact these disks to smaller disks or contract this region to a very small disk around these points a 1, through a n.

Then apply this lemma repeatedly to each of these disks to show that the above holds the same lemma holds with many points having such a such a condition limit z goes to a i z minus a. If of z is equal to 0, so even in this case, even in this case integration over gamma f of zeta d zeta is equal to 0, where gamma is a simple closed curve in B.

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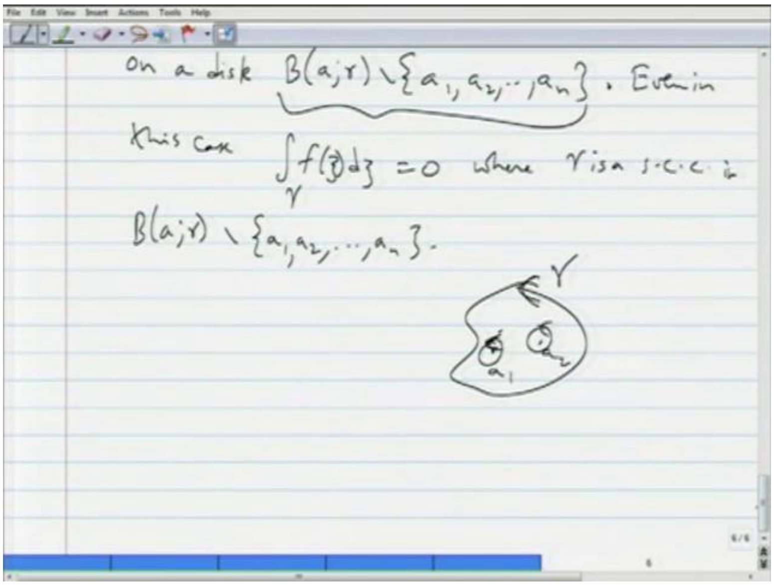

a r minus a 1 a 2 so on till a n . So, we have the slightly modify the technique of the of the proof of lemma to to consider the case where more than one points a 1 through a n are inside gamma. What you can do is actually, then take very small circles around these points. So, that the integration or gamma equals the integration for the smaller circles which do not contain any other points. Then one of the a i s inside them and then by using Cauchy's theorem the integration on gamma.

So, may be a schematic will help, so here is gamma. Suppose, it contains a 1 and a 2 you can consider two small circles around a 1 and a 2. The integration over gamma will equal the integration on these kind of circles and then we can use Cauchy's or the limit on a, the limit condition to show that the integration on the smaller circles of f is 0. Hence, the integration on gamma is 0, okay? So, so that is a remark on the theorem and we need one more lemma.

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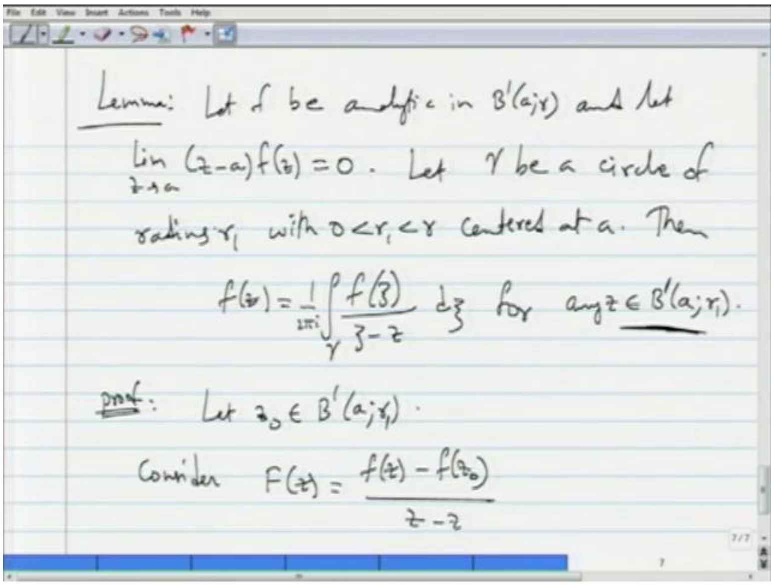

Here, so let f be analytic what we are going to do is with the same condition limit z goes to a z minus a f of z is equal to 0. We are going to say that the Cauchy's integral formula holds. So, in B prime a r and let limit z goes to a a minus a f of z is equal to 0. So, if this condition holds, let gamma be circle of radius r 1 less than r. So, I will write this as gamma be a circle of radius r 1 with 0 r less than r 1 less than r centered at a. Then f of z is equal to the integration 1 by 2 pi i times the integration of f of zeta by zeta minus z d z d zeta for any z belongs to b prime a r 1.

Note that, we use the Cauchy's integral formula to show that f of z is equal to this particular thing this particular integral on the right hand side. That was a kind of representation formula for the value of f at points inside the circle. So that, we that we have emphasized, while showing I mean, while showing the version of Cauchy's integral formula. Here this lemma says that, that kind of representation formula for f of z for z inside a circle of radius r 1 like this is still valid, if z is not equal to a z belongs to B prime a r 1 provided that this condition limit z goes to a z minus a f of z equals 0.

So, once again we will we will use this condition to actually show that all the previous results are the the Cauchy's integral formula still holds. So, let z naught belong to B prime a r, so I am picking an arbitrary point in B prime a r 1, sorry B prime a r 1 and then I will show that this is representation formula holds. So consider, consider capital F of z is equal to f of z minus f of z naught by z minus z naught.

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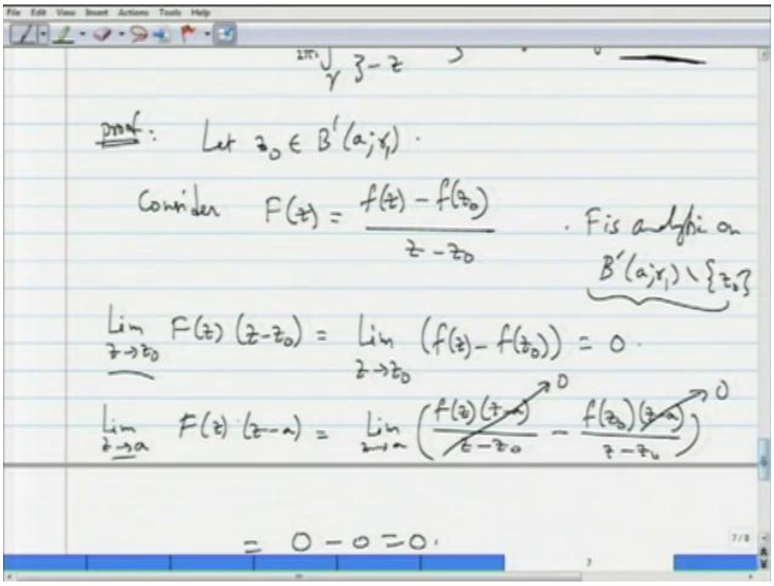

Then notice that the limit z goes to z z naught of capital F z times z minus z naught, so firstly capital F is analytic on B prime a r 1 minus the point z naught. So, we are in a situation like this a remark like in this remark. So, there are finitely many points and f is analytic in a disk minus some point removed, okay? So, we are in that kind of situation and F is analytic over there and limit z goes to z naught f of z times z minus z naught. What is this? This is the limit as z goes to z naught of f of z minus f of z naught. Well z naught belongs to B prime a r 1 where F is analytic.

So, it is continuous at least, so this limit is 0 also for the other ambiguous point a capital F of z times z minus a. This, what is this? This is the limit as z goes to a of f of z times z minus a by z minus z naught minus f of z naught times z minus a by z minus z naught. Each of these is 0, so the first one is 0 because of the given condition by the hypotheses and the second one is 0 because the limit as z goes to a z minus a 0. So, all in all this is 0 minus 0, this is 0. So, the limit exists and this is 0. So, B prime a r 1 in B prime a r 1 minus z naught f capital F is analytic. At the two points z naught and a, this kind of condition is satisfied.

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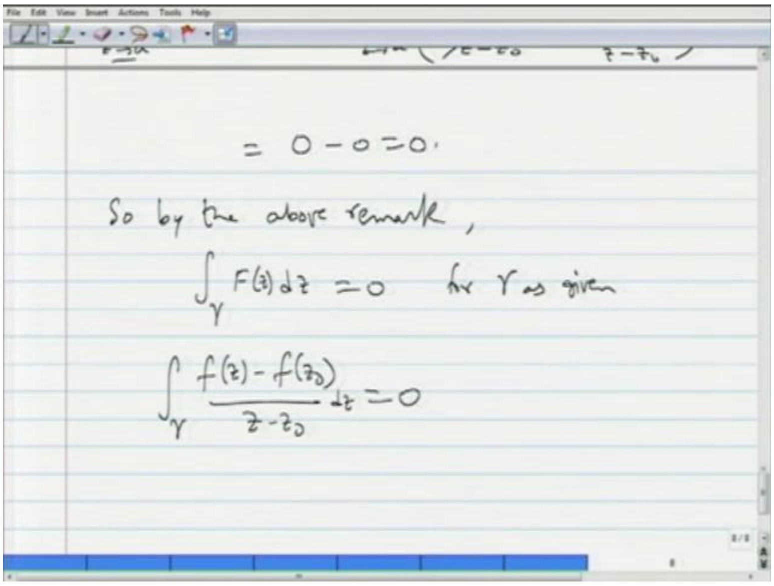

So, by above lemma, so by the above remark, following the lemma, following the proof of the previous lemma, what we can say is that integration over gamma capital F of z d z is 0 for gamma has given. Since, since we are picking this point z naught inside the circle of radius r 1, there is a no danger of z naught itself lying on the circle of radius r 1. Also a is, a is not on the circle of radius r 1, centered at a. So, we have this integral 0. So, what this means is that the integration over gamma of f of z minus f of z naught by z minus z naught is equal to 0, d z is equal to 0. So, that is the definition of capital F of z.

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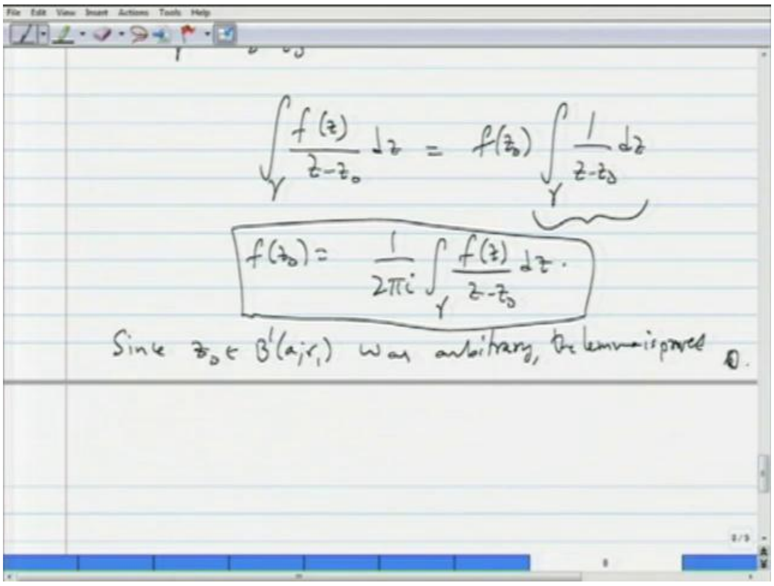

So, this tells that integration of f of z by z minus z naught d z is equal to over gamma is equal to f of z naught times the integration over gamma of 1 by z minus z naught d z. We know that that integration is 2 pi I, so this tells us that f of z naught is equal to 1 by 2 pi i times integration over gamma of f of z by z minus z naught d z, which is what we want, okay? So, since z naught is arbitrary point inside B prime of a r 1, so since z naught belongs to B prime a r 1 was arbitrary, that lemma is proved. So, this representation formula holds for any such z naught, so that is the proof of this. Once again there is there can be a remark following this, that there can be more points than a itself, where this kind of condition can be satisfied.

Even then we can have the representation formula, where we avoid such points as a. So, now we are ready to prove the theorem. So, let us go back to the statement of the theorem, so this theorem says that if we have this really handy condition that z minus a times f of z is equal to 0, in which case the two lemmas following this hold. Then there is an analytic extension to f of z at the point a, so one direction is pity easy. Also I, I will slightly add an important term to this theorem sorry, I will I will actually add there, there is a unique analytic function.

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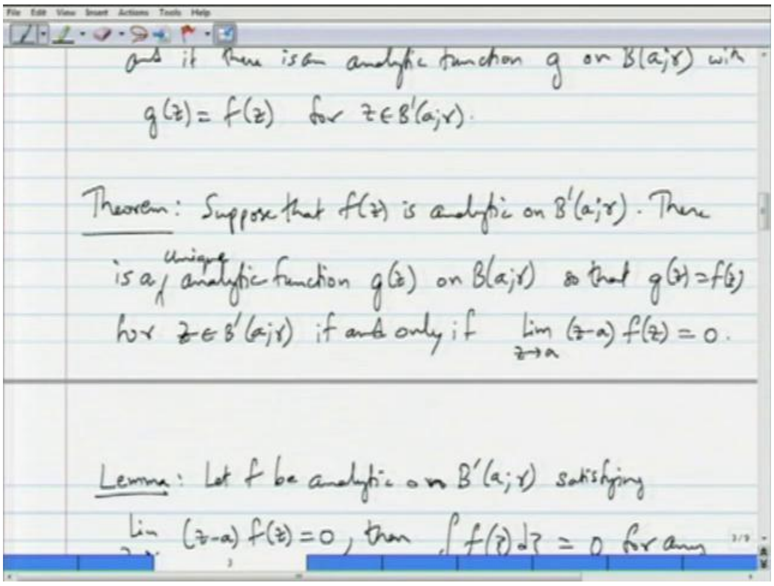

So, this function is also unique which gives more rigidity, okay? So, this should, this function such an extension is unique well the uniqueness is immediate because g and f agree on a set with a limit point. So, by that identity theorem the, the uniqueness automatically follows, so that is a run much of a punch, but nevertheless uniqueness follows.

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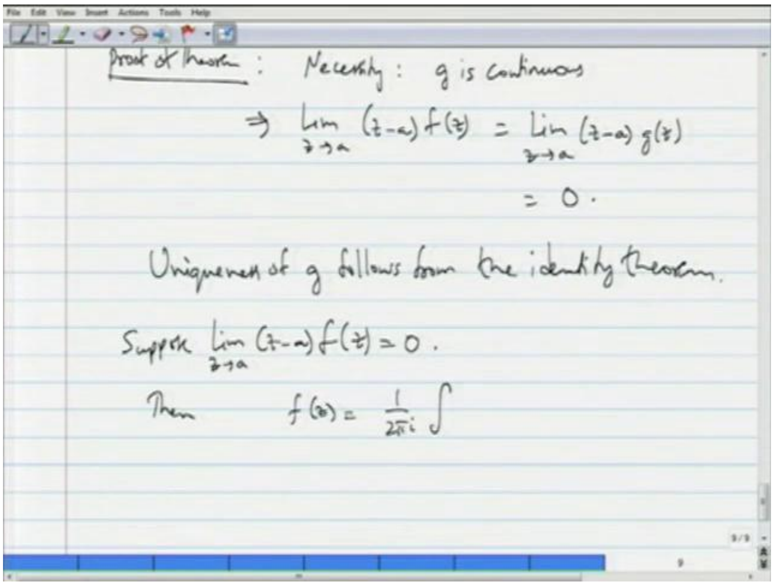

So, proof of theorem, so well the necessity condition is easy one direction of it is easy necessity says that, well requires that g is continuous. So, which implies that limit z goes to a z minus a f of z is equal to limit z goes to a, there is unique extension to f and that is g. So, we are assuming that, so then this is z minus a times g of z f is equal to g. Then this is equal to limit z goes to a, well I mean this is 0 and that is 0. So, this is equal to 0, so the necessity is really easy uniqueness follows from of g follows from the identity theorem.

So, let me be more clear here what I mean is, if we know that there is a function g which extends f to the point a as well, then it has to be unique by the identity theorem. Because f and g have to agree that is the condition on g, so f and g have to agree on B prime a r, which is a set with limit point. So, now I have to assume the condition that limit, limit z goes to a z minus a f of z exists and I have to show that f function exists. Now, suppose limit z goes to a z minus a f of z exists or is equal to 0 rather f of z is equal to 1 by 2 pi i integration over C r 1 of f of zeta by zeta minus z d z, where C r 1 is a circle of radius r 1 centered at a with with 0 strictly less than r 1 strictly less than r and z naught equal to a z belongs to B prime a r B prime a r 1.

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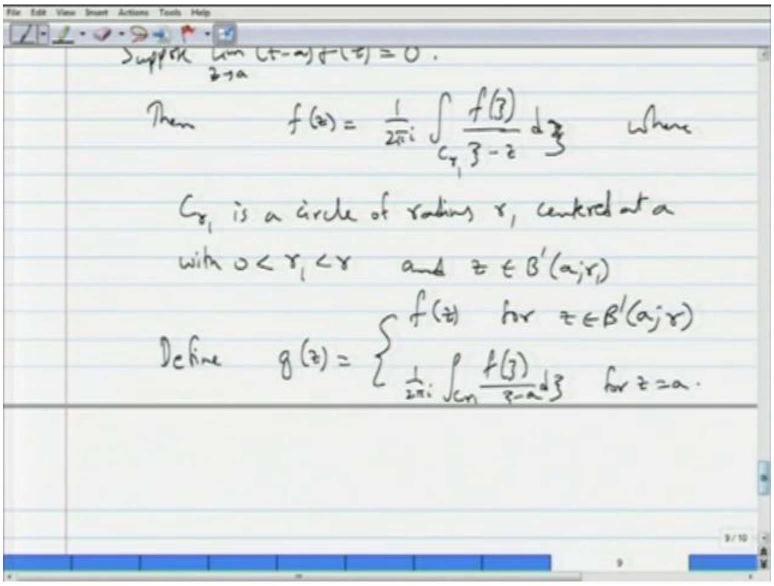

So, by considering large enough circle we can actually include all the points in B prime a r. So, this actually suggests the definition of the new function g. So, g of z, so define g of z is equal to well, we are force to define it to be f of z for z belongs to B prime a r. Define this to be 1 by 2 pi i integration over C r 1 if you place of f of zeta by zeta minus a d zeta. I apologize this should be d zeta zeta d zeta for z equals a. So, will force it to be equal to this representation formula g to be equal to this representation formula, when, when z is equal to a.

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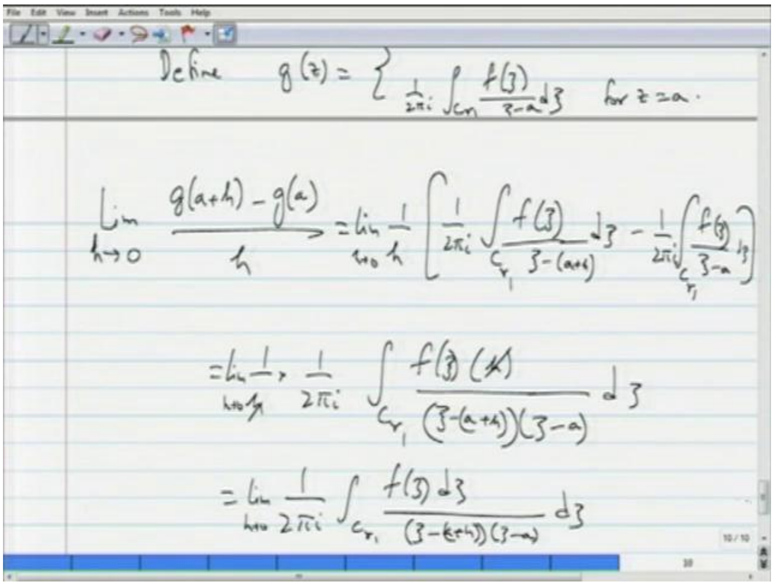

We will show that g is analytic, so the limit as z or we can say limit as h goes to 0 of g of a plus h minus g of a by h. There is no doubt that g is analytic in B prime a r because f is… So, we will calculate this limit, this is equal to 1 by h times 1 by 2 pi i times the integration over C r 1 of f of zeta by zeta minus a plus h d zeta minus 1 by 2 pi i times integration over C r 1 of f of zeta pi zeta minus a d zeta. This is equal to 1 by h times to 1 by 2 pi i times integration over C r one of f of zeta.

I will combine these terms can clear the denominator I will get zeta minus a minus zeta minus a plus h. So, that will give me an h in the numerator divide by zeta minus a plus h times zeta minus a, so this h cancel and I have this is equal to 1 by 2 pi i. So, I have a limit hanging in there limit h goes to 0 limit goes to 0 etcetera. This is limit as h goes to 0 of 1 by 2 pi i integral over C r 1 f of zeta d zeta by zeta minus a plus h times zeta minus a d zeta.

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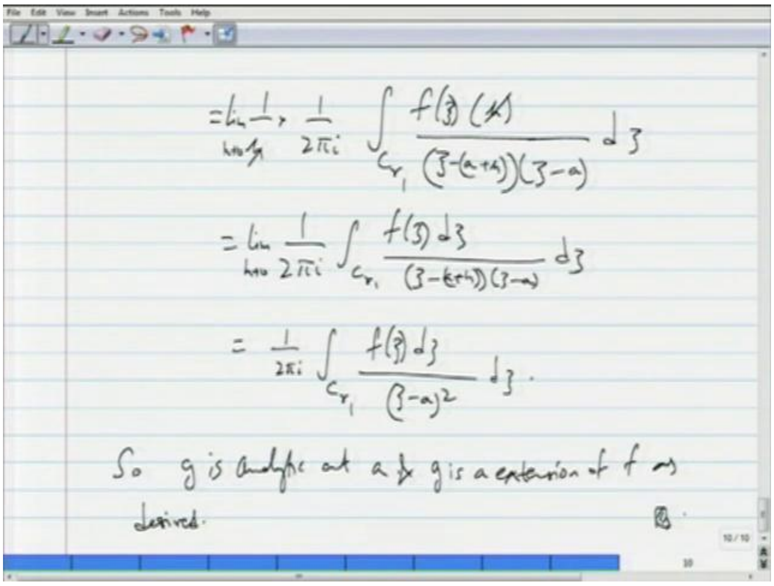

So, now I can take the limit into the integral, and then this is equal to 1 by 2 pi i integral over C r 1 of f of zeta d zeta divide by zeta minus a square d zeta. So, this limit exists, limit h goes to 0 g of a plus h minus g of a by h it exists and it equals 1 by 2 pi i C r 1 integration over C r 1 of f of zeta d zeta by zeta minus a square d zeta, whatever that value is. So, g is analytic at a, and g is a extension of f as desired. So, that completes the proof of this theorem and will see other kinds of singularities, next time.