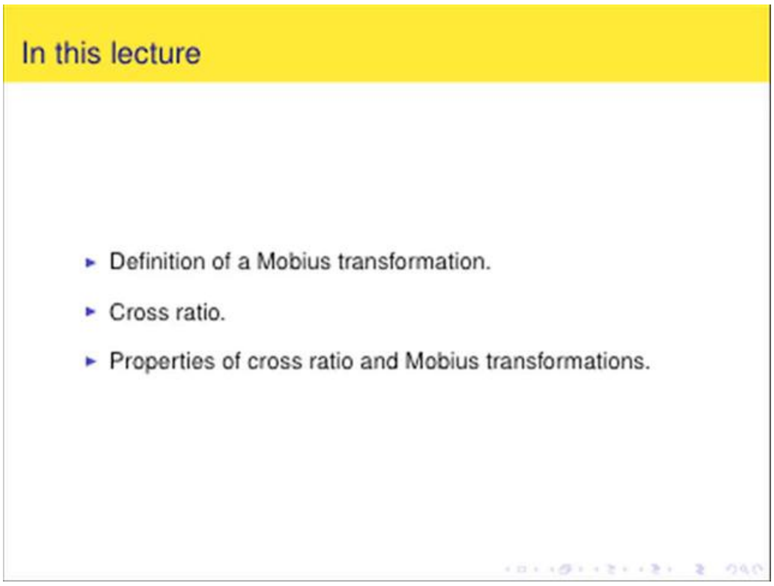
Complex Analysis

Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

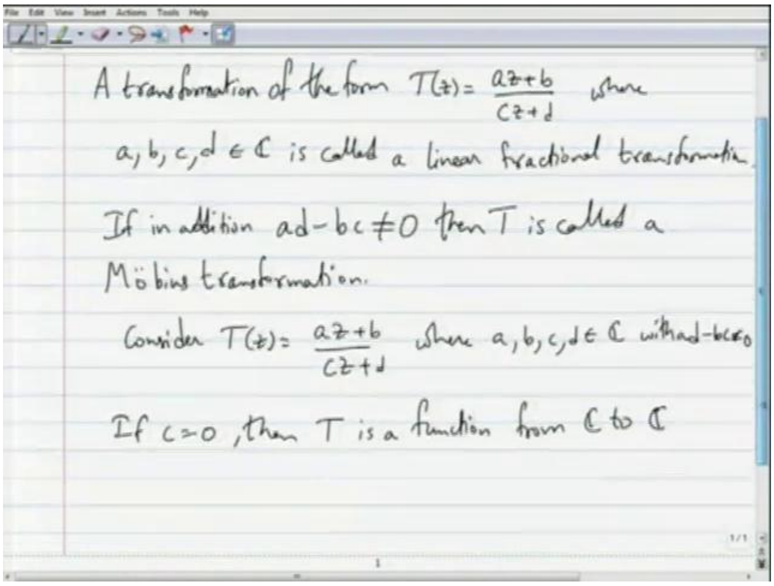
Module - 5
Mobius Transformations
Lecture - 1
Properties of Mobius Transformations Part I

(Refer Slide Time: 00:15)



Hello viewers, in this session, we will discuss about Mobius transformations and some of their properties. So, let start with the definition.

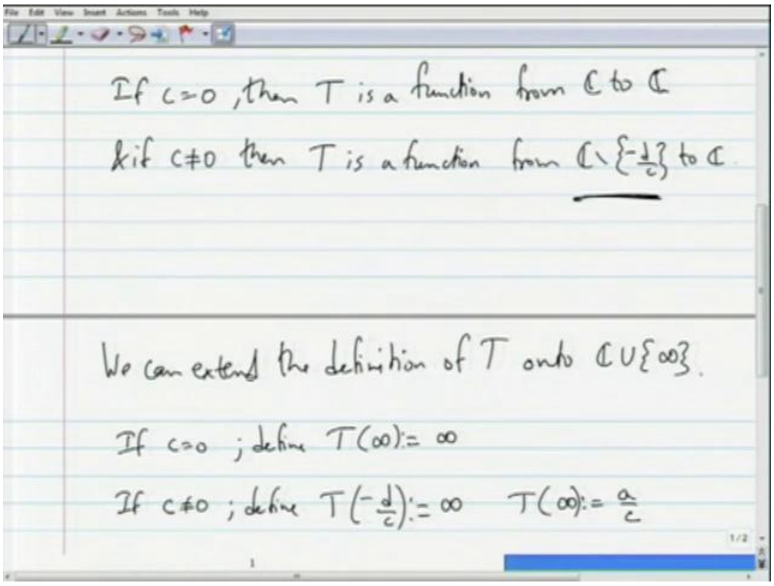
(Refer Slide Time: 00:33)



A transformation of the form T of z is equal to a z plus b by c z plus d, where a, b, c, d belongs to complex numbers is called a linear fractional transformation. And I have deliberately with held specifying, the dominant range of this function. Well these depend I mean, the domains depends on the numbers c and d, if an addition a d minus b c is non zero, then T is called Mobius transformation.

So, now let us consider a Mobius transformation consider T of z is equal to a z plus b by c z plus d, where a, b, c, d are complex numbers with a d minus b c non zero so, that will be or subject of discussion. And if c is 0, if the constant c, there is 0, then is a function from c to c either domain is all of the complex numbers and then the co-domain is the complex numbers of course.

(Refer Slide Time: 02:44)

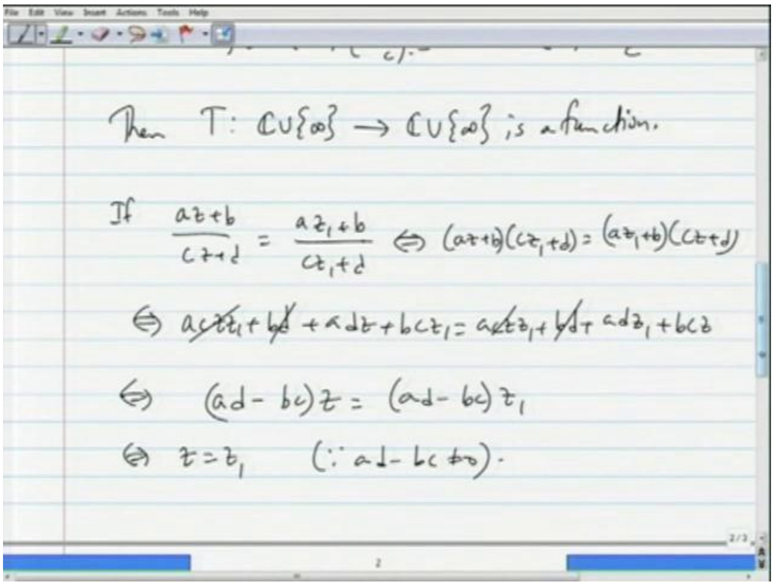


And if c is non zero then it has to miss a point, then the function T is defined on c minus the point minus d by c, because at minus d by c, this function denominator the function has value 0. So, and then this match to c so, it is sought of split case here. So, what we can do is we can do better, we can extent the definition. So, what we can do is, we can extend the definition of this function on to the complex plane, union the point at infinity

So, using the point at infinity we will see that this is the most natural setting for these functions. So, I will see actually that all of these functions are actually bijections, first I should define the extension. So, if c is equal to 0, define T of infinity is equal to infinity because now our domain contains infinity, I have to specify where infinity goes to. So, in the event that is 0 will define T of infinity and if c is non zero, then I have modified two definitions.

So, define T of minus d by c, which we omitted earlier defined that to be infinity define T of infinity to be a point a by c. In the earlier case that point a by c would never have been taken, if we restrict ourselves to the domain c minus d by c, c minus d by c with this extension, what we are going to show that.

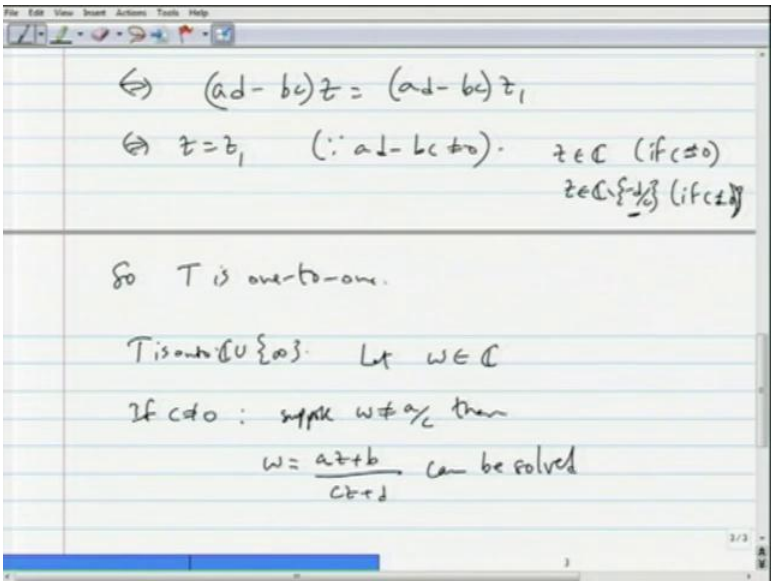
(Refer Slide Time: 04:58)



Firstly, then with this extension T from c union infinity to see union infinity is the function and we are going to show that this is actually bijection. So, if it is well defined firstly, if a z plus b by c z plus d is equal to a z 1 plus b by b plus divided by c z 1 plus d this happens, if and only if a z plus d times c z 1 plus d is equal to a z 1 plus b times c z plus d. And that happens if and only if let us multiplied out and they will be cancelations.

So, a c z z 1 plus b d pluc a d z plus b c z 1 is equal to a c z z 1 plus b d plus a d z 1 plus b c z. When we multiply this out and then they will be cancelations, and after cancelation this is if and only if cancels with this, this cancels with this, then we are left with a d minus b c times z is equal to a d minus b c times z 1, this is if and only if this z is equal to z 1 because a d minus b c is not equal to 0.

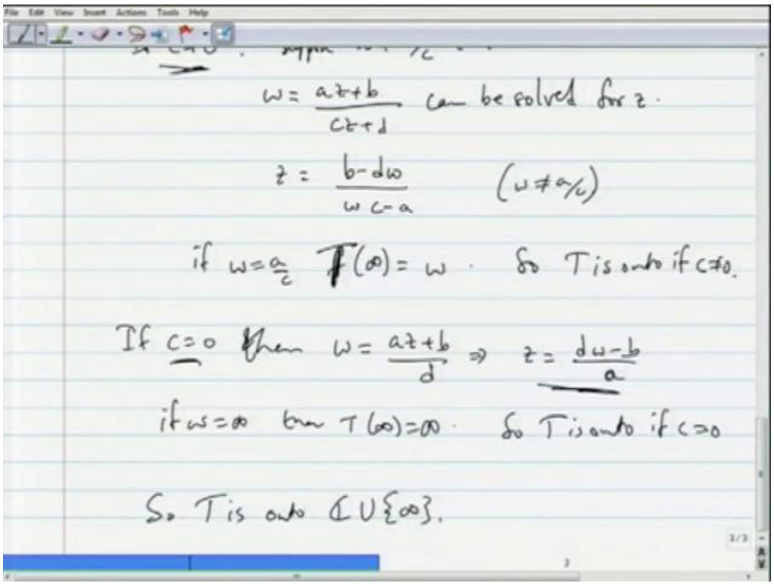
(Refer Slide Time: 06:55)



So, this works for all of these works for z belongs to well z belongs to c, if c is not equal to or c is equal to 0 and z belongs to c minus, minus d by c. If c is not equal to 0 if c is equal to 0 for infinity or if c is equal to c is not equal to 0, for the case of minus d by c etcetera and infinity, we show this property separately. We can show that this holds the image of T is equal holds. So, all in all so except for some cases, we have shown that T is 1 to 1. So, T is 1 to 1. So, the viewer is informed here that there are some cases I have already spoken of which I have covered, but you know that is an easy exercise one can show that then T is 1 to 1 can conclude T is 1.

So, then we can also show that T is on to, T is on to 1 is belong to c is union infinity that is what I mean. So, we will do this in two cases, let w belong to c union infinity or let me take w belong to complex plane. If c is not equal to 0, if in first case c is not equal to 0 and suppose w is not equal to a by c. So, then in this event that w is not a by c then we can work out what w is by setting is w equals a z plus b by c z plus d, we can this can be solved, this can be solved for z.

(Refer Slide Time: 09:16)

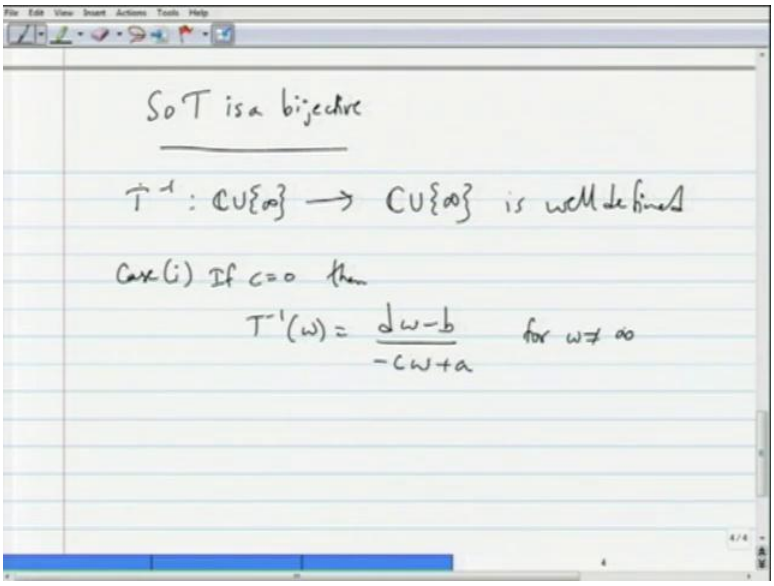


What we get is z, z is equal to b minus d w multiplying without and factoring out a z we get z is b minus d, w divided by w c minus a. And we assume the w is not a by c w is not equal to a by c, in this event z is this particular number if w is equal to a by c, we already know that in this event I have that f of infinity is equal to w. In the case that c is non zero we have shown that I apologies. I am using this functions T, T of infinity is w. So, we have shown that so T is on to.

So, T is on to if c is non zero in the event that c is 0 then also we can show that, this is on to c is and if w belongs to c, we have let w belong to c then we can directly say, that w is equal to a z plus b by by d which implies, that z is equal to d w minus b by a. Notice that a cannot be 0 if c 0 because a d minus b c is non zero. So, this happens and this z gives you T of z for this z, T of z is equal to w, and if w is equal to infinity.

In this case, when c is equal to 0 we already know that we defined T of infinity to be infinity T like standard definition, refer to the extended definition they said T of infinity if c is 0. So, in this case T of infinity is infinity so T is on to in this case as well c is equal to 0. So, in either case so T is on to to either case T is infinity on to T is 1 to 1 and on to. So, it is a bijection from c union infinity to c an infinity.

(Refer Slide Time: 11:58)

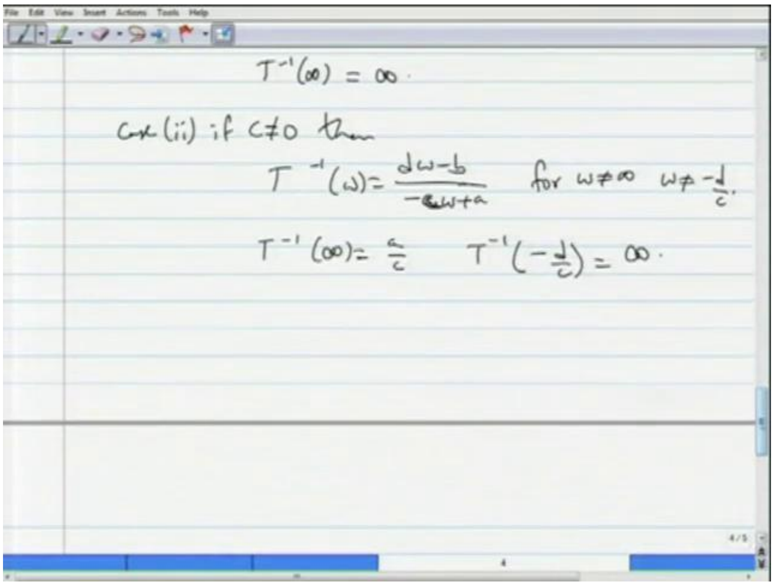


So, T is bijection so we will see that these a Mobius transformation are not only by bijections from c union infinity to c union infinity, but they also preserve the complex structure of c union infinity. So, what one means by that is at least to start with Mobius transformation maps, we will see that is map is circle in c union infinity, which we called Riemann's pear to circles to c union infinity.

So, circles the images of circles are circles and not only that we also have the fact that it maps the disk bounded by this circle on to so t maps the disk bounded by circle ion to a disk bounded by circle in a very nice passion so, that we are going to see now. Firstly, I have to show that T maps circles upto circles, in order to do that let me first begins with some more facts about Mobius transformation.

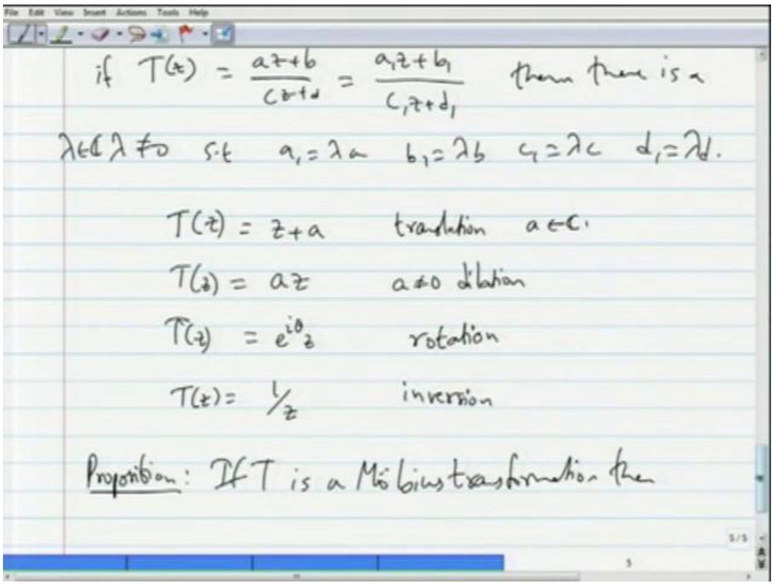
Since, T is the bijection T inverse from c union infinity to c union infinity is well defined so case 1 if c is equal to 0. Likewise I mean corresponding to the case for T will have a case for inverse, if c is equal to 0 then, then T inverse of w is d w minus b by minus c w plus a for w not equal to infinity and T inverse of infinity is of course, infinity.

(Refer Slide Time: 14:09)



And case two if c is not equal to 0, then T inverse of w is once again d w minus b by minus c w plus a and T inverse of this is true for w naught equal to infinity, and w naught equal to minus d by c and T inverse of infinity is a by c and T inverse of minus d by c is infinity. So, we just write down the corresponding cases for T inverse. So, that is the formula we have for T inverse, given T. Also we will note that this notation for Mobius transformation is a z plus b by c z plus d is not unique.

(Refer Slide Time: 15:22)

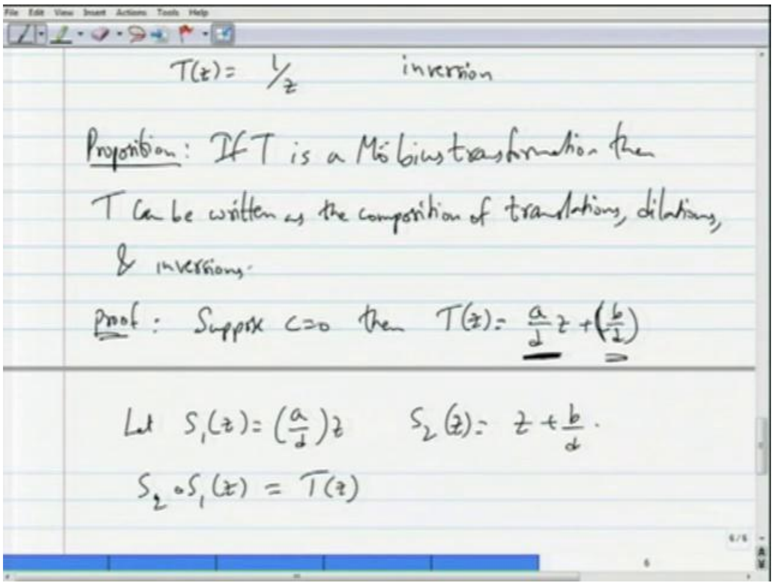


So, T of z you will notice is a z plus b by c z plus d, it can also be written as a lambda z plus b lambda divided by c lambda plus d lambda where, where lambda is non zero complex number, not only that it is actually true well one can solve this an exercise. So, one can show that if T z is a z plus b divided by c z plus d is a 1 z plus b 1 divided by c 1 z plus d 1.

So, you can specify a Mobius transformation in two different ways. Notice here we have a a b minus c is non zero and so a 1 d 1 minus a 1 c 1. So, then there is a, there is a lambda, lambda not equal to 0, lambda belongs to c such that a 1 is equal to lambda a b 1 is equal to lambda b c 1 is equal to lambda c and d 1 is equal to lambda d. So, one can try that either exercise.

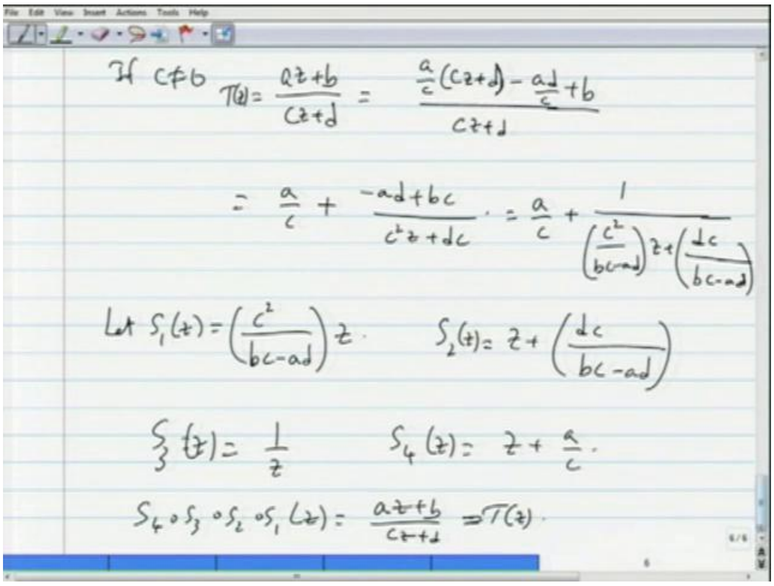
So, the specification of T in this form is not unique. So, also we can decompose a given Mobius transformation of a composition of certain elementary kinds of Mobius transformations for that purpose, define for kinds of elementary transformations T of z is z plus a is called a translation. So, for some a belongs to c and T of z is equal to a z a times z a non zero is called a dilation. And T of z is equal to e power i theta z is called a rotation and T of z is equal to 1 by z called inversion. So, all these are actually Mobius transformations themselves special kinds of Mobius transformations.

(Refer Slide Time: 18:28)



And what we are going to show is we are going to give, the following proposition that if T is a Mobius transformations, then T can be written as composition. The composition of translations dilations and inversion notice that rotation is actually a form of dilation, where the constant a has modulus 1. So, proof of this fact is easy, what we can do is suppose c is equal to 0 then it is easy then T of z. What does it look like? T of z is looks like a by d z plus b by d, which we can immediately see is composition of dilation and then are translation. So, let S 1 of z is equal to a by d z and S 2 of z is equal to z plus b by d the S 2 compose with S 1 of z is going to be equal to T of z.

(Refer Slide Time: 19:57)

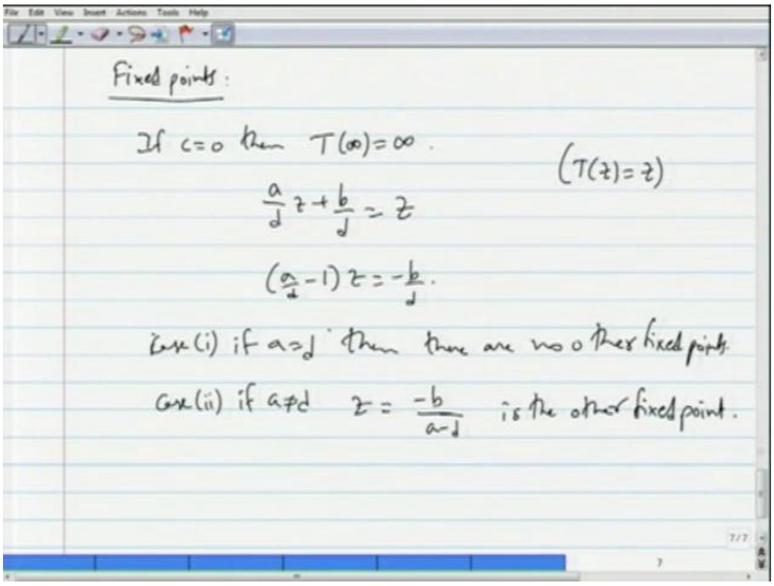


If c is non zero, then let us a tinker with the formula for the Mobius transformation and get a particular form. And then we will see how we can write T as compositions of these elementary Mobius transformation. So, what I am going to do is do is the numerator, I am going to get a c z plus d, so that I can divide some dividing the denominator into the numerator. So, I will write a by c since, c is not 0 I can write the numerator as a by c times c z plus d plus b divided by c z plus d apologies, I need a minus I have added a a d by c. So, I subtract a d by c and I have plus b.

So, then now what I can do is I can write this as a by c minus or plus minus a d plus b c, I am clearing the fraction in the numerator, and then I am multiplying the c into the denominator to get c square z plus b c. Now, I see what the form is so this is this can be written as a by c plus 1 divided by c square by b c minus a d. Notice b c minus a d is non zero z plus d c divided by b c minus a d.

So, if we let S 1 of z is equal to the dilation by c square divided by b c minus a d and z and then, let S 2 is equal to translation by d c divided by b c minus a d and S 3 to be the inversion S 3 of z to be 1 by z and S 4 of z once, again the translation by a by c. What we have is S 4 compose with S 3 compose with S 2 compose with S 1 of z is going to be or a z plus b by c z plus d which is your T of z. And notice that these compositions also work for those other exceptional points minus T by c or infinity etcetera. So, one can verify by substitution that this compositions also respect those special definitions. So, that proofs these propositions. So, next we are going to see some other properties of Mobius transformations, so namely fixed points.

(Refer Slide Time: 23:29)

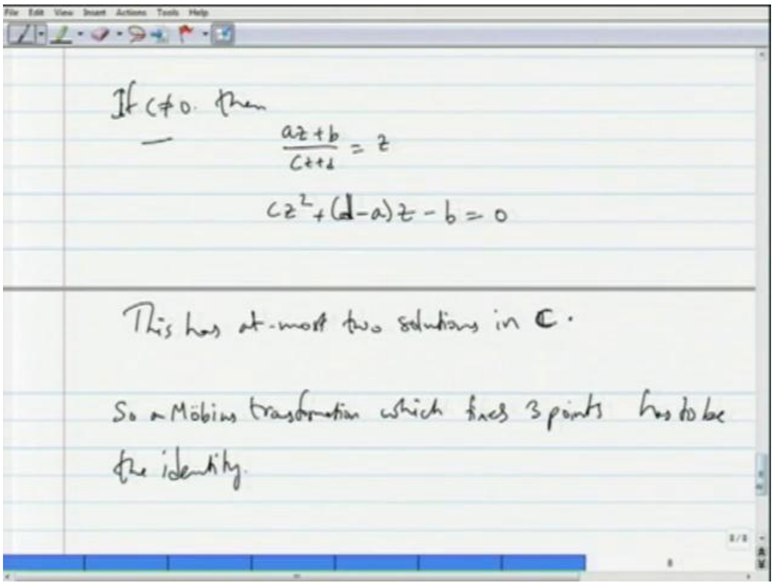


So, for the purpose of this discussion I am actually, the book by Convoy one can refer to book by John V Convoy, which is listed in the text books for references and so notice that a Mobius transformation by the following analysis will have some fixed points. So, firstly if c is equal to 0 then T of infinity, we know infinity. So, all c union infinity this there is one fixed point, if c is little c is equal to 0 and then also a by d z plus b by d is equal to z, let us equate the Mobius transformation T of z is equal to z. so, T of z.

Now, since c is equal to 0 it looks like a z plus b by d then that is equal to z, we have this equation which up on solving gives us a by d minus 1 times z is minus b by d. So, if now there are cases. Case one if a is equal to d then you get 0 equals minus b. So, then there are no other fixed points. So, in the event c is equal to 0 and the event a equals d then there are no other fixed points.

Case two if a is not equal to d, then we can solve this equation to get z equals minus b by a minus d. Since, a is not equal to d the denominator is not 0 is this is the other fixed point only, one other fixed point we get from solving T of z is equal to z. So, there are 2 at most 2 fixed points in the case c equals 0.

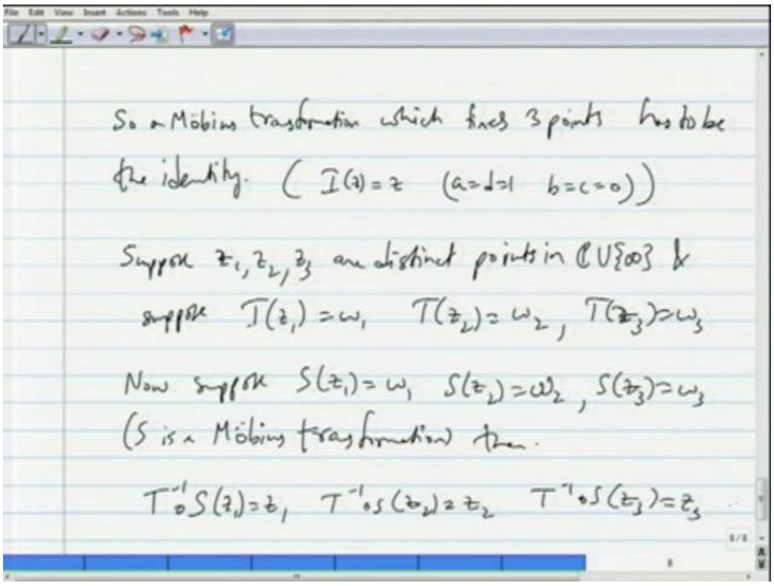
(Refer Slide Time: 26:07)



If c is not equal to 0, what we have is then a z plus b by c z plus d equating, this 2 z what we get is c z square plus d minus a times z plus or minus b is equal to 0. So, we get a quadratic equation. So, this has at most 2 solutions in C. So, the summary is there are at most 2 points whether c 0 or c is non zero.

So, a Mobius transformation, which fixes 3 points has to be the identity. Wise that the identity, fix the every point and by the above analysis, you can have at most 2 fixed points for the Mobuis transformation. So, the only Mobuis transformation, which fixes three or more points three or more points has to be the identity Mobuis transformation. So, the identity, what is that?

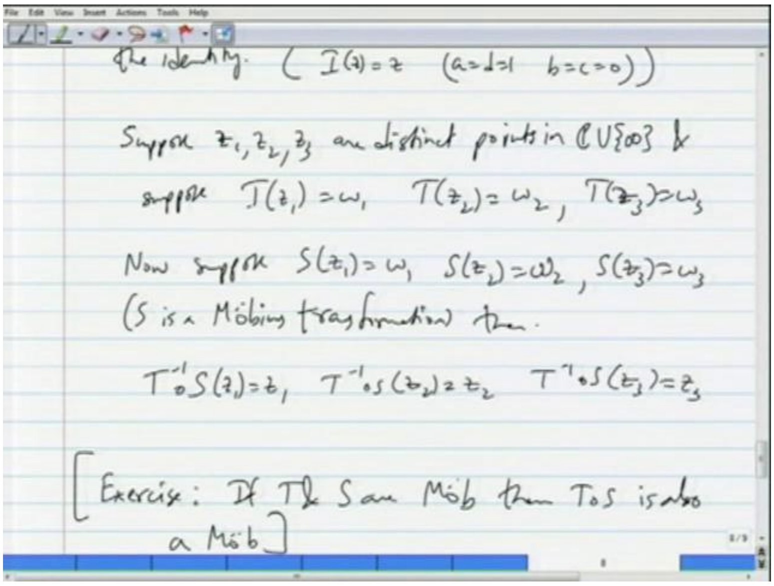
(Refer Slide Time: 27:56)



I of z is equal to z this case a d are 1 and b and c are 0 a equals d equals 1 b equals c equals 0 in that event, every point in the finite complex plane and the point at infinity or all fixed. Suppose z 1, z 2, z 3 are distinct points in c and or you could actually pick c union infinity point in infinity, and suppose T of z 1 is equal to w 1 and T of z 2 is equal to w 2, T of z 3 is equal to w 3. Now, suppose S also have this property.

Suppose, S of z 1 is also equal to w 1 and S of z 3 is equal to w 3, S is the Mobius transformation, then you notice that apply T inverse circle S to any of the point z 1, z 2, z 3. What you get is this is of z 1 is equal to z 1 and T inverse of circle S of z 2 is equal to z 2 etcetera. T inverse of circle S of z 3 is equal to z 3. So, it fixes 3 point T inverse circle S is also a Mobius transformation and so, well actually that is an exercise.

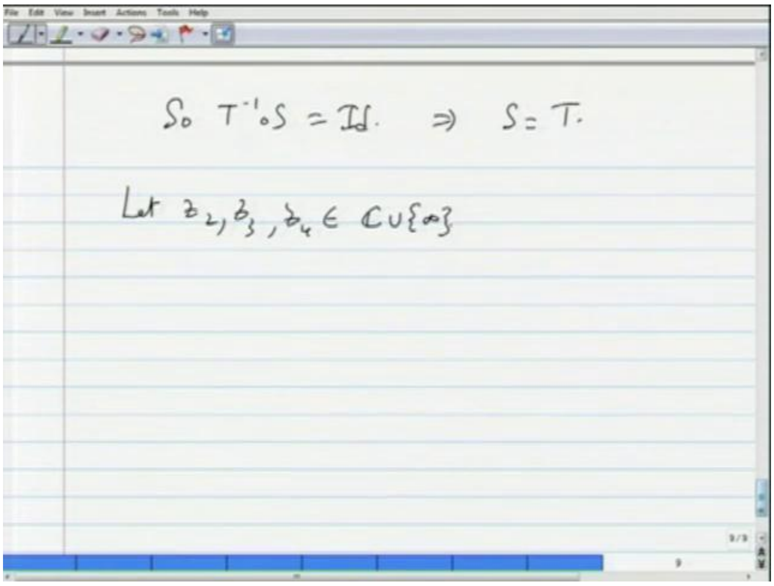
(Refer Slide Time: 30:10)



Exercise that if T and S are Mobius transformation, move for short then the composition of these is also a Mobius transformation that is easier exercise, one to the viewer is advice to call this exercise is an easier exercise, just compose and show that the condition a d minus b c for the new constant, which arise when you compose also holds.

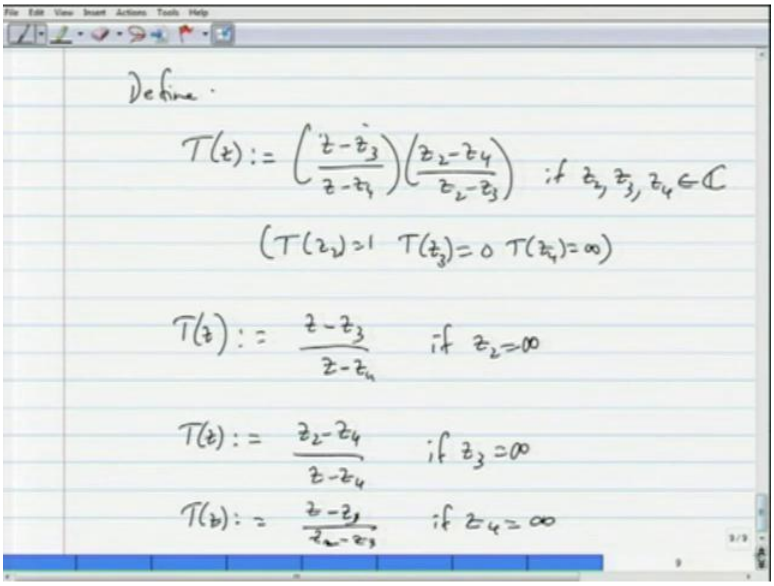
And when you show that T inverse circle S will also be a Mobius transformation and it fixes z 1, z 2, z 3. So, what that says is that so T inverse circle S is equal to the identity transformation. So, which implies S is actually is equal to T. So, if you know the image 3 points, where Mobius transformation then the Mobius transformation is determined, that is what the, this means? So, going to this property, what we are going to do is we are going to get another representation of Mobius transformation. So, here is definition.

(Refer Slide Time: 31:38)



So, let z 2, z 3, z 4 belongs to c union infinity. Since, the image of 3 point determines Mobius transformation.

(Refer Slide Time: 31:56)

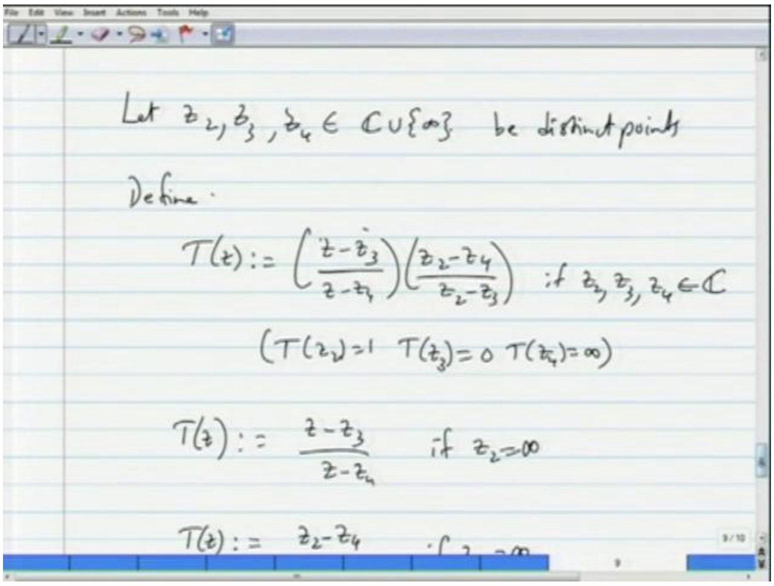


What we are going to do is we are going to make the following definition, define T of z as z minus z 3 divided by z minus z 4 time z 2 minus z 4 divided by z 2 minus z 3, this additional adjustment factor. If all z 2, z 3, z 4 belong to the finite complex plane not infinity, finite complex plane. So, notice that when you substitute I mean, this is the function of z the right hand side is the function of z and when you substitute z 3, the numerator is going to be 0 in the first factor. So, you get 0.

When you substitute z 4 of the denominator is going to be 0. So, you get infinity and when you substitute z 2 there is cancel in factor, the second one, second one cancels first factors to give you 1. So, what this definition does is T of z 2 is 1 T of z 3 is 0 T of z 4 is infinity or T of z 4 is infinity. And likewise you can for other cases when z 2, z 3, z 4 are not in the finite complex plane for other case.

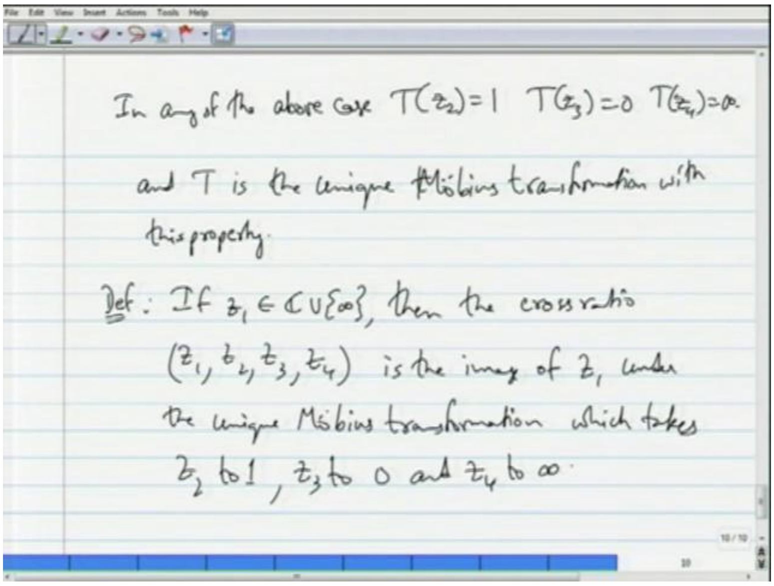
You can make equivalent the definitions z minus z 3 divided by z minus z 4, if z 2 is infinity T of z define that to be z 2 minus z 4 divided by z minus z 4, if z 3 is infinity and T of z define that to be z minus z minus z 3 divided by z 2 minus z 3 if z 4 is infinity. So, any of these cases well not two of them can be infinity only one of them are allow to be infinity at a time. So, if you pick z 2, z 3, z 4 belongs to c union infinity.

(Refer Slide Time: 34:15)



Let these be distinct points that is the main assumption and then you define T to be like that.

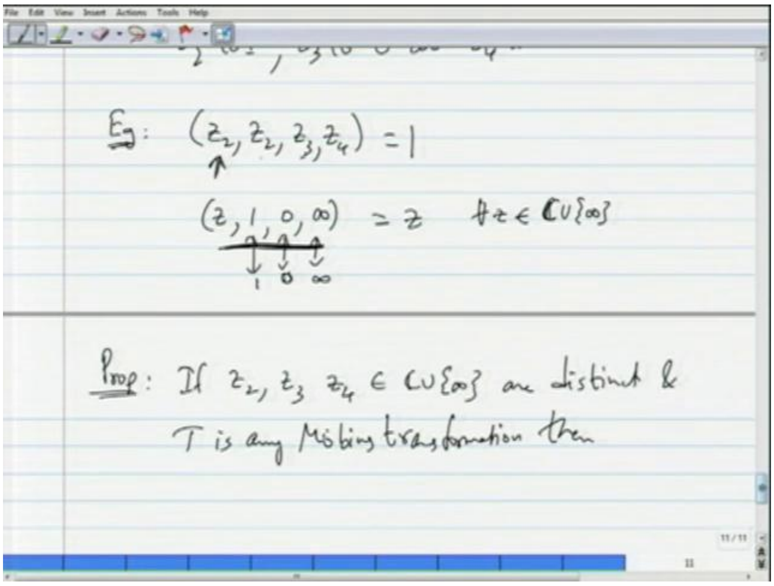
(Refer Slide Time: 34:30)



In any of the above case, what we observe is T of z 2 is 1 and then T of z 3 is 0 and T of z 4 is infinity. And T is the unique Mobius transformation with this property, from what we said above if there is any other Mobius transformation, which sets 2 to 1 and z 3 to 0 and z 4 to infinity. Given 3 fixed point z 2, z 3, z 4 then it has to equal S or T rather. So, by that property T is the unique Mobius transformation.

So, going to this we are going to make definition, we will say that if z 1 belongs to c union infinity then, the cross ratio of the or I will say the cross ratio z 1, z 2, z 3, z 4. So, that is the notation is the image of z 1, under the unique Mobius transformation, which takes z 2 to 1 z 3 to 2 and z 4 to infinity. So, we have define such a T then the cross ratio z 1, z 2, z 3, z 4 the essentially the image of z 1 under this Mobius transformation. So, we will see this notation or this quantity is very useful, capture some properties of this cross ratio and we will see that it is useful.

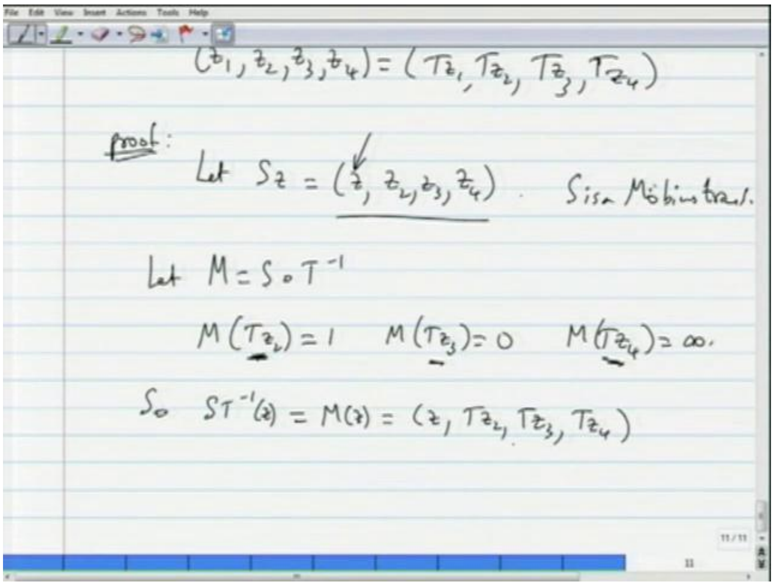
(Refer Slide Time: 37:19)



So, firstly a some trivial example if we take z 2, z 2, z 3, z 4 for example, the cross ratio of z 2, z 2, z 3, z 4 is going to be 1 because the image of z 2, under the Mobius transformation, which takes z 2 to 1 and z 3 to 0 and z 4 to infinity has to be 1. And likewise the image of the identity image of a point z, under the identity Mobius transformation is itself, for all z belongs to c union infinity.

So, the identity transformation you know now denoted y, the cross ration 1, 0 infinity because in this case, this represents 1 goes to 1 and 0 goes to 0 and infinity goes to infinity. There is only one transformation, which fixes three points namely the identity all right. So, these are some trivial examples and here is a proposition if z 2, z 3, z 4 belongs to c union infinity are distinct points and T is any Mobius transformation, then the cross ratio.

(Refer Slide Time: 39:12)

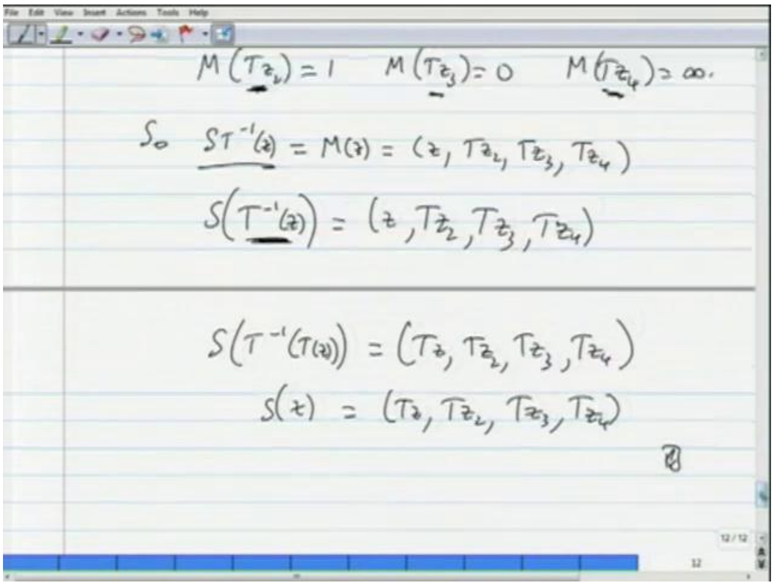


Z 1, z 2, z 3, z 4 is equal to the cross ratio T z 1, T z 2, T z 3, T z 4. So, for any Mobius transformation, we T we have this property so, that is the relation between any other Mobius transformation cross ratio. The proof of this is very simple, but we what we are going to do is we are actually going to use this proposition to explore some properties of Mobius transformation. So, let S z be the Mobius transformation, which is given by z 1, z 2, z 3, z 4 and by the exercise, which I left to the viewer M is let M equals S circle T inverse T is the Mobius transformation. T is any Mobius transformation, but the inverse to the any Mobius transformation exists.

We have seen that T inverse is also bijection between from c union infinity itself. So, defined new Mobius transformation, which is compositions of S circle, c inverse composition is also a Mobius transformation by this exercise. So, let M equals S circle T inverse and notice that M of T z 2, which is S circle T inverse circle, T of z 2 this is equal to 1 because S of z 2 is 1.

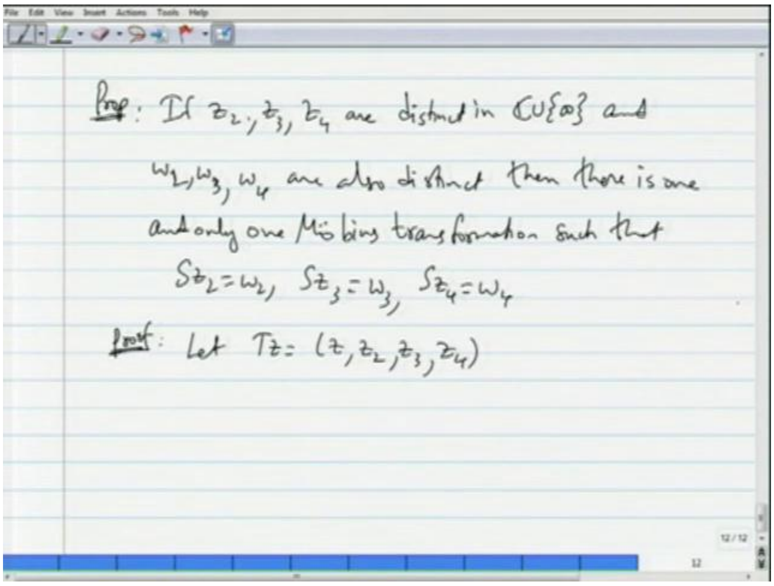
And likewise M of T z 3 is equal to 0, and M of T z 4 is equal to infinity. So, of course S T inverse of z which is M of z, M of this quantity is 1. This quantity is 2 and this quantity is infinity. So, M of z itself can be written as comma T z 2, T z 3, T z 4 because the image of T z is 1 and T z 3, 0 and that of T is infinity Y R M. So, from here, what we have is S T inverse of z is this.

(Refer Slide Time: 41:57)



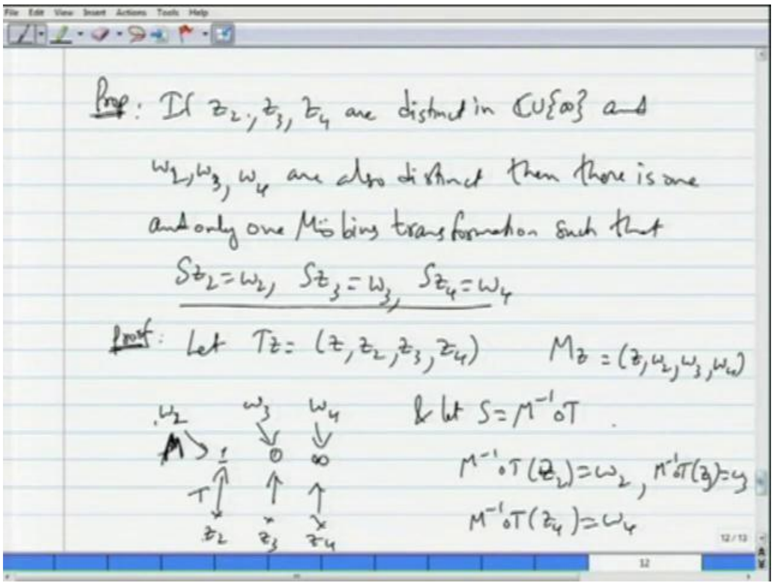
So, S of quantity c union infinity is equal to z comma, T z 2, T z 3, T z 4. So, what is this means is that if I substitute for z, if I substitute T of z what I get is T of z, T of z 2, T of z 3 and T of z 4. So, actually I am writing the image as T z, I been using that before. So, what distances is T inverse circle, T is identity. So, this gives us the LHS is sub z this is T z, T z 2, T z 3 and T z 4 and this is what we wanted. So, it completes the proof of this proposition. And there is yet another property that we will need.

(Refer Slide Time: 42:55)



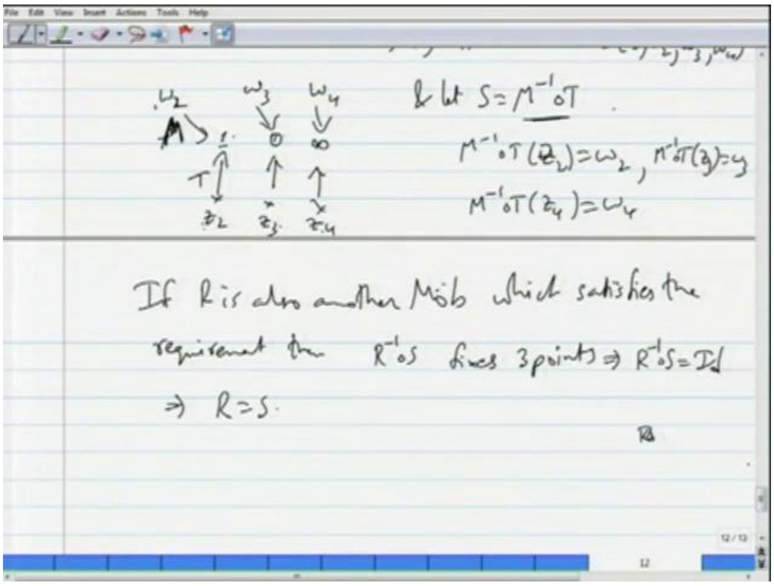
So, here is another proposition if z 2, z 3, z 4 are distinct, in c union infinity and w 1, w 2, w or w 2, w 3, w 4 are also distinct then there is 1 and only 1 Mobius transformation. Such that, S z 2 is equal to w 2, S z 3 is equal to w 3, S z 4 is equal to w 4 and the proof is really easy. If you let T z is equal to z, z 2, z 3, z 4.

(Refer Slide Time: 44:15)



So, the idea is z 2, z 3, z 4 go to 1, 0 and infinity and you have w 1, w 2, w 3, w 4 distinct these go to 1, 0 and infinity, this is the schematic. So, T is this I am going to call this as M, M of M of z is equal to z w 2, T 3, w 4. So, M is the one which takes the w 2 to 1 w 3 to 0 and w 4 to infinity. Then you notice that let S and S equals M inverse circle T. So, M inverse circle T takes and takes w 2 or z 2 to w 2, z 3 to w 3 and z 4 to w 4. So, satisfies the requirement for S.

(Refer Slide Time: 45:39)



And if R is also another Mobius transformation, which satisfies the requirement namely it takes z 2 to w 2, z 3 to w 3 and z 4 to w 4. Then R inverse circle S fix, fixes 3 points by you know the previous kind of argument R inverse circle fixes 3 points, z 2, z 3, z 4. So, implies R inverse circle S is the identity transformation, which implies R is equal to S. So, this is unique and your M inverse circle T is what is wanted so, that is the idea you take z 2 to 1, z 3 to 0, z 4 to infinity and go back via M 1 goes to w 1, 0 goes to w 3 and infinity goes to w 4 so, that way you can achieve, what is wanted in this proposition that S.

So, that gives a way to represent Mobius transformations using a cross ratios; and our goal is to actually show that these functions Mobius transformations not only bijections from c union infinity to c union infinity, but they also preserve complex structure i e that takes to circle to circle. And that take the disk bounded by these circles to disk bounded by the image circle. So, these properties will help us achieve that goal, these propositions that we proved will help us achieve that goal. So, we will continue to explore the properties of Mobius transformations, I will stop here.