Complex Analysis
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Module - 5
Mobius Transformations
Lecture - 2
Properties of Mobius Transformations Part II

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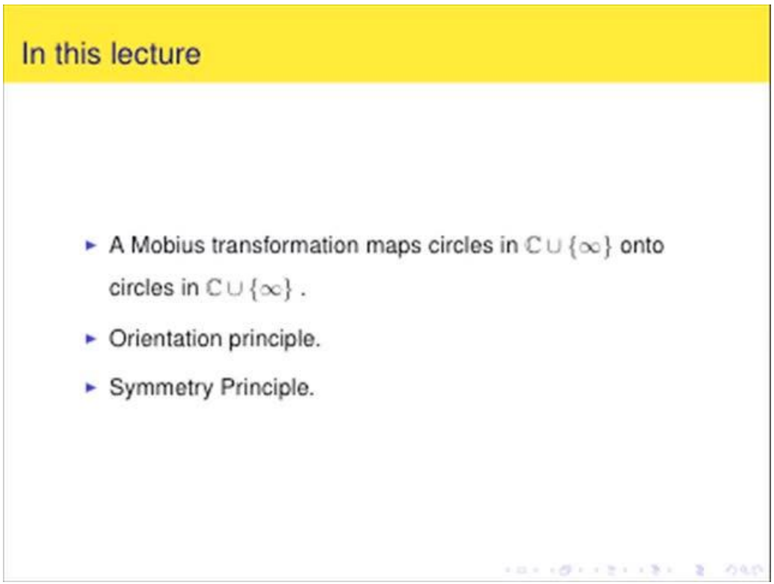

Hello viewers, in this session we will continue with the properties of Mobius transformations. So, last time we have defined the cross ratio and we have seen, how we can use the cross ratio to prove some properties of Mobius transformations? So, we will continue with that and we will show that Mobius transformations in the session, we will show that Mobius transformations take circles to circles, so images of circles on the Riemann's sphere via Mobius transformations are precisely circles; so in order to show that, we will prove the following important preposition.

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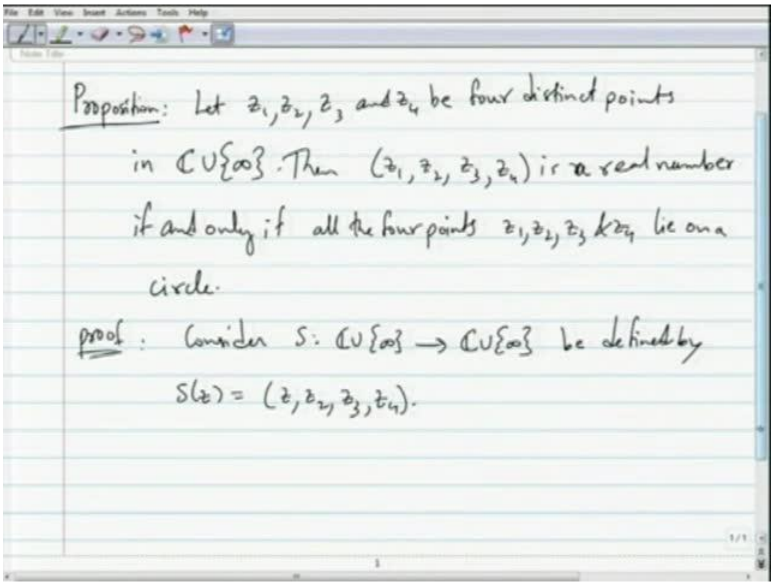

So, let z 1, z 2, z 3 and z 4 be four distinct points in C union infinity. Then the cross ratio z 1, z 2, z 3, z 4 is real, is a real number, if and only if all the four points z 1, z 2, z 3 and z 4 lie on a circle. So, this preposition is useful in the following sense, it it tells it gives criterion in terms of the cross ratio, when four points, four distinct points lie on a circle, okay? So, here is the proof of this fact, so consider S from C union infinity to C union infinity defined by this cross ratio, we know that these are bijections. So, be defined by S of z is equal to z, z 2, z 3, z 4.

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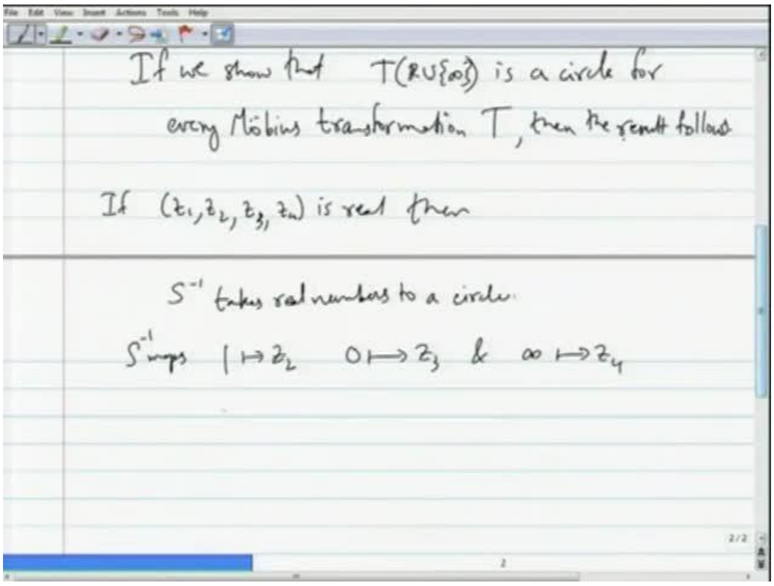

So, we will first see that if we show that, T of R infinity or union infinity is a circle for every Mobius transformation T, then we are done then the follows. I will explain why? So, if we are able to show that the image of the R union infinity, the real line in the complex plane union the pointed infinity is a circle, for every Mobius transformation. Circle in the sense of course, it is a circle on the Riemann's sphere, so you also includes straight lines union pointed infinity, okay?

So if the cross factor is real z 1, z 2, z 3, z 4 is real then S inverse takes real numbers to a circle, right? So, because every Mobius transformation, so if we show that every Mobius transformation takes R union infinity to a circle S inverse takes real number union infinity to a circle. In particular it takes real numbers to the, to to the circle and S inverse maps 1 to z 2, 0 to z 3 and infinity to z 4, right? Because S maps the other way round z 2 to 1 z 3 to 0 and z 4 infinity respectively, okay?

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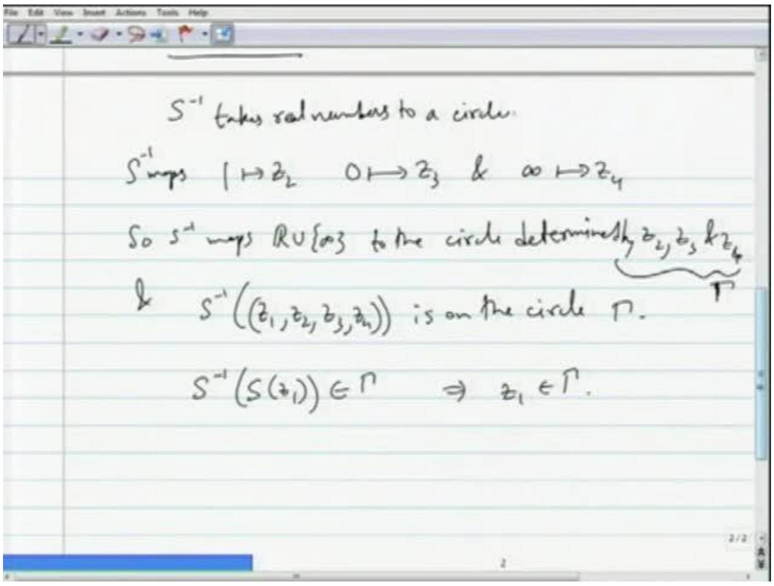

So, so S inverse maps you know 1 0 infinity to those points, so S inverse maps R union infinity. We already assumed that it maps R union infinity to a circle, so it maps R union infinity to the circle determined by the three points z 2, z 3, z 4. They are distinct determined by z 2, z 3 and z four and three points always determine a circle in the complex plane union infinity. S in I mean the cross ratio we know is real, so S inverse of a this real number z 1, z 2, z 3, z 4 of this real number has to be is on the circle on the circle gamma, where gamma is the circle determines by these three things.

So, let us call that gamma is on the circle gamma. So, what that means is S inverse of S of z 1, right? S of z is the cross ratio, so S inverse of S of z 1 belongs to gamma, which implies z 1 belongs to gamma. So, if the cross ratio is real and if he assume that every Mobius transformation takes R union infinity to a circle then z 1 has to be on the circle determined by z 2, z 3, z 4 which means z 2, z 3, z 4 and z 1 lie on a circle.

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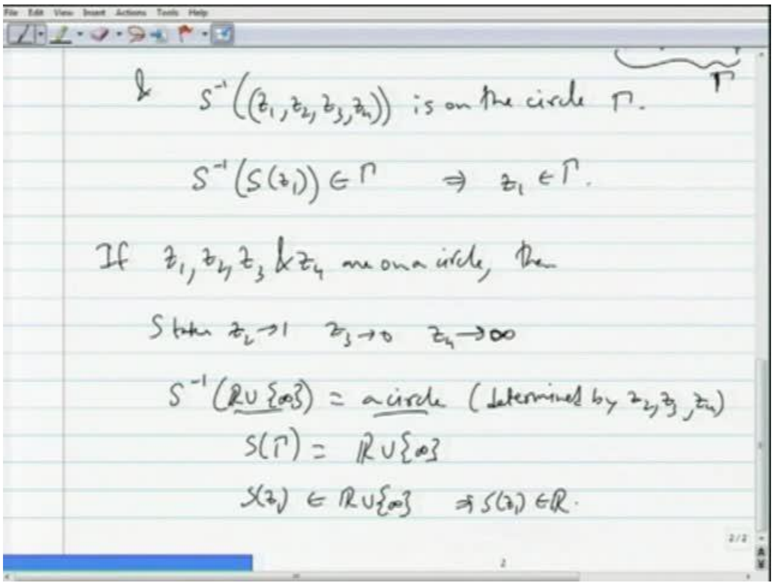

So, now in the other direction if z 1, z 2, z 3, z 4 are on a circle, then what we have to show is the cross ratio is a real number. Well, S takes z 2 to 1, right? Like above z 2 to 1 z three to 0 and z 4 to infinity, infinity. Then S inverse of R union infinity is going to be a circle like as I already circle it is going to be a circle and then it is, this is, this is determined by z 2, z 3, z 4 rather, okay? So, S of that circle S of gamma is equal to R union infinity.

So, S of z 1 which is on the circle gamma will belong to R union infinity. Well it cannot be infinity because these are distinct points and Mobius transformations are one to one functions and z 4 already goes to infinity. So, S of z 1 belongs to it is a real number, which means cross ratios are real number. So, if we assume or if we prove that T of R union infinity is a circle for every Mobius transformation T, then where so we will we will show that part here.

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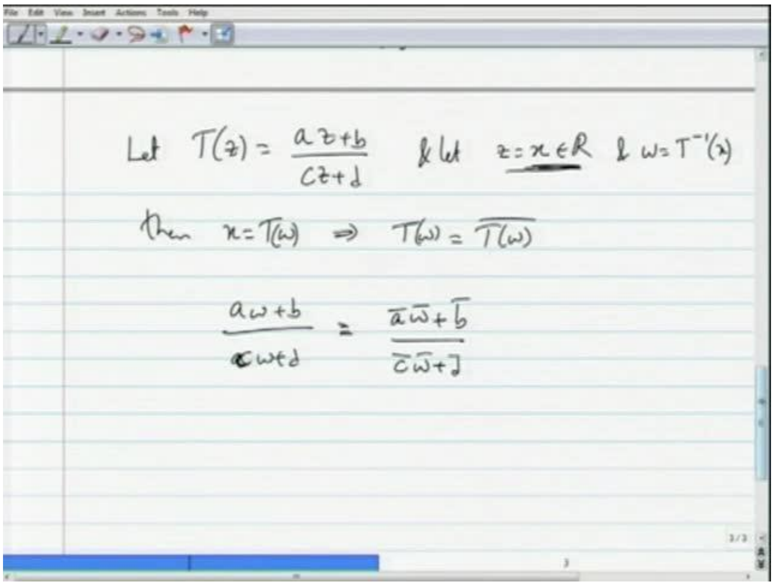

So let, let T of z be a z plus b divided by C z plus b, it is a Mobius transformation and let z equals x belongs to R. So, we will we can treat infinity separately. So, let us say x x z is x belongs to R and w is S T inverse of x. Then x is equal to T w, which implies T of w T w or T of w is equal to its own conjugate, because it is a real number. x is a real number, so T is equal to its own conjugate T of w is equal to conjugate. So, let us write down what that means a w plus b divided by a C w plus d is equal to a bar w bar plus b bar divided by C bar w bar plus d bar by the properties via the properties of conjugation.

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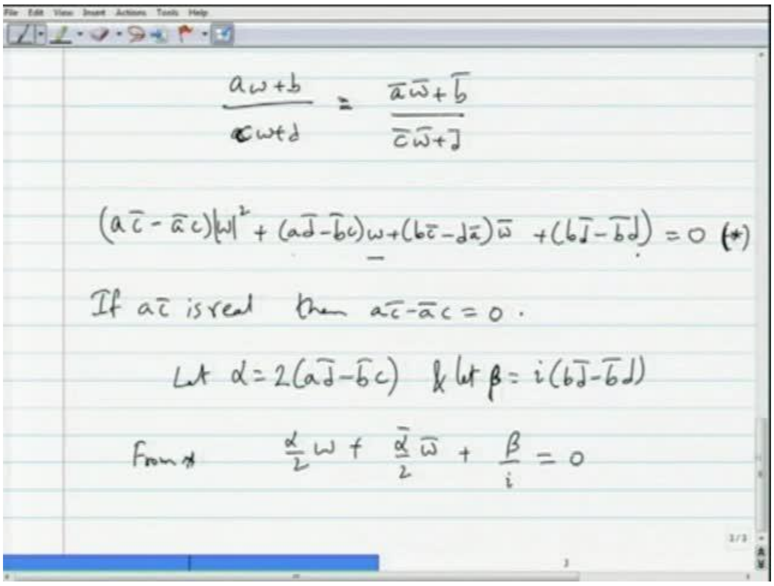

So, after rearrangement this gives us a c bar minus a bar C modulus of w square plus a d bar minus b bar c times w plus b C bar minus d a bar times w bar plus b d bar minus b bar d is equal to 0. So, let us call this equation star looking at this equation, what we want to show is that w is a point on the circle. So, the locus of w it satisfies this equation is actually a circle, so we are actually showing this statement T of R union infinity is a circle by showing that T inverse of R union infinity is a circle for every Mobius transformation T. But since every Mobius transformation is an inverse of yet another Mobius transformation, we get the result we get the result that we desire.

So, firstly there are two cases to consider if a c bar is real this is the first case then, a c bar will be equal to a bar c. It will be equal to its own conjugate. So, in this case then a c bar minus a bar c is equal to 0. So, this coefficient here is going to be 0, so in this event we observe that we have conjugates here. So, 1 is the conjugate of the other two terms, which are conjugate to each other. So, let alpha equals 2 times a d bar minus b bar c, that 2 is just to avoid a multiple of 2. We will see in a moment and let beta is equal to i times b d bar minus b bar d.

So, once again I have multiplied that by an i to put this other term here in a nice form. But notice that this number is a difference of conjugates. So, that gives an purely imaginary number, so it gives the imaginary part of b d bar two times that, so that is that is imaginary number. So, beta here is a a real number, so then what we get by substituting alpha and beta like this. What we get is is alpha by 2 w plus alpha bar w bar by 2 plus beta divided by is equal to 0.

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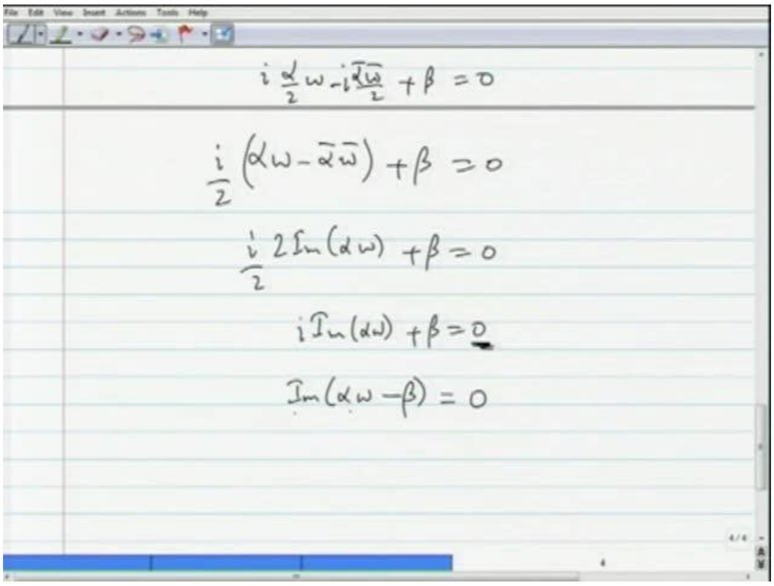

So, then I will multiply by i to get i times alpha by 2 w plus alpha bar w bar by 2 plus beta is equal to 0. I apologise this should be a minus this is b c bar minus b a bar, which is minus of this. So, I will remove this underlines, so this is a minus alpha bar w bar this is a minus and then I get alpha w minus alpha bar w bar. Then I had an i here, I had to multiply an i here i by 2 times all this plus beta is equal to 0. Then alpha w minus alpha w bar is 2 times the imaginary part of alpha w times i by 2 plus beta is equal to 0. This is i times the imaginary part of alpha w plus beta is equal to 0. So, so we can sum these up by saying imaginary part of alpha w minus beta. So, by equating the real parts we have that is equal to 0. So, we recognise this as I mean the locus of these points is a is a line, right?

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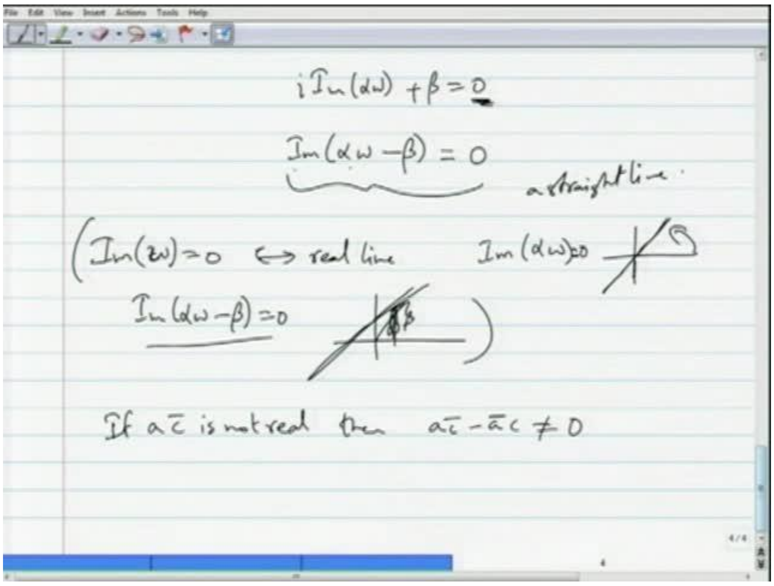

Imaginary part recall from section of geometry, we said that imaginary part of w is equal to 0 gives us real line and likewise imaginary part of alpha times w. What it gives is, it is a line passing through origin of that sort a is equal to 0 and imaginary part of alpha w minus beta is equal to 0, which gives a line which is that sort everything is transferred by a vector beta, okay?

So, that is beta. So, every point is moved by a vector beta and here you multiply by number alpha. So, the real line actually turns and then alpha w minus beta will transfer the line by beta. That signifies a straight line, so the locus here is a straight line. So, in the event that a c bar is real what we have is a straight line, so in the other case, if a c bar is not real if a c bar is not real in the coefficient of mod w square in in star is not 0. Then then a c bar minus a bar c is not equal to 0, so that is the point.

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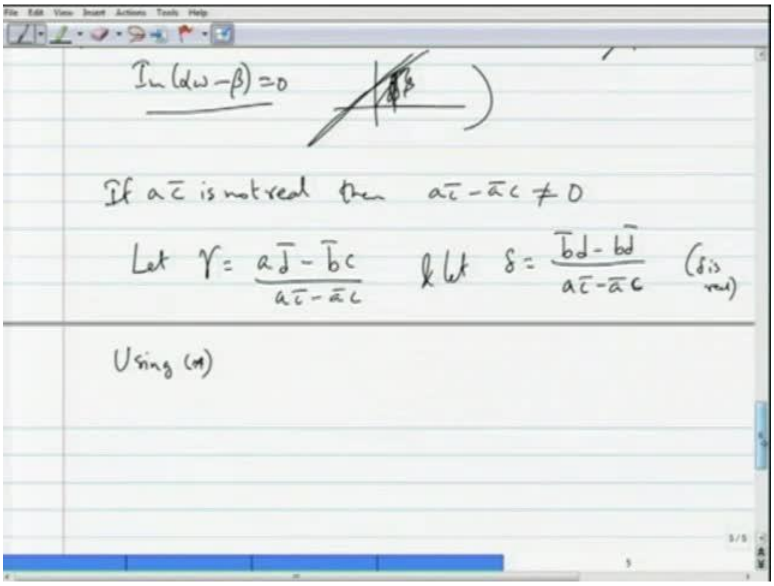

So, then that let gamma be a d bar minus b bar c divided by a c bar minus a bar c. So, we can now divide a c bar minus a bar c and let delta be equal to b bar d minus b d bar divided by a c bar minus a bar c. So, then notice that delta is a is the real quantity because the numerator is a purely imaginary quantity it is two times the imaginary part of certain complex number and likewise the denominator is a purely imaginary quantity.

So, i s cancel as the two cancel etcetera so you have a real quantity. So, delta is real, so real number and then by letting this and substituting this star using star what we have is let me go back to star, sorry so this is I am dividing equations star by a c bar minus a bar c. Then the coefficient of w and w bar show up something in terms of gamma because i am dividing by a c bar minus a bar c, so that is the coefficient and this last coefficient this last constant will be in terms of delta.

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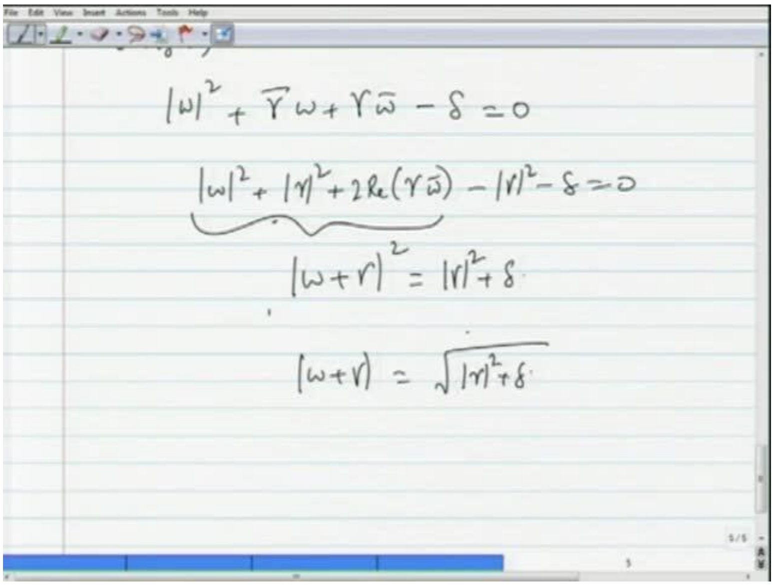

So, what I get is, I have modulus of w square plus gamma bar w minus gamma w bar like we observed earlier. These are conjugates of each other, these two terms are conjugates of each other and then finally, I have minus delta is equal to 0. The way I have assumed delta is b bar d minus b d bar that is the negative of what I have there, we will see why in a moment, okay? So, then this is the same as modulus of w square plus modulus of gamma square. I am adding a modulus of gamma square and these are conjugates of each other.

So, when I add them, I get 2 times real part of gamma w bar. Then, minus modulus of gamma square I have added a modulus of gamma square there. So, I get that minus delta is equal to 0, so this we recognise the modulus of w plus gamma whole square. That equals modulus of gamma square plus delta and modulus of gamma square is a real number plus delta is a real number. So, this looks like the equation of a circle, so this is modulus of w plus gamma. I will take this part is equal to square root of modulus of gamma square plus delta it is a positive real number we will show, okay? So, one can check, okay?

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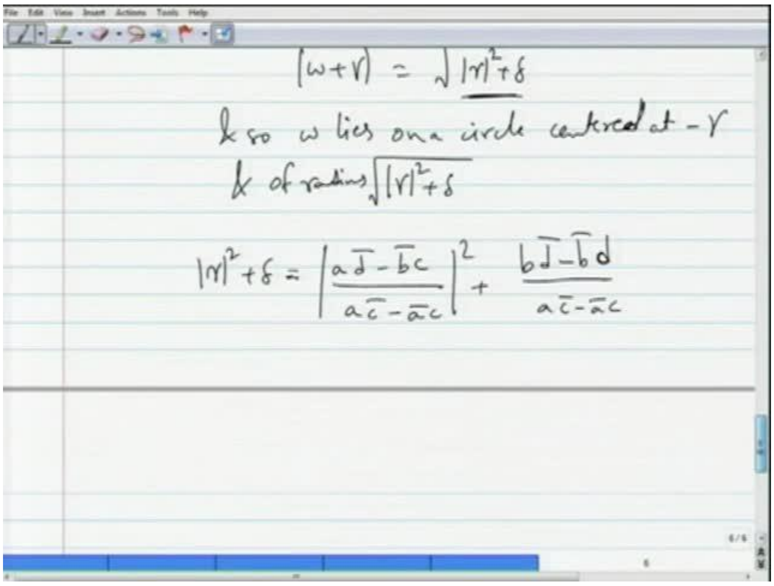

So, w lies on a circle centred at minus gamma and of radius modulus gamma square plus delta square root. So, only if modulus of gamma square plus delta is a positive real number, but that is easy so you can check that modulus of gamma square plus delta. Well that is equal to modulus of a d bar minus e divided by a c bar minus a bar c whole squared plus delta modulus of delta. Well, delta is b d bar minus b bar d divided by a c bar minus a bar c, when I take the common denominator.

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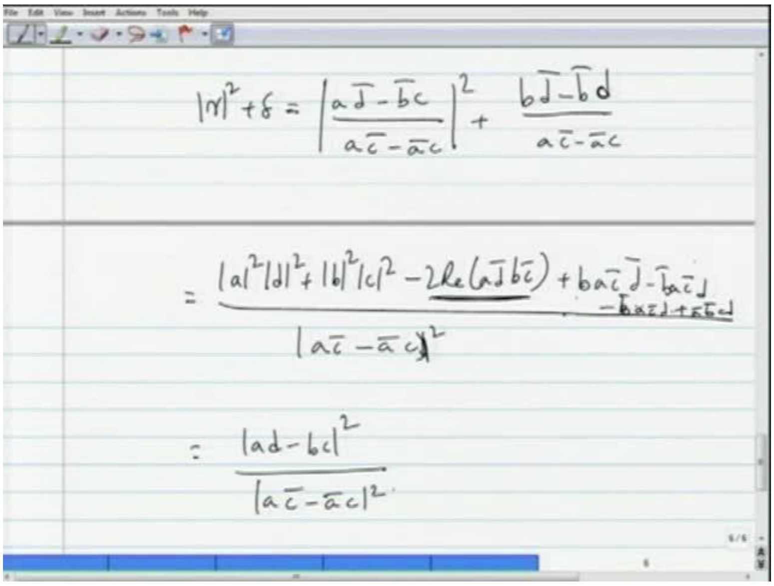

What I have is modulus of a square modulus of d square plus modulus of b square modulus of c square minus two times the real part of a d bar minus b bar c a d bar minus c b b c bar rather. Then divided by modulus of a c bar minus a bar c square plus b ad bar so we can actually directly add this, okay? So, I get plus b a c bar d bar etcetera of minus b bar a c bar d minus b a bar c bar d plus a bar b bar, so this two times real part of a d bar b c bar cancels with this plus this.

So, this I mean terms with the plus sign in front of them that cancels with that b bar and then the terms with the minus sign in front of them they remain. That gives me modulus of a d minus b c that gives me that divide by modulus of a c bar minus a bar c square, okay?

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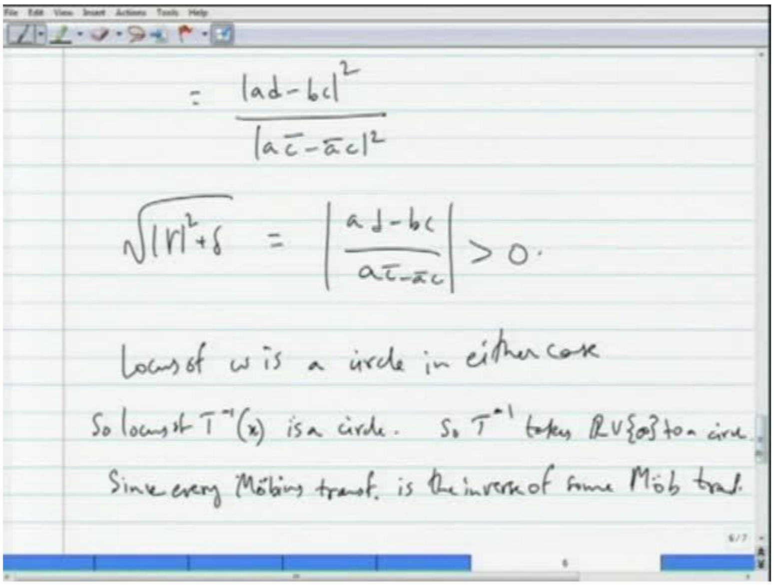

So, the square root of modulus gamma square plus delta is equal to modulus of a d minus b c by a c bar minus a bar c which is a positive quantity. So, that shows that T inverse in in either of these two cases, what we have shown is that locus of w according to star the locus of w. According to star is a circle, so locus of w is a circle in either case it is a generalised circle on the Reimann's sphere. So, you have straight line from the complex plane in an infinity or you have a circles is a circle.

So, T inverse takes r union infinity to a circle since, there was no assumption about x the real number x t inverse of r union infinity is a circle. Since, every Mobius transformation is inverse of some Mobius transformation, it is through that the image of R union infinity is a circle for Mobius transformation. So, since every Mobius transformation is the inverse of some Mobius transformation and every inverse has this property every Mobius transformation takes R union infinity to a circle and that completes the proof of this proposition.

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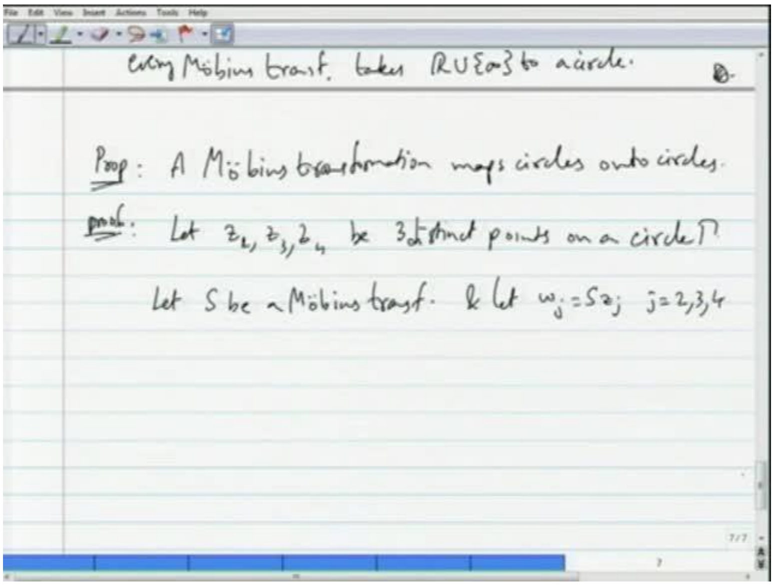

Next, we will show the following proposition based on this one which we have been aiming at a Mobius transformation maps circles on to circles. Now, this is easy or it follows easily from the previous proposition. So, let z 1, z 2 or z 2, z 3, z 4 be three distinct points on a given circle on a circle gamma. So, let S be Mobius transformation, so you take the general Mobius transformation and let w 2 I will just say w j is equal to S z j for j is equal to S z j or j is equal to 2, 3, 4. So, let w j be the image of z 2, z 3, z 4 or z j in under S, okay?

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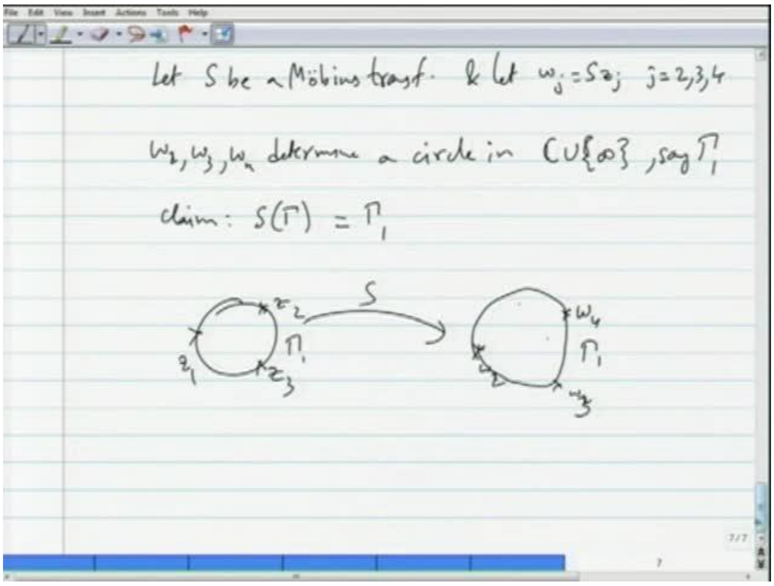

Now, w 2, w 3, w 4 determine a circle any three points. Determine a circle these w 2, w 3, w 4 they have to be distinguished because z 2, z 3, z 4 were distinct. So, and S is a Mobius transformation, so these determine a circle in c union infinity. We will call that say gamma 1. So, we will claim the, claim is that S maps, so S for now maps three points z 2, z 3, z 4 to w 2, w 3, w 4. There is a circle determined by z 2, z 3, z 4 and 1 by w 2, w 3, w 4, okay?

So, the claim now is that f takes all of gamma on to gamma 1. So, here is z 1, z 2 and z 3 on a circle and then all we know is that the image of these three points is is w 1, w 2, w 3 or w 2, w 3, w 4 and there is a circle which passes through these three gamma. One the claim is that s maps exactly the circle under that circle now the proof is very easy using that other proposition.

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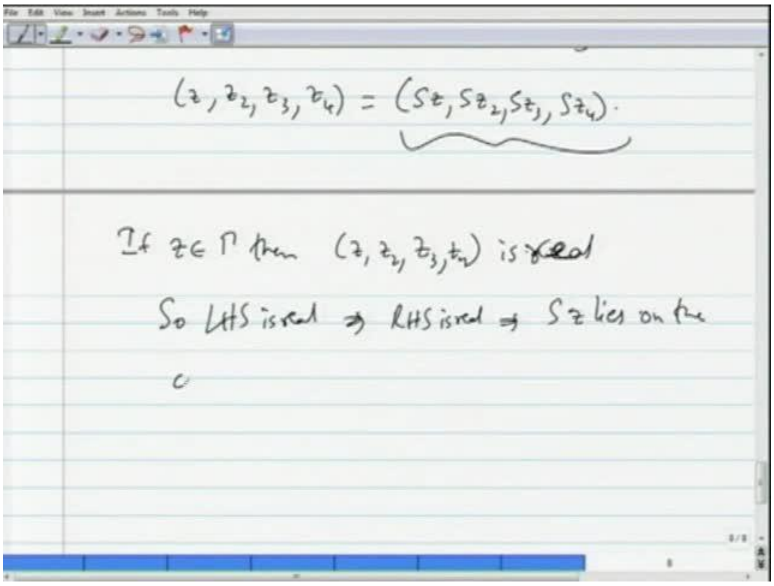

The cross ratio z z 2, z 3, z 4 is equal to cross ratio S z, S z 2, S z 3 and S z 4 by a previous preposition. We saw this, this tells us that if z belongs to gamma, then by the previous preposition this cross ratio z, z 2, z 3, z 4 z is a point on the circle determined by z 2, z 3, z 4. So, this is real, is a real number. This is real and so LHS is real, so this number is also a real number. What that tells is that implies RHS is real and what that implies is S z lies by the previous preposition S z lies on the circle.

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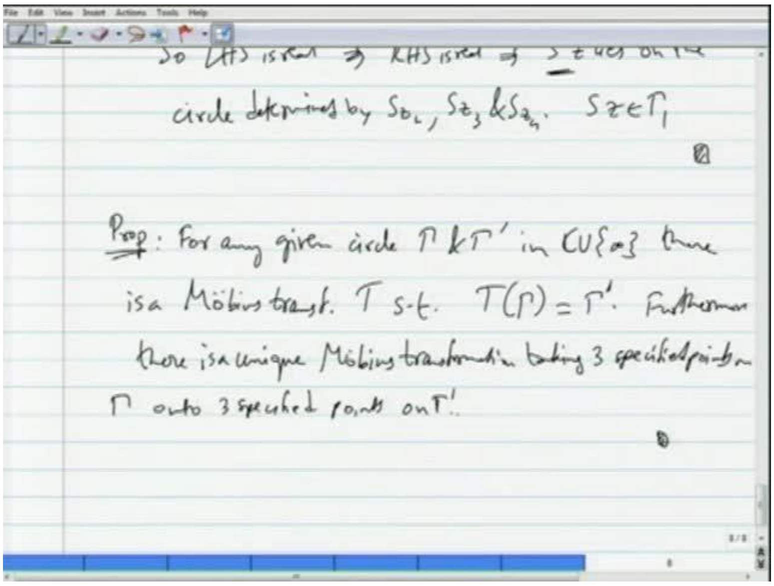

Determined by S z 2, S z 3 and S z 4, so S z belongs to gamma 1, so that completes the proof of this preposition. So, it follows directly from the previous preposition. Now, then we can also do the following for any given circles gamma and gamma 1 or gamma prime. Let us say in C union infinity, there is a Mobius transformation that is easy Mobius transformation, T such that T of gamma is equal to gamma prime, okay? Furthermore, there is a unique Mobius transformation taking three specified points on to on gamma on gamma onto three specified points.

These points have to be distinct, distinct points on gamma prime. The proof is it just follows from what we have done already. So, the proof is an exercise for for the viewer. It just follows from what we have done, so given three points from the circle and then three point from that circle that is unique Mobius transformation, which takes these three points to those three points. Well if you give a circle, there are many ways to specify circle as an by many triples of points specify this circle.

So, if you want a circle to be mapped on to another circle, there are many Mobius transformations doing the job. So, that is this preposition. What we have shown, so far is that circles on C union infinity on the Riemann’s sphere are actually mapped onto circles via Mobius transformations. So, we will rough roughly state some theorems, which will tell us that the inside and outside of the circles all are also mapped onto inside and outside of circles, via Mobius transformation. So, we will make that notion a little concrete, okay? So, firstly let me make a diagram like this, okay?

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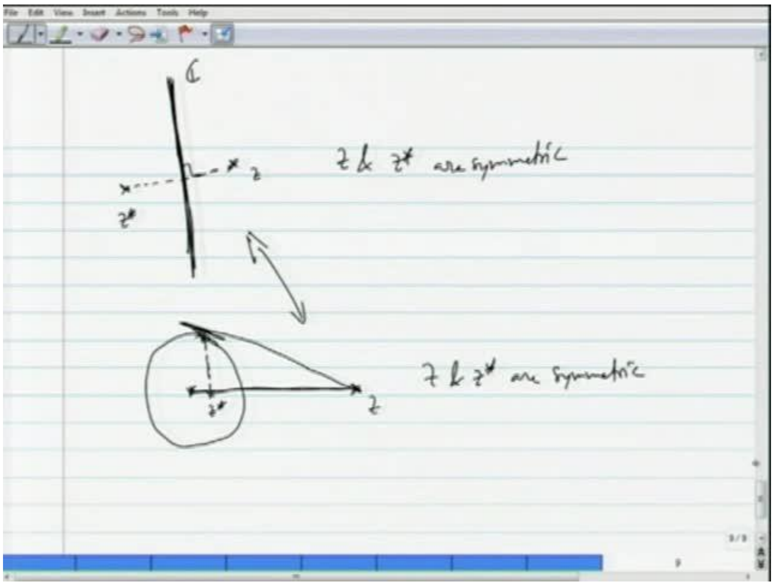

So, the point z here is a straight line in C union infinity or a circle in C union infinity, which is a straight line in the complex plane. So, if you have a point z from high school geometry we know that where we call a point z star symmetric to z with respect to this line. If the line joining z z star is perpendicularly bisected by this one, so z and z star we call a symmetric and there is a similar notion, when it comes to circles. So, this notion actually transfers to circle from the Riemann's sphere you take a point, which is outside the circle here is z should draw tangent from the circle to the point to the circle. You drop a perpendicular onto the line joining z in the centre of the circle and this point z star that you get is called symmetric. So, this is a notion, which equates to this notion for a straight line, when you look at it on the complex on the on the Reimann's sphere.

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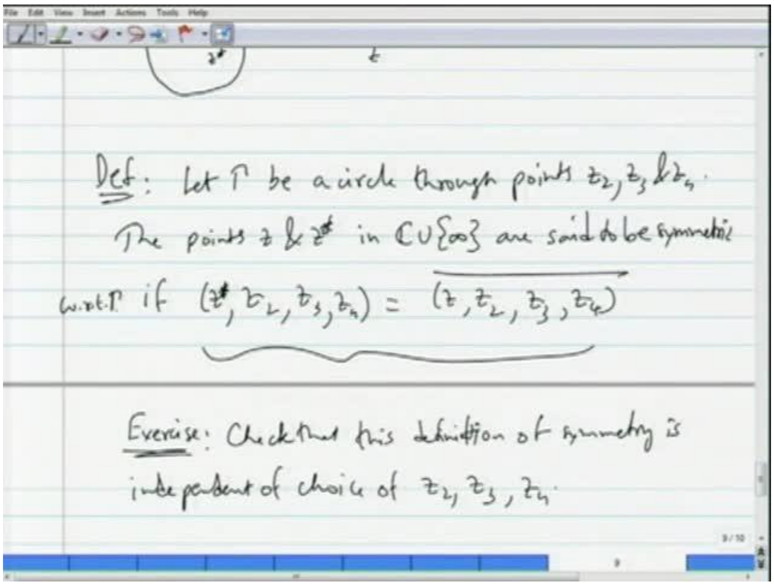

So, z and z star then are called symmetric and when we write in the language of cross ratio we have the following definition. So, let gamma be a circle through points z 2, z 3 and z 4 the points z and z star are in C union infinity are said to be symmetric. If z star z 2, z 3, z 4 is equal to z, z 2, z 3, z 4 conjugate a conjugate of this is equal to the cross ratio with z star, okay?

So, in the, this is essentially the symmetry the geometric symmetry. Here in the language of cross ratio and the equivalence of this is proved in the textbook. I am following for this discussion namely in John Conway's book functions of one complex variable. So, interested viewer can actually work this out as an exercise or look into the book. One one thing that one should observe is that check that here z 2, z 3, z 4 determine a circle and I am defining symmetry apparently by choosing a particular z 2, z 3, z 4 on the circle. So, check that this definition of symmetry is independent of the choice of z 2, z 3, z 4, okay?

So, in the definition I should say said to be symmetric with respect to gamma that is important, sorry for that so, this is are said to be symmetric with respect to gamma. If that happens there are other ways of specifying gamma. You can take other z 2, z 3, z 4. So, if you take z 2 prime z 3 prime and z 4 prime, try to show that this equalities will holds if it holds for z 2, z 3, z 4, okay?

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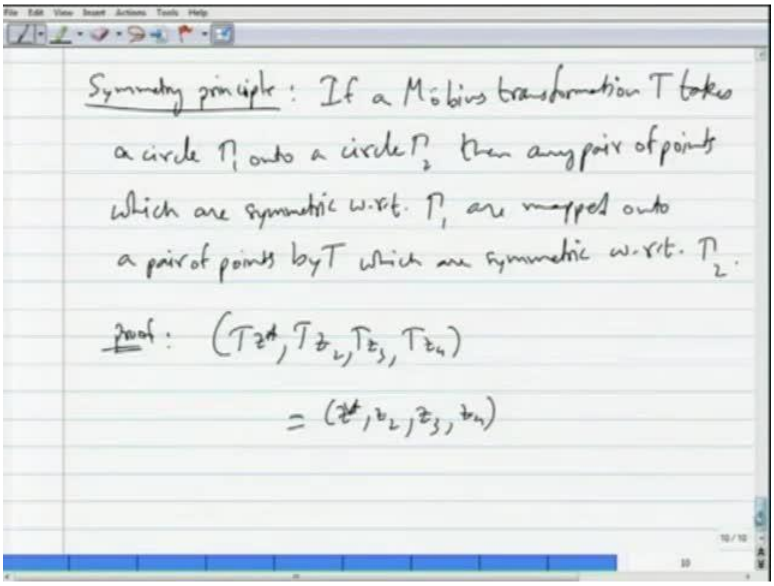

So, then I am going to state what happens to points inside and outside of a circle via a Mobius transformation, via the mapping this is the Mobius transformation, okay? So, here is symmetric principle I will not prove this, but I will just state this if a Mobius transformation T takes a circle gamma 1 onto a circle gamma 2. Then any pair of points which are symmetric with respect to gamma 1 are mapped onto a pair of points by T. They are mapped onto two points by T, which are symmetric with respect to gamma 2. So, the image of a circle is a circle and image of two points are symmetric with respect to the initial circle are now symmetric with respect to the image circle via T.

So, that is the picture we have and the proof is it is very easy. So, T z star, T z 2, T z 3, T z 4, that is going to be we have to show that this is equal to T z, T z 2, T z 3 T z 4 conjugate, but this is equal to z star, z 2, z 3, z 4 via property, we had earlier that the cross ratio remains in variant under when the when all the points are moved via Mobius transformation.

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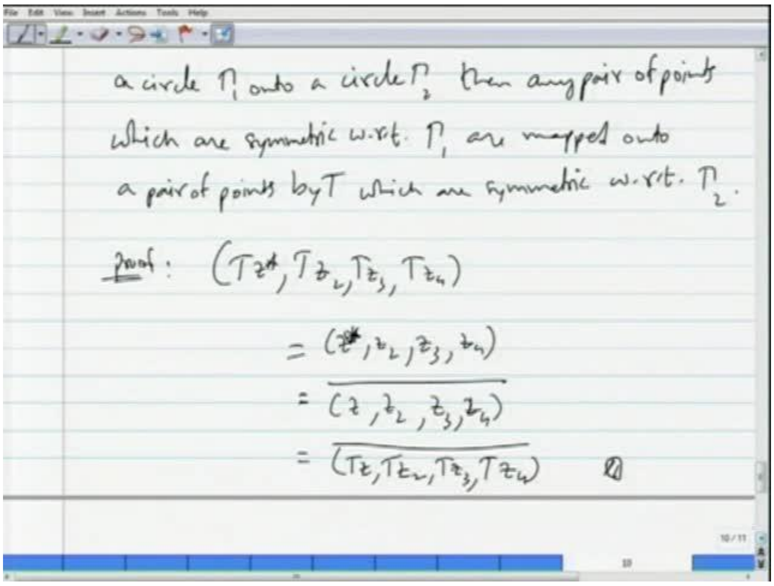

This is equal to z, z 2 T z 3 T z 4 because z star and z are symmetric with respect to gamma 1. That is equal to once again by invariance by r Mobius transformation. We have T z, T z 2, T z 3, T z 4 conjugate, okay? So, since z, z 2, z 3, z 4 is equal to the cross ratio T z, T z 2, T z 3, T z 4, so it completes, okay?

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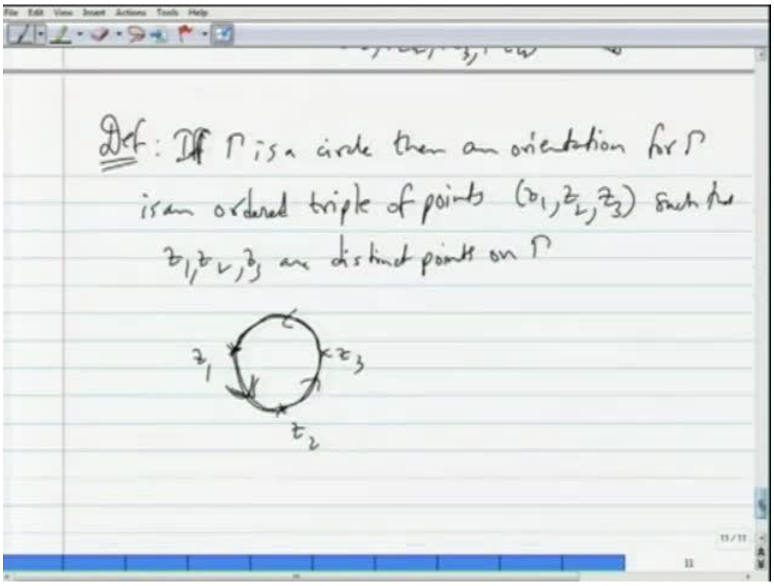

So, then we are going to make a a definition of orientation. Now, well if you have a circle its stays unoriented, but in order to identify the inside and outside we will orient the circle first. So, if gamma is a circle, if gamma is a circle, then orientation for gamma is an ordered example of points z 1, z 2, z 3. This is an ordered triple such that z 1, z 2, z 3 are distinct pints on gamma.

So, what you are doing is you are picking three points z 1, z 2, z 3 on a circle and specifying them as an ordered triple. So, the order tells you that you go from z 1, to z 2 and then z 2 to z 3 and comeback to z 1. So, that specifies a particular order, when z 1, z 2, z 3 are distinct. So, it specifies an orientation on that circle and when you imagine walking on that circle in the complex plane or in C union infinity. So, with your with your head up, so to say then you have a region to the left and you have a region to the right. Then what we have is the following we have the orientation principle.

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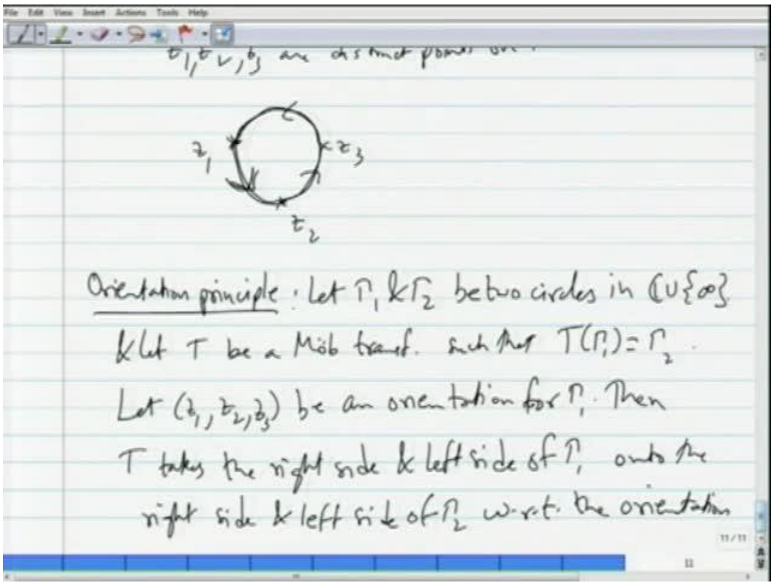

So, let gamma 1 and gamma 2 be two circles in C union infinity. So, let T be a Mobius transformation such that T of gamma 1 is equal to gamma 2. Let z 1, z 2, z 3 be an ordered be an orientation of gamma 1 or for gamma 1. Then T takes the right side and left side of gamma 1. Onto the right side and left side of gamma 2 with respect to orientation the orientation of gamma 2 given by T z 1, T z 2, T z 3.

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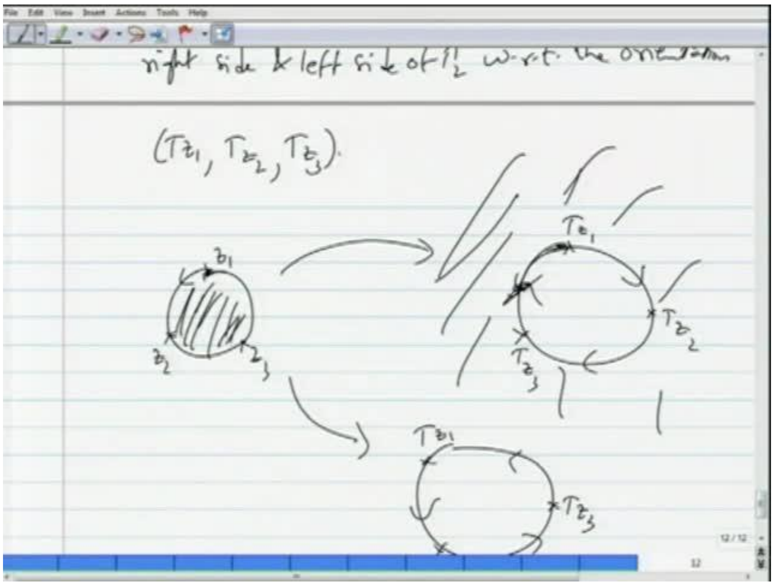

So, a circle can be mapped onto, so you pick a circle with a orientation z 1, z 2, z 3, okay? It can be mapped onto a circle in two ways looking at the orientation it could be T z 1, T z 2, T z 3 or you could have a circle T z 1, T z 2, T z 3. So, it depends upon the Mobius transformation this circle will give you, sorry this kind of orientation and this circle will give you this kind of orientation. So, there are two ways two different orientation used on the same circle.

That orientation depends on T and this orientation principle which we are which we are stating here without proof that tells you that the left side here. For example, this is the left side on on this, in this picture that is mapped onto the left side. Here in this case the left side happens to be outside of this circle and in this case it happens to be inside of the circle, okay? The left sidedness and right sidedness are preserved by Mobius transformation and that is stated by the orientation principle. So, once again for rigorous proofs one can consult convey.

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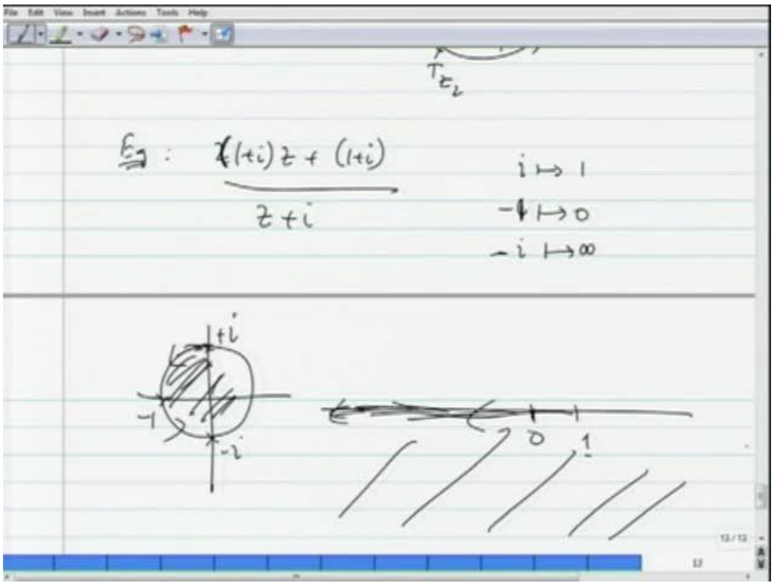

So, here is a quick example, the transformation 1 plus i times z plus 1 plus i divided by z plus i, this is the Mobius transformation. It takes it maps i to 1 minus 1 to 0 and, and i to infinity or minus i to infinity. It maps so, it maps the unit circle here is the minus i and minus 1 sorry, here is plus i here is minus 1 and here is minus i, it maps them to 1, 0 and infinity.

So, infinity is direction, think of it in this direction. So, it maps this circle onto this line in this direction with that orientation. So, what it does is it maps the unit disc, which appears to the left of the circle now onto the lower half. So, that is an example, so we will conclude this discussion on Mobius transformations with this example. I will stop.