Complex Analysis

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Module - 5
Mobius Transformations
Lecture - 3
Problem Solving Session

Hello viewers, in this session we will solve some problems based on the theory we have seen so far. So, like before I encourage the viewer to solve as many exercises from the text book, as it takes to gain confidence in the subject matter. So, here I will be able to solve a few problems, which will sort of use the theory, which I have discussed so far. So, after each question that I present here the viewer is encouraged to pause the video and try to solve it by himself or herself. Then you can look at the solution that I present here. So, let start with the following question.

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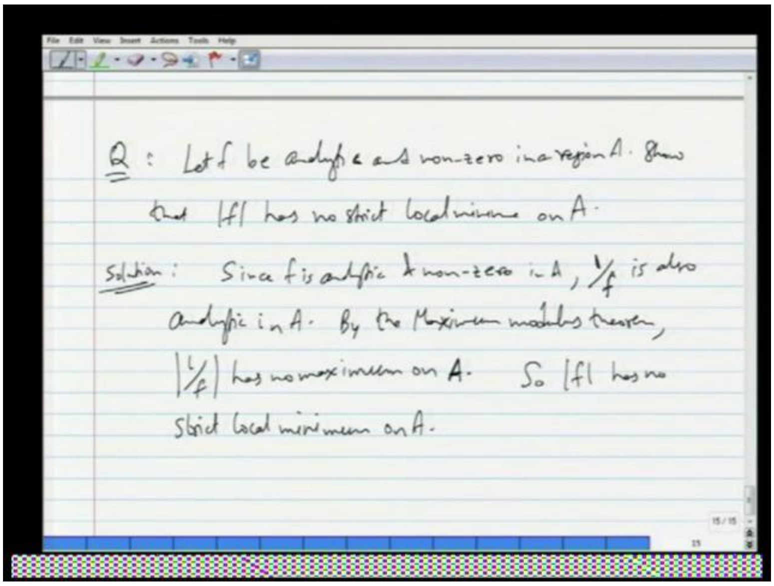

So, this question is sometimes called minimum modulus theorem, so or a version of it. So, let f be analytic and non-zero in a region A, show that the modulus of f has no strict local minima on. So, the solution as here, since f is analytic and non-zero, more importantly non zero in a 1 by f is, so you can consider the function 1 divided by f, which is non non-zero, there is no 0 in the denominator 1 by f is also analytic in a, okay?

Now, by the maximum modulus theorem we know that the, modulus of 1 divided by f has no maximum on the open set. On the open connected set a if the modulus of 1 by f has a maximum, that has to occur at the boundary of a not at any interior point of that we have showed, so this is no maximum on a. So, what this means is that, so f has modulus of f has no strict local minima.

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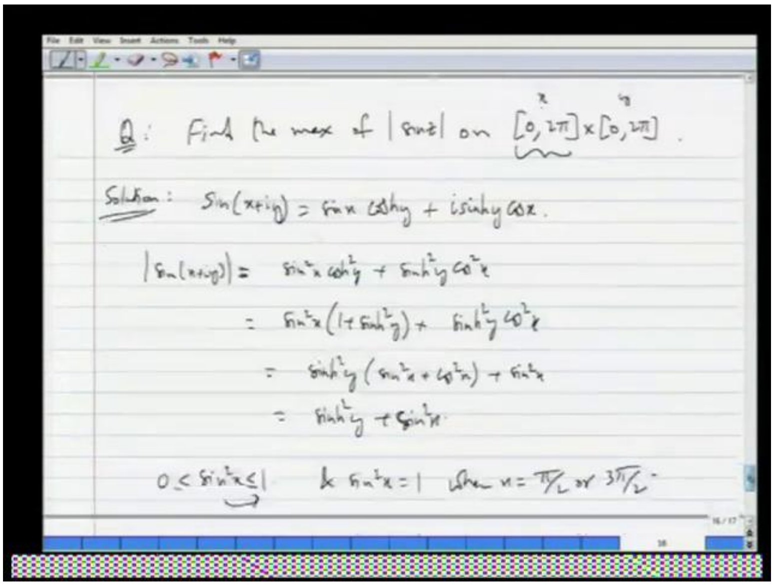

So, this is easy so the next question is as follows by the maximum modulus theorem, we know that if we consider a entire function and a bounded set, then the entire function will have its maximum modulus somewhere on the boundary of the bounded set. So, here is a question to find so find the maximum of the modulus of sine z on 0 to pi cross 0 to pi, what that means is, the x variable within 0 and 2 pi and the y variable within 0 to pi.

Try to answer this question and I will present the solution here the solution is as follows. So, when you look at the sine z sine of x plus i y. This is equal to sine z cosine h y it should be cosine i y, but we know that cosine i y is cosine h hyperbolic cosine y and plus cosine x sine i y, which is i sine hyperbolic y. So, i sine hyperbolic y times cosine z, right? So, the modulus of sine x plus i y is going to be a simple calculation shows this is going to be sine square x cosine h square y plus sine h square y cosine square x. So, this is equal to sine square x times 1 plus sine h square y plus sine h square y times cosine square x, which is equal to sin h square y times sine square x plus cosine square x plus sine square x which gives us sine h square y plus sine square x, okay?

So, now sine square x we know is between 0 and 1 sine square x is at most 1 and sine square x is equal to 1 when x is equal to pi by 2 or 3 by 2 in this interval 0 to pi. Now, the maximisation problem just boils round to the maximisation of this function.

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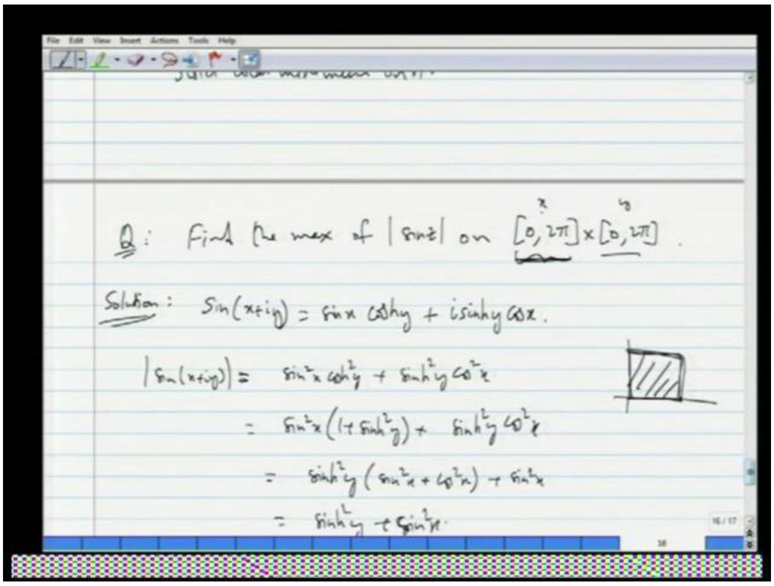

And we know that the maximum occurs only on the boundary of this region this square region 0 to pi, 0 to pi; so we are only interested in inspecting on the boundary region, where this expression attains its maximum. So, we can look at sine x square y sin x square y is an increasing function.

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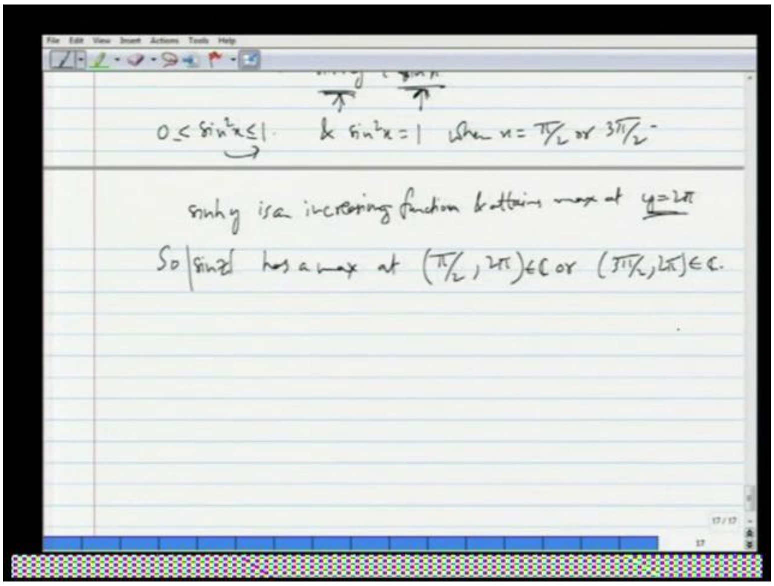

So, sine h y is an increasing function that is because cosine h y its derivative is never 0. It is actually positive strictly positive. So, sine h y is always an increasing function and attains maximum at y equals 2 pi in the interval 0 to pi its maximum is obviously at 2 pi. So, y equals 2 pi you have maximum, so sine z on the modulus has a maximum at the point pi by 2 comma 2 pi or 3 pi by 2 comma 2 pi. So, what I mean by this is, this is a point in the complex plane 2 pi belongs to C.

So, that is the maximum of sine z modulus. So, the next question is the extended Liouville's theorem. So, suppose that f is an entire function and if for some integer k greater than or equal to 0, there are positive constants such that a and b such that modulus of f of z is less than or equal to a plus b modulus of z power k that inequality is true for any z belongs to C.

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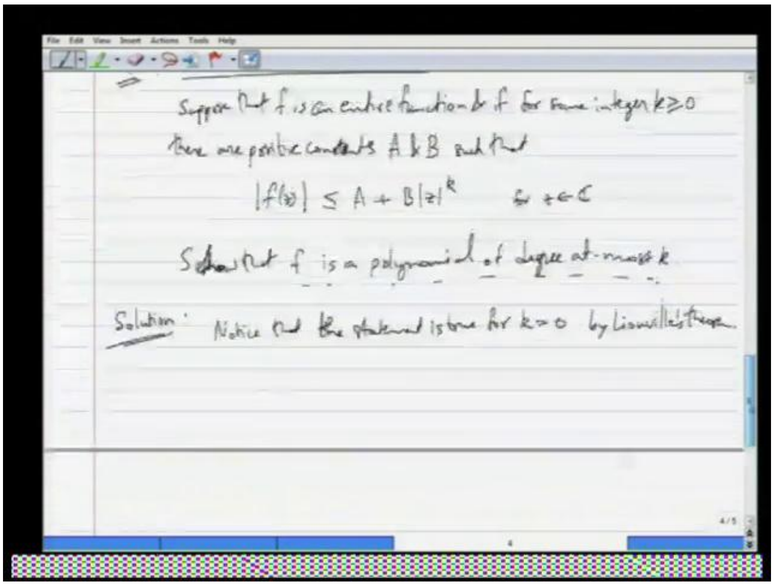

Show that f of z is a polynomial of degree at most k, so we are calling this the extended Liouville theorem because when k is equal to 0 you are saying that the modulus of bounded by positive real number. So, then we know by Liouville's theorem that this is a constant function. So, f is definitely is polynomial of degree 0, now we can extend it to any k any integer k, which is non negative please try to solve this exercise. I will present the solution here, solution what one does is that 1 proceeds induction because we need to do this for every non negative integer k. So, notice that k equals 0, the statement is true for k equals 0, the statement that f is a polynomial of degree at most k is 0 for k equals zero by Liouville's theorem, okay? Now, that is nothing but the Liouville's theorem.

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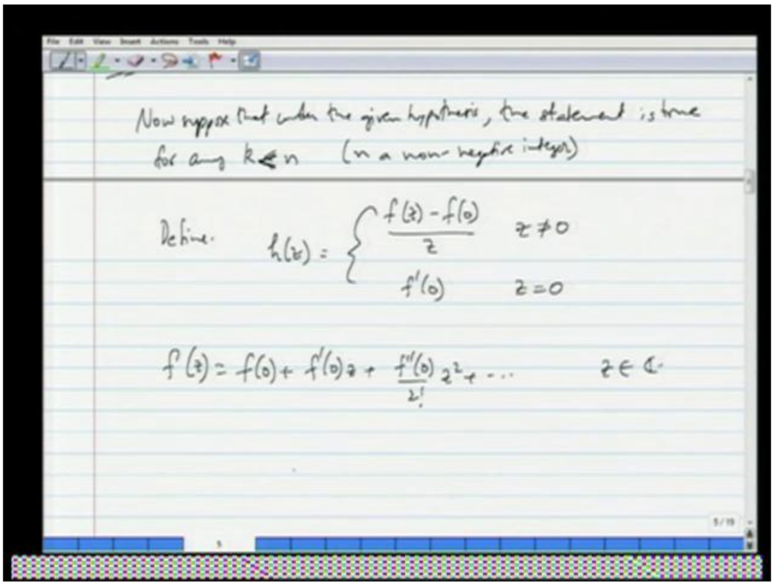

Now, suppose that under the given hypothesis the statement is true, this statement is true of course, only under the given hypothesis for any k less than or equal to or strictly less than a particular integer n. So, n non negative integer, so we are going to show that the statement is true for n itself. So, define h of z to be a split function minus f of 0 by z for z naught equal to 0 and define this to be f prime of 0 at z is equal to 0 h of z. We define it to be this kind of split function.

So, we will show that f h is entire, so first notice that f has a Taylor series expansion around 0, which is valid on all of the complex plane because f is entire f of z can be written as f of 0 plus f prime of 0 z plus double prime of 0 by 2 factorial times z square etcetera. This is valid this expression is valid for all z belongs to C because f is entire.

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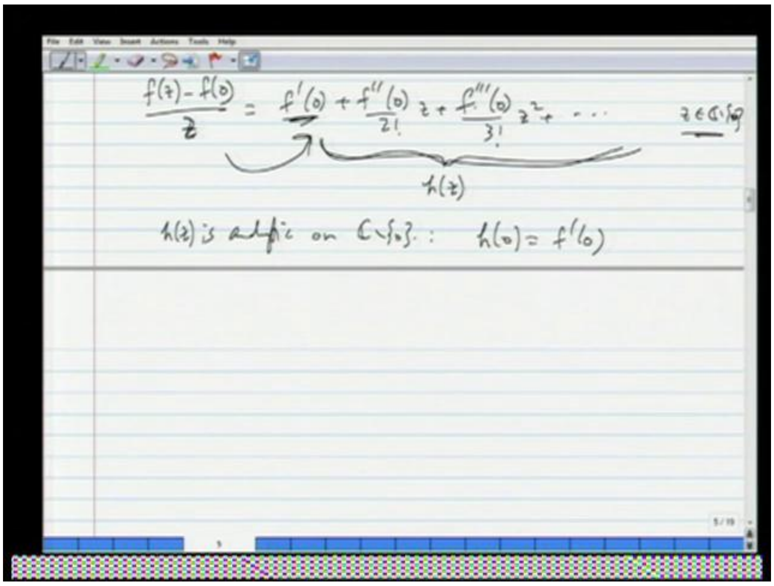

From this we can calculate that f of z minus f of 0 by z is equal to f prime of 0 plus f double prime of 0 by 2 factorial z square plus 1 sorry, z here plus let me go one more expression f triple prime of 0 by 3 factorial times z square plus. So, on for z belongs to C minus 0 for C minus 0 because we are dividing by z f of z minus f of 0 is this expression. So, this power series, this is a power series, so this is what we are defining h of z to be is analytic. We showed that the power series are analytic. So, analytic on C minus 0 in fact by defining h by 0 to be f prime of 0 we have made it analytic on all of C. So, h of z is actually analytic. So, h of 0 is equal to f prime of 0, if we consider this h.

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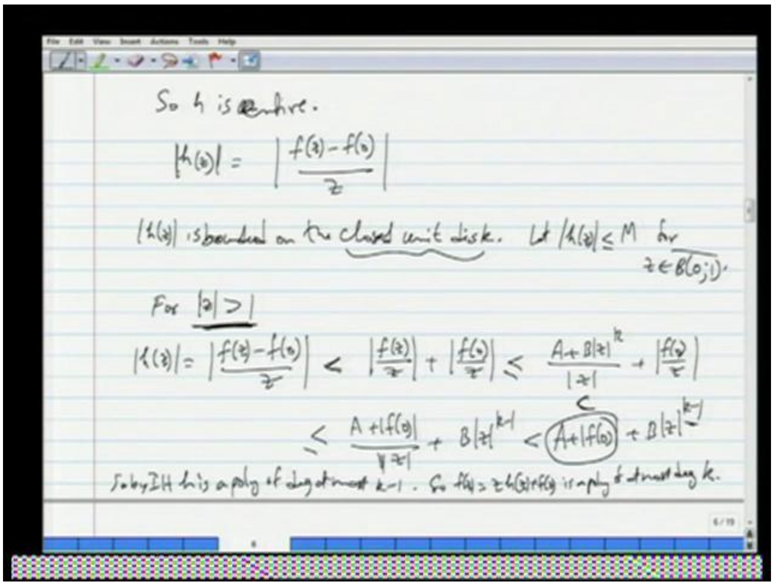

So, so h is analytic in entire actually, so the definition of h of z as f of z minus f of 0 by z is precisely this kind of expression except at 0 and at 0 we have a remedied the situation by defining it to be f prime of 0, okay? So, h of z is precisely this power series which is valid on all of the complex plane. So, h is entire we can use the induction hypothesis. So, h is the modulus of h notice is now going to be modulus of f of z minus f of 0 by z. So, firstly the modulus of h of z is bounded on the, unit disk the closed unit disk the closed unit disk, right?

That is because h is continuous h is entire function its analytic, so h is continuous. So, modulus of h is continuous function a continuous function on compact set is a is bounded, okay? So, modulus of h of z is bounded, so let modulus of h of z be less than or equal to m for z belongs to the closure of b 0 1 for z such that the modulus of z is strictly greater than 1. What we have is the modulus of h of z here is equal to the modulus of f of z minus f of 0 by z. So, the modulus of this is a less than the modulus of f of z by z plus the modulus of f of 0 and we know that the modulus of f of z by z is less than the modulus of a plus. So, here let me go to the question. The question says the modulus of f of z is less than or equal to a plus b mod z power k, so I will use that.

So, the modulus of f of z by z is less than or equal to a plus b mod z power k divided by mod z plus modulus of f of 0 by. Since, this is strictly less than or simply less than or equal to a plus modulus of f of 0 by modulus of z. Then b modulus of z power k minus 1, so which is I mean this is a constant and modulus of z is greater than one tells that 1 by mod z is less than 1. So, this is less than a plus modulus of f of 0 plus b times modulus of z power k minus 1 and by the induction hypothesis this is some other constant C.

So, by the induction hypothesis this statement here we know that h has to be polynomial of at most degree k minus. So, by induction hypothesis h h is a polynomial of degree at most k minus 1. So, going back to the definition of h, so z times h of z plus f of 0 gives f of z. So, f has to be a polynomial of at most degree k, so f of z is equal to z times h of z plus f of 0 is a polynomial of at most degree k, that proves what contention is, so that shows that f is a polynomial of at most degree.

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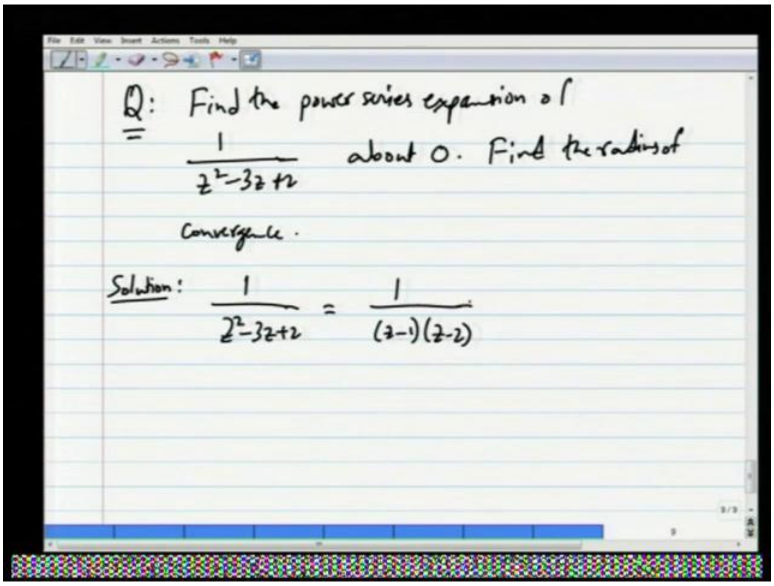

So, the next question is as follows. So, find the power series expansion of the function 1 by z square minus 3 z plus 2 about 0, find the radius of convergence you can pause the video here to answer the question yourself. Here is the solution, so 1 by z square minus 3 z plus 2 can be written as well we will factorise z square minus 3 z plus 2 as z minus 1 times z minus 2. We observe immediately that the denominator is 0 at 1 and 0 also at 2.

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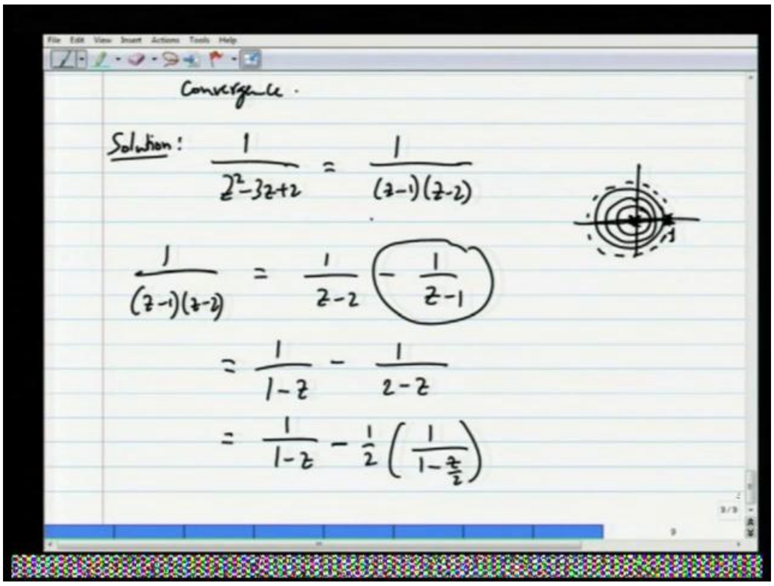

So, if you are going to determine a power series expansion around 0, the function encounters this point 1. So, when you go in circles outwards of 0, there is a certain resistance at this point 1, where the function is not defined later. When we study singularities we will call such a point a pole in this case a simple pole in any case a, this the Taylor series expansion about 0 can have radius of convergence at most one from this kind of picture.

So, we will also show that directly without resorting to this kind of picture, okay? So, 1 by z minus 1 times z minus 2 in order to write in the form of power series about 0. What we will do is split this into partial fractions, so when we split this into partial fractions, we get 1 by z minus 2 minus 1 by z minus 1. What we will do is expand 1 by z minus 1 and 1 by z minus 2 independently as power series. So, this gives us this is equal to 1 by 1 minus z. So, I am taking this term first and converting that to 1 by 1 minus z plus. Rather minus 1 by 2 minus z, which can be written as 1 by 1 minus z minus half times 1 by 1 minus z over 2, okay?

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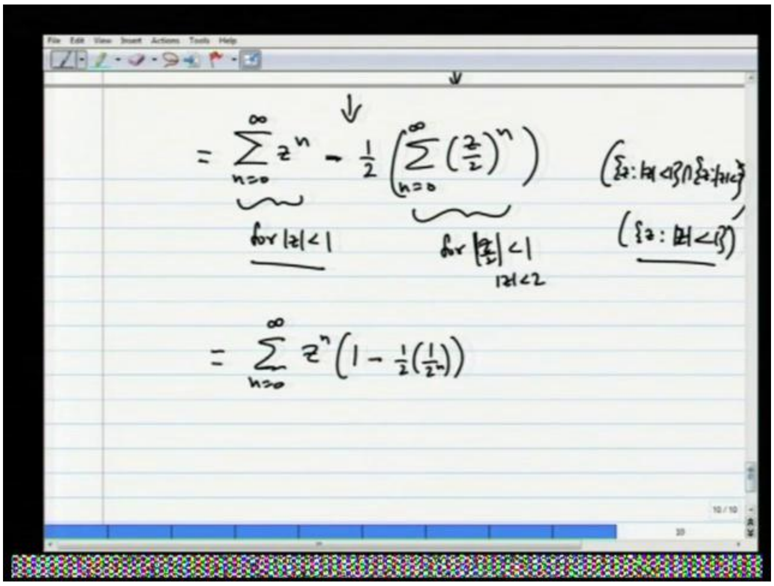

1 by 1 minus z we know has an expansion around 0, taking the series expansion around 0. It is a geometric series z power n equals 0, through infinity minus half times the expansion for 1 minus z by 2. Likewise it is a geometric series n equals 0 through infinity of z by 2 raise to n. This series is equal to this function this is valid only for modulus of z strictly less than 1, okay?

Like we have remarked here in the picture, so this picture appears in the statement and likewise this expansion this function is equal to this expansion only, if the modulus of z by 2 is strictly less than 1, which means the modulus of z is strictly less than 2. So, this difference, so the difference of these two functions is equal to the difference of these two series, if and only if both the series converge, which means converge to these functions, which means this series expansion is valid only on the intersection of these two sets.

Set of all z such that mod z is strictly less than one intersection set of all z such that mod z is less than 2, which is simply that mod z is strictly less than 1 on the disk of radius 1 on the on the unit disk open unit disk this expansion is valid n equals 0 through infinity z power n minus half times this. Now, let us try to put these two series together by collecting the coefficient of z power n in either. So, this gives me 1 minus 1 by 2 times 1 by 2 power n, that is the coefficient of z power n.

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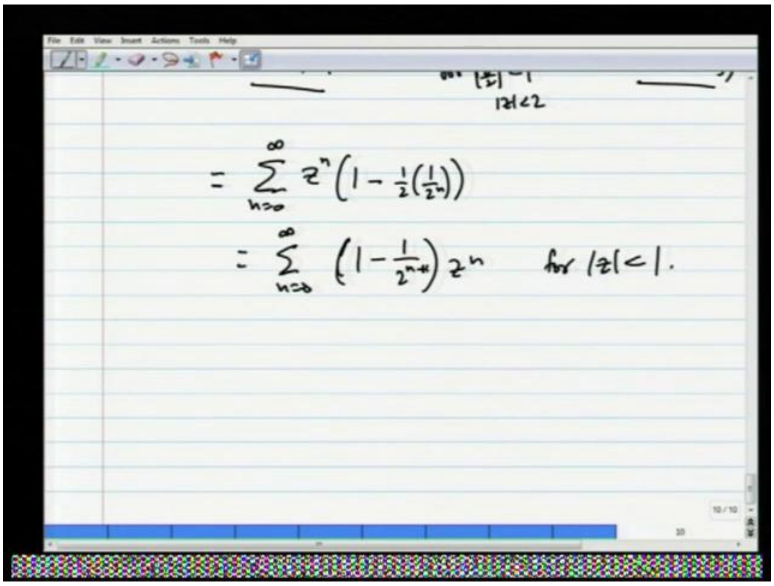

So, this is sigma n equals 0 through infinity z power n times 1 minus 1 by 2 power n. So, that is the power series expansion of this function around 0 and this is valid for mod z strictly less than 1. So, that is the solution to this problem onto the next question.

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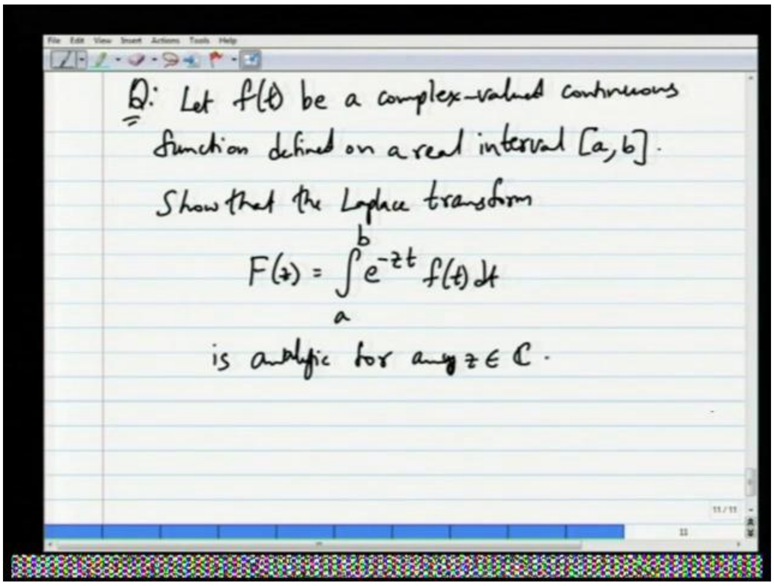

So, let f of t be a complex value continuous function defined on a real interval a comma b. So, it is a function of real variable complex valued function of a real variable show that the Laplace transform capital F of z is equal to integration from a to b e power minus z t f of t d t is entire analytic for any set belongs to C. So, you can pause the video here to answer the question yourself.

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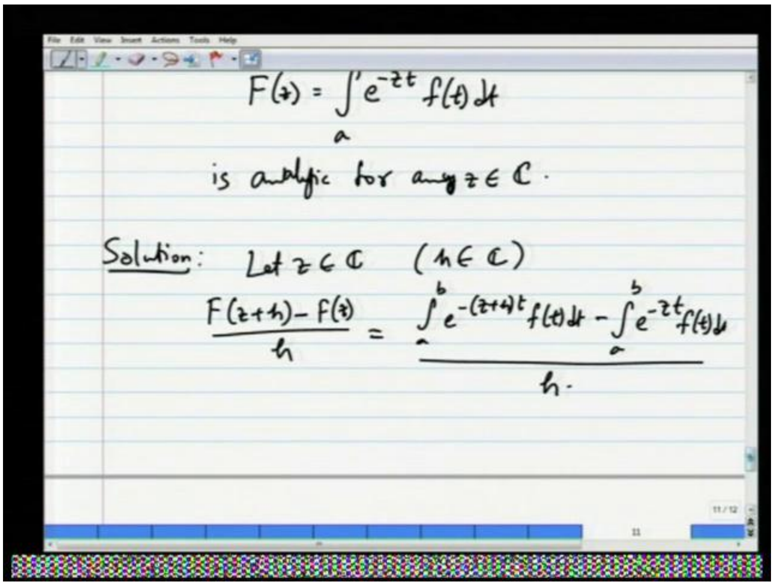

And here is the solution. So, what way we will show is capital F is differentiable for every z in the complex plane and that will show that capital F is analytic. So, let us consider the difference quotient, let z belong to C. So, we will fix the particular z f of z plus h minus f of z divided by h, here h is a complex number h. We will later make it tend to 0, so h belongs to C we can take we can assume h has a small modulus if needed because we will eventually take the limit of this quotient as h tends to 0, okay? So, this is equal to the integration from a to b of e power minus z plus h t f of t d t minus integration from a to b of e power minus z t f of t d t divided by h.

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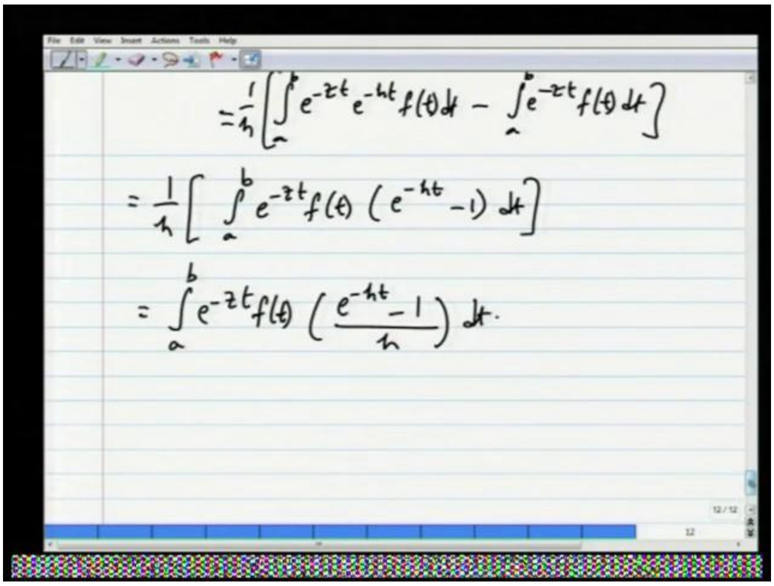

We will simplify this, well firstly this is integration from a to b a to b e power minus z t e power minus h t f of t d t minus integration from a to b e power minus z t f of t d t divided by h. So, this is 1 by h times all of this. So, this is 1 by h times, well I will combine these two integrals a to b e power minus z t f of t times e power minus h t minus 1 times d t. I will write this again as since h is independent of the integration h inside of integration. Then e power minus z t f of t e power minus h t minus 1 by h d t, okay? So, here now one can after after this basic calculation, once can see that this expression, which is inside the integration can be likened to the derivative of the function e power minus z.

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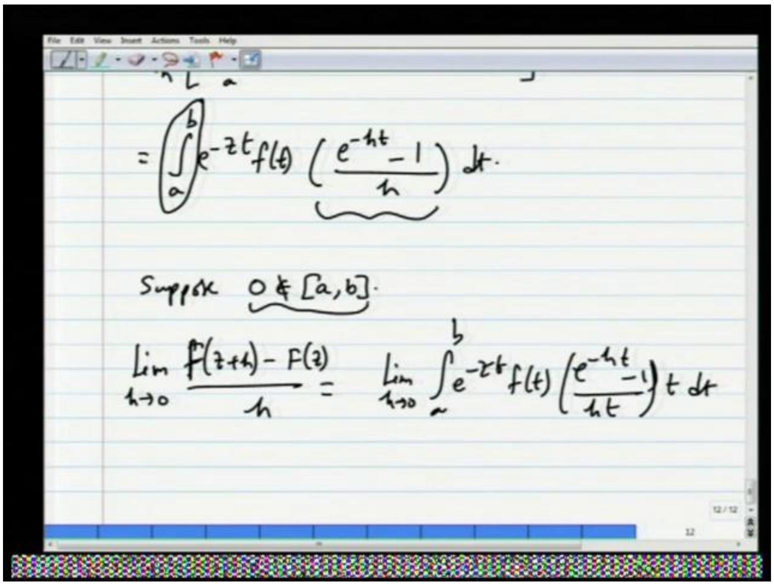

Let us let us say provided we have a t in the denominator here etcetera. So, one then encounters a problem because t could be 0, in this interval a b. So, there are two cases, so for time being let us take this simpler case. So, suppose suppose 0 does not belong to interval a b this is the easier case. So, if the interval a b is half of 0, what we can do is, then this is equal to... Then the, then let me take the limit as h goes to 0 of f of capital F of z plus h minus capital F of z by h, that is going to be the limit as h tends to 0 of this integral a to b e power minus z t f of t e power minus h t minus 1, okay? So, then I am going to multiply and divide by t, I can do that because now 0 does not belong to the interval a b. So, t cannot be 0, so I can multiply and divide.

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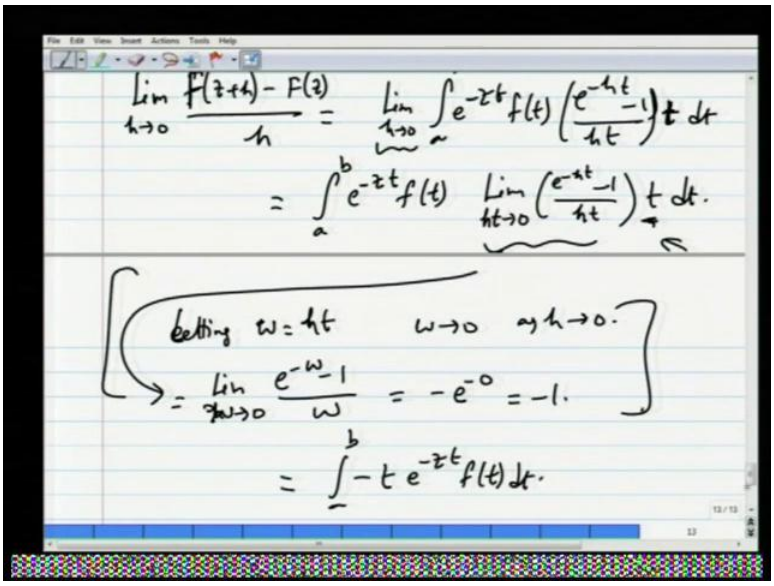

Then this since the limit h goes to 0 does not depend on t, I can take the limiting process inside the integral. So, this is integration from a to b e power minus z t f of t minus limit as h goes to 0, well if h goes to 0 h t also goes to 0 of e power minus h t minus 1 divided by h t times t d t. We know that well e power letting z or I think, I use z w equals h t w tends to 0 as h tends to 0 we have limit as h t goes to 0. This limit is equal to, so this limit I will just say that limit is equal to limit as w goes to 0 of e power minus w minus 1 by w. That is the derivative of e power minus w at at 0.

So, this is nothing but minus e power minus 0 which is minus 1, so with that I have this is equal to, so this is all in parenthesis. So, this is equal to integration from a to b minus t e power, so I am taking this t here minus t e power minus z t f of t d t. So, the derivative exists and so if 0 does not belong to this interval a b, I have shown that f is entire.

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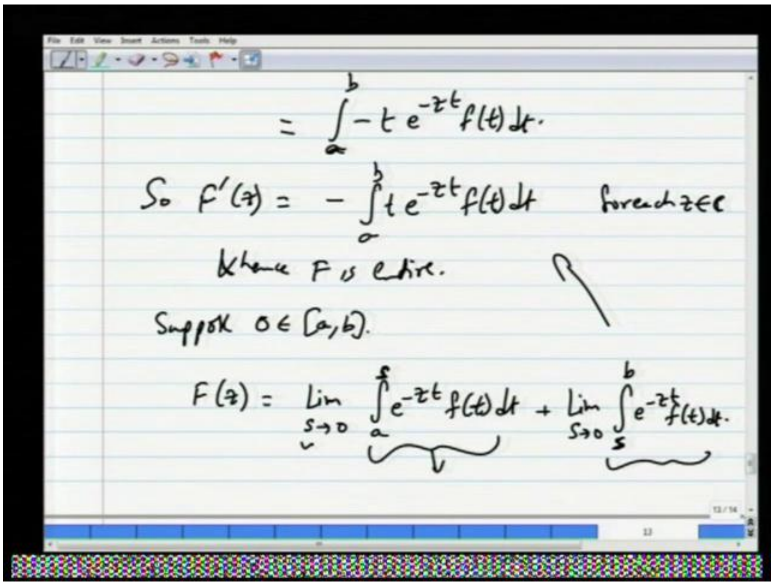

So, F prime of z is equal to minus integration from a to b t e power minus t f of t d t and for each z belongs to C. Hence, capital F is entire. Now, in the other case where suppose 0 belongs to a b, then what one does is one considers this integral by splitting the integral at 0. Taking the limits of, so I will I will explain that f of z is equal to... So, suppose 0 is in there, so f of z can be written as limit as I want to pick s tends to 0 of integration from a to s of e power minus z t f of t d t.

Plus limit as s tends to 0 of integration from s to a b e power minus z t f of t d t. One of them use this what we have already done to conclude that this is the this, write here and this once again is a this write here with s and b as limits. Then once can club them, club the two integrals because now in the end after the derivative process you can have, I mean you can club these two limits to get integration from a to b of this expression, okay?

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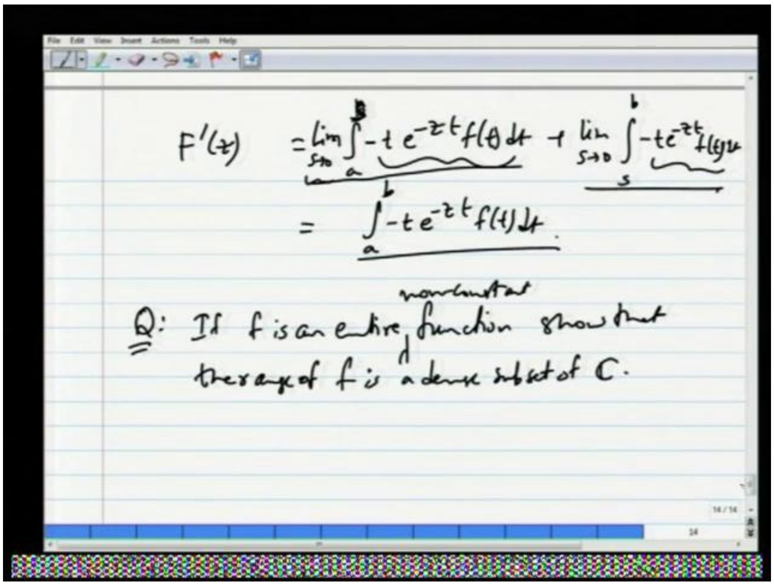

So in deep, what I mean is this is independently integration from a to b minus t e power minus z t f of t d t limit as, sorry a to s, I apologise limit a to s, s goes to zero plus limit s s goes to 0 integration from s to b minus t e power minus z t f of t d t. So, we will delay that limits. So, f prime of z will be that and that can now be combined a to b minus t e power minus z t f of t d t because now there is nothing improper about about combining these two definite integrals into this 1.

So, that is the other case, so that shows that F is entire in any case capital F is entire. So, that is and I will put one more question to the viewer, okay? So, if F is an entire function entire non constant function, show that the range of F is a dense subset of C, so to show that it is a dense subset of C requires a some more work.

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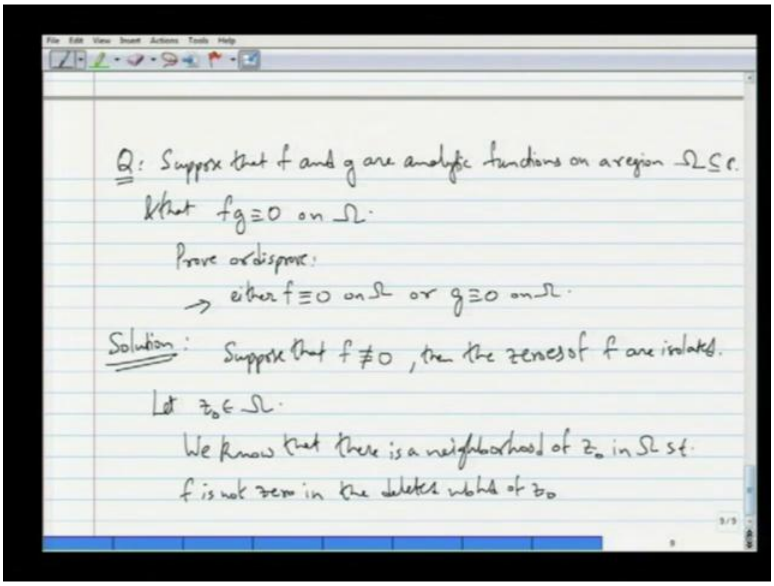

So, it is an exercise, so the next question is as follows, suppose that f and g are analytic functions on a region omega contained in C and that f times g is identically the 0 function the constant 0 function on the region omega, then prove or disprove the following either under these circumstances either f is identically 0 on omega or g is identically 0 on omega. So, that is the question and try to solve it yourself and I will provide the solution here. So, the idea is to use a the identity theorem. So, what is important is that if f if not identically zero, then its zeros are actually isolated.

That is the identity theorem, notice that such a feature is not available for functions of real numbers a differentiable function can be 0 for a while on the real line and then suddenly have a non zero value or the pickup from there. Have non zero values beyond a point, so but that is not the case for a for functions of complex numbers or analytic functions of complex numbers. If they are 0 on a set containing a limit point, then they are identically 0 on the whole region of analyticity on region of analyticity as long as that is connected, okay?

So, so that is a true for analytic functions here, now what we will do is, we suppose that f is not identically 0, if f is identically 0, then this statement is already true. So, let us suppose that f is not identically 0, then let us see what happens then the zeros of f arise solitude. We have examined the zeros of analytic functions and this is our conclusion from that the zeros of f arise, since they are isolated, okay?

So, let z naught belong to omega f possibly could be 0, there or it is possibly non zero there. Whatever that is we know that there is a neighbourhood of of z naught in omega such that f is not zero in the in that neighbourhood in the deleted neighbourhood. So, in the deleted neighbourhood of z naught more specifically we know that we know that there is an r positive by properties of the isolated zeros of analytic functions that or by continuity of f .

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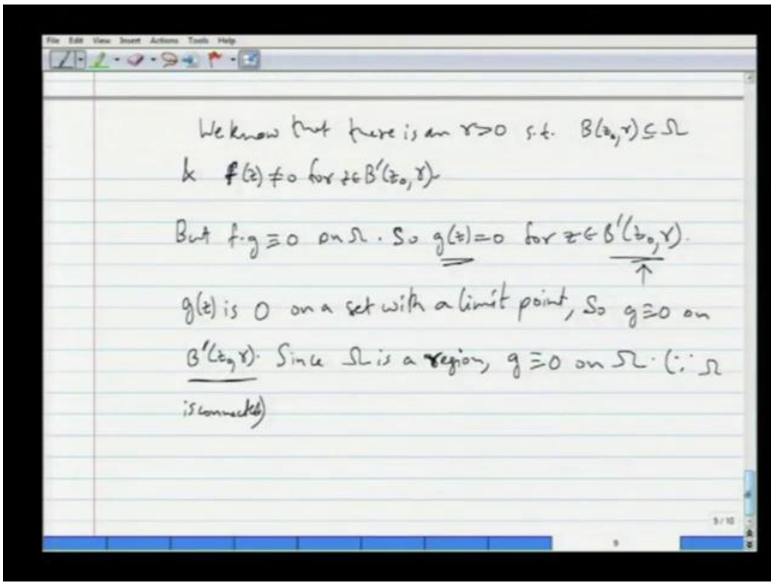

So, there are two cases if f of z naught is 0, then we know this by the isolated zeros of f. If f of z naught is non zero, then we know by continuity of f there is a r positive such that b z naught r is contained in omega this is by the openness of omega and B prime f of z is non zero for z belongs to B prime of z naught r. So, that is true if f is 0 at z naught, then this is true because zeros are isolated. If f is non zero at z naught then, this is true by the continuity of f in b z naught r its analytic. So, it is definitely continuous in B z b prime z naught r, okay?

So, that is true we know there is such an r and since f is non zero in there, but f times g is identically zero. But f times g is identically 0 in on omega. So, in particular we are left with no choice, but to say that g of z is equal to 0 for any z belongs to B prime z naught r because f is non zero there. So, what that tells us is that g the function g is 0 on a set with a limit point. So, notice that this is an open set, so every point in B prime z naught r is actually limit point, okay?

So, in particular g of z is 0 on a set with a limit point, so by the identity theorem we know that, so g is identically 0 on B prime z naught. Since omega is a region we know that it is corrected and on a on a portion of it set containing limit point g is identically 0 on it which is contained in omega. So, since omega is the region g is identically 0 on all of omega by connectedness of omega omega is connected. So, that shows that this if we suppose, that f is not identically 0, then g has to be identically 0 which proves this statement, so that is the solution to this problem.

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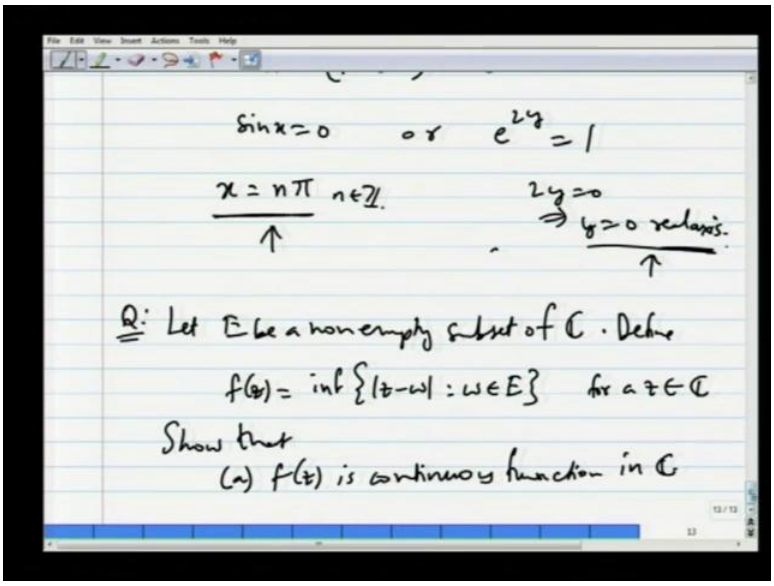

Let e be a non-empty subset of C defines f of z to be the infimum of modulus of z minus w, such that w belongs to e. So, the infimum is taken over all the points belonging to e for, for a, for a z belongs to C for a given z belongs to C. You define f of z to be this show that a f of z is continuous function continuous function on all of C in C and d.

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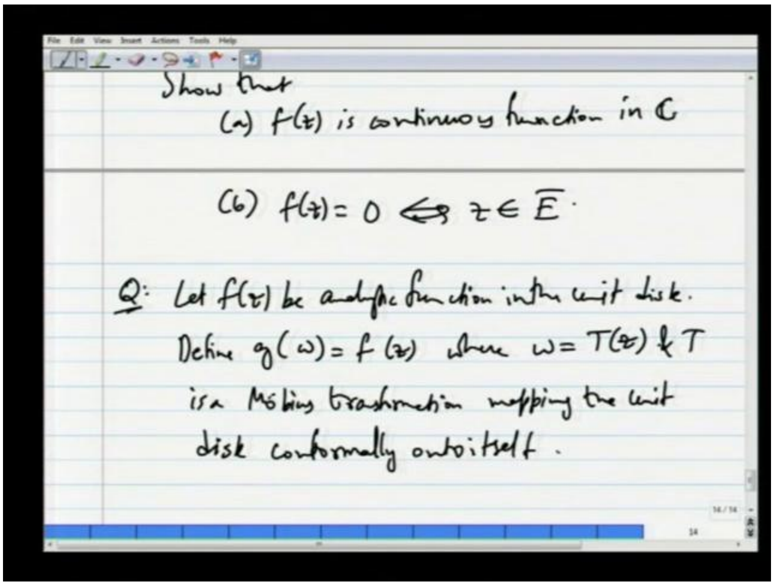

Show that f of z is 0 if and only if z belongs to the closure of... So, this is a question the viewer is asked to call and then there is yet another question that I will put here. So, suppose f is analytic let f of z be analytic function in the unit disk define g of w to be f of z, where w is t of z and t is a Mobius transformation mapping the unit disk to unit disk or rather unit disk conformally onto itself, then show that the 1 minus the modulus of w square times modulus of d g by d w is equal to 1 minus the modulus of z square times modulus of d f by d z, okay?

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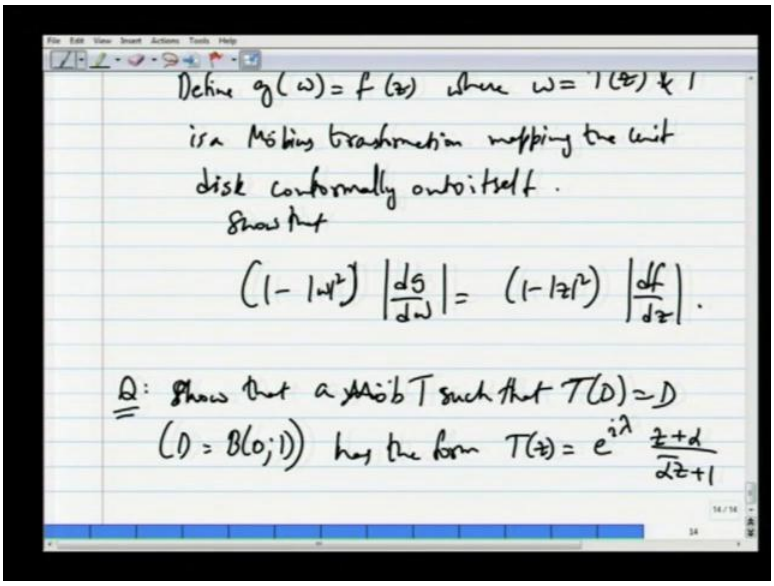

So, you have to use the chain rule to arrive at the solution. So, please try this problem also of interest is the following its very useful and important problem find or give an expression. So, I will may be give it as a show that show that mob T such that T of d is equal to d, where d is the unit disk unit b 0. One has the form has the general form T of z is equal to e power i lambda times z plus alpha by alpha bar z plus 1 where lambda is a real constant and alpha belongs to b 0 i.

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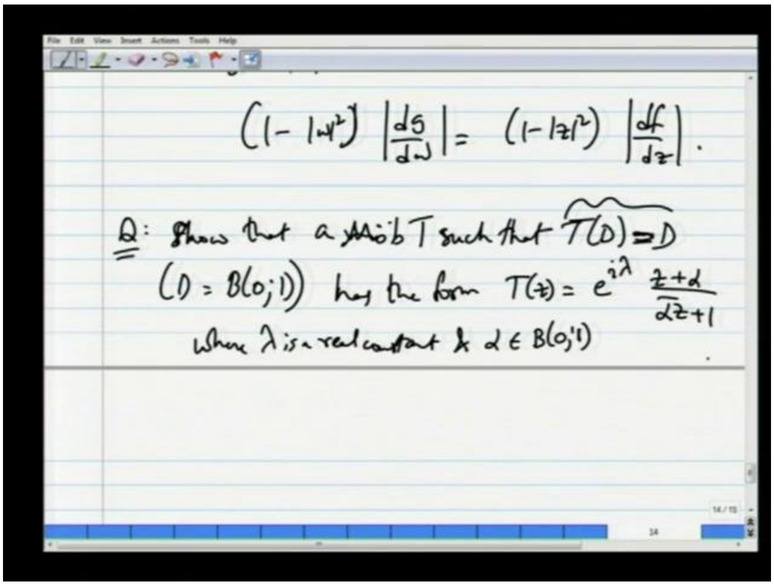

So, the set of all transformations, which take the unit disk onto the unit disk, that equality means that it takes unit disk onto unit disk has the form e power i lambda times z plus alpha divided by alpha bar z plus 1. So, this is also an exercise for the viewer. So, with this I will stop this session here. So, the viewer is asked to solve more exercises and try to gain good understanding of the theory via the problems as well. I will stop here.