Complex Analysis
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Module - 1
The Arithmetic, Geometry and Topological Properties of the Complex Numbers
Lecture - 2
De Moivre's Formula and Stereographic Projection

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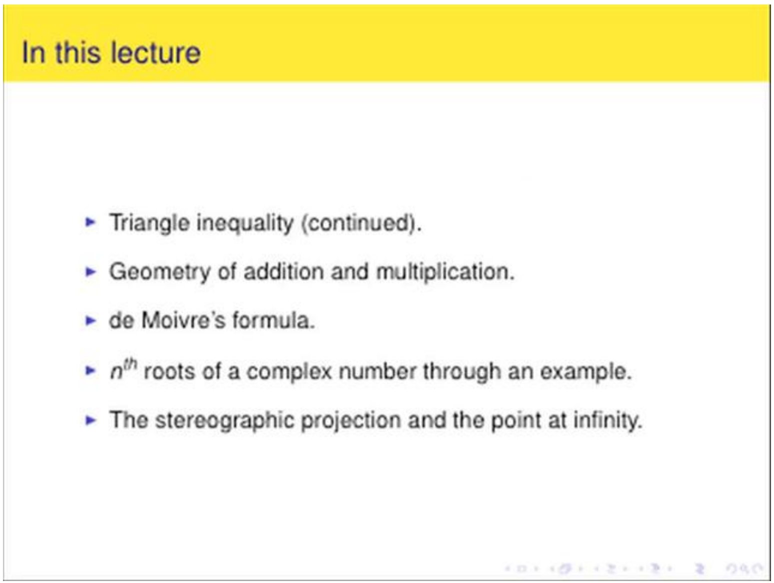

Hello viewers, in this session, we will continue with the properties of complex numbers, geometric, arithmetic, etcetera. So, firstly last time, we saw the the modulus of a complex number, and we also said what the argument of a complex number is, so we will see a couple of more properties of the modulus.

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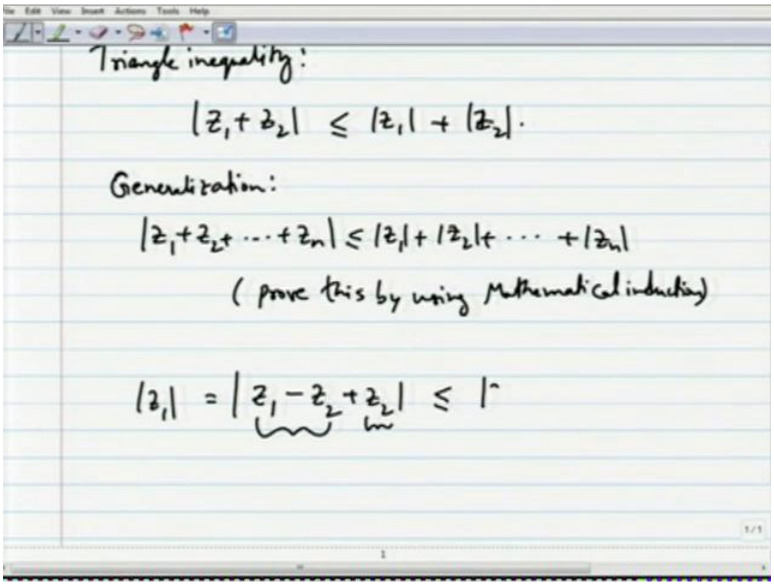

So, last time recall we proved the triangle inequality for the modulus of complex number. So, it said that the modulus of z 1 plus z 2 is less than or equal to modulus of z 1 plus the modulus of z 2. And it was an exercise to show that equality occurs only when z 1 and z 2 lie on the same line passing through the origin, and on the same side of the origin. So that is, that is your triangle inequality. So, it can be generalized, so it can be applied to n numbers.

So, may be well using induction one can show that using the principle of mathematical induction one can show that the modulus of a sum of n complex numbers like that is less than or equal to the, the sum of the moduli, one can prove this using mathematical induction. So, exercise prove this by using mathematical induction. And it is convenient to record one other form of triangle inequality, it is as follows. So, the modulus of, so the modulus of z 1 plus z 2 or let me say modulus of z 1 minus z 2, so let us estimate this, plus z 2, which is equal to the modulus of z 1.

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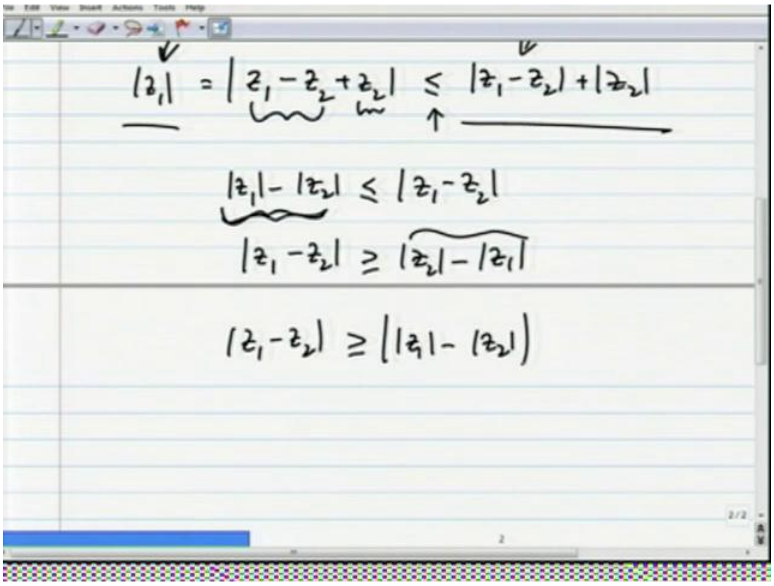

So, I am adding and subtracting z 2, by the triangle inequality if I treat this as one number, complex number and this as the second complex number by the triangle inequality, this is less than or equal to the modulus of z 1 minus z 2 plus the modulus of z 2. Now, using this and this on 2 sides of this inequality, what we get is that the modulus of z 1 minus the modulus of z 2 is less than or equal to the modulus of z 1 minus z 2. And this is symmetric in z 1 and z 2 by that I mean, the modulus of z z 1, so the modulus of z 1 minus z 2 is likewise greater than or equal to we can show the modulus of z 2 minus the modulus of z 1.

By doing this using doing this using z 2 instead of z 1, I mean doing the same procedure here you will get modulus of z 2 minus z 1, but that is the same as a modulus of z 1 minus z 2. Then so you get this other part, and so in conclusion the modulus of z 1 minus z 2 is greater than or equal to since it is greater than or equal to this and this it is greater than or equal to the absolute value of the modulus of z 1 minus the modulus of z 2. So, this form of triangle inequality is also useful sometimes and I have proved it here for records. Next what we are going to do is, we are going to give a geometric interpretation using or geometric locus using the modulus. So, we know the geometric interpretation of the modulus as the distance of the complex number z from the origin.

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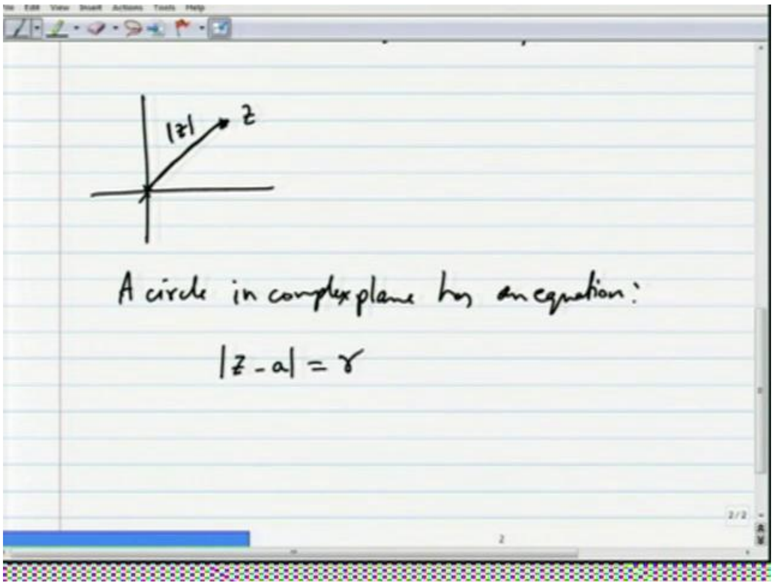

So, the modulus of z for example here here is z let us say, so the modulus of z represents the length of this line segment from 0 to z, that is that is your modulus. Since, a circle is the set of all points which are at a distance, which are at a constant distance from a fixed point, so we can say that a circle in complex plane has an equation, the modulus of z minus a is equal to r.

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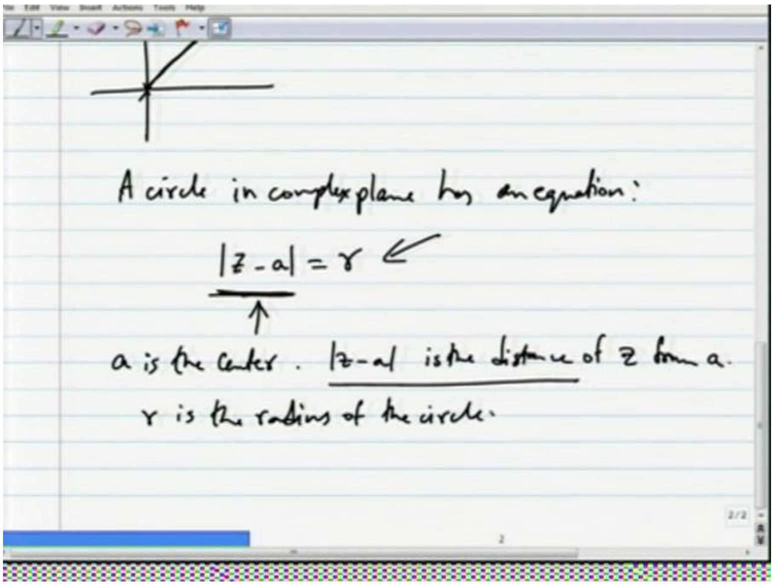

So, the modulus of z minus a measures the distance of z from a; so a is the fixed point, a is the centre and modulus of z minus a is the distance of a varying point z from the fixed point a and r is where I mean, this equation is telling you that this distance has to be constant r, so r is the radius of the circle.

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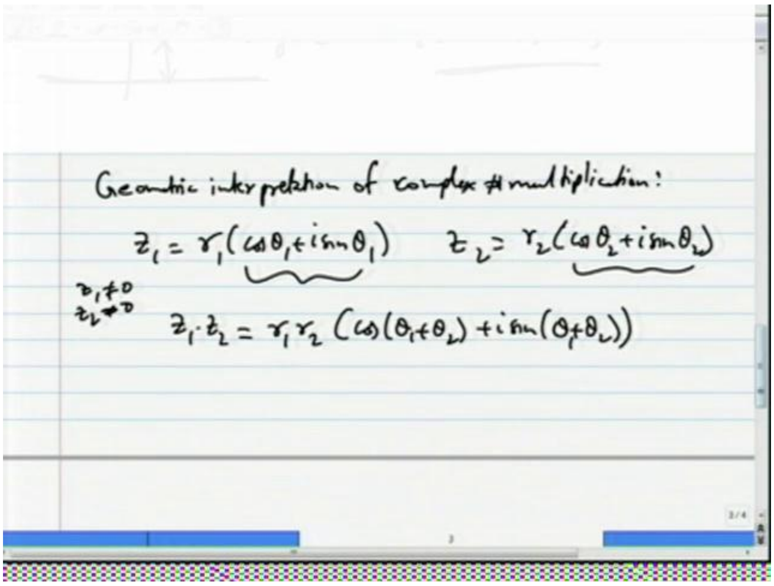

I will first talk about the geometric. So, consider the geometric interpretation of multiplication, of of complex number multiplication and then I will come back to the straight line business. So, if you have z 1 equals r times, r 1 times cosine theta 1 plus i sine theta 1 and z 2 equals r 2 cosine theta 2 plus i sine theta 2 for some r 1 theta 1 and r 2 theta 2. Let us assume for the time being z 1 not equal to 0, z 2 not equal to 0, so that they have a polar representation like that.

Then z 1 time z 2 we can work out is r 1, r 2 times when you multiply this expression with this expression you get cos sine theta 1 plus theta 2 plus i sine theta 1 plus theta 2. So, this complex number z 1 times z 2 has modulus r 1 times r 2 and an argument for z 1 times z 2 is theta 1 plus theta 2. Where theta 1 is an argument of z 1 and theta 2 is an argument of z 2.

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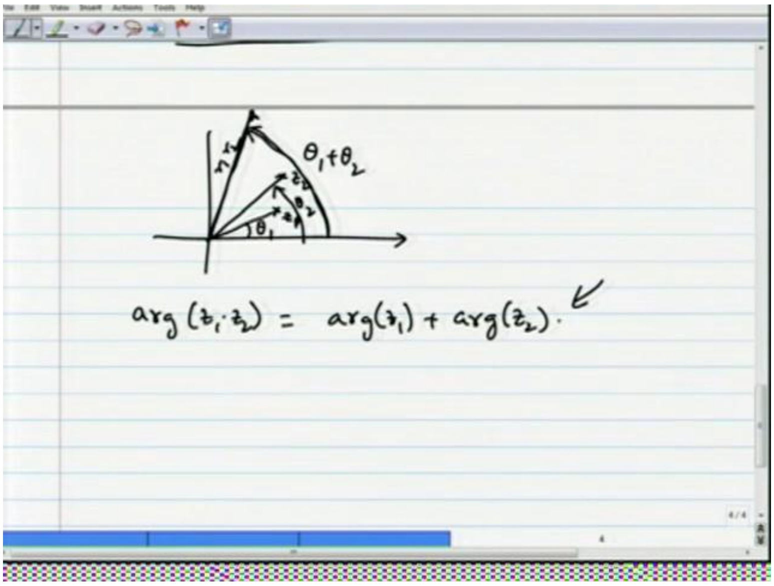

So, the interpretation one can give is, let us take a simplistic picture, suppose this is z 1 and then it opens with an angle theta 1 from the positive x axis. Then suppose this is z 2 and it opens with an angle theta 2 from the positive x axis, then your z 1 times z 2 is going to have a modulus. So, the length of this line segment is going to be modulus of z 1 times modulus of z 2, like we see here. So, it is going to have r 1 times r 2, as it is modulus and then it is going to open with the x axis, positive x axis with an angle theta 1 plus theta 2. So, that is your geometric interpretation of multiplication and note that argument of n argument of z 1 times z 2 is equal to an argument of z 1 plus an argument of z 2.

So, if you take a particular argument of z 2, z 1, any argument of z 1 for that matter and any argument of z 2, recall there are many possible values of theta 1 and theta 2, that you can take because cosine and sine are 2 pi periodic. So, any choice of theta 1 and theta 2 will give you this equation, this multiplication rule dictates that equation and so you get this, this equality. So, the n argument, n argument for z 1 times z 2 is argument of z 1 plus the argument of z 2 or plus n argument of z 2.

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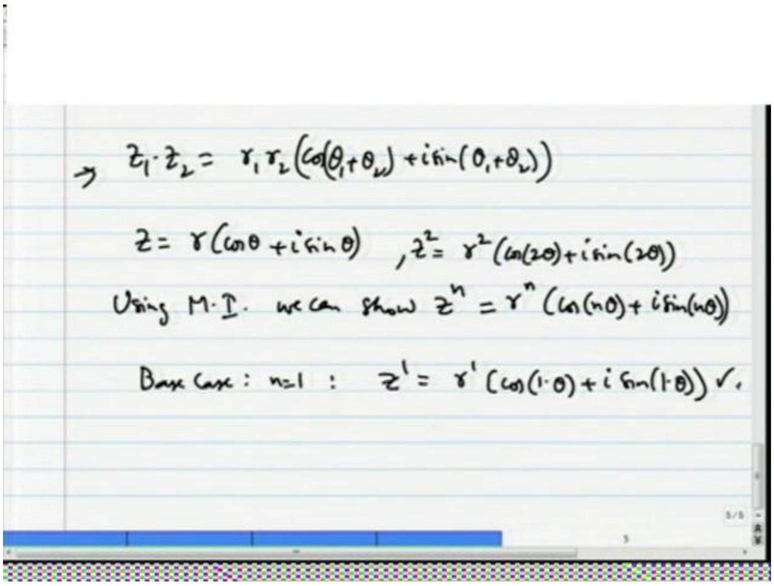

Next, what I want to do is, I want to look further into a multiplication. So, we saw that z 1 times z 2 is equal to r 1 r 2 cosine theta 1 plus theta 2 plus i sine theta 1 plus theta 2, where z 1 is r 1 cosine theta 1 and z 2, sorry r 1 cosine theta 1 plus i sine theta 1 and z 2 is r 2 cosine theta 2 plus i sine theta 2. So, this we saw. So, as a consequence if you consider z equals r cosine theta plus i sine theta, I will drop the subscripts then z squared is equal to r squared cosine theta plus theta.

So, that gives me 2 theta plus i sine 2 theta so likewise we can show using mathematical induction. So, using the principle of mathematical induction just say M I we can show z power n, n is a positive integer is equal to r power n cosine n theta plus i sine n theta. The base case is clear base case, n equals 1 so then z power 1 is equal to r power 1 cosine 1 times theta which is theta plus i times sine 1 times theta which is sine theta, so this is true.

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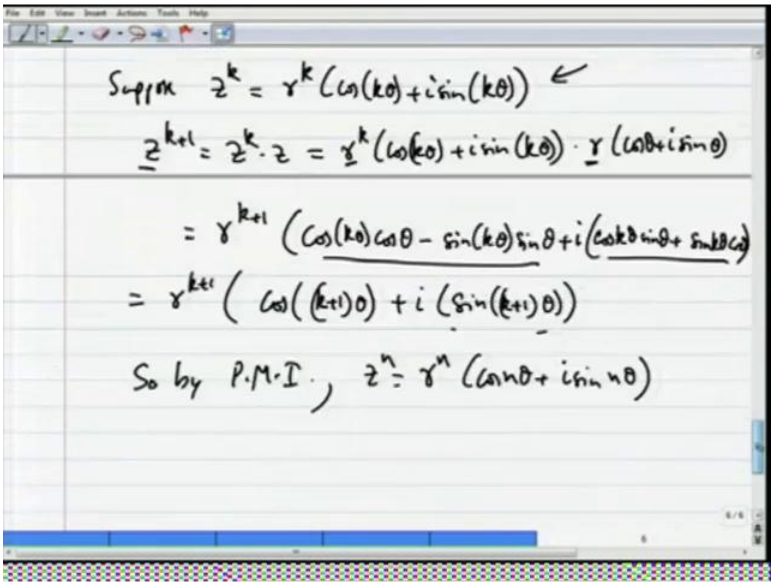

Now, suppose z power k for some positive integer k greater than 1 is equal to r power k cosine k theta plus i sine k theta. Then z power k plus 1 will be z power k times z and then by using this supposition which is the induction hypothesis I can say this is equal to r power k cosine k theta plus i sine k theta times z, z is once again r times cosine theta plus i sine theta. So, this is equal to r power k plus 1, i will club this r power k and r and then multiply cosine k theta plus i sine k theta with cosine theta plus i sine theta. So, you get cosine k theta cosine theta i times i will give you a minus and then sine k theta sine theta. That is the real part plus the imaginary part is i times cosine k theta sine theta plus sine k theta cosine theta.

So, that gives you r power k plus 1 times of course, this is cosine k plus 1 theta cosine k theta plus theta, which is cosine k plus 1 theta plus i times this is sine k theta plus theta, which is sine k plus 1 theta sine k plus 1 times theta. So, indeed you get z power k plus 1 is r power k plus 1 cosine k plus 1 theta plus i sine k plus 1 theta. So, by the principle of mathematical induction so by the principle of mathematical induction z power n is r power n cosine n theta plus i sine n theta. So, this statement is true for for positive integers, for any positive integer n.

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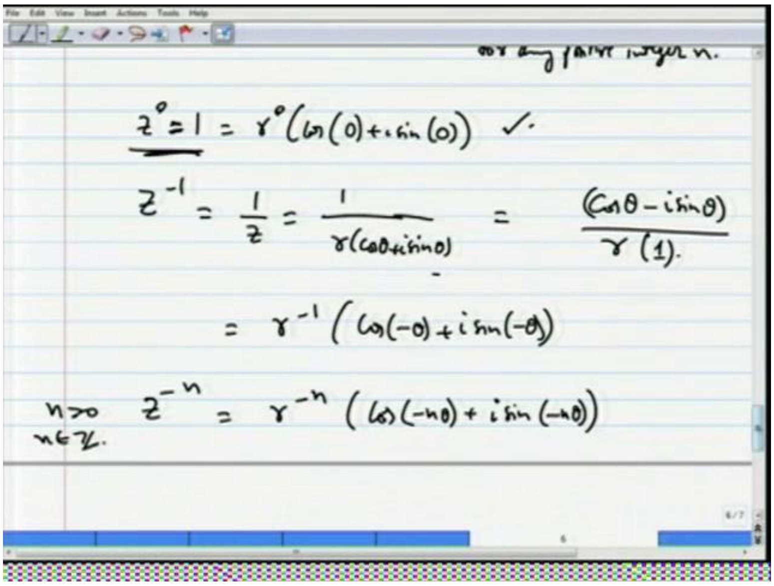

It is also true for negative integers and 0, well 0 by convention, z power 0 by convention is 1 and so that will tally with r power 0 cosine 0 plus i sine 0. So, this is by convention and so this is true for n equal 0. So, this statement that z power n is r power n cosine n theta plus i sine n theta is true for n equals 0. By the way, of course we need that this is z is not 0 otherwise we do not have, an we do not have the argument or an argument for z defined. So, we need all this is true only in the case of z not equal to 0. Likewise, we do not define 0 for 0 it is undefined and z power minus 1 is, is indeed, well it is 1 by z which is 1 by r times cosine theta plus i sine theta and by multiplying and dividing by cosine theta minus i sine theta we get, we get r in the denominator.

So, cosine theta plus i sine theta times cosine theta minus i sine theta gives you cosine squared plus i plus sine squared so that is times 1. So, this can be written as 1 by r which is r power minus 1 times cosine minus theta plus i sine minus theta. So, it is true that z power minus 1 likewise is r power minus 1 times cosine minus 1 theta plus i sine minus 1 times theta. Now, either by using induction or by using what was already proved we can show that z power minus n, n positive integer, z power minus n is r power minus n times cosine minus n theta plus i sine minus n theta. So, you can do this directly by using the fact that already this is true for positive integers, so this is true for positive integers.

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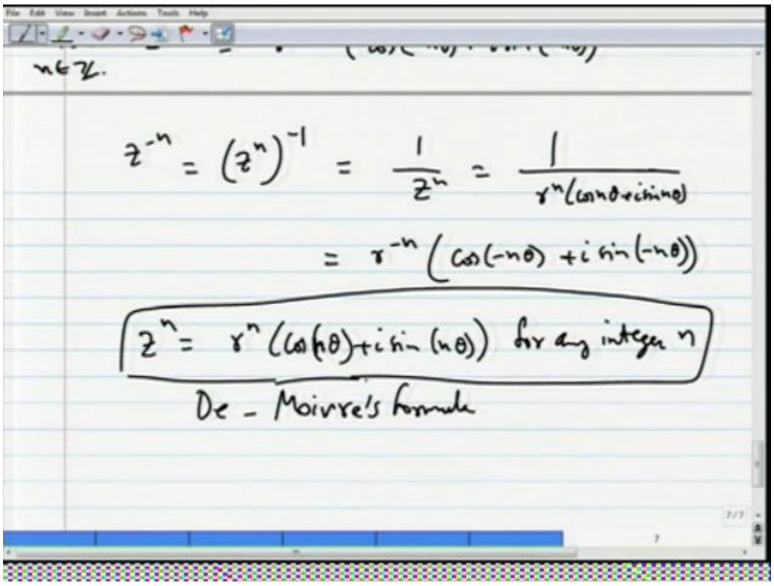

So, z power minus n is z power n power minus 1 and then by using this loss of exponents which will work and then this is equal to 1 by z power n etcetera. So, this is, this gives you r power minus n, I mean if you do what we have done already here you, have to use the De Moivre's formula first or sorry, you have to use the fact that z power n is r power n cosine n theta plus i sine n theta.

This is what we have done already, n is a positive integer and then write this as 1 by r power n which is r power minus n r power minus n and then multiply and divide by cosine n theta minus i sine n theta to to get cosine minus n theta plus i times sine minus n theta. That denominator will give you cosine squared n theta plus i sine plus sine squared n theta which will give you 1. So, z power minus n will also be this. So, it is true that z power n is r power n cosine n theta plus i sine n theta for all integers (( )) for any integer and this is referred to as a De Moivre's formula. So alright.

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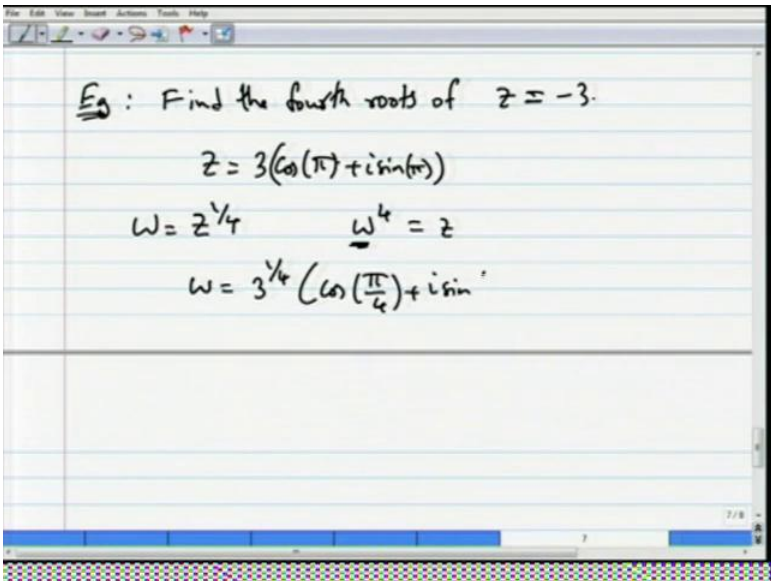

So, De Moivre’s formula can be put to use at least in the following way. Here is an example, so find the fourth roots of z equals minus 3. So, consider this example z equals 3 cosine pi plus i sine pi. So, we can write if we take the argument of z to be pi we can write z like that 3 times cosine pi plus i sine pi. So, De Moivre’s formula suggests that if you want if you want if you say w equals z power 1 by 4 you want the fourth root, w power 4 is equal to z.

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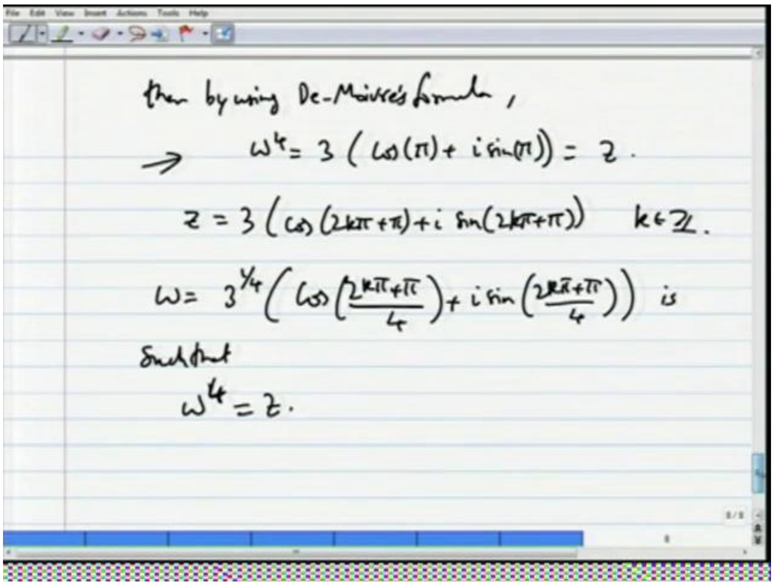

So, w power 4 you imagine w in the place of z here; so if w is, if w is taken to be 3 power 1 by 4 times cosine pi by 4 plus i sine pi by 4, then definitely, then by using De Moivre’s formula we get, we get w power 4 is 3 power 1 by 4 power 4 which is 3 times cosine pi plus i sine pi, which is your minus 3 which is your z. So, definitely this value of w will work, but is this all? Well, the answer is we are missing out on other values of other possible values of w by considering one particular argument for z.

We can do better by considering a general argument for z, so if we consider z to be 3 times cosine 2 k pi plus pi plus i times sin 2 k pi plus pi which it is because sine and cosine are 2 pi periodic, k belongs to a integers. Then by doing the same what we can say is that w equals 3 power 1 by 4 times cosine 2 k pi plus pi now divided by 4 plus i times sin 2 k pi plus pi divided by 4 is such that w power 4 now gives you z it it gives you the z. So then, now the question is how many different values does w give? Notice that now this 2 k pi has been divided by 4 so you get for example, when k is, k is 1 you get 2 pi by 4 pi by 2 plus pi by 4.

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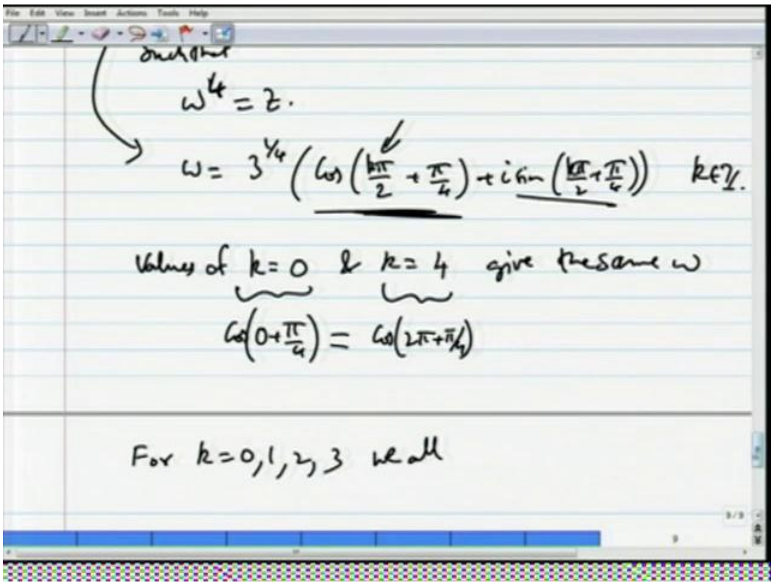

So, for for k equals 0,1,2 and 3 we notice that we get different values of w. So, let me be slower, so maybe I will say w is 3. Now, 3 by, 3 power 1 by 4 times cosine k pi by 4 plus pi by 4 sorry k pi by 2 plus pi by 4 plus i sine k pi by 2 plus pi by 4 where k belongs to integers. Now, notice that if values of k equals 0 and k equals let us say 4 give the same w, that is because, when you substitute 4 in there you get 4 pi by 2 which is 2 pi, and then I mean, k pi by 2 is equal to 4 pi by 2 which is 2 pi and cosine and sine are 2 pi periodic. So, cosine k pi by 2 plus pi by 4 will be cosine 2 pi plus pi by 4 which will once again give you cosine pi by 4. So, k equals 0 gives you the this part gives you 0 plus pi by 4 and then for k equals 4 you get cosine 2 pi plus pi by 4 etcetera and these are equal.

Likewise, sine gives you same values when k equals 0 and k equals 4, so you get the same w as a result. So, for k equals 0, 1, 2 and 3 you get all the distinct values because k equals 1 and k equals 5, k equals 2 and k equals 6 etcetera they will all give you the same values for w.

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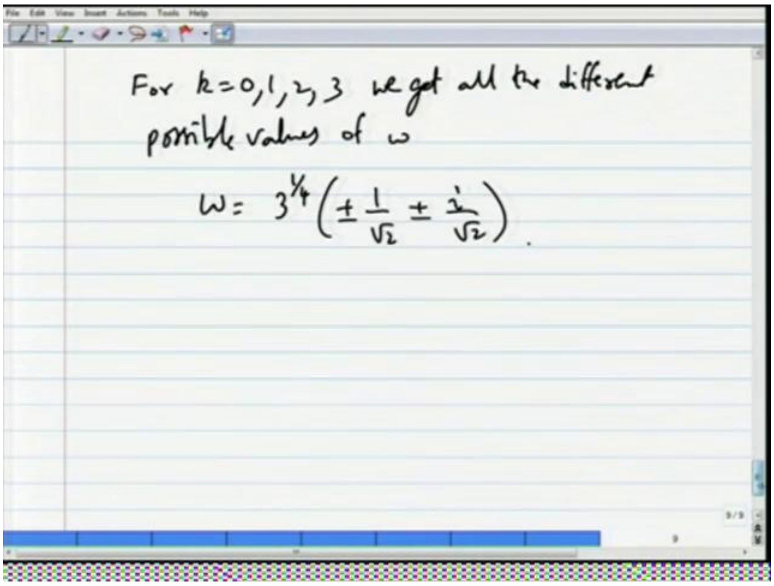

So, pair wise for k equal to 0, 1, 2, 3 we get we get all the possible values, all the different possible values of w. So, when we work it out, well when we substitute and see what the values are we get w equals let us say 3 power 1 by 4 times plus or minus 1 by root 2 plus or minus i by root 2. So, all the possible combinations 1 by root 2 minus i by root 2 or 1 by root 2 plus i by root 2 etcetera. So, minus 1 by root 2 plus i by root 2 minus 1 by root 2 minus i by root 2 so these are different possible values of a w which give you w power 4 is equal to z minus 3, so those are the fourth roots of minus 3. So, De Moivre's formula can be used to solve this example or to find nth roots of a certain complex number in general.

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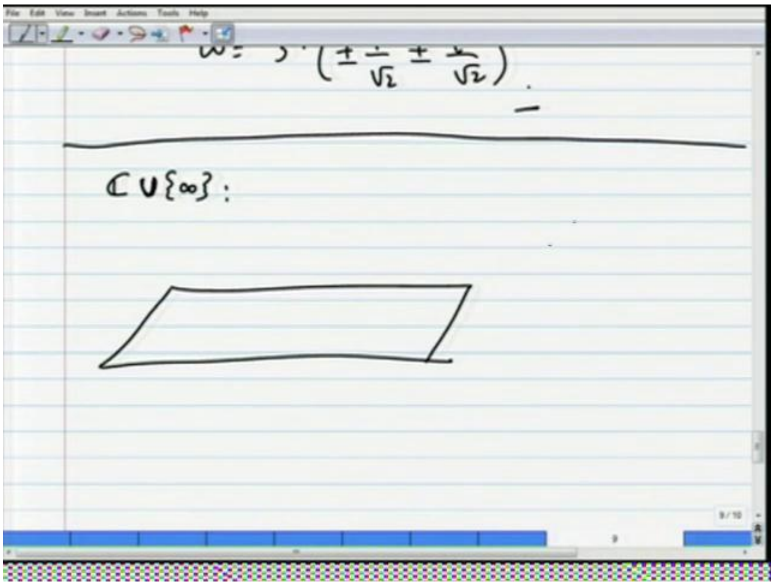

So, next we are going to meet the notion of the complex plane with the point at infinity. So far we have this complex plane and then what we can do is we can add 1 point at infinity and this is to discuss, this will be useful this gadget will be useful to discuss continuity of certain functions as these functions tend to 0 tend to infinity. So, and in the discussion for Mobius transformation for example, so we are we are going to give this c union infinity, this set c union infinity a concrete picture. So, what we are going to do is we will consider the complex plane as contained in r 3.

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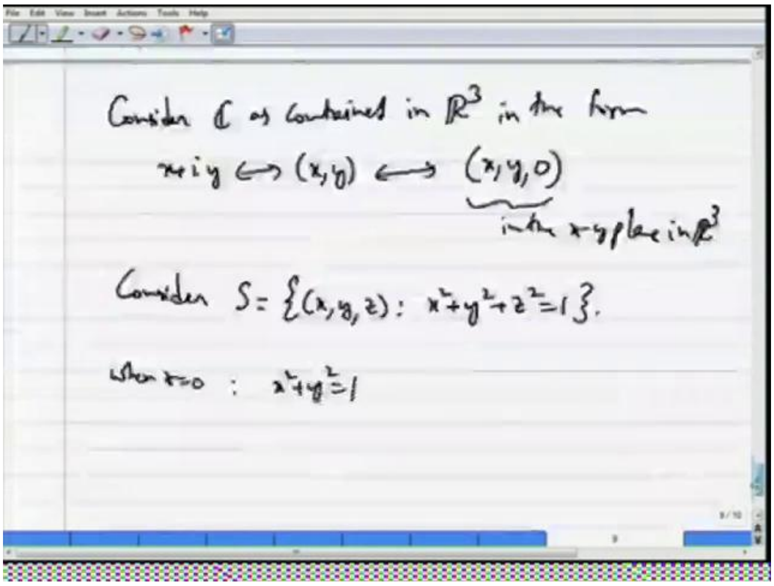

So, consider ,I will write here, so consider C as contained in r 3, triples of real numbers in the form x plus i y or which is now in the plane form it is x comma y. It is contained in r 3 in the form x comma y comma 0, this is a point in the x y plane in r 3.

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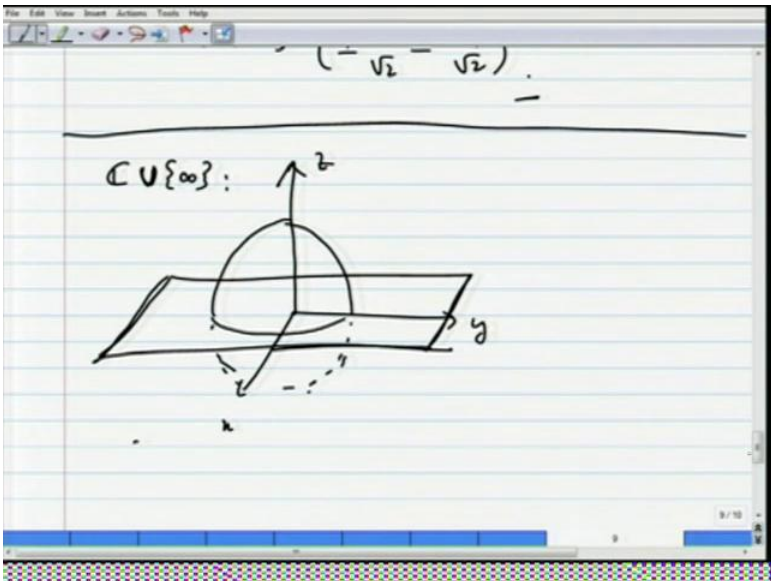

So, here is the plane x y plane in r 3 and we have this z axis coming out, so this is the x axis y axis. So, what you can do is now you consider the unit sphere, so so consider the unit sphere S equals set of all x y z such that x squared plus y squared plus z squared is equal to 1. So, this set intersects the x y plane in the unit circle, so when z equals 0 this equation is just x squared plus y squared is equal to 1. So, that is the unit circle in the complex plane.

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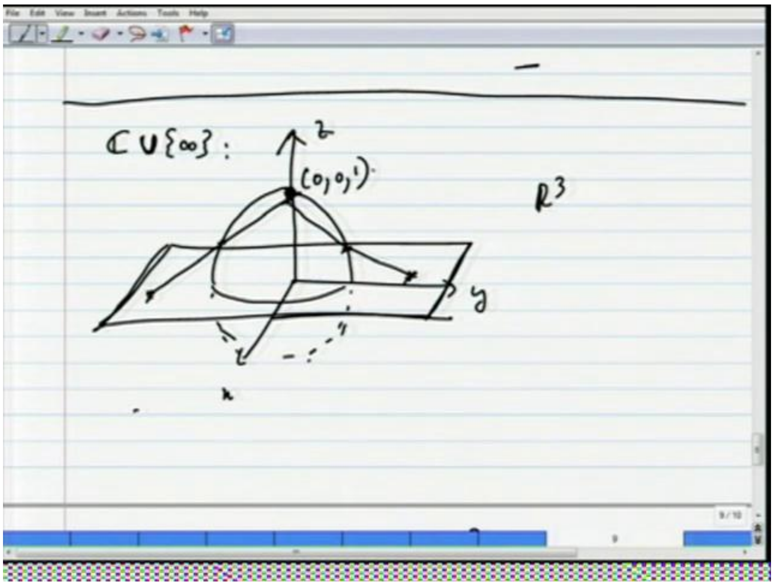

And then complex plane contained in r 3, now what you can do is you can you can take any point in the complex plane here which is now sitting in r 3 and join it to the north pole of this sphere by a straight line and that straight line hit is the unit sphere S at some point. You notice that if you take any complex number on the x y plane corresponding to it, there is a unique point on this a sphere. By joining, which is obtained by joining the number on the complex plane to this point at this north pole which is the point 0, 0, 1.

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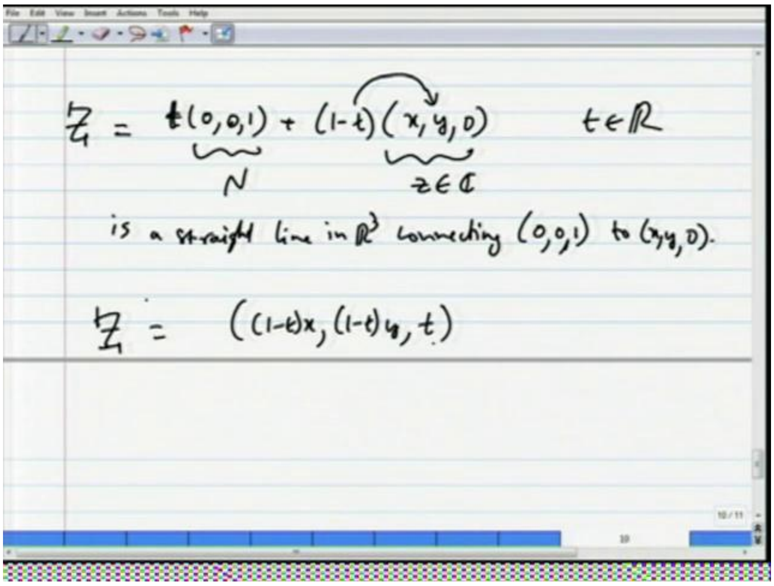

So, we can we can find out what that point of intersection of that line is, so to do that you consider t times 0,0,1 plus 1 minus t times x,y,0. So, this is the point z in the complex plane and then this is the point which is the north pole the north pole and so this this t times 0,0,1 plus 1 minus t times x,y,0 when t is allowed to be any real number gives you a straight line connecting 0,0,1 and x,y,0. We know how to construct equations of lines in r 3, so t times a point plus 1 minus t times another point will give you a line passing through these 2 points. So, so this is a point in r 3, let us call that capital z. So, capital z equals this is a straight line in r 3 connecting 0,0,1 to a complex number to the complex number x,y,0.

Now the complex numbers are on the x y plane, now the point of intersection of this line we can calculate with the sphere is obtained by well S is set of all points I already wrote it here, so S is the set of all points such that x squared plus y squared plus z squared is equal to 1. Let us look at the x y z coordinates of this capital Z capital Z. Let me rewrite that as, it is t 1 minus t times x so that will give me x, sorry t times 0. So, that gives me 1 minus t times x and then t times 0 times 1 plus 1 minus t times y that gives me 1 minus t times y likewise and t times 1 plus 1 minus t times 0 gives me t.

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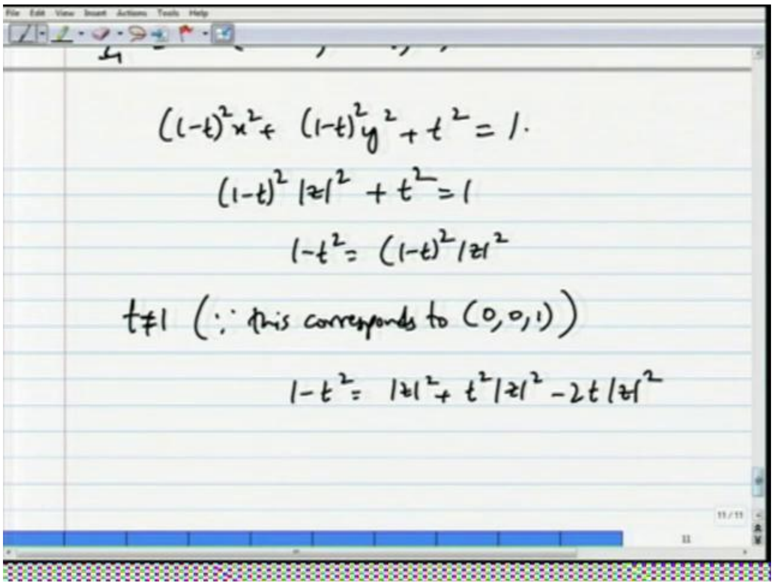

Now that is your point z and when we put it on the unit sphere, we get 1 minus t squared x squared plus 1 minus t squared y squared plus t squared is equal to 1, but x squared y squared is the modulus of a complex number z. So, if we call this number as z belongs to

C we already called this z belongs to C, so this is 1 minus t squared modulus of z squared plus t squared is equal to 1. Now, let us solve for t, so we get 1 minus t squared is equal to 1 minus t whole squared modulus of z squared. So, t from here, we can solve for t. Well notice that t does not equal 1 because when t is equal to 1 when you substitute t equals 1 in here you get 0, 0, 1.

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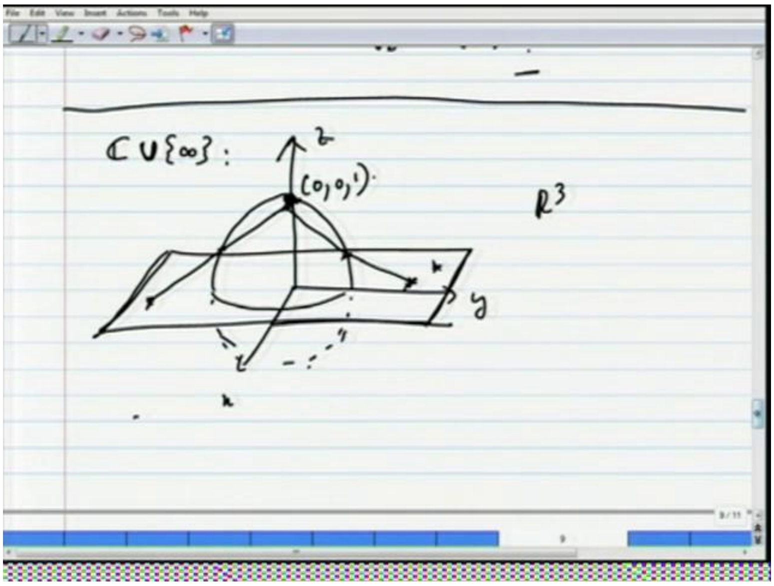

So, there is no, well there is no complex number which intersects this sphere in this point 0,0,1 itself. After all you are, you are connecting the complex number to this point 0,0,1 so that will not intersect this sphere in 0, 0, 1 itself. So, t can t will not be 1 because since this corresponds to the point 0, 0, 1, so you can divide by 1 minus t squared for example, and then if we solve this for t we can solve this for t, well let us do it. So, 1 minus t squared is modulus of z squared plus t square modulus of z squared minus 2 t modulus of z squared.

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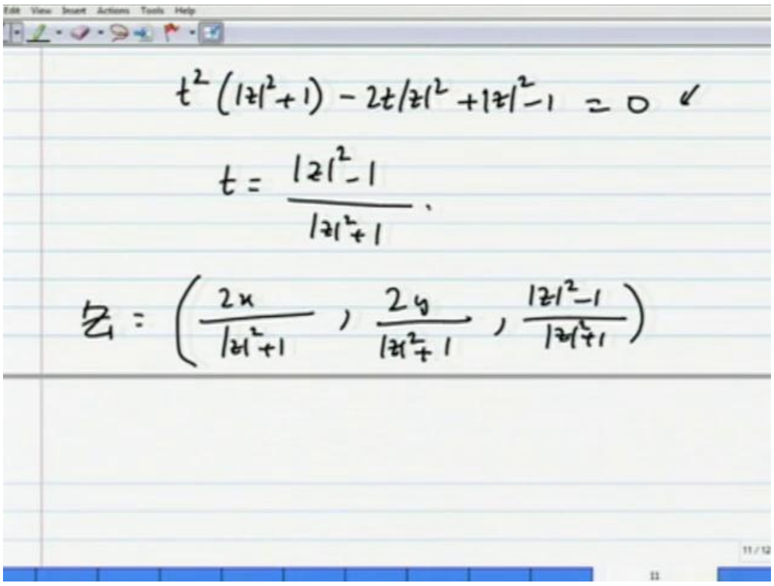

So, this gives you t squared times modulus of z squared plus 1 plus minus 2 t times modulus of z squared plus modulus of z squared minus 1 is equal to 0 and so excuse me so t then is by using the quadratic formula what we can get is t is modulus of z squared minus 1 by modulus of z squared plus 1. So, substituting t in this point capital z we get, we get the following, capital Z is equal to 2 x by modulus of z squared plus 1 comma 2 y by modulus of z squared plus 1 comma modulus of z squared minus 1 by modulus of z squared plus 1.

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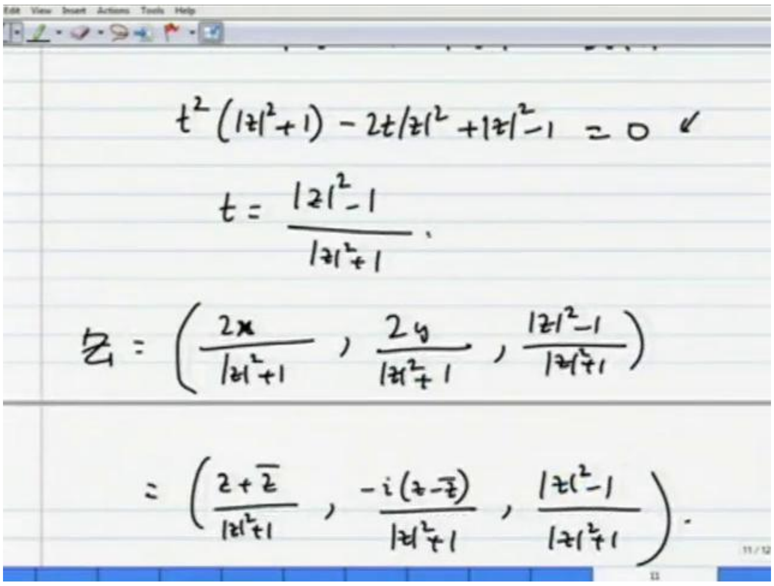

And if you want to write this completely in terms of the complex number z we will remove the appearance of x, so this is z plus z bar by modulus of z squared plus 1. So, that represents the conjugate (( )) z bar represents the conjugate of z, then minus i times z minus z bar by modulus of z squared plus 1. And then this is modulus of z squared minus 1 by modulus of z squared plus 1, so that is your point of intersection of the line joining the north pole and point z on the complex plane with the unit sphere.

So, that is the point z and then so if you if you wish to find if we are given a point, the opposite is if you are given a point capital Z on on the sphere on S then then we can find the point z on the complex plane, then z on C can be found recall what we are doing is we are drawing a straight line connecting 0,0,1 and the point capital Z and that gives you and when you project it on to the complex plane it gives you a point of intersection of the complex plane and we can calculate that, so that can be found.

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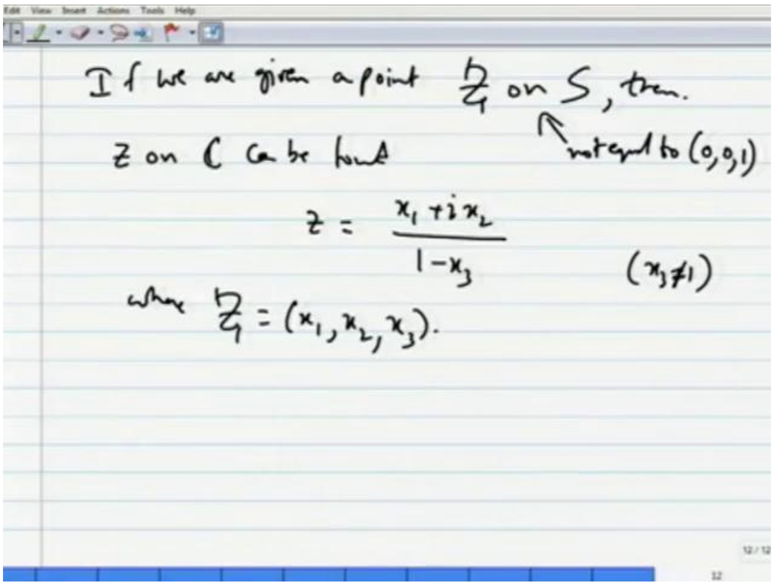

We get z equals x 1 plus i x 2 by 1 minus x 3 where capital Z is assumed to be x 1 x 2 x 3. Once again x 3 is is not 1, x 3 is not 1 because x 3 equals 1 will correspond to the north pole. So, I should say that if we are given a point Z on S not equal to 0, 0, 1, then we can find z on the complex plane. So, by this kind of association corresponding to each point on the sphere there is a point on the complex plane and corresponding to each point on the complex plane there is a point on the sphere. So, there is a 1 to 1 correspondence between points on the sphere and the complex plane, we can we actually have the equations for them.

Then these points, the one to one correspondence is given by drawing geometrically drawing a straight line passing through the north pole and the point on the, on the complex plane or in the other direction by the by joining the north pole and any other point on the, on the sphere on the sphere, which will be projected on to the complex plane to get the complex number.

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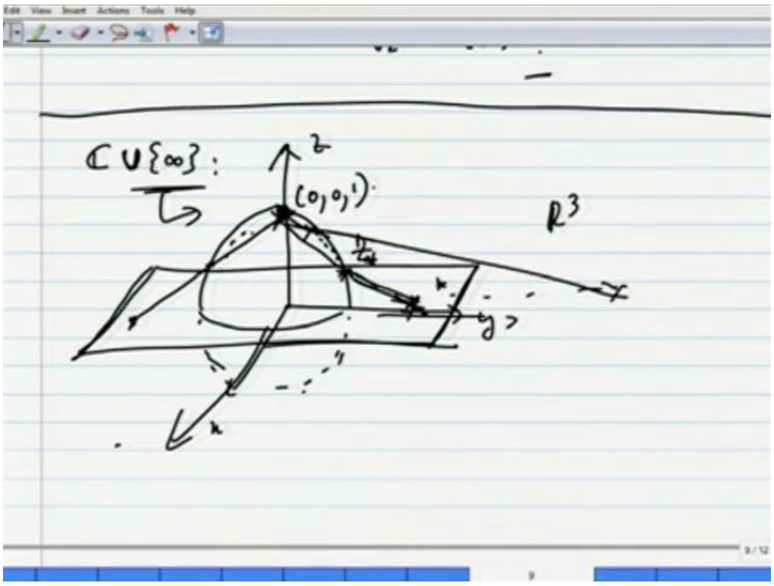

So, in this way there is a 1 to 1 correspondence, but notice here that we have accounted for all the points on the sphere except the north pole and and let us look at the picture once again and as you move farther away in the complex plane your moving higher up on the sphere. The point of intersection of the line joining that farther away point on the complex plane and the north pole will be higher up here. So and so you keep approaching the north pole, but you never reach the north pole by these lines which are intersecting the sphere. So, in that sense when you move faraway your getting closer to the north pole in that sense the north pole represents the point at infinity. So, you call you you you think of the point at infinity has the north pole.

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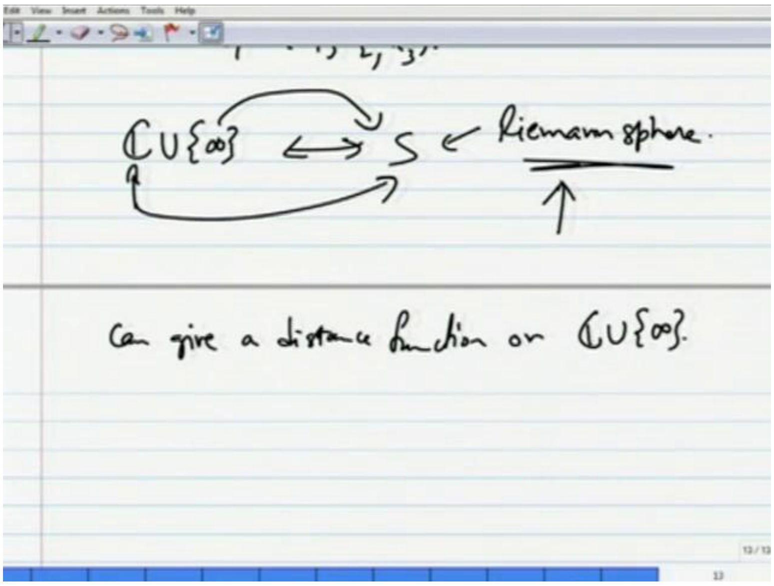

And then so that is the correspondence of points in C union infinity with the points in S, points in C go to points in S which are anything but the north pole by that straight line connecting north pole and the complex number and the point infinity it iself goes to the north pole. So, that is the correspondence and this sphere is often called the Riemann sphere when we put some distance also on this, so we can put a distance on a can give a distance function on C union infinity.

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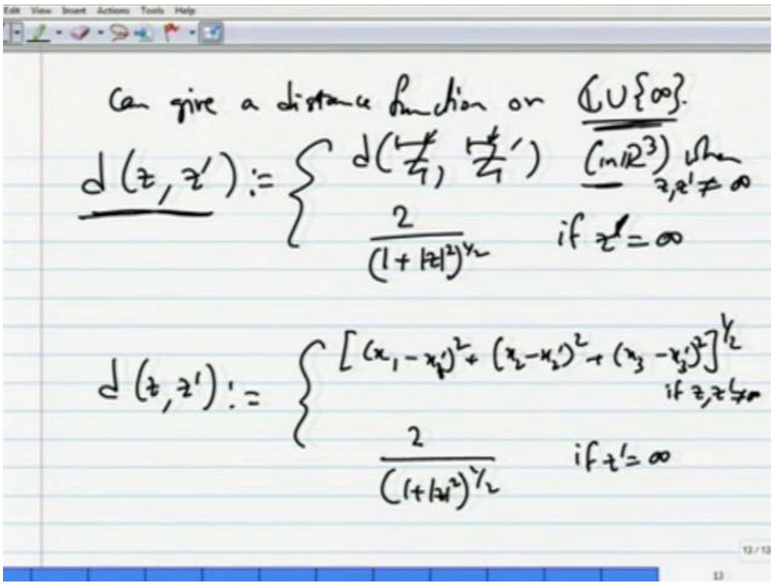

We can find the distance between any 2 numbers z and z prime in C union infinity, so this is defined as define this as the distance between, so actually it is a split definition so define this to be the distance between the corresponding point z and z prime on the sphere in r 3, distance in r 3. When z and z prime not equal to infinity, we are defining distance function for C union infinity, so forget infinity for the time being if you take any 2 complex numbers. The the new distance on the in C union infinity is defined to be the distance between the corresponding points z and z prime in r 3.

And then if you have infinity in picture so then you can also define distance what you do is define this to be 2 by 1 plus modulus of z squared power half if z prime is equal to infinity. If z prime is infinity, then this is nothing and if z is infinity of course, you have 2 by 1 plus z prime modulus of z prime squared power half is raised to power half. So, let me rewrite this as the distance between z and z prime is x 1. So, in terms of x 1 x 2 x 3 representing z and x 1 prime x 2 prime x 3 prime representing z 1 z prime we have x 1minus x 1 prime squared plus x 2 minus x 2 prime squared. Of course, that is the distance in r 3 plus x 3 minus x 3 prime squared power half.

And then this is equal to 2 by 1 plus mod z squared power half if z prime is equal to infinity, if z or z prime not equal to infinity. So, that way we can measure the distance between any 2 points the the only new point is infinity, but actually we we measure distances using the distance in r 3.

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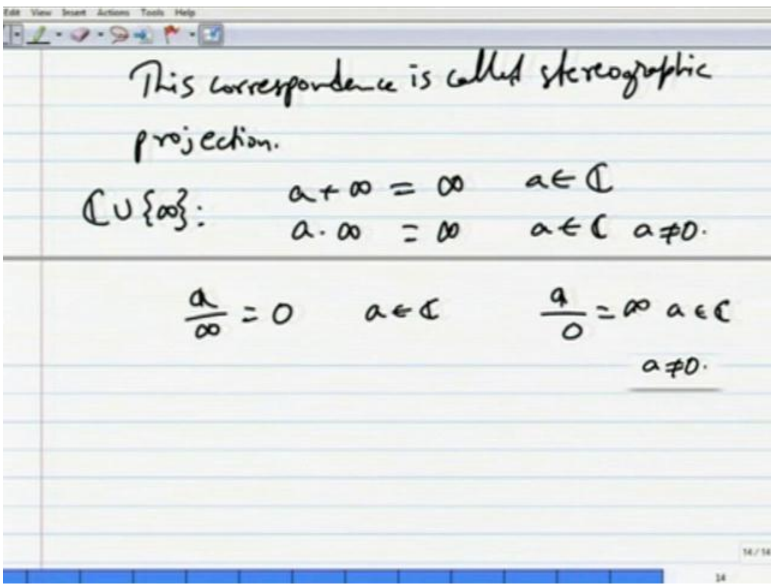

So, this correspondence, this, this structure on C union infinity is often called the Riemann sphere structure, and then correspondence along with this distance is called stereographic projection. So, using the stereographic projection, we, we have this point at infinity being added to this, this complex plane. And then in C union infinity we have arithmetic as well. So, a plus infinity is equal to infinity for a belongs to C and then a times infinity is infinity a belongs to C a by a not equal to 0; we do not define 0 times a infinity. Then 1 by infinity is equal to 0 and then or a by infinity is equal to 0 a belongs to c, and then 1 by 0 or a by 0 a belongs to C is infinity.

So, we have this additional arithmetic, and then so this is a discussion about the Riemann sphere. So, we have added a point at infinity in addition to the complex numbers, and then there is a distance measuring gadget as well. So, we will put this to use when we study a Mobius transformations.