Complex Analysis
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Module - 4
Further Properties of Analytic Functions
Lecture - 5
Counting the Zeroes of Analytic Functions

Hello viewers, in the last session, we have proved the identity theorem, which says that two functions f and g, analytic function f and g, which are defined on a region, if they agree on a set containing a limit point, then they have to be identically equal. So, that was the identity theorem. So, today we are going to see some consequences of the identity theorem, and see some applications of these consequences. So, first in order is the uniqueness theorem.

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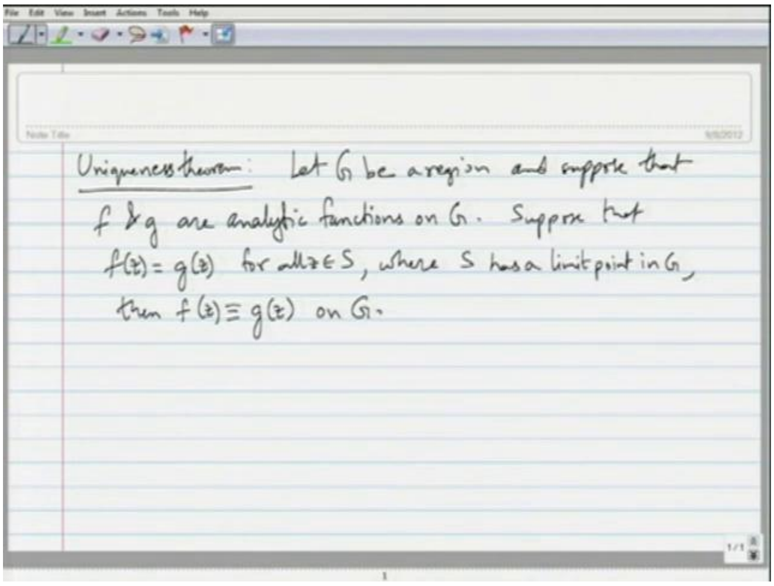

So, we can state the identity theorem in different format called the uniqueness theorem. So, it says that let G be a region and suppose that f and g are analytic functions on G. Suppose, further suppose that f of z is equal to g of z for all z belong to a certain set S where S has a limit point in G then the f of z is identically equal to g of z on G. So, f of z is identically equal to little g on capital G. So, this is nothing but the identity theorem in disguise. We are applying the identity theorem to the analytic function f minus g if f and g are analytic, f minus g is analytic. So, we are just applying identity theorem on that function.

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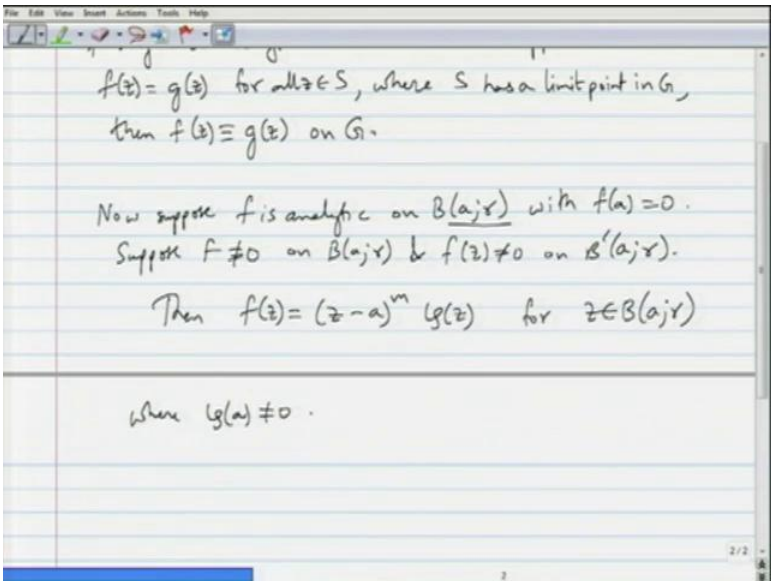

So, there is nothing new here the uniqueness theorem. So, then now we are going to further analyze the zeroes of the analytic function and note some consequences of the identity theorem. So, here now we can suppose f is analytic on B a, r on a ball of radius r with, with f having a 0 at the centre a. Also assume that, and suppose that suppose f is not identically 0 on B a, r which means the zero of f at a isolated. So, we are sought of assuming that f is non-zero on the whole, whole disk B a, r. So, without loss of generality you can contract r contract r to such a number positive real number such that f has no other 0 in B a, r.

So, let us assume that f is and f of z is not equal to 0 on B prime of a, r. We can always assume that as long as f is not identically 0. The zeroes of f are isolated. So, let us now notice that then f of z, we saw can be written as z minus a power m times phi of z, where, where firstly this is valid for for z belongs to B a, r all of B a, r, where phi of a is not equal to 0. Here so f has a zero of order m at a. So, z minus a power m we can factor out we found out. And then the the remaining power series the the when factor out z minus a power m where m is order of 0 the remaining power series is an analytic function which we called as phi of z. So, we saw this form earlier.

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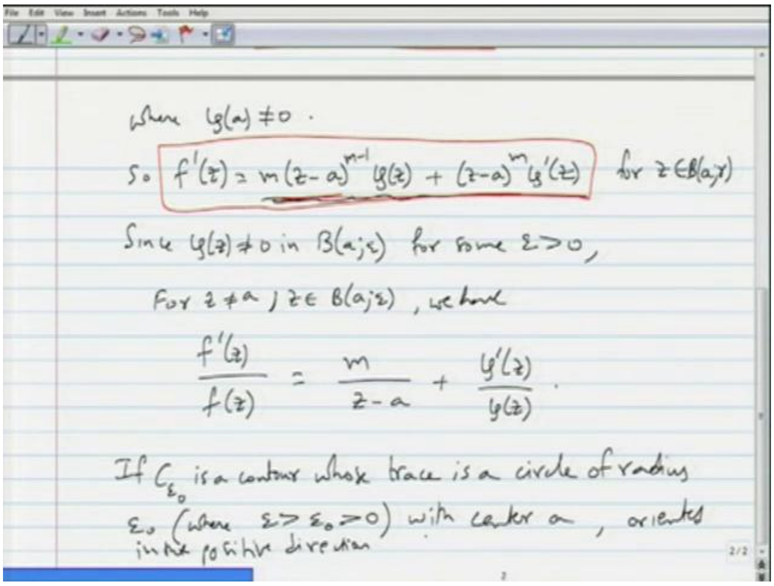

And then we also know that the leading coefficient there c m is non zero. So, we are calling we are saying that phi of a is non zero. And then what we can say further that so f prime of z we can have a form for f prime of z. This is m times z minus a power m minus 1 times phi of z plus z minus a power m times phi prime of z and this is valid for z belongs to B a, r. We are using the product rule. And then since, since phi is not identically 0. We can also say that phi of z is not equal to 0 in B a, epsilon. Now, I have contracted the disk further phi of z is not to be 0 for some epsilon positive. This we know is possible by continuity of the function phi of z.

This saw in last session any way. So, for z not equal to a and z belonging to B a, epsilon we have, what do we have? We can divide f prime by f and notice something. Then we divide f prime by f, what happens is here is an expression for a f prime, so I may be showed you some other colour. So, here is as expression for f prime and the first term the first term in that expression when I divide that by f of z whose expression is above. What happens is the phi of z cancels phi of z is non zero, phi of z cancels and then m z minus a power m minus 1 cancels z minus a power m to give me m by z minus a.

Likewise the second factor cancels with z z minus a power m z minus a power m cancels z minus a power m to give me phi prime of z by phi of z. So, for z not equal to a we have this expression. In particular if C epsilon naught is a contour whose trace is a circle of radius epsilon naught where epsilon greater epsilon naught greater than 0. So epsilon is between epsilon naught is in between epsilon and 0. And with centre so this is circle of radius epsilon naught with, with centre a oriented in the positive direction.

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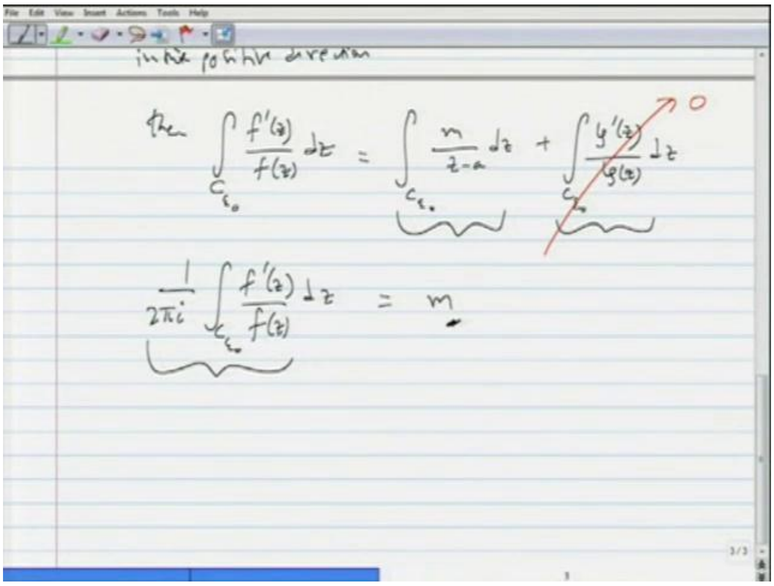

Then we know that the integral of the left hand side, integral of left hand side on C epsilon naught f prime of z by f of z the contour integral of that on C epsilon naught exist for C. Because f is non zero on any point on this on the trace of C epsilon naught. And this is equal to the integration the contour integration of m by z minus a d z plus the contour integration of phi prime of z by phi of z on C epsilon naught. Notice that the first integrand on the right hand side is the fundamental integral. It is a multiple of the fundamental integral 1 by z minus a. And the second integral is the integral I forgot a d z there.

Integral of of an analytic function, notice phi of z is never 0 on B a, epsilon. So, in particular it is non zero on a inside this contour C epsilon naught. So, and also phi prime is analytic since phi is an analytic. So, phi prime by phi analytic on on an inside C epsilon naught; so the second integral vanishes. So, this is 0 and then what we have is what we have is integral C epsilon naught f prime of z by f of z d z. Or more trace if I divide this by 2 pi i what I get on the right hand side is simply m. So, the order of this 0 is captured by this kind of integral which sometimes called logarithmic integral. So, the integral of f prime by f modular constant may be 1 by 2 pi i gives me the number of zeroes of f inside this little disk or you know counting its multiplicity.

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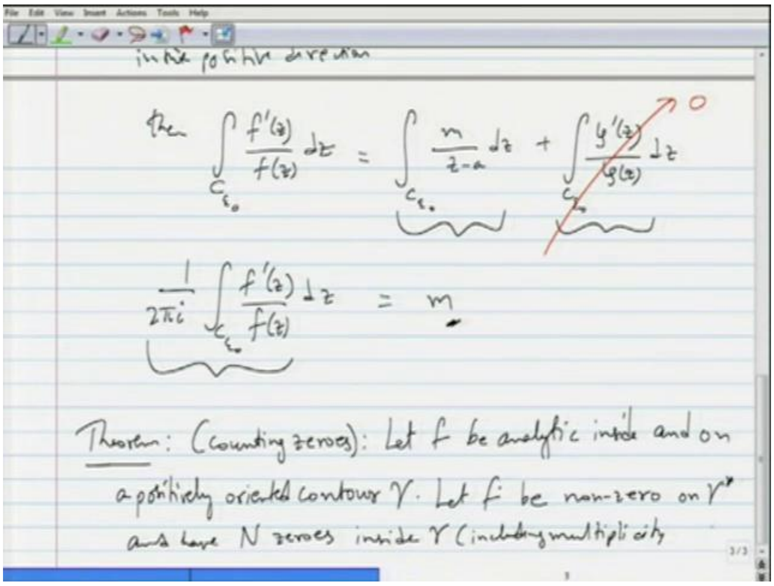

So, the m is actually multiplicity of the 0, so we can call them m zeroes at a. So, likewise we generalize this if the function, if, if the function is analytic on a certain region or on a certain disk and if it has you know more than few zeroes inside, inside or may be none of them. Then a certain integral will actually capture the number of zeroes of f inside that contour. So, here is the more general form. So, here is theorem more generally counting zeroes. Let f be analytic inside and on a positively oriented contour gamma. Let, so that is the simple closed contour. So, let f be non zero let f be non zero on gamma on the trace of gamma gamma star and have capital N number of zeroes inside gamma inside gamma including multiplicity of zeroes.

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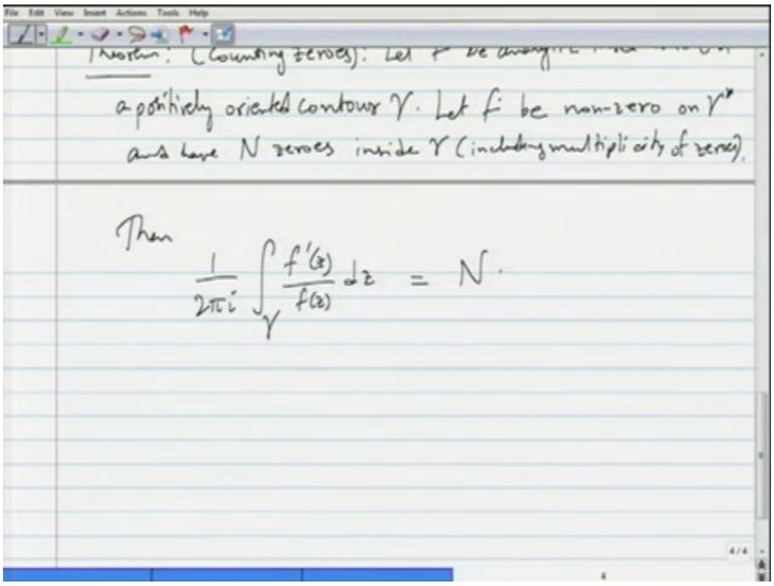

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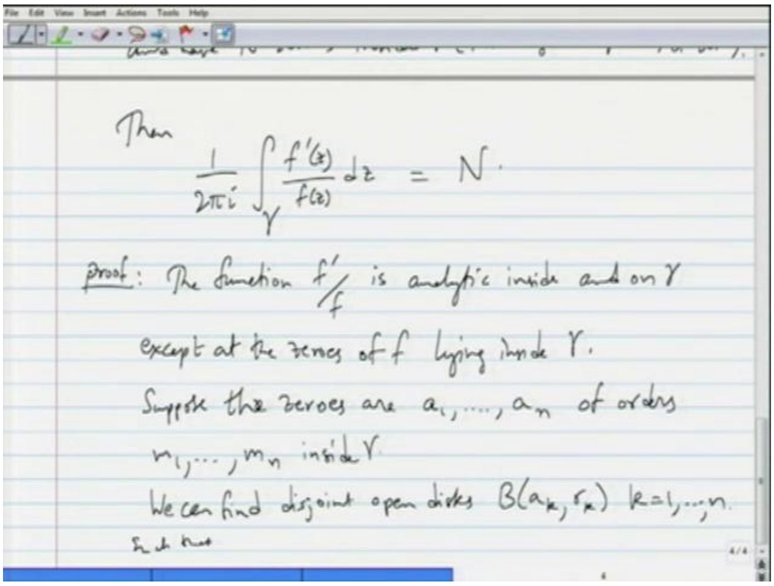

Then the conclusion is then this is integral 1 by 2 pi i times integration the contour integration on gamma of f prime of z by f of z d z is actually equal to we capture this number capital N; so what is the proof? Here is the proof. The function f prime by f is analytic inside and on gamma except at the zeroes of f lying inside gamma, except at those points analytic everywhere else inside this on an inside gamma. So, suppose there are zeroes, suppose the zeroes are a 1 so on until a p or a p. So, there are n zeroes, let me say a little n of orders m 1 through m n. So, there are a 1 through a n zeroes of orders m 1 through m n inside gamma.

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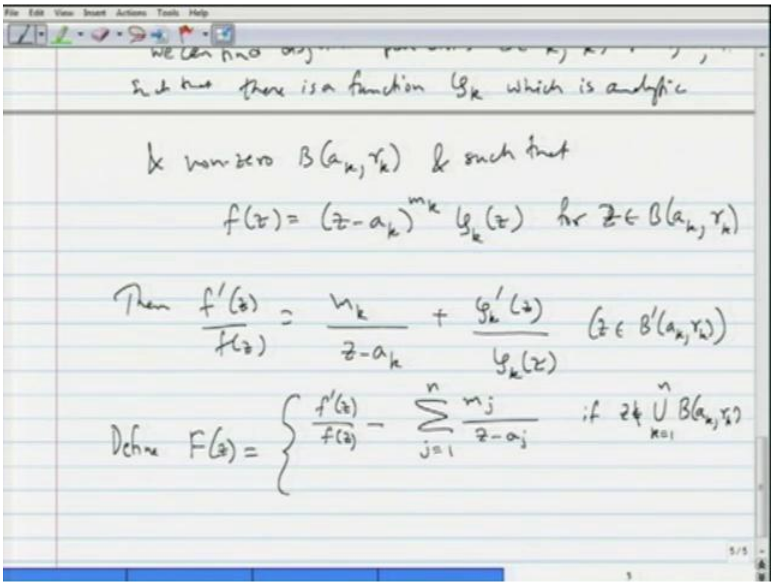

So, we are assuming that f has no zeroes on the trace of gamma itself. So, we can find since the gamma is an open set. We can find disjoint open disks B a k, r k, k equals 1 through n such that there is a function phi k for this each of k's which is analytic and non zero on this set B a k, r k. So, essentially I am just redoing what I have done here allow me to go back. So, I am just redoing what I have done here or here. So, I am considering this function phi for each of those zeroes. I am considering this kind of factorization and I am considering that function phi. I am call indexing them by k. So, there is a function phi k which is analytic and non zero in B a k, r k and such that f of z is z minus a k over m k times phi k of z for z belong to z belonging to B a k, r k.

So, this is true locally or in a small neighbourhood of these zeroes a k. So, this is essentially what we have done earlier. So, we are locating small disks in which we can find this functions phi k. Now, that trick is to actually join all these together. So, what we will do is we will define. So, then firstly on the disk itself on the little disk itself f prime z by f of z like we have done earlier is m k divided by z minus a k plus phi k prime of z divided by phi k of z z belongs to B prime of a k, r k. So, this is true locally. Now, the trick is to define the function capital F of z. We define capital F of z to be f prime of z by f of z minus the summation j equals 1 through n of m j by z minus a j.

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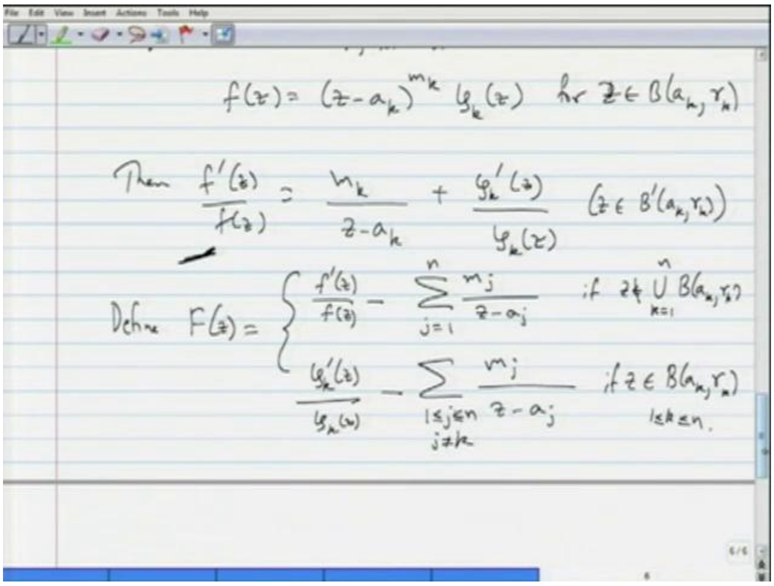

And if this is for z not belonging to any of these disks, so outside the union of these little disks that we found union of B a k, r k k equals 1 through n. So, outside of this disks we define capital F to be f prime by f minus the summation of m j by z minus a j. And we define this to be phi k prime by phi k of z minus sigma j equals 1 or I should write 1 less than or equal to j less than or equal to little n, j not equal to k. So, we are when j equals k we are observing the m j minus z m j divided by z minus a j into here. Notice that f prime by f minus m k by z minus a k from this expression here will give you phi k prime by phi k.

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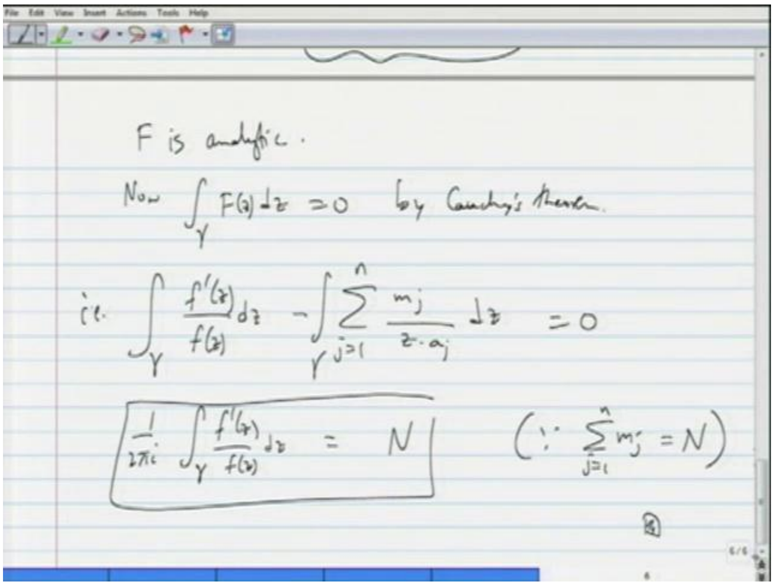

So, phi k prime by phi k minus the sum of the remaining m j by z minus a j if z belongs to B a k, r k, k equals 1 through n. By doing so, we are making capital F, the definition of capital F continuous on the boundary of this B a k, r k this disks B a k, r k. So, the definition match up and and hence this function capital F you could say well by identity theorem capital F is analytic. The definition of capital F you know on the boundary of any B a k, r k agrees with this function here, this function here. And inside B a k, r k this is analytic and outside of the union of B a k, r k this function this other portion of capital F is analytic.

So, F is analytic F is analytic. So, and then the now conclusion follows by now integral f. What I should say is now, integral of gamma of f of z d z is 0 by Cauchy's theorem. Since we are out you know outside of all these disks, capital F has this definition integral over gamma i e integral over gamma of f prime of z divided by f of z d z minus you know sigma m j. So, the integral j equals 1 through n of m j by z minus a j d z is equal to 0, the integration on gamma. So, so integration over gamma f prime of z by f of z d z is equal to the summation of, so if I divide everything by 1 by 2 pi i. So, what I get is this summation m j. So, summation m j is nothing but your capital N since summation j equals 1 trough n.

When you add the orders of zeroes of all the zeroes f inside gamma we agreed that is the number capital N. So, this is equal to capital N. so, that is completes the proof of this theorem. So, so that is way to count the zeroes of f inside a contour gamma. We are going to put this counting zeroes theorem to use and prove the following important result and useful result Rouche's theorem.

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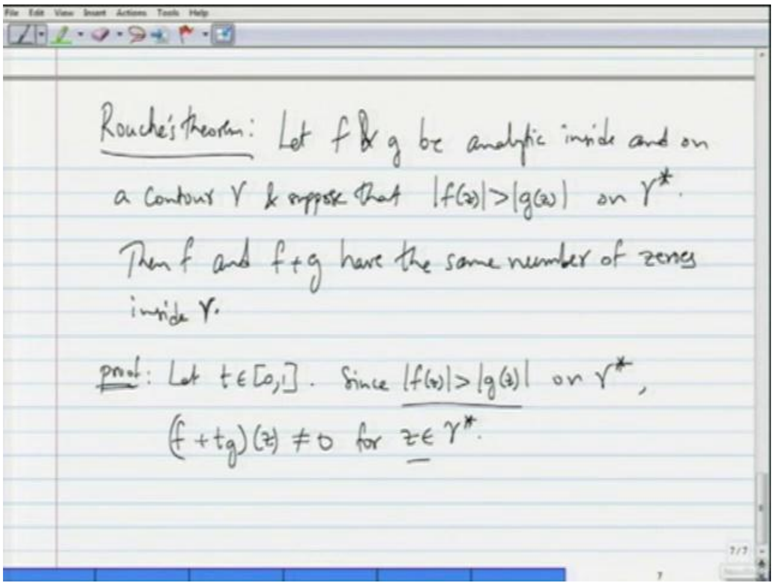

Let f here is the statement, let f and g be analytic inside and on a contour gamma simple close contour gamma. Suppose that modulus of f of z is strictly greater than modulus of g of z on on the trace of gamma. Then f and f plus g have the same number of zeroes inside gamma. Once again note that here we count zeroes including multiplicity. So, when we including multiplicity f and f plus g will have the same number of zeroes. So, let t belong to 0, 1. So, for any t in this unit interval, since modulus of f of z strictly greater then modulus of g of z on gamma star what we can do is we we can say f plus t times g of z is not equal to 0 for g belongs to gamma star. This follows from a certain kind of triangle inequality.

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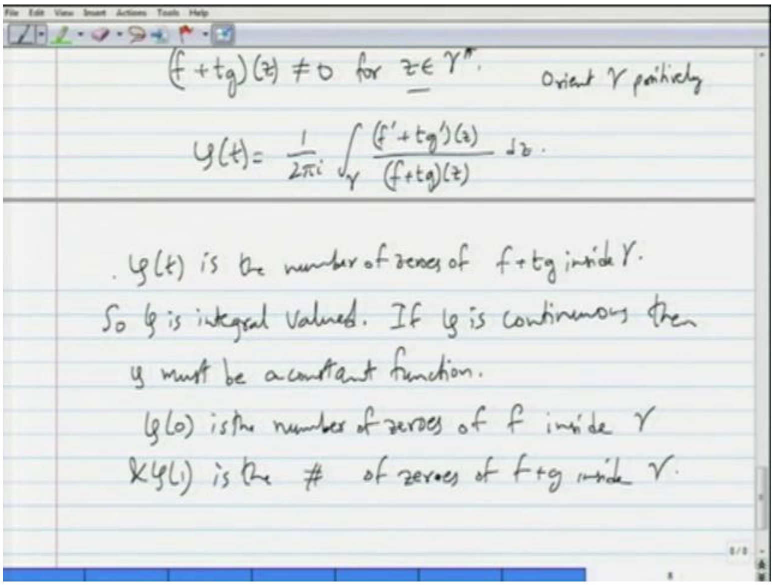

Well, if so from this modulus we can say that modulus of f plus t g is never equal to 0. So, f plus t g itself is never equal to 0 on z belongs to gamma star. So, that is easy to see. So, then the function we will define a function. Phi of t is equal to 1 by 2 pi i. Well, we will orient gamma positively. So, orient gamma positively. It does not hurt if it is oriented otherwise, but nevertheless we can orient gamma positively. Phi of t is 1 by 2 pi i integration over gamma f prime plus t g prime of z. So, it is the derivatives of f plus t g divided by f plus t g of z d z. This we know counts the zeroes of the function f plus t g inside the contour gamma.

So, what we are going to do is we are going to claim that phi is a continuous function, but before that notice the following. So, note phi of t firstly is the number of zeroes of number of zeroes of f plus t g inside gamma. So, phi is integral value number of zeroes. So, it has to be integral value if phi is continuous. If we show that this is continuous then the phi must be a constant function. And what is of interest is the value of phi of at 0 phi of 0 is the I mean t is equal to 0 gives f prime by f as the integrand. Phi of 0 is the number of zeroes of f inside, inside gamma and phi of 1 is t equals 1 gives us f prime plus g prime by f plus g in the integrand here.

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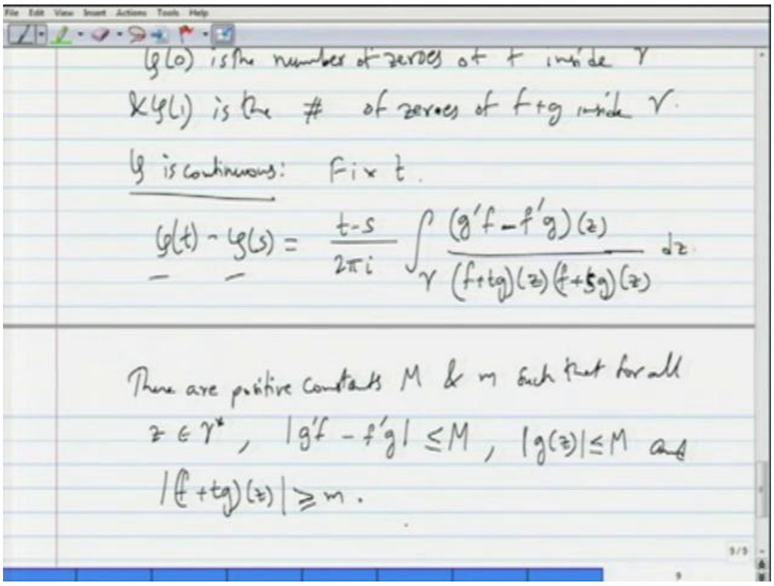

So, so phi of 1 is the number of zeroes, zeroes of f plus g inside gamma. So, if phi is continuous and phi is a constant then phi of 0 equal to phi of 1. And so the number of zeroes of f will be equal to number of zeroes of 1. So, all the hard work is actually inside proving that phi is actually continuous function. So, here is is how we prove that phi is continuous. So, phi is continuous is a prove. So, fix t then phi of t minus phi of s. What is this? By definition this is t minus s divided by 2 pi i times integration over gamma of g prime f minus f prime g of z divided by f plus t g of z times f plus t s g of z. So, this is obtained by you know clearing denominator and etcetera.

So, I mean writing expressions for phi of t and phi of s clearing the denominators. Now, we note the following. There are positive constants capital M and little m such that for all z belongs to gamma star. Modulus of g prime f minus f prime g is less than or equal to M. That is the extreme value theorem which says that a function continuous function attains its maximum. So, there is a constant M such that this is bounded, this f prime g g prime f minus f prime g modulus is bounded. g prime of, sorry g of z likewise is a function which is bounded, it is continuous.

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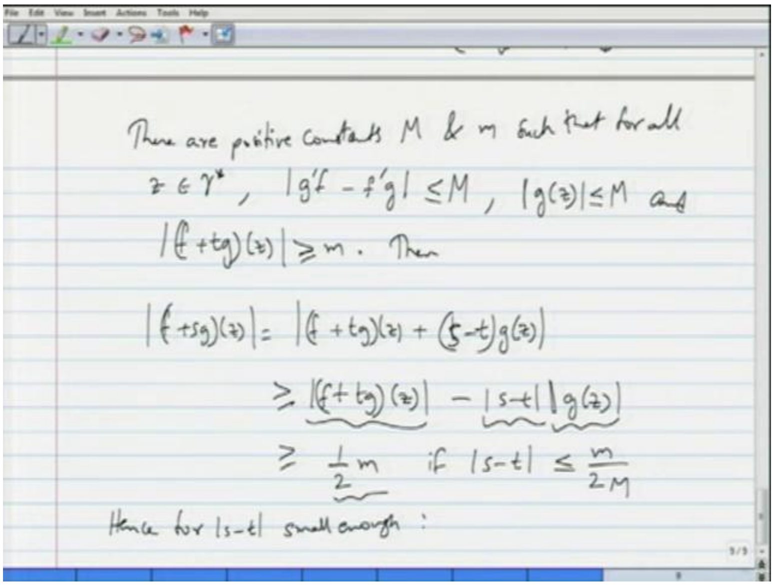

So, we are picking the maximum of them. So, a single capital M works also and modulus of f plus t g. We know that this is never 0, the function of f plus t g is never 0 on the trace of gamma. So, this has to be have a minimum by a extreme volume theorem. Once again so the modulus of this function is greater than or equal to M on gamma star. So, then by once again form of triangle inequality the modulus of f plus s g of z which is equal to modulus of f plus t g of z plus t minus or s minus t g of z. This is greater than or equal to modulus of f plus t g of z minus modulus of s minus t times modulus of g of z by triangle inequality.

This is greater than or equal to well, this is greater than or equal to half times little m if modulus of s minus t this choose to be less than or equal to m by 2 M. If modulus of s minus t is less than or equal to m by 2 M this quantities less than or equal to m by 2 M, so this times this is less than or equal to this times g of z in modulus less than or equal to m by 2. So, when we subtract something which is at least m or we subtract m by 2 from something which is at least m we have something which is at least half m. So, it is easy to see; so hence for modulus of s minus t sufficiently small, small enough.

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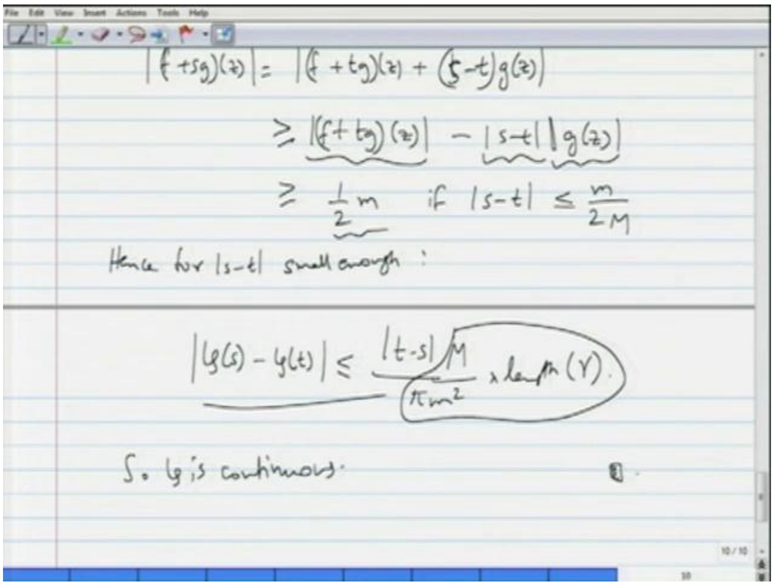

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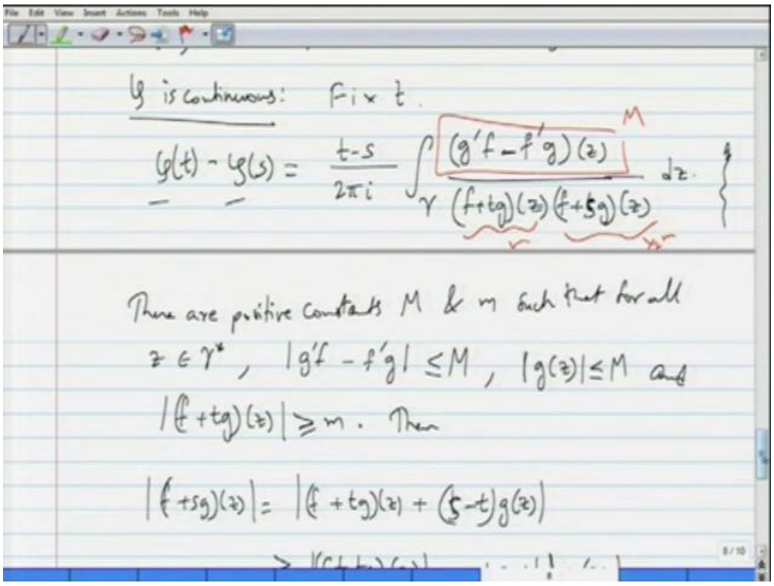

What we have is phi of s minus phi of t in modulus is less than or equal to modulus of t minus s times capital M by pi m. I am using this expression here this particular expression here. So, g prime f minus f prime g is less than or equal to the numerator less than or equal to little m capital M, sorry. And then in the denominator we have proved that this is at least little m and this is at least half m. So, in summary what we have is this is less than or equal to modulus of t minus s by 2 pi times capital M by m times half m. So, 2 into 2 cancel I have pi m in the denominator. So, I apologise this should be pi m squared. So, I will say it should be pi m square.

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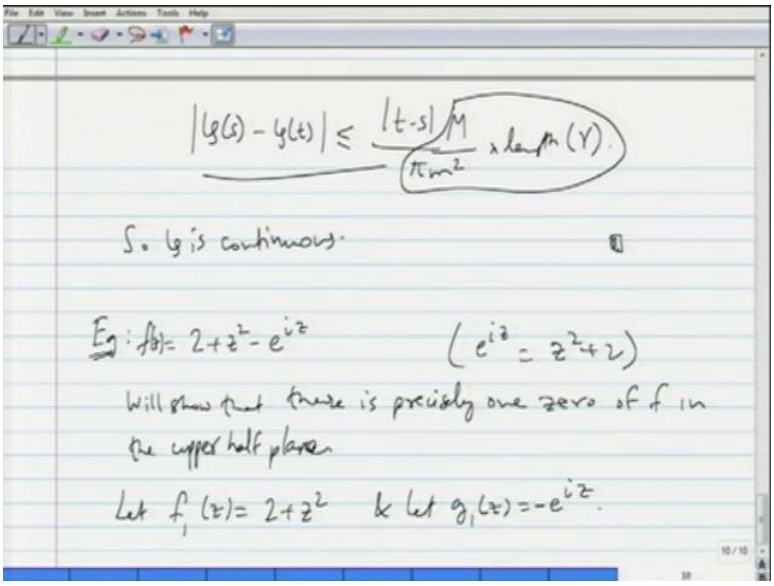

And then and then I have times of course, the length of gamma itself of the contour gamma. So, all these are constants, point is all these are constants. And so I can say that phi of s or phi of t or phi is continuous at the point t. So, phi is continuous. So, that completes the proof of Rouche's theorem. We can apply Rouche's theorem for example, count zeroes of some functions to look at zeroes of some functions here is an example. So, so consider the function 2 plus z square minus e power i z. The zeroes of this function f of z is this. Well, it as an entire function. The zeroes of this function are precisely the solutions to the equation e power i z is equal to z square plus 2. So, we are trying to locate the solutions to this equation.

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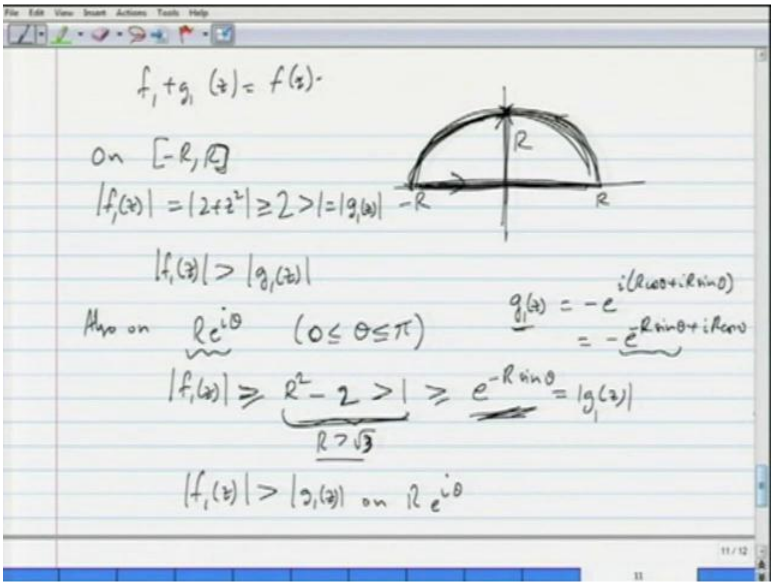

So, in the upper half plane we will show that there is precisely one zero. We will show that there is precisely one solution or I will say one zero of f in the upper half plane. So, how do we do that, we use Rouche's theorem. We will let so I will change the name of does not matter. So, let f 1 of z equal 2 plus z square and let g 1 of z equal to e power i z. Or minus e power i z so that the sum will give me f. So f 1 plus g 1 of z is f of z. So, if we consider the following contour the we will exercise k in selecting a contour. So, if we take a contour like that, so it is this portion of real line starting from minus R to R and then we will take contour a semi circle of radius capital R.

So, if we consider this contour so we go from minus R to R on the real line and then along this along this semi circle. Then what we have is modulus of f 1 of z on on this real line on the interval minus R to R that is the notation for the complex numbers on the real line minus R to R. We know that modulus of f 1 of z is modulus is of 2 plus z square well that is at least 2. And because, so that is at least 2 and this is strictly greater than 1 which is the modulus of g 1 of z. Modulus of g 1 of z on on real line is essentially 1 because modulus of e power i z for z real number is a 1. We know that for real number z it is 1. So, in particular numbers between minus R and R real numbers between minus R and R modulus of g 1 of z is 1.

So, on minus R R on this piece modulus of f 1 of z is greater than modulus of g 1 of z, strictly greater that is important. And also on the circle R e power i theta theta from minus R or i should say 0 to pi theta from 0 to pi (( )) R e power i theta. What is the modulus of f 1 of z modulus of f 1 of z is well, z could be opposite to 2, so on the negative direction. So, the modulus of this is at least R squared minus 2 right. Modulus of 2 plus z square for any R e power i theta this can be this has to be at least R squared minus 2 in the worst case. Then this is strictly greater than 1 and 1 is greater than or equal to e power minus R sin theta which is the modulus of g of z.

So, well g 1 g 1 of z so g 1 of z noticed is minus e raise to i z. So, if I write z as a x plus i y or or at least when z equals r e power i theta what I have is i times R cos theta plus i R sine theta. So, this gives me minus e arise to i or minus R sin theta plus i R cos theta. So, the modulus of g 1 of z on the contour R e power i theta will give me it is real part namely e power minus R sine theta. Whatever that, is when R is greater than R is greater than 1 at least for for any R this is less than 1, less than or equal to 2 for R sine theta positive. So, this gives me modulus of g of z. And notice this inequality true only if R is greater than let us say root 3.

So, I I wrote this in a (( )) actually R square minus 2 is greater than 1 only if R is greater than some number, R is greater than root 3. So, of course, we can keep this contour high enough we can choose R as big as we like. So, we choose R to be at least root 3 and then that will give us modulus of f 1 of z is strictly greater than modulus of g 1 of z on R e power i theta as well. So, in summary modulus of f 1 of z is greater than modulus of g 1 of z from all of this contour.

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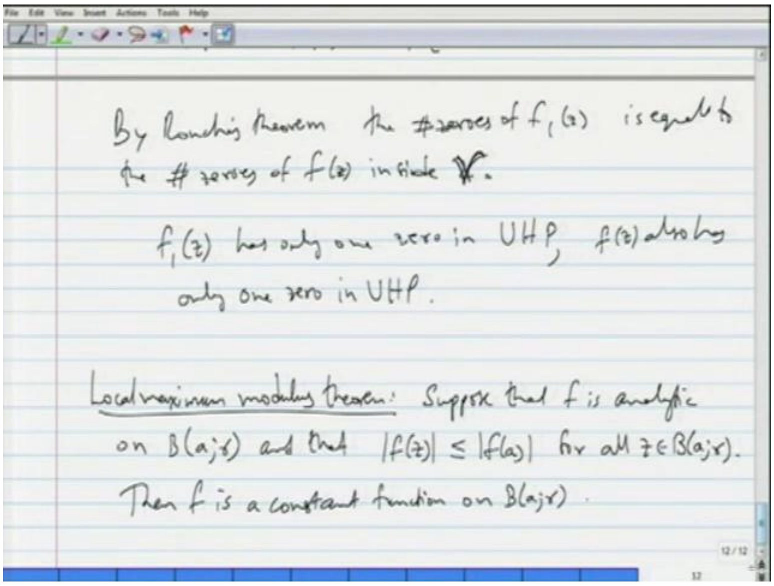

So, we can conclude by Rouche's theorem that by Rouche's theorem the number of zeroes of f 1 of z is equal to the number of zeroes of f 1 plus g 1 which is f of z of f of z inside gamma z gamma. So, we can take gamma as large as we like. So, you can increase R as much as we like. So, making R criteria to infinity, since we know that f 1 of z has only one zero namely minus root 2 i or plus root 2 i. One zero in a the upper half plane, f of z also has only one zero in the u h p. So, let R tend to infinity and make this conclusion. So, we can count zeroes, common zeroes using Rouche's theorem. So, next we will consider some applications of Rouche's theorem.

In particular we will see maximum modulus theorem and and using to prove the open mapping theorem for analytic functions for non constant analytic functions. So, in that line first what I want to do is that I want to consider a local version of the maximum modulus theorem. So, here is local maximum modulus theorem. Suppose that f is analytic on a ball of radius r around a and that modulus of f of z is less than or equal to modulus of f of a for all z belongs to B a, r which means the the function f is such that the modulus of f is lagest at its centre. Then, then it has to be that f is a constant function on B a, r. So, in particular this is telling you or converse of this is counter positive of this tells that if f is not a constant function then it cannot have maximum modulus at it centre.

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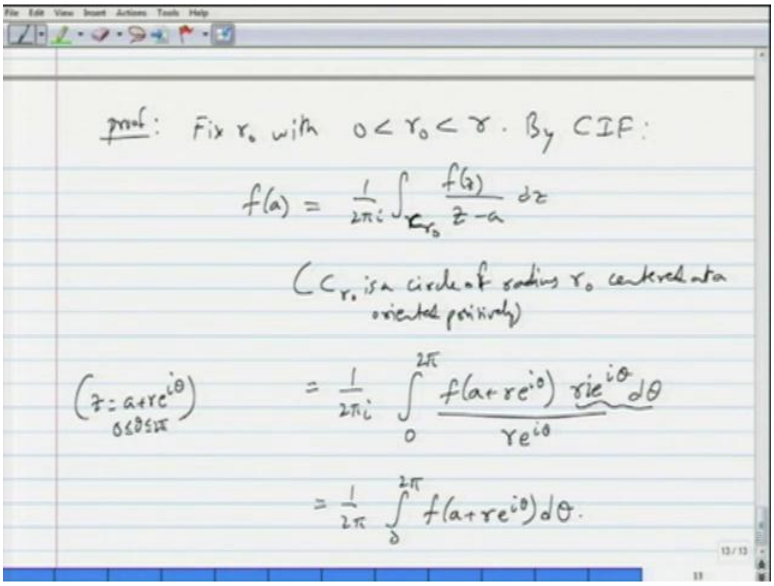

So, here is a proof of this fact. So, firstly we want a little r or r naught fix r naught with 0 less than strictly r naught strictly less than r. By Cauchy's integral formula by Cauchy's integral formula we know that f of a the value of the f at the centre is 1 by 2 pi i integration over C r naught where C is C r naught is a circle of radius r naught centred at a of f of z divided by z minus a d z. C r naught is a circle of radius r naught centred at a oriented positively, that we know from Cauchy's integral formula. So this is the simple circle there and then this is 1 by 2 pi i. This can be written as 1 by 2 pi will parameterize this circle.

So, we get 0 to 2 pi f of a plus r e power i theta times r i e power i theta d theta, this is the d z. And then denominator we have z minus a which is r e power i theta, z is a plus r e power i theta. That is the parameterization, theta from 0 to 2 pi. So, this is d z, this little p is d z and then we have that. So, this is equal to 1 by 2 pi i or now 1 by 2 pi times integral 0 to 2 pi. After cancelation this you have f of a plus r e power i theta times times d theta, after all the cancelations.

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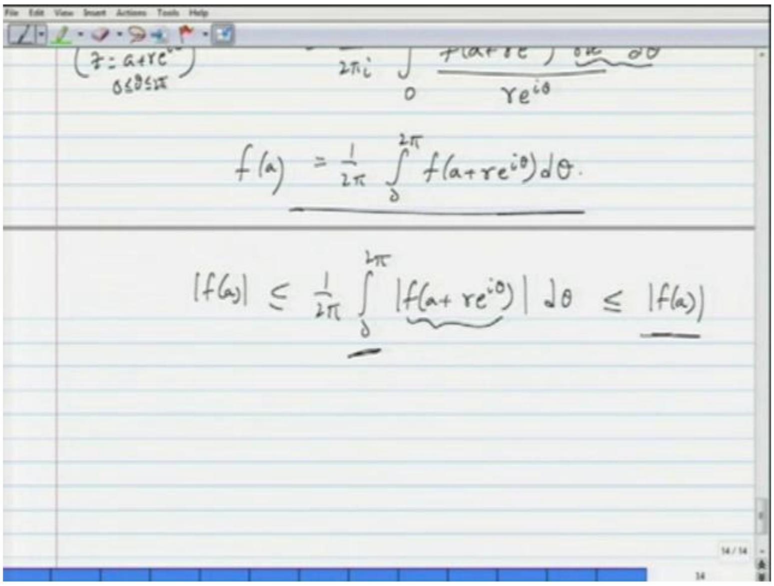

Now, from using the hypothesis that modulus of f of z is less than or equal to modulus of f of a. Modulus of f of a firstly from this formula has to be less than or equal to 1 by 2 pi integral 0 to 2 pi modulus of f of a plus r e power i theta. That is from, that is from this f of a equals this. And then and then by hypothesis we have this is less than or equal to modulus of f of a. That is because well this is modulus of f of a plus r e power i theta is modulus of f of z some z on the circle. So, that I am writing is less than or equal to modulus of f of a, f of a is constant integration from 0 to 2 pi of d theta will give me 2 pi, 2 pi 2 pi cancel to give me this modulus of f of a. So we have this inequality.

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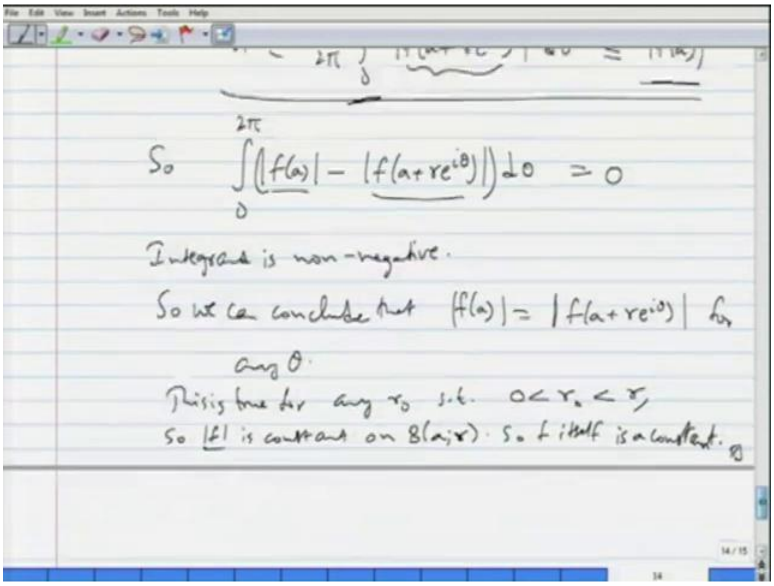

So, integral 0 to 2 pi of modulus of f of a minus modulus of f of a plus r e power i theta is d theta is 0. From this, we can conclude that this integral is 0. Also since modulus of f of a is constantly greater than or equal to modulus of f of a plus r e power i theta, the integrand is non-negative. When we have a non-integrand, and then its integral is 0, we get that. So, we can conclude, conclude that the integrand itself modulus of f is identically 0. So, f of a is equal to modulus of f of a plus r e power i theta for theta, for any theta. Now, this is true for any r less than r naught less than r. So, this is true for any r naught such that 0 less than r naught than strictly less than r.

So, we conclude that so modulus of f is constant, it has to be constant on the whole disk B a, r. So, f itself and we know that an analytic function if its modulus constant, then the function itself is a constant. So, f itself is a constant. That was an exercise way back using Cauchy's Riemann equations. So, when the modulus is constant the function itself is constant and that proves the local version of this maximum modulus theorem, which says that a function, non-constant analytic function cannot have maximum at the centre the of a disk of analyticity. So we will use this, and Rouche's theorem to proceed further and prove one of the important results namely the open mapping theorem for non constant analytic functions. So, I will stop here.