Complex Analysis
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Module - 4
Further Properties of Analytic Functions
Lecture - 7
Open mapping theorem – Part two

Hello viewers, in the previous session, we have proved the following theorem. This theorem that, if f of z is analytic at z naught and f of z naught equals w naught and that f of z minus w naught has order n at z naught.

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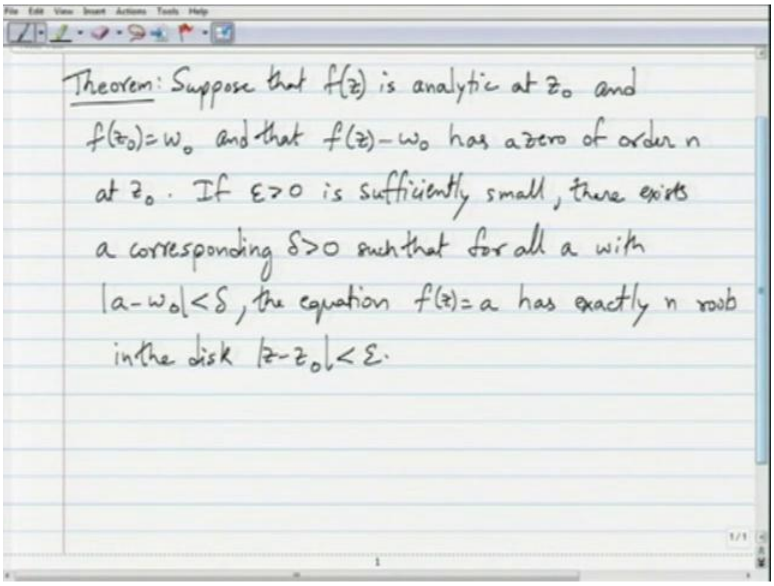

Under these circumstances, if epsilon is positive then there is a corresponding delta positive, such that all the values in the delta neighbourhood of w naught are taken exactly n times by f, in the epsilon neighbourhood of z.

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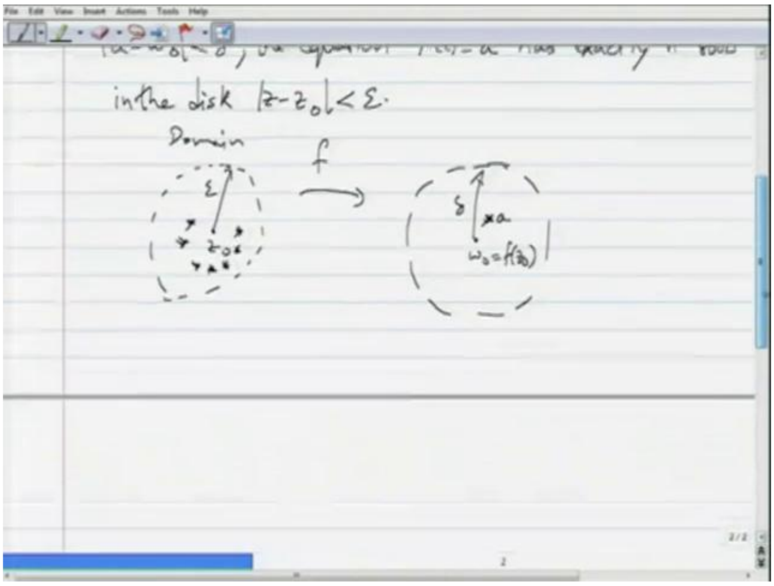

So, a picture for this is definitely in order. So, here is an epsilon neighbourhood of z naught, so this is the domain and this is tlhe range. So, here is w naught equals f of z naught and f of f takes z naught to w naught, okay? So, now if this epsilon is sufficiently small though, then also the 0 of f of z minus w naught is of order n, then each point in this neighbourhood. So, let us call a, let us pick an a, in this delta neighbourhood, this is a delta neighbourhood of w naught. So, each point a is assumed n times here there are n points here counting multiplicity such that f of z is equal to a for these points in the, for these points in the epsilon neighbourhood. That is what this theorem asserts.

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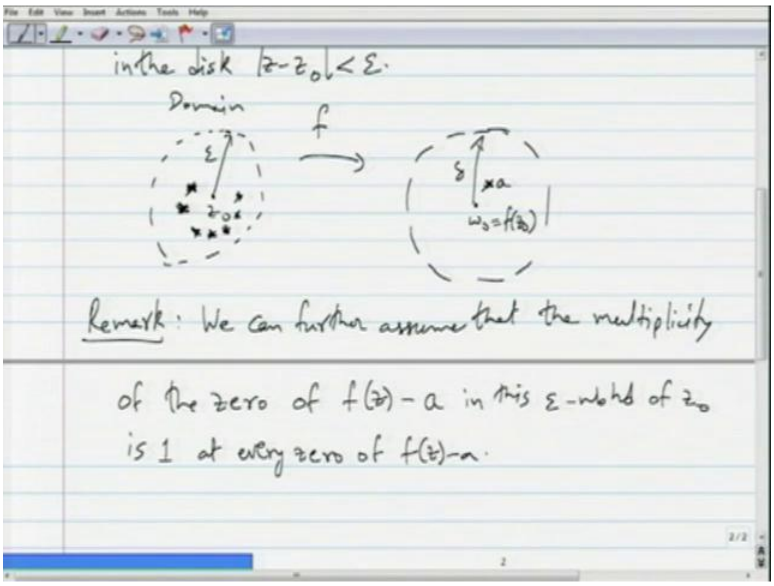

So, a remark is in order here, we can further assume that the multiplicity of of assuming a at each of these points is actually 1. What I mean by that is remark, so I will remark that we can further assume that the multiplicity, or I will say the multiplicity of the 0 of f of z minus a in this epsilon neighbourhood. So, I am referring to the theorem under the conditions of the theorem in this epsilon neighbourhood of z naught is 1 at every 0 of f of z minus a.

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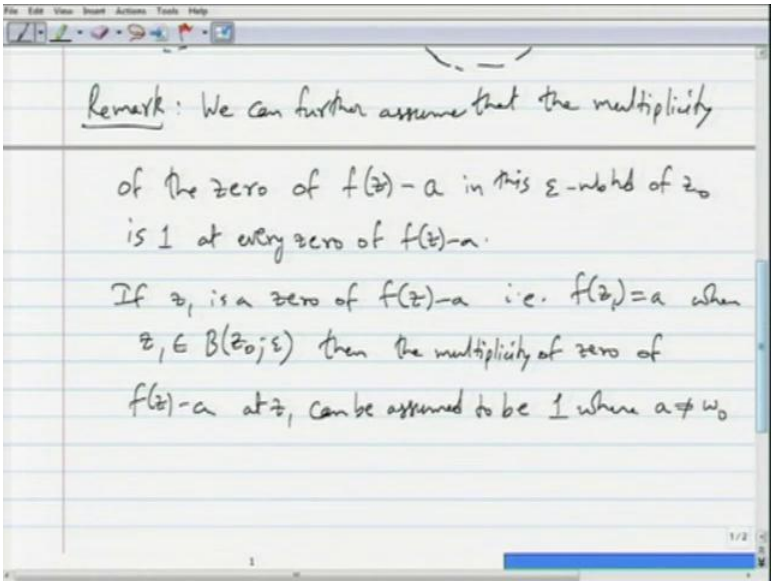

What I mean by that is if z 1 is a 0 of f of z minus a i e f of z 1 is equal to a, where z 1 belongs to B z naught epsilon. So, z 1 is a solution to f of z equals a and z 1 belong to epsilon neighbourhood around z naught. Then the multiplicity of 0 of f of z minus a at z 1 can be assumed to be 1 or it can said to be a simple 0, where there is an assumption where a is not equal to w naught, okay? So, for for anything other than w naught, so if I pick any a here, I am referring to this picture here now, if I pick any a here which is not w naught itself, then then each of these points which hits a it is 0 to f of z minus a. Then this the multiplicity of 0 at at that point z 1 of f of z minus a can be assumed to be 1.

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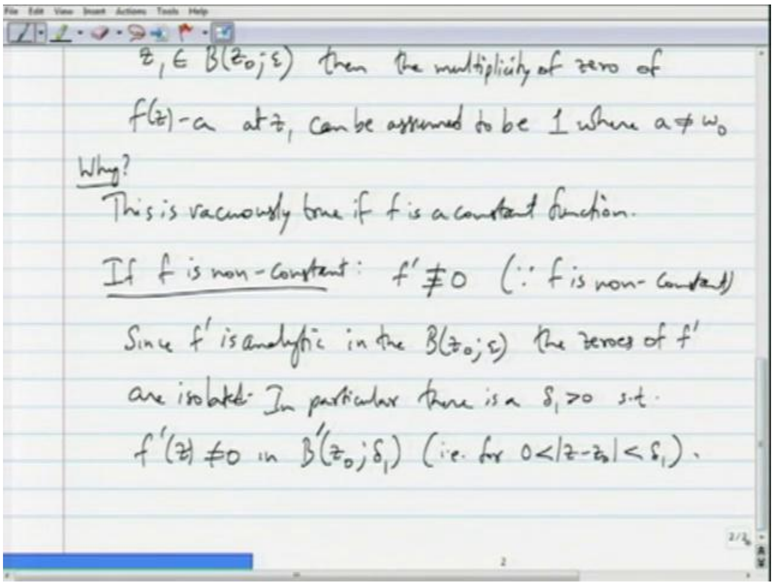

Why is it so? Well firstly this why? The answer is as follows, this is vacuously true, if f is a constant function, right? If f is a constant function, there is no other a that we can pick other that w naught because all the all the points are mapped to this w naught itself. It is a constant function. So, we can assume that f is a non constant function, if f is non constant, we can do the following. Notice that f prime f prime of, f prime is not identically the 0 function. Why? Well because if f is analytic, f prime is analytic and f prime identically 0 on an open set gives you that f is constant on the components on which f is analytic.

So, f prime is not not identically 0, if f prime is identically 0 f is constant on in the ball at least. So since f is non constant and since f prime is an analytic function, the zeros of f prime are isolated, by the identity theorem if you wish. So, since f prime is analytic as well, in in the in the in the epsilon ball if you wish f of z naught sorry B of z naught epsilon the zeros of f prime are isolated. In particular there is a, there is a let us say delta on positive such that f prime of z is not equal to 0 in B z naught delta 1, okay?

So, even if f prime is 0 at z naught, f there is a small neighbourhood around z naught such that f prime is not 0 in I should say this is the deleted neighbourhood, B prime z naught delta 1 so i e, this is i e for for 0 strictly less than mod z minus z naught strictly less than delta 1. That is a deleted neighbourhood of z naught.

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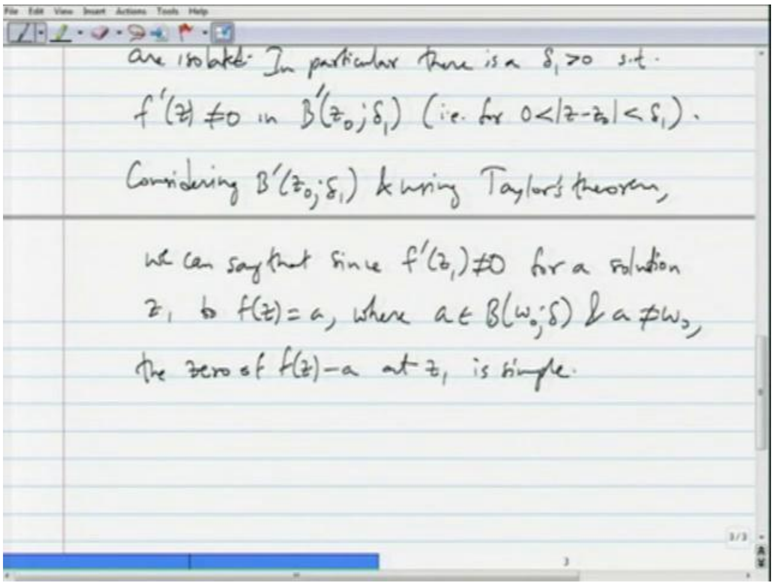

So, what this gives us is that then considering B prime or B prime z naught delta 1 which is an open set and using Taylor's theorem if you wish, we can say that since f prime of z 1 is not equal to 0, well here z 1 is the 0 of f of z minus a, okay? So we can say that since, f prime of z 1 is not equal to 0 for a solution z 1 to f of z is equal to a, where a belong to the delta neighbourhood of w naught and a not equal to w naught. The 0 of f of z minus a at z 1 is simple you expand f of z minus a f of z minus a is also an analytic function. So, you expand f of z minus a around z 1.

So, since f prime of z 1 is non-zero, you have that the the 0 at z 1 of f of z minus a has to be simple. That is by the Taylor's expansion for f of z minus a around z 1, so that says that you are, so these zeros can be assumed to be simple the zeros of f of z minus a can be assumed to be simple, okay?

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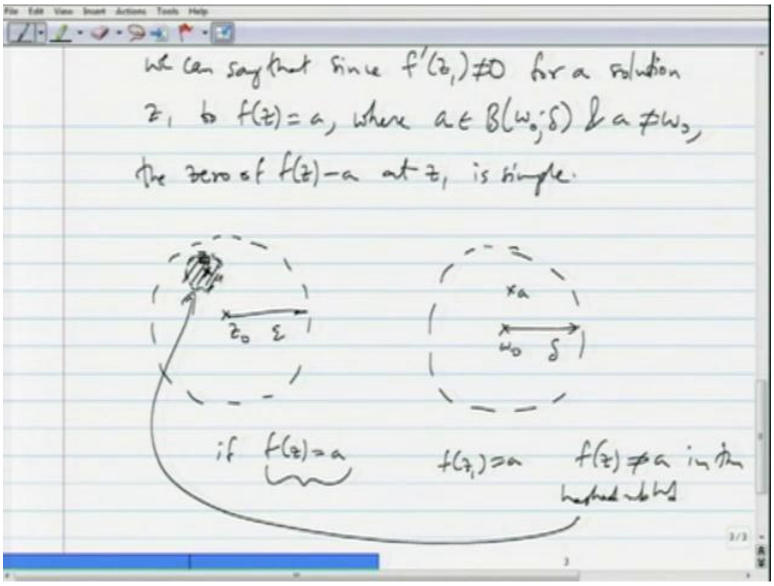

So what that is telling us is that here once again the picture here is, here is w naught here is z naught. If I pick any z which gets mapped on to some a in the delta neighbourhood of w naught, this is the delta neighbourhood of w naught, this is the epsilon neighbourhood of z naught as in the theorem. And if f of z is equal to a then there is a small neighbourhood of this z, such that z is the only well I do not need a neighbourhood here, this is a.

So, in this small neighbourhood here this small neighbourhood here of z z is the only solution to f of z is equal to a. So, this is z 1, so then f of z 1 is equal to a and f of z is not equal to a in the hashed neighbourhood hashed as in this this picture, this this pictured neighbourhood, okay? Now, we are a ready to prove the open mapping theorem as a corollary to the we had last time.

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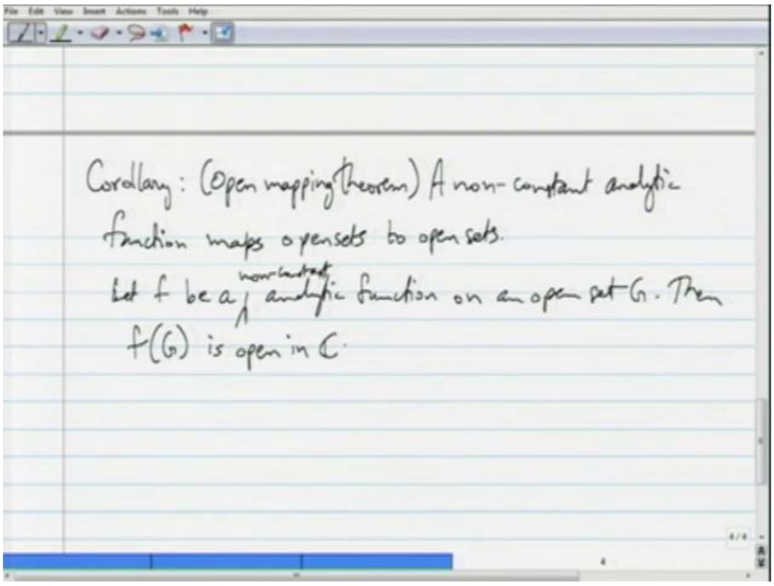

So, a non constant analytic function maps open sets to open sets i e, so stated in another way let f be an analytic function, non constant. Let f be a non constant analytic function function on an open set g, then f of g is open in C g is open open set in C then f of g is open set in C as well. So, this is another way of stating the sine all right?

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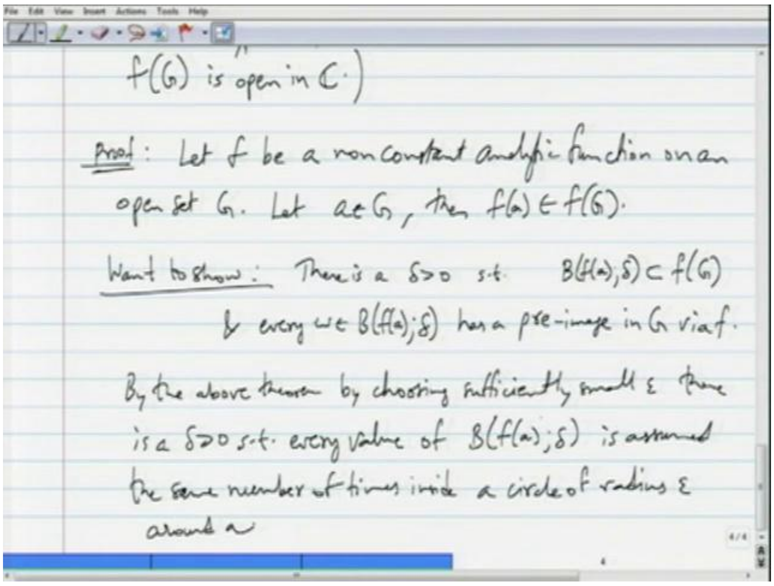

So, what is the proof? Well the proof follows from the theorem above. So, what what do we need to show, let let f be a non constant analytic function on an open set g. We have to show that f of g is open f of capital G is open. So, let a belong to G, and then f of a belongs to f of G. So, we want to show that there is a delta positive such that B of f of a delta is contained in f of G and every w belongs to B of f of a delta, has a pre image in G via f. This is what we want to show.

So, this comes directly from the previous theorem. So, by the above theorem by choosing sufficiently small epsilon there is a delta positive, such that every value of B of f of a. Well f of a itself is assumed by a by by f at the point a. So, B of f of a delta is assumed the same number of times inside a circle of radius epsilon around a. So, f of a itself like a remarked is is assumed by f at a. So, all the values in this ball are assumed by f at least once.

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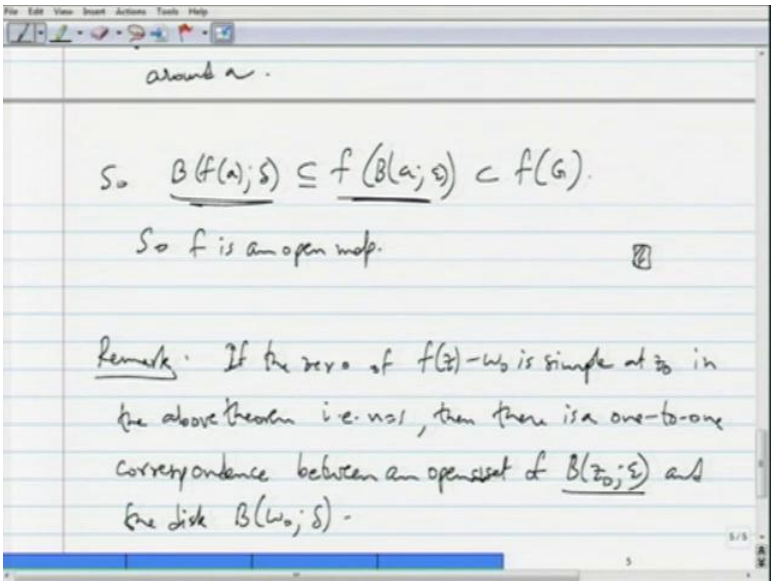

So, that is the point. So B of f of a delta is contained in the image of the ball of epsilon radius around a via f. So, this is the point and then this is contained in f of G. So, f is an open map. So, that completes the proof of this theorem. And that is a very important result. We will see that the maximum modulus principle follows as a consequence, there is a remark following this corollary if the 0 of f of z minus w naught is simple at z naught in the above theorem i e n is equal to 1.

So, in this theorem here, in the first theorem that I quoted; so if n is equal to 1 the 0 f of z minus w naught is simple. Then there is a one to one correspondence between an open set, open set of f of z minus sorry, between open set of B z naught epsilon; so between open subset rather, subset of this ball and the disk B f of z naught namely w naught delta, right? Because each each value in B w naught delta is assumed exactly one time by by some point in a B z naught epsilon.

So, there is some open subset here by the open mapping theorem. We can say that an open set is carried to this here and then so an open set open subset of B z naught epsilon is is is mapped in one to one fashion to B z naught delta.

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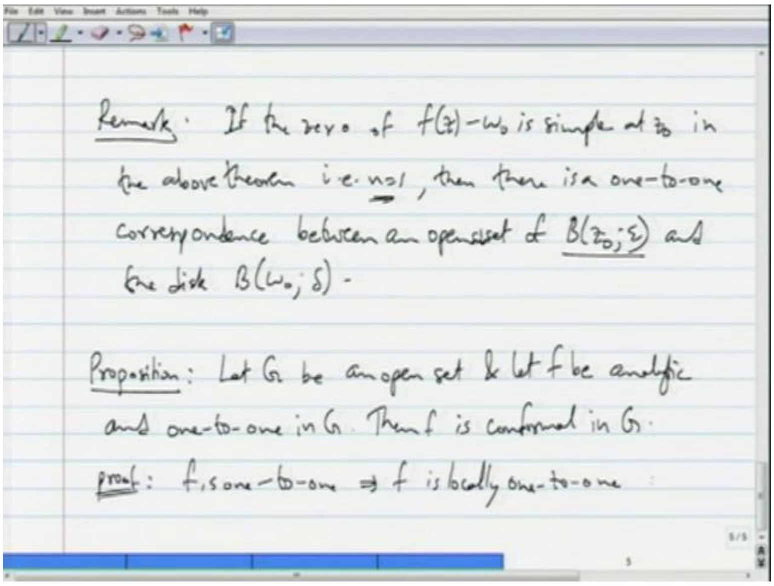

That is a remark and then we will see some some more results following from this important theorem we we saw in the last session. So, here is a case where we are assuming n is equal to 1. So, here is a proposition let G be an open set. And let f be a analytic and one to one in G then f is conformal in G. Here is the proof in detail, so f is one to one implies f is locally one to one. What I mean by locally is that, there is a neighbourhood around every point, where it is one to one.

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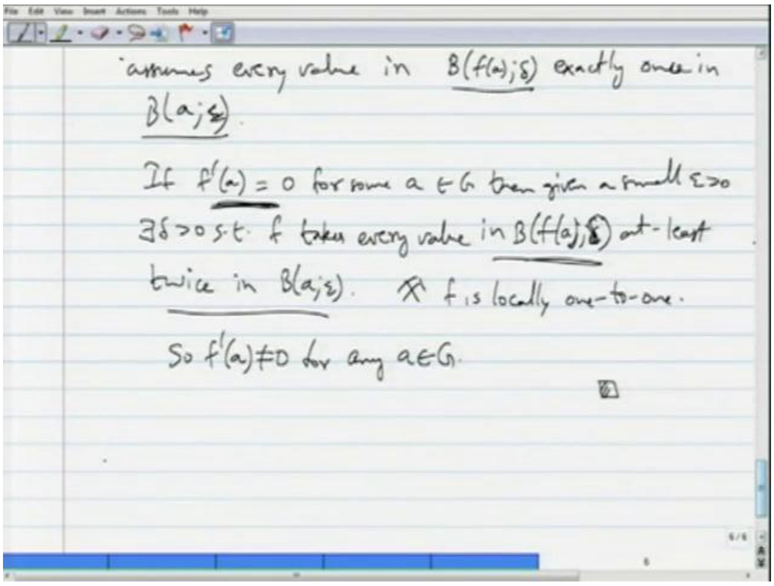

If f prime of a is some arbitrary point in G if f prime of a is 0 for some a belongs to G, then given a small epsilon positive. There exists a delta positive such that f takes every value in B f of a epsilon or sorry, delta at least twice in B a epsilon that is because if f prime of a is 0 then the order of 0 of f of z minus f of a. At a is at least 0 so, by the remark earlier we can assume that, in a small n of neighbourhood a, we can assume that the number of number of zeros of f of z minus some w is simple or is equal to 1, okay? Or the multiplicity of such zeros is equal to 1, so what that means is, that every value in B f of a delta has to be taken at least twice in B a epsilon in in B a epsilon. So, that is the contradiction since f is locally one to one. So, that says so f prime of a is not equal to 0 for any a belongs to G.

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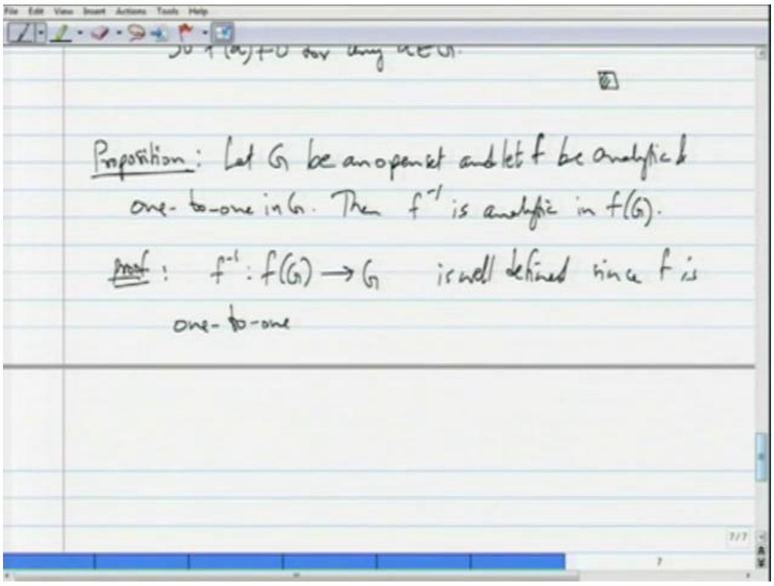

Another consequence that we can speak here is another proposition. So, let once again here we are dealing with one to one analytic functions, let G be an open set and let f be a analytic and one to one in G. Then f inverse is analytic in f of G, so firstly note that since f is one to one on G f inverse from f of G to g is well defined. Since, f is one to one further as a consequence of open mapping theorem; we can note that f inverse is firstly continuous, okay?

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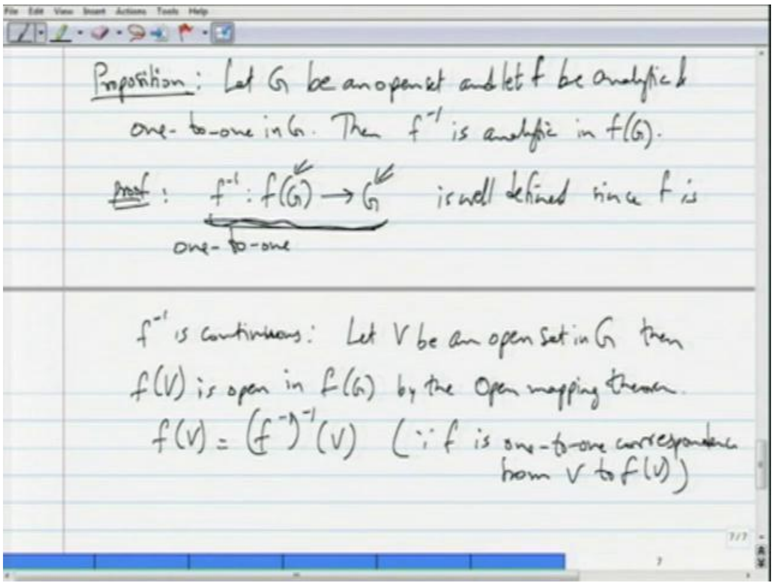

So, f inverse is continuous here is the argument. So, let so what I will show is the inverse image of an open set in G. So, here is f inverse from f of G to G. If I pick an open set here I will show that the inverse image of that is open in f of G. So, let V be an open set in G then f of V is open in f of G, because of the open mapping theorem, by the open mapping theorem. So, also notice that f of V is f inverse inverse of its f inverse inverse of V. What I mean by that is, it is the inverse image under f inverse of the set V. Since, this is because f is one to one, is one to one correspondence more. So, it is a bijection one to one correspondence from V to f of V.

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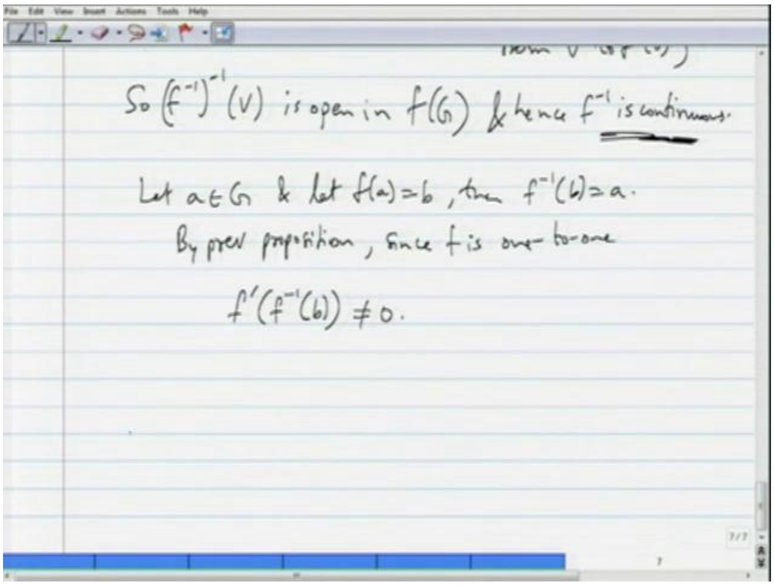

So, the inverse image of f inverse of V is nothing but f of V; so so f inverse inverse so f inverse inverse of V is open in f of G and hence that that tells that f inverse is continuous. We have shown that the inverse image of an arbitrary open set in G is open in f inverse in f of G under f inverse. So, f inverse is continuous. Let a belong to G and f of a, and let f of a is equal to B. Then f inverse of B is equal to a. So, I want to show that f inverse is analytic at f of a and every point in f of G looks like f of a.

So, I will be done if I show that f inverse is analytic at the point f of a by previous proposition. We have shown that since f is one to one, we have shown that once we have a one to one function on open set f prime is non zero or f is conformal. So, f prime of f inverse of B namely f prime at a is non-zero. Now, here is where I will actually use a continuity which I showed separately here of f inverse, okay?

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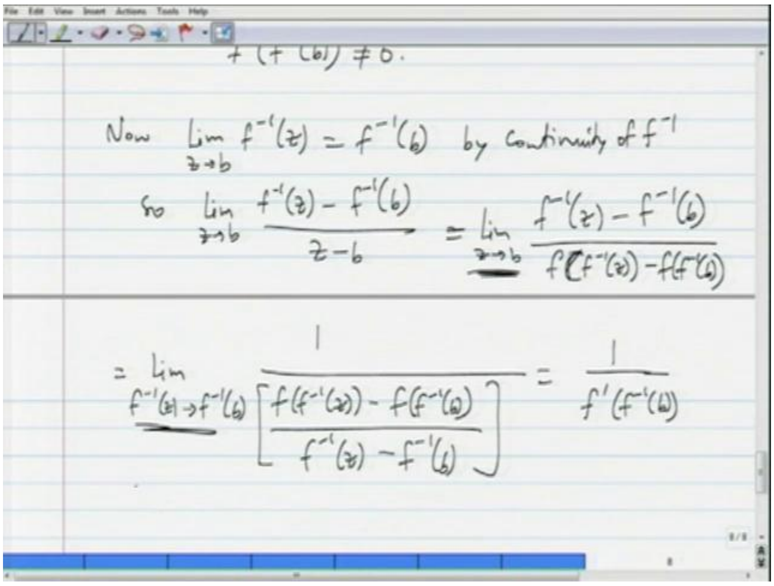

So, now since the limit as z goes to B of f inverse of z is equal to f inverse of B. So, this is why I need continuity in the first place and then I can show differentiability in the standard order, by a continuity of f inverse. So, the limit as z goes to B of f inverse of z minus f inverse of B by z minus B. This is the difference quotient and the limit of the difference quotient. So, I want to show that this limit exists and and then I will be able to conclude that f inverse is differentiable.

So, this is equal to the limit as z goes to B of f inverse of z minus f inverse of b. I am preserving the numerator and I am writing the denominator as f inverse f of f inverse of z minus f of f inverse of B, okay? So, this is equal to, so what I will do is, I will say this is limit as f inverse of z goes to f inverse of B limit as z goes to B is the same as limit as f inverse of z goes to f inverse of B because f is one to one and f inverse is continuous. So, this is 1 divided by f of f inverse of z minus f of f inverse of B divided by f inverse of z minus f inverse of B. So, notice that the denominator is nothing but the differentiation of f at f inverse of p and we know that is non zero. So, this is equal to one by f prime of f inverse of B. So, this makes sense this thing makes sense because you know f f f is conformal at f inverse of p, okay?

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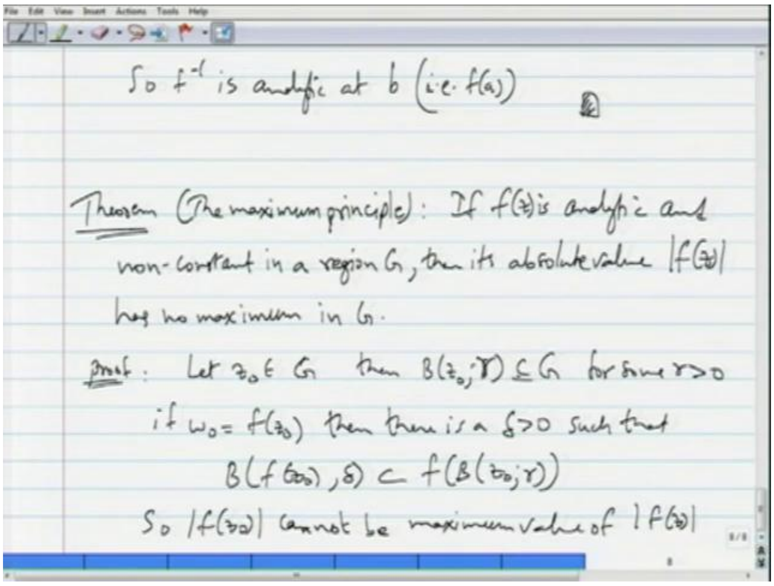

So, f inverse is as a conclusion we can say that f inverse is analytic at B i e f of a and since f is on to f of G f of capital G every point in f of capital G looks like f of a for some a in G. So, f inverse is essentially analytic on all of f of G. So, that completes the proof of this proposition. So, in the case that f is one to one, we can say more we can on an open set G we can say more we can say that f is conformal on that open set. We can also say that the inverse function is analytic.

So, these are two conclusions we can make from the theorem that I wrote at the beginning of the session. In the case where n is equal to 1 all right? So, next we will see that we can deduce the maximum principle as a consequence of the open mapping theorem. So, theorem the maximum principle, so if f of z is analytic and non constant in a region G, then its absolute value modulus of f of z had no maximum in G. So, we want G to be a region when modulus of f of z has no maximum in G, okay?

So, we stated the maximum principle before in a slightly different version and there I remarked that it can be directly proved using the local version of the maximum modulus principle that we have already seen. So, but we wanted to take a different root we wanted to prove the open mapping theorem first and deduce this theorem as a corollary to that. So, there is a there is a merit to it this shows that I mean this process of showing the maximum modulus principle tells you more about the local property of the analytic functions.

That that open sets are actually mapped open sets. So, you cannot have that the interior point is mapped on to some boundary point as a consequence of open mapping theorem; so, hence the maximum modulus principle, okay? So, let us prove this theorem using open mapping theorem. So, here is a proof belong to G then b z naught r is contained in G for some r positive. If w naught is equal to f of z naught, then there is a delta positive such that B of f of z naught is delta is contained in f of B of z naught r by the earlier theorem, which we had.

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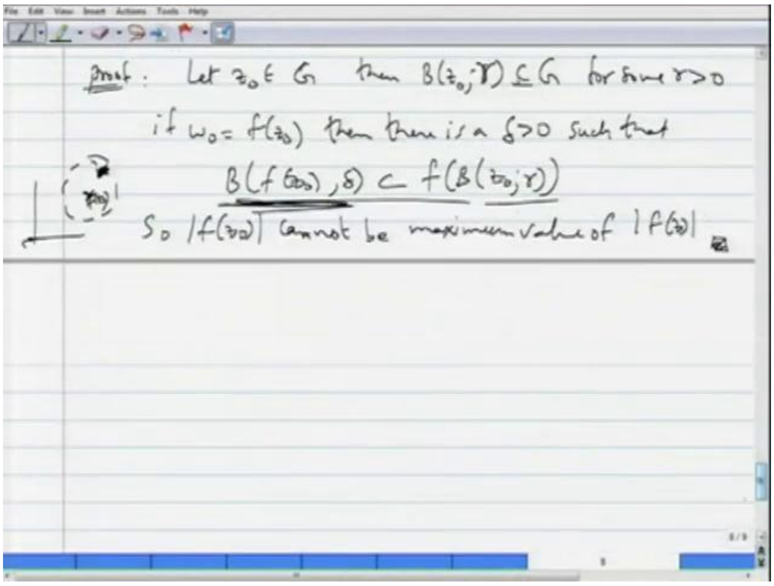

So, modulus of f of z naught clearly cannot be maximum value of modulus of f of z that is because in this ball there are always points, which have modulus greater than the modulus of f of z naught. So, in any ball the points further appear if origin is here. If 0 or the complex plane is here, then there are always points appear, which greater modulus than f of z naught itself. So, that is the idea. So, it proves this theorem. So, maximum modulus principle follows easily by using in the previous theorem.

Notice that, I am calling all these as a consequences of open mapping theorem, but I am desorting to a, this theorem. Well actually the hard work of the open mapping theorem is captured in this theorem. So, this is more general, but this theorem is the fundamental idea behind the open mapping theorem. So, coming back to here, now we can prove the Schwarz Lemma, which is actually a important consequence to the maximum modulus theorem, okay?

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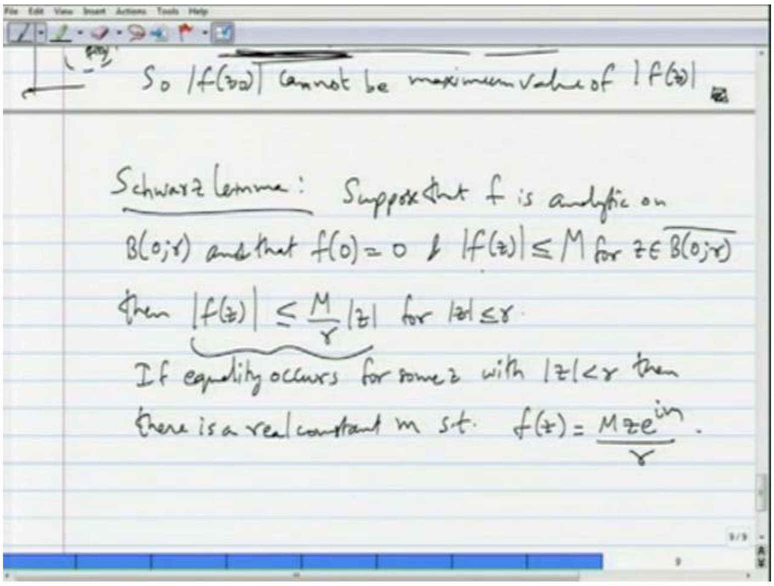

So, we can say something more about the function f and its bounds when when we have more conditions as follows. The Schwarz Lemma is an application. So, here is Schwarz Lemma, so suppose that f is analytic on B 0 r a disk of radius r centred at 0 and that f of 0 is equal to 0, okay? That and modulus of f of z is at most capital M for z belongs to B 0 r bar. Then modulus of f of z is less than or equal to M by r modulus of z for mod z less than or equal to r. So, if the modulus of f of z has a maximum of capital M on the one closed disk B 0 r bar the closure of B 0 r.

Then the modulus of f of z is at most m by r mod z. So, before proving this let me quickly remark also that this maximum principle tells you that if G is a bounded region. If G is a bounded region, it tells you that mod f of z has no maximum inside in the interior of G closure. So, modulus of f of z has to have a maximum on the boundary of G, if G is bounded. Why? Well modulus of f of z firstly is a continuous function from the, from G closure into the complex plane. So, a continuous function on a compact set G I am assuming is a bounded set, okay?

So so on a compact set G closure f modulus of f has to have a maximum and the maximum cannot occur in the interior of G bar namely G. So, the the the the modulus of f has to have a maximum on the boundary whenever G is bounded. So, Schwarz Lemma is a something in that line. It is telling that if I assume that the modulus of f of z is bounded by M on, on this closed disk B 0 r bar and f of 0 we can relax that in some sense at least for this version f of 0 is 0 f is analytic on B 0 r. Then modulus of f of z is less than or equal to M by r modulus of z for mod z less than or equal to r.

So, further actually there is more to this Lemma, we can say that if if equality occurs for some z with mod z less than r. So, in this inequality if equality occurs then there is a real constant M such that f of z is actually equal to M z e power r i m by r, okay? So, it is actually the function M z by r up to some rotation e power i n since f of 0 is equal to 0, we have this as well, okay?

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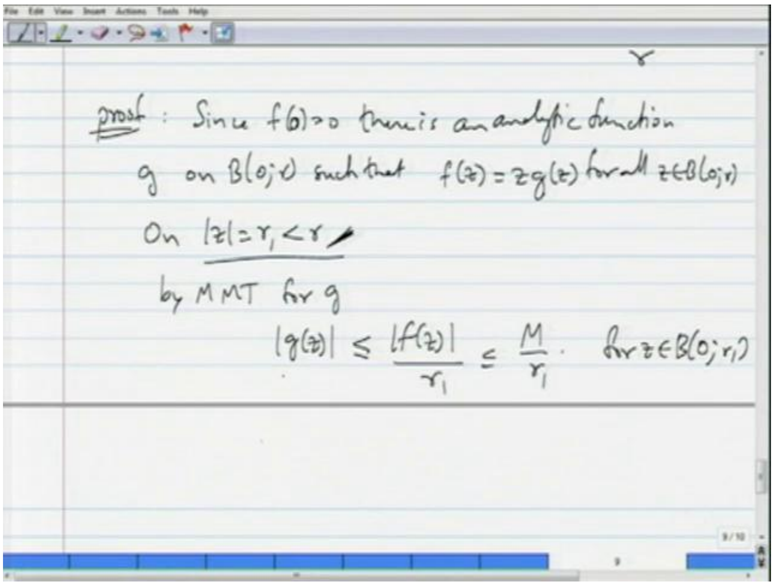

So, proof so since f of z is 0. Now f of 0 is 0 there is an analytic function G on b 0 r. Such that f of z is equal to z G of z for all z belongs B 0 r. So, by considering the Taylor's series expansion around around 0, if you wish you can say that there is a function G like that the order of 0 at 0 is at least 1. So, we have this so on a mod z equals r 1 strictly less than r, modulus of G of z is less than or equal...

So on a circle of radius r 1 strictly less than r by using by maximum Modulus theorem for the function G, what we can say is that the modulus of G of z is less than or equal to modulus of f of z by r 1, which is less than or equal to M by r 1; so, for for z belongs to B 0 r 1.

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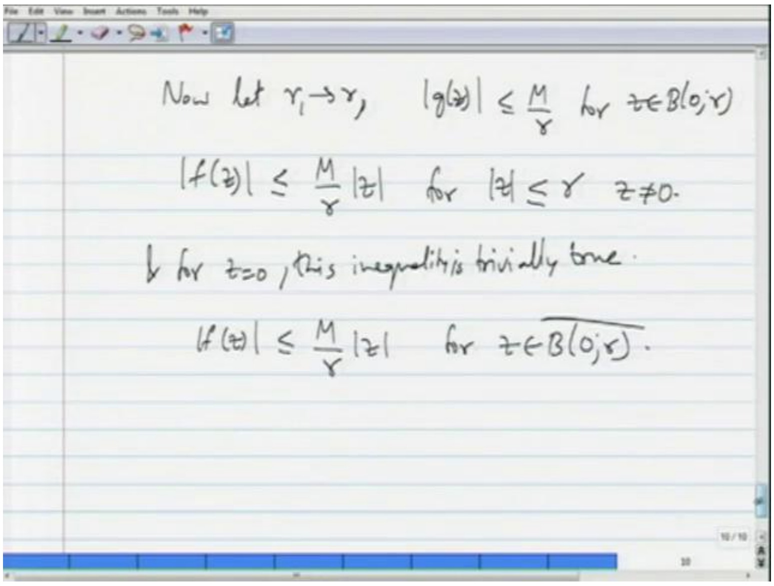

For all the z belongs to b 0 r 1 by maximum Modulus theorem, we have this and now letting letting r 1 tend to r we can conclude that modulus of G of z is less than or equal to M by r for z belongs to B 0 r. So, from this we can say that modulus of f of z is less than or equal to M by r modulus of z. By substituting what G of z is we get this. For modulus of z less than or equal to r z not equal to 0, okay? Since, we are multiplying by modulus of z. And for z equals 0, this inequality is true, is trivially true, because f of 0 is 0. So, all in all Modulus of f of z is less than or equal to M by r mod z for z belongs to b 0 r bar.

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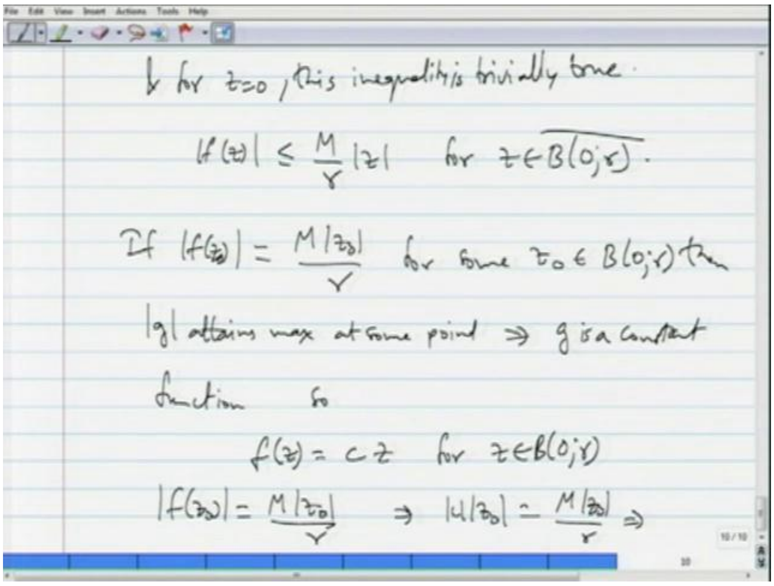

Now, f equality occurs if modulus of f of z happens to be equal to m mod z naught by r f of z naught is equal to m mod z naught by r, for some z naught for some z naught in B 0, r then mod G attains maximum at some point, implies in inside the disk. So, implies that G is a constant function that is the only way an at an interior point, you can have a maximum modulus, so G is a constant function.

So, so G of z naught is or sorry, G of z. So, f of z looks like C times z for z belongs to B 0 r also modulus of C modulus of f of z naught is equal to M times modulus of z naught by r, because equality occurred, which implies modulus of C times modulus of z naught is equal to M times mod z naught by r, which implies modulus of C is M by r.

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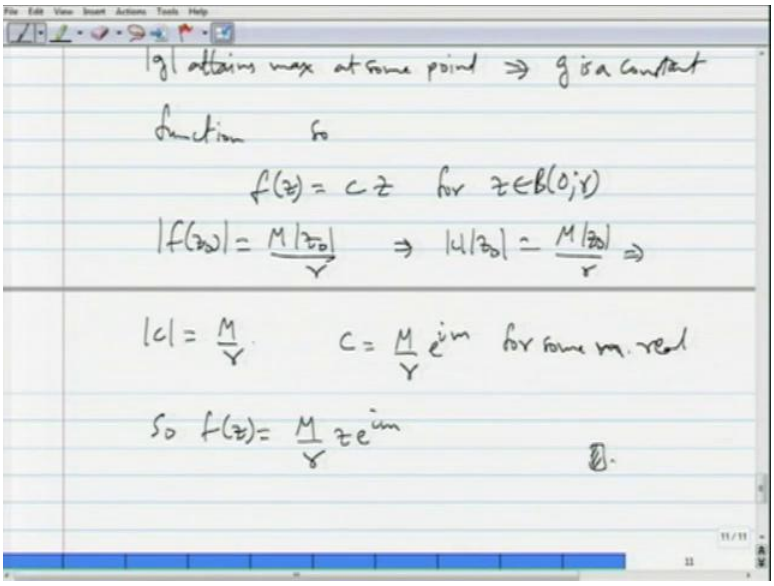

So C looks like M by r e power i M for some M. It is on M real, so f of z then is equal to M by r z e power i M as claimed. So, that completes the proof of this lemma. So, we can say something more about f of z, when we know it is bound on the boundary of B 0 r and if f of f is 0. So, that is an application of the maximum Modulus theorem.