Complex Analysis
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Module - 4
Further Properties of Analytic Functions
Lecture - 6
Opening Mapping Theorem Part II

Hello viewers in the last session we have seen the theorem on counting zeroes of an analytic function inside closed curve and we have seen as the consequences the Rouser's theorem or one version of Rouser's theorem and we used that to prove the local; version of maximum modulus theorem we use the counting zeroes theorem to prove the local version of the maximum modulus theorem. So, today we are going to see yet another consequences of the theorem on counting zeroes important one very important one namely the opening mapping theorem.

So, first I will begin with stating the global version of the maximum modulus thermo and instead of proving it directly I will take an indirect root I will I will prove the open mapping theorem and show that maximum modulus theorem of the global version or the local version can also be seen as consequences and corollary of the open mapping theorem. So, I will take the root. So, we need to build some tools before that so. Firstly, let me begin by stating the global version of the maximum modulus theorem.

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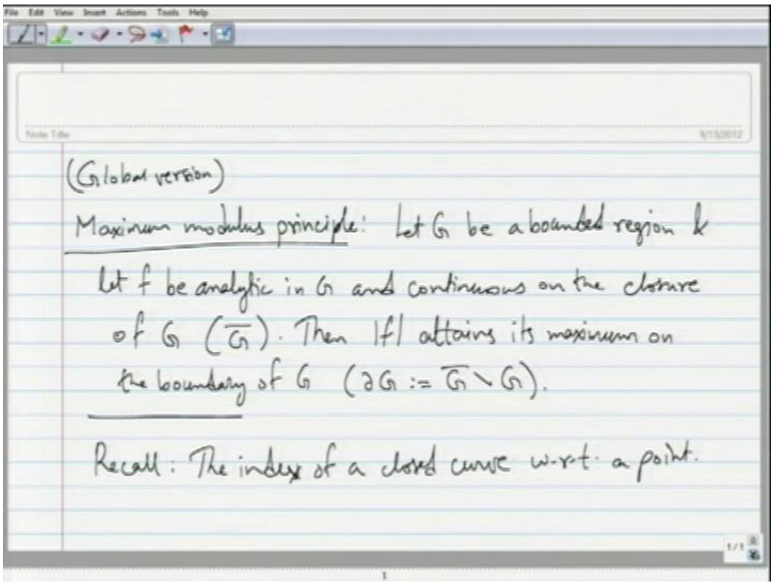

So, maximum modulus principle. So, let G be a bounded region and let f be analytic on or say in G and continuous on the boundary continuous on the closer actually on the hold close closer of G closer of G. So, we call it is denoted by G bar. So, then the modulus of f attains its maximum on the boundary of G. So, recall we denote the boundary of G by dou G which is the closer of G minus the interior of G which is G itself because G is a region. So, its a maximum of a modulus of f is attained on the boundary of g. So, that the statement of the maximum modulus principle. So, I said we can directly use topological argument and use the local version of maximum modulus thermo to prove this theorem.

But what we are going to do is we are going to develop tools for proving the open mapping theorem as a result of counting mapping theorem and then we will see the proof of this maximum modulus principle as a consequence. So, recall. So, in that direction we will recall something. So, recall the index of a closed curve with respect to a point. So, sometimes when discussing Cauchy’s theorem or various versions of Cauchy’s theorem we had the concept of the index of a closed curve with respect to a point what this was as fallows we define the index of.

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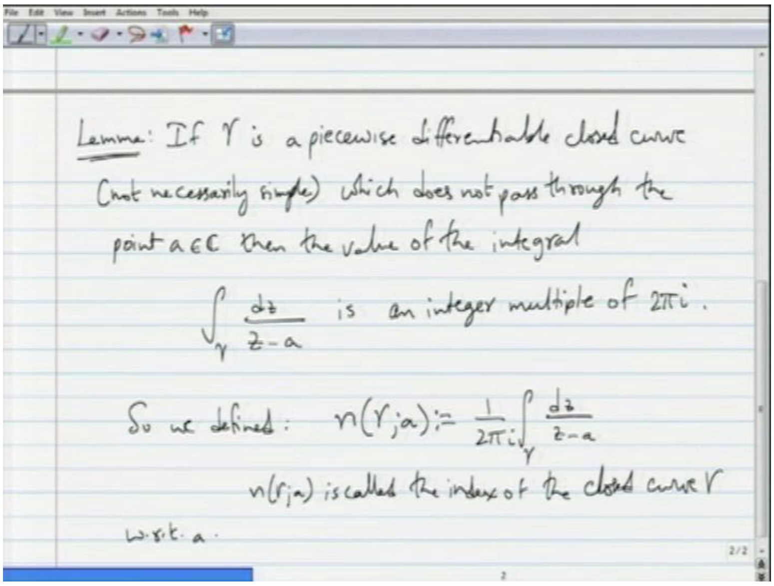

So, let me recall lemma which actually had the set something. So if gamma is a piece wise differentiable closed curve we can prove easily a smooth curve we can prove the lemma using smooth curve and generalise to piece wise differentiable closed curve like we have seen earlier. So, few points or a finite number of points on the curve having I means where gamma is not differentiable will not actually effect the proof of this lemma. So, gamma is piece wise differentiable closed curve not necessarily simple that is important which does not pass through the point a a belongs to c.

Then the value of the integral of the contour integral gamma d z by z minus a is an integer multiple of 2 pi i. So, this lemma we prove sometime back and then based on this lemma since this integral evaluates to a integer multiply 2 pi i we name that integer as the index of gamma with respect to a. So, define. So, we defined based on this lemma n gamma a to be 1 by 2 pi i. So, n gamma a is an integer which is defined to 1 by 2 pi i times the integral contour integral over gamma of d z by z minus a called n gamma a is called the index of the closed curve gamma with respect to a.

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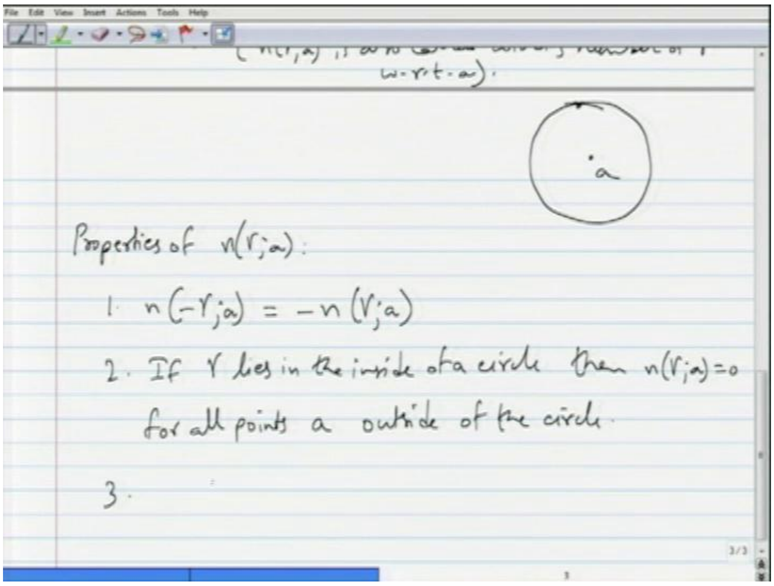

This is also called as winding number going to the geometric significance that if you if a is the centre of a circle or if a lies inside a simple closed curve then the index of that circle with respect to the point a is one times to pi i or if you have a circle which goes around n times. So, if you have a parameterization of a circle which goes around the point a n times visually then the index of that curve gamma which close around n times with respect to the point a is n is the integer n. So, owning to that geometric significance also called n gamma a is also called winding number. So, that is the suggested of the geometric significance of gamma with respect to a all right.

So, some properties are in order. So, properties of n gamma a. So, once again the assumption are that gamma is s as closed curve which does not pass through the point a and its piece wise differentiable curve as well gamma a gamma. So, then the properties of gamma under these assumptions one n gamma a is equal to or n minus gamma a rather is equal to minus of n gamma a that is the first property second if gamma lies in the inside of a circle because we mode that notion very precise what the inside of a simple closed curve is. So, in this case we take a circle and we assume that gamma laying inside of a circle.

Then a n gamma a is equal to 0 for all points a outside of the circle three. So, for three la’s notice that when you have a closed curve. So, before stating three let me make remark.

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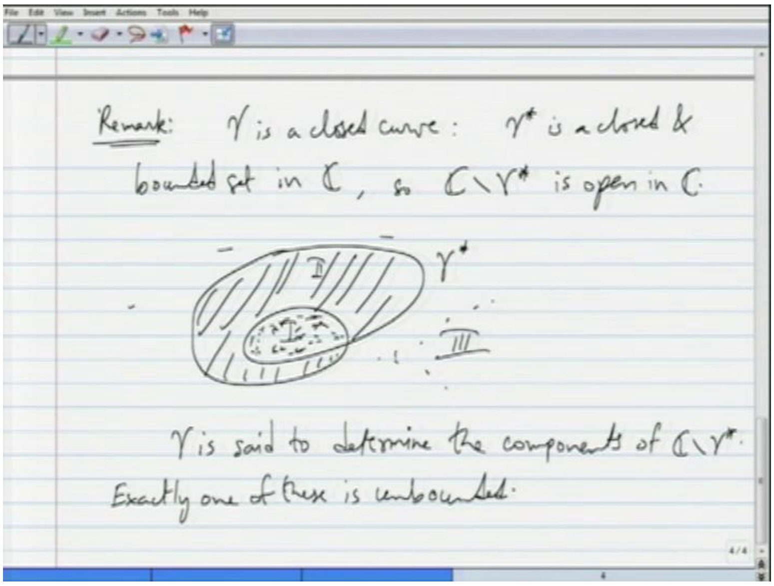

So, this is remark gamma is a closed curve what; that means, is our closed curve are functions from intervals to the complex plane and we are calling the image of gamma as the trace gamma gamma star.

So, gamma star is a closed and bounded set in the complex plane. So, c minus gamma. So, it is a compact set in c. So, c minus gamma is gamma star is open in c. So, what that tells us is that c minus gamma star is actually divided into components which are the maximal open sets whose union is c minus gamma star. So, so here is a visual example suppose your gamma star in c then you have component one which is this open set which does not include the boundary and then you have component two which is right here the hashed area and then there is component three which is sought of you know outside the visual boundary here. So, which is everything else in c minus gamma star. So, it is the unbounded component.

So, this is the visual example, but we can say that c minus gamma star is an open set. So, it is the union of its components recall are the maximal open sets maximal open connected sets. So, you take the connected component containing a point in c minus gamma star and then take the maximal search then form there union and then you get c minus gamma star and then on these component. So, property three says that before that I will also want say that. So, gamma is said to determine the components of c minus gamma star and exactly one of these is unbounded of course, a compact set. So, I mean gamma star is a compact set in c the complement of it will contain some points which have very large modulus. So, take because compact set is bounded. So, you take a point which is very far off from the origin in terms of modulus.

And then you take the connected component containing it and it is exactly one because you are any way connecting the connected component containing points which have very large modulus all right.

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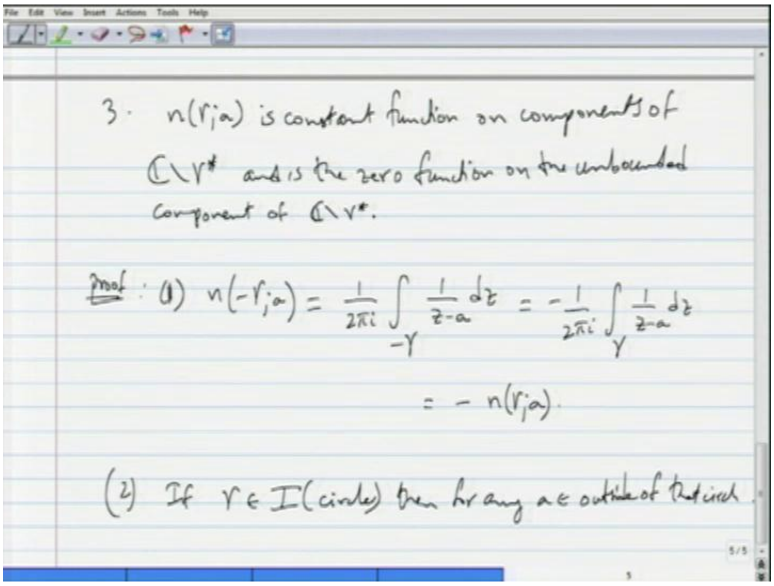

So, its unbounded and here is the third property n gamma a is constant is constant function on components of c minus gamma star and is a 0 and is the 0 function on the unbounded component of c minus gamma star.

So, what; that means, is any gamma can be different constants on different components, but on a given component it is a constant function. So, that is what is the is meaning of this statement. So, here is the proof of this properties well the proof of one and two are very easy the proof of one is that n minus gamma a recall what minus gamma is it is not minus gamma of t for t belonging to the interval 0 1 it is the inverse I mean it is gamma star traced in opposite direction. So, recall that. So, this is equal to 1 by 2 pi I by definition integral over minus gamma of 1 by z minus a d z. So, gamma is being traced in the opposite direction from the ending point to staring point for gamma. So, that is minus gamma. So, by definition we already capture this property this is equal to minus 1 by 2 pi i of integral contour integral over gamma of 1 by z minus a d z and this is precisely minus n gamma a. So, that is easy that proof is easy and also property two the property two can be proved easily.

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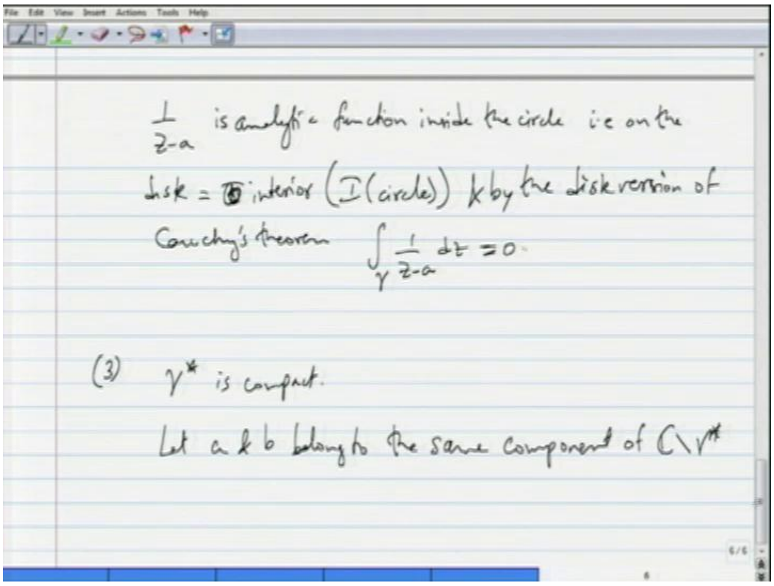

So, if gamma lies on the inside of a circle gamma belongs to the I of a circle then for any a belongs to outside of a circle of that circle one by z minus a is analytic function inside the circle I on the disk the interior of which is the interior of I of circle the interior of the inside of a circle is a is a disk and by the disk version of the Cauchy's theorem we know that and by the disk version of Cauchy's theorem we know that the integral over gamma. So, any closed curve gamma of 1 by z minus a d z is equal 0 all right. So, that is the proof of property two.

Property three gamma star is compact. So, so let a and b belong to the same region I mean to say the same component of c minus gamma star what I want to show is that n gamma a is equal to n gamma b. So, what. So, I am going to use the topology of the component. So, recall that if you have a component the maximal connected set open connected set then given any two points in this component there is a polygonal path connecting this two. So, that is from the topological topology of the complex plane.

So, using that fact there is a there is polygonal path connecting these particular a and b. So, if I can show that n gamma of x is constant on a straight line segment then you can extend this fact to the polygonal path I am claiming that n gamma x is constant for any polygonal path all together. So, then I can conclude that n gamma a is equal n gamma b here is the path that I will take.

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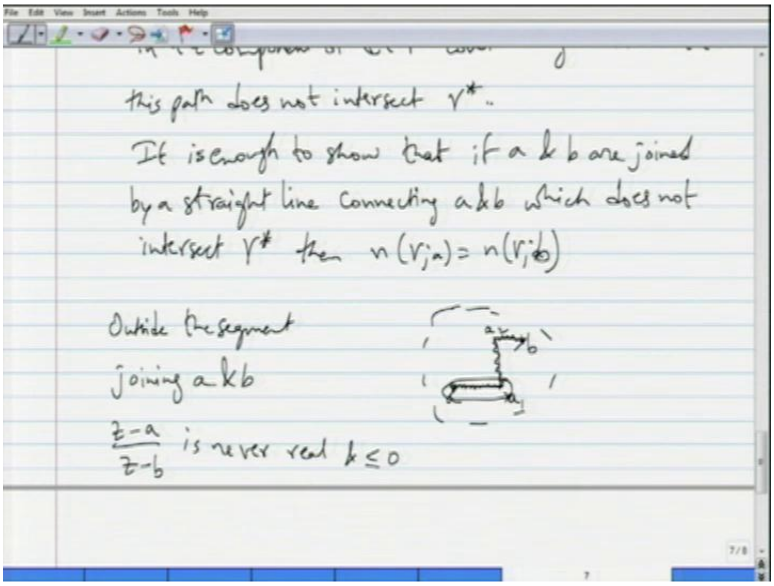

So, there is a polygonal path connecting a and b laying completely in the component of c minus gamma star containing a and b. So, the point a it is clear of gamma star it is the polygonal path does not intersect gamma star. So, I e this path does not intersect gamma star.

So, it is enough to show that if a and b are joined by a straight line connecting a and b which does not intersect gamma star then n of gamma a is equal to n of gamma b. So, outside the segment joining a and b this is the standard thing that one does z minus a divided by the z minus b for z outside the segment a and d is never real and less than or equal to 0 simultaneously.

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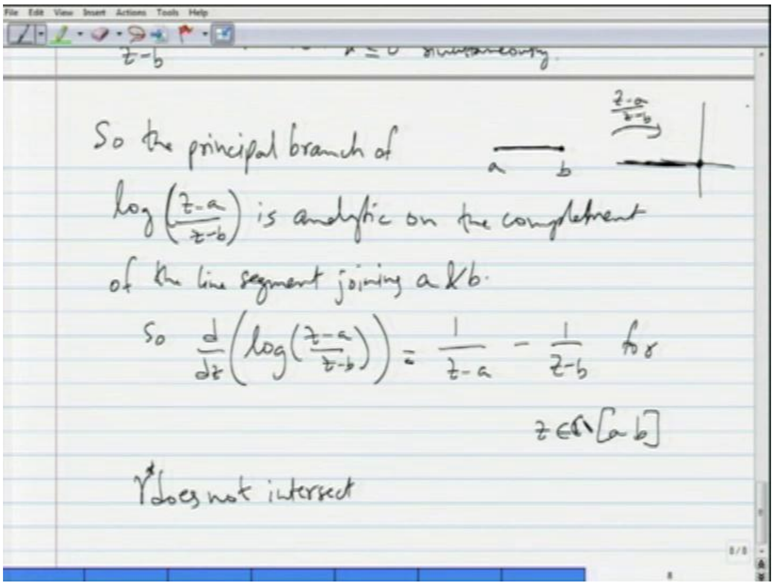

So, here is a b if you take the straight line joining the a b. So, z minus a by z minus b `takes the map z minus a divided by z minus b takes b to infinity and then it take a to 0.

So, you have a line joining 0 and infinity well it actually takes this line this map f of z equals z minus a by z minus b takes the line segment joining a b to this negative real axis to this whole negative real axis. So, if we ignore the line segment joining a and b the point outside it are precisely mapped on to the complement of the negative real axis where the map z minus a by z minus b and we know that when we remove the negative real axis we can define a branch of logarithm an analytic branch of logarithm on the complement of the of the negative real axis. So, that is the point.

So, what we will do is. So, the principle branch you can take a branch of logarithm analytic branch. So, I will take the principle branch of logarithm of log z minus a by z minus b is analytic on the complement of the line segment joining a and b. So, since this principle branch is analytic we can differentiated we know what it is differentiation is d by d z of log of z minus a by z minus b is your one by z minus a. So, log z minus a by z minus b I am writing that as log z minus a minus log z minus b for all z in the appropriate place.

So, then this is the differentiation of this is one by z minus a minus one by z minus b for z belongs to the complement of the line segment. So, I will just use a shorter notation I will just say the a b c minus a b what; that means, is the complement of the line segment joining a b I am just using a short cut. So, then gamma does not need the segment. So, gamma star does not intersect. So, recall we are being sloppy about gamma and gamma star what I mean by that is we agreed that we confuse between the curve itself and its case.

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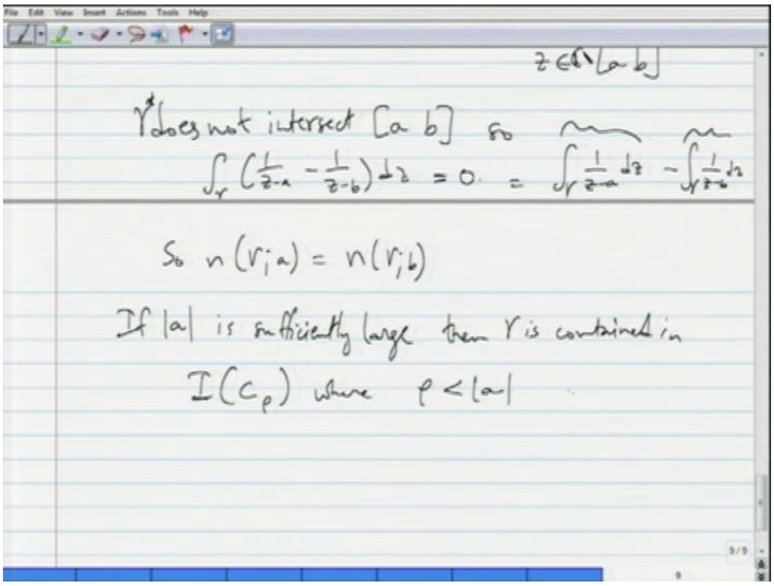

So, I will right gamma star or gamma gamma star does not intersect this line segment joining a b. So, the integration over gamma of 1 by z minus a minus 1 by z minus b which is which has an anti derivative now the integrand has an anti derivative in the region of interest. So, if you take any closed curve in that in that region may be which slice entirely in the region you know that by the anti derivative theorem this has to equal 0 and gamma a is equal to n gamma b right because this is also integral over gamma 1 by z minus a d z minus integral over gamma 1 by z minus b d z and each of these are 2 pi i times the index of gamma with respect to a or b.

So, this is a. So, these are equal. So, that close property three also I claim that the unbounded component has a 0 as a its index. So, if a is sufficiently large then gamma is contained in the interior of circle of radius row where rho is strictly less than the modulus of a then by the earlier property that if gamma lies in the interior of a circle then the points outside the circle this is 0 the I mean the index is 0.

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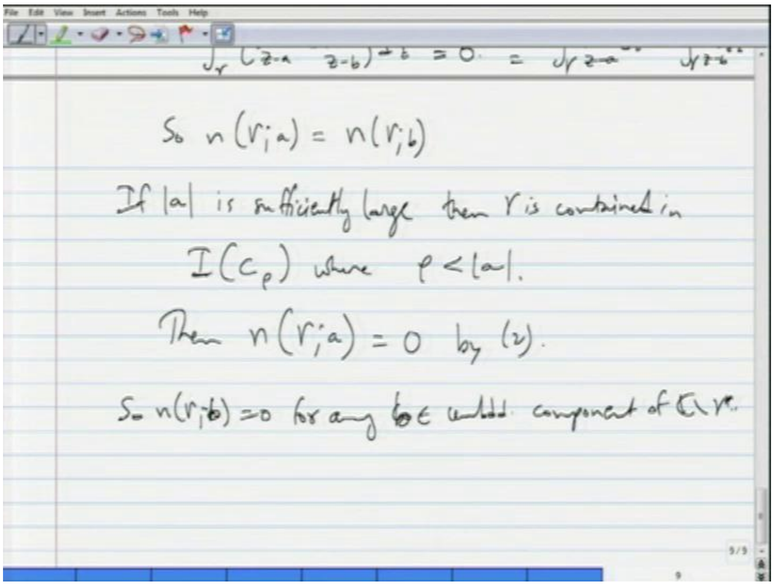

Then we have that the index of gamma with respect to a is 0 definitely by 2 and since we just proved that the function n of gamma x is constant on components of c minus gamma star you know the value of n gamma a for any point in the unbounded component has to be 0.

So, so n gamma a I will say b is 0 for any b belongs to unbounded component of c minus gamma star all right. So, that is the property properties of this index. So, now, we are going to use this index and try to sort of reinterpret the counting zeroes theorem and we will see that that interpretation has good consequence by counting zeroes theorem we had that if gamma is a simple closed curve.

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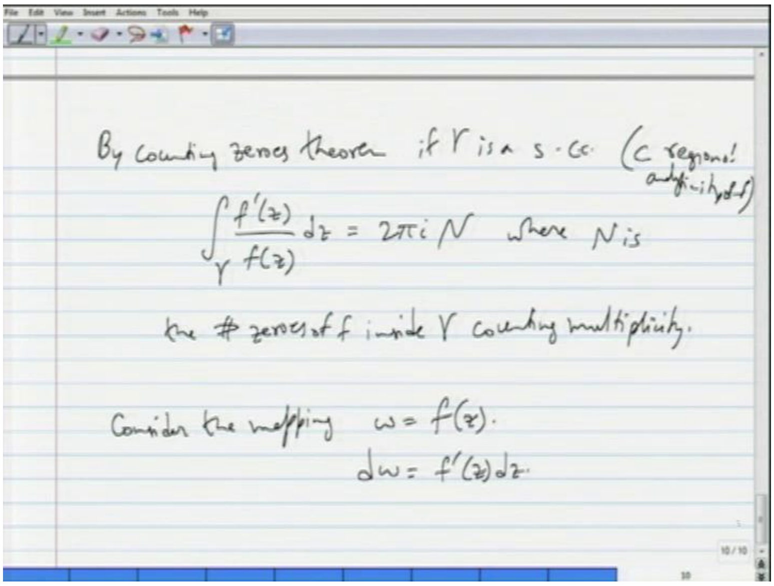

Then the integration the contour integration over gamma of f prime of z by f of z of course, gamma is contained should be contained in region of analyticity I am not going to state this precisely, but you already had this statement of counting zeroes theorem. So, I will say that if gamma is a simple closed curve contained in the region of analyticity of f then integration over gamma of f prime z by f of z d z gives you 2 pi i times N where N is the number of zeroes of f inside gamma counting multiplicity . So, all right. So, by considering the map. So, consider the mapping w equals f of z. So, then d we equal f prime of z d d z also.

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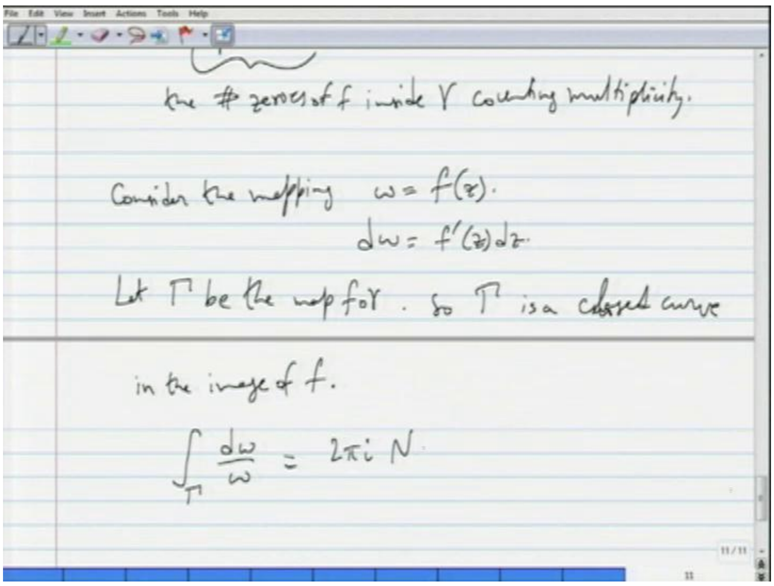

Let capital gamma be the map of be the be the map gamma trace over composed with f. So, so capital gamma is a curve is a closed curve now gamma is a curve and f maps gamma the simple closed curve gamma to somewhere else. So, capital gamma is a closed curve in the image of f it is it is contained with the image of f. So, so this left hand side this left hand side integral over gamma f prime z d z by f of z can be reinterpreted as the integration contour integration over a capital gamma of d w by w that is now equal to... So, by substitution really this contour this new contour integral is equal to 2 pi i times N.

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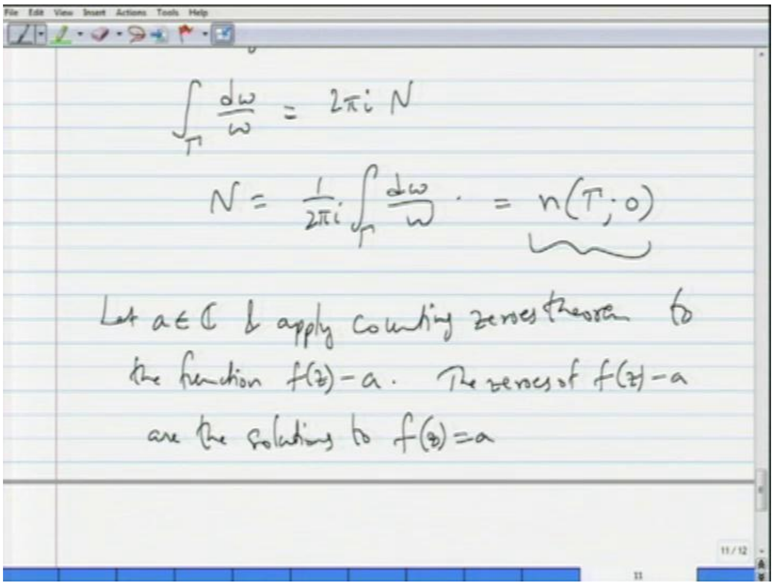

But so, rewriting this is 2 pi i times integral over gamma of d w by w which is nothing, but the index of capital gamma with respect to 0. So, this is the very useful interpretation tell is that the index of the of the target curve with respect to 0 target curve via f of little gamma with respect to 0 gives actually the number of zeroes f of inside little gamma this is very useful.

So, this is just a reinterpretation of that counting zeroes theorem, but we will put this to use. So, let what we will do is let a b an arbitrary complex number. So, let a belongs c just say a belongs to c and apply counting zeroes theorem to the function to the function f of z minus a. So, the zeroes of. Firstly, the zeroes of f of z minus a are the solutions to the equation f of z equals a f of z minus a equals 0 in place f of z minus a equals a.

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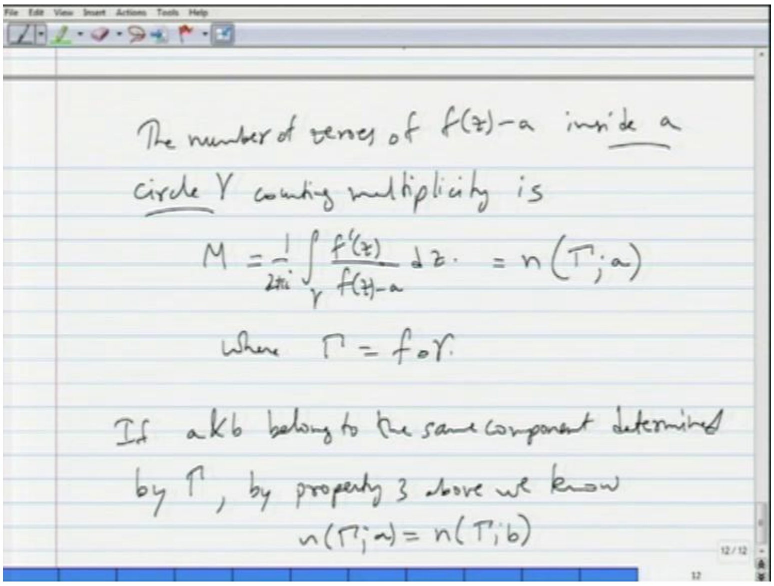

So, the number of zeroes of f of z minus a inside a circle let us say a simple closed circle gamma counting multiplicity is say M call that m this is integral over gamma of G prime of z divided by or rather I am using f. So, this is 1 by 2 pi i times f prime of z divided by f of z minus a because the derivative of f of z minus a is f prime of z still and this we said is the index of capital gamma with respect the point a now where capital gamma is f circle little gamma where we used the simple closed curve use simple circle around a. So, we have that.

So, if a and b belong to the same region same component actually determined by capital gamma by property three above. So, you know that the index function is constant on components. So, if you pick two pints in components determined by capital gamma is also closed curve. So, it is determines some components.

So, if you pick two points in the in the complement of capital gamma star then you know that the index of those two points has to be equal the interpretation here is that the number of times f of takes a we will have to equal the number of times f of z takes b it might I might not take a at all. So, in which case it will not take b at all. So, that the number of times it takes a should equal the number of times it takes b. So, the property three above we know that n gamma a is equal to n gamma b.

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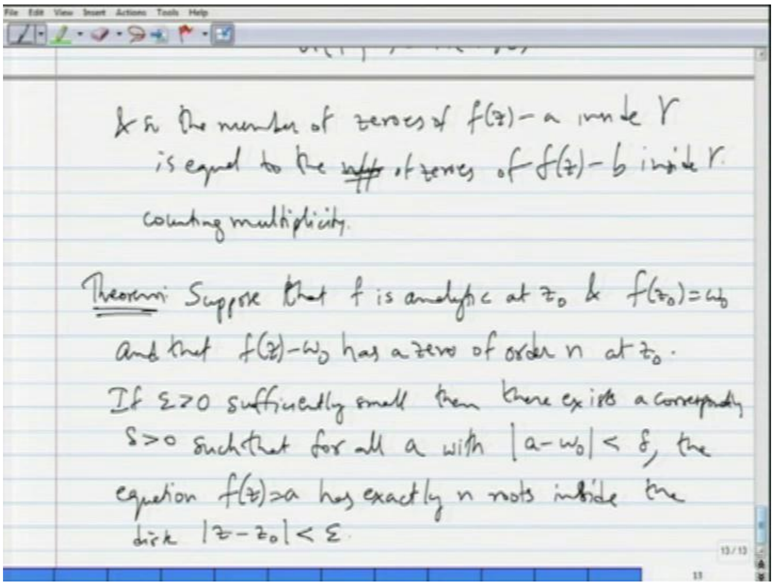

So, that has an interpretation that and. So, the number of zeroes of f of z minus a inside little gamma is equal to the number of zeroes I will just put hash of zeroes of f of z minus d inside gamma and said in. So, said otherwise the number of times f takes a inside gamma little gamma should be equal to the number of times f takes b inside little gamma.

So, of course, counting multiplicity. So, now, here is the theorem which is from which the open mapping theorem will follow. So, suppose that f of z f is analytic at z naught a function f if analytic at a point z naught and f of z naught is equal to w naught and that f of z minus w naught has a 0 of order n at z naught not a z then I say that f has a f of z minus w naught has a 0 of order n and automatically assuming that f of z is not identically w naught in a neighbourhood of z naught.

So, it has 0 of order n n finite. So, if here is the conclusion epsilon naught is sufficiently small then there exist a corresponding deltas positive such that for all a with modulus of a minus w naught strictly less that delta the equation f of z equals a has exactly n roots inside the disk modulus of z minus z naught less than strictly less epsilon. So, so this is the very important theorem it says that there is there is some delta around w naught there is delta neighbourhood of the w naught. So, which is map on to by a sub set of a neighbourhood of a z naught.

And in such way that it is an end to 1 correspondence if the where n is order of the 0 at z naught at z naught of f of z minus w naught. So, f of z minus w naught is definitely 0 at z naught. So, if n is the order of 0 then any there is a there is small neighbourhood around w naught such that any point in the neighbourhood is actually taken up by f in some neighbourhood of z naught and it does. So, n times exactly.

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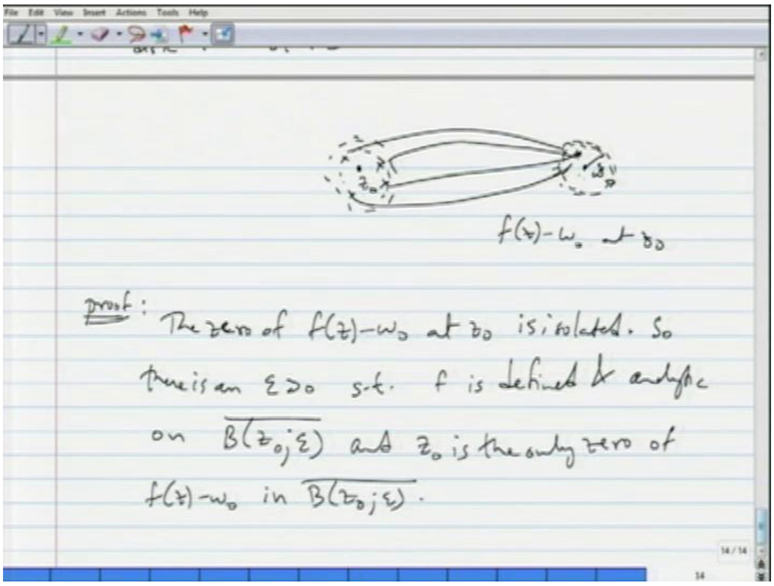

So, I will I will sort of draw a picture to elaborate that. So, here is w naught here is z naught. So, there is small claim is that the theorem claims that there is small neighbourhood delta neighbourhood of this is delta such that if you take any point here then there are exactly n free images counting multiplicity of course, you have to entertain multiplicity there'll n free images here which all map to that point where n here reverse to the order of 0 of f of z minus w naught. So, this is interesting.

So, here is the proof I mean as consequence will actually have the opening mapping theorem because now we can say that if you take z naught and the target w naught y r f then there is a open set around this there is a neighbourhood around this such that you know you have you have something from here from the neighbourhood of z naught which is mapped to this neighbourhood. So, every point in this is a target in the from the neighbourhood of z naught. So, that actually gives the open mapping theorem. So, let see the proof of this theorem. So, we saw that the zeroes of an analytic function are isolated of a function which is not identically 0 are isolated.

So, the 0 of a f of z minus w naught is also isolated. So, the 0 of f of z minus w naught at z naught is isolated. So, there is. So, there is an epsilon positive such that f of z f is defined and then analytic on modulus of z minus z naught strictly less than or equal to epsilon. So, this is better side as b z naught epsilon bar. So, by contracting that neighbourhood of the isolated 0 enough I can assume that there is a closed you know inside on which f is defined and analytic. So, f is defined and analytic on that. And z naught is only 0 only 0 of f of z minus w naught in b z naught epsilon bar.

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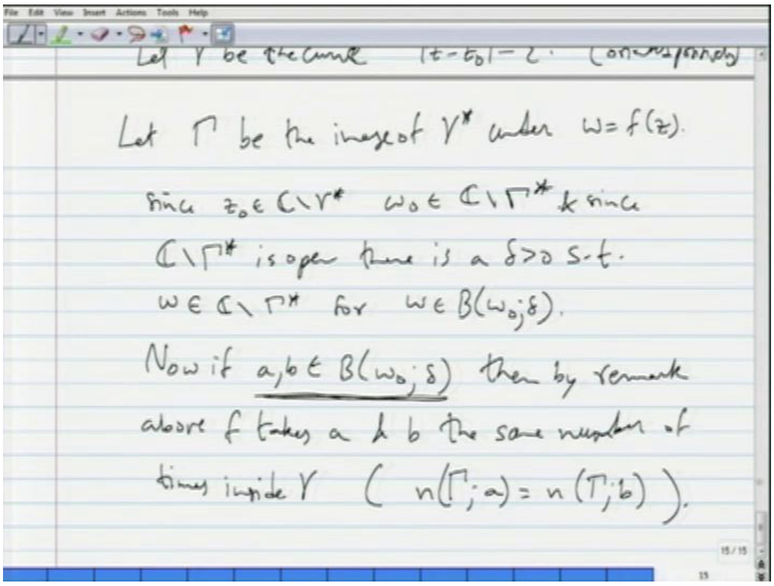

So, what that gives is let gamma be simple closed curve whose stress. So, let gamma be the curve I will just say the curve modulus of z minus z naught is equal to epsilon. So, it is it is a circle of radius epsilon around z naught oriented positively we will assume oriented positively that does not matter for us, but what we will do is now let capital gamma be the image of gamma star under f under the mapping w equals f of z. So, since w z naught belongs to c minus gamma star of course, w naught will belong to c minus capital gamma star.

so and since this is open is open there is a neighbourhood of w naught which is in c minus gamma star. So, there is delta positive such that w belongs to c minus gamma star for w belonging to a boll of radius delta around w naught. So, this is the our candidate which we are seeking. So, now, if you take any two points in this boll since this boll clear of gamma. So, there is no intersection of this boll with gamma star then by a remark above f takes a and b the same number of times.

What is the remark we said that n capital gamma a. So, I should complete this inside I will say inside of little gamma. So, we said that n gamma a is equal to n gamma b and I mean the index function on the components of determined by capital gamma is a constant function. So, n gamma a is equal to n gamma b for a b belonging to the same component determined by capital gamma. So, in particular manner when they are in this boll they are in the same component. So, n gamma a should equal n gamma a should equal n gamma b. So, this happens, but we know the specific point which is already in this boll namely w naught.

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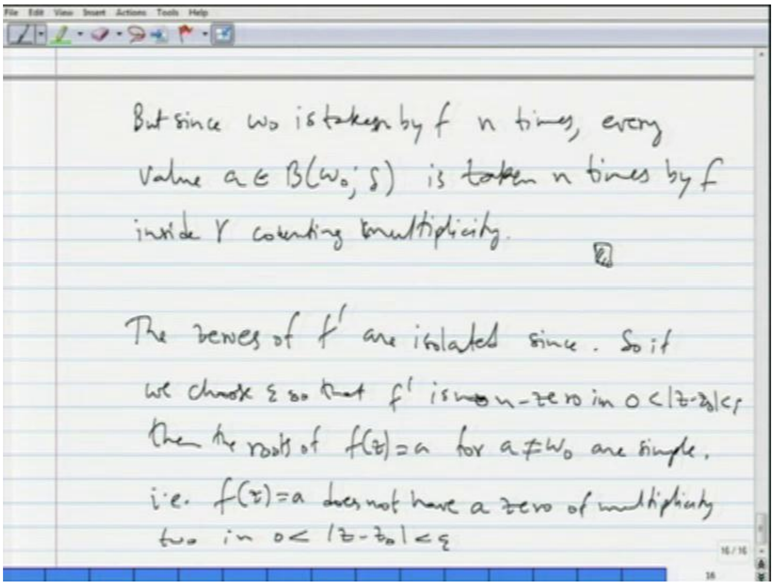

But since the order of, but since the value w naught is taken by f n times every value a belongs to B w naught delta is taken n times by f inside gamma counting multiplicity this is exactly what we wanted. So, that is the. So, that is the conclusion f of z equals a has exactly n roots inside the disk modulus of z minus z naught is strictly less than epsilon the inside of gamma is nothing, but inside of little gamma is nothing, but the interior of the disk which precisely this region. So, that is the conclusion. So, that completes the proof of this theorem.

One more remark. So, the zeroes of f prime are isolated since f prime is also when f is analytic f prime is an also analytic function. So, the zeroes of f prime are isolated if we choose epsilon. So, that f prime is non 0 in 0 strictly less than mode z minus z naught strictly less than epsilon then the roots of f of z equals a for a not equal to w naught are simple I e f of z equals a does not have does not have a 0 of multiplicity two in 0 strictly less than mode z minus z naught is strictly less than epsilon epsilon. So, that follows from you know Taylor's theorem using local power series expansion if you like. So, so we will see some consequences of this theorem next time I will stop here.