Complex Analysis
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Module - 4
Further Properties of Analytic Functions
Lecture - 4
Zeroes of Analytic Functions

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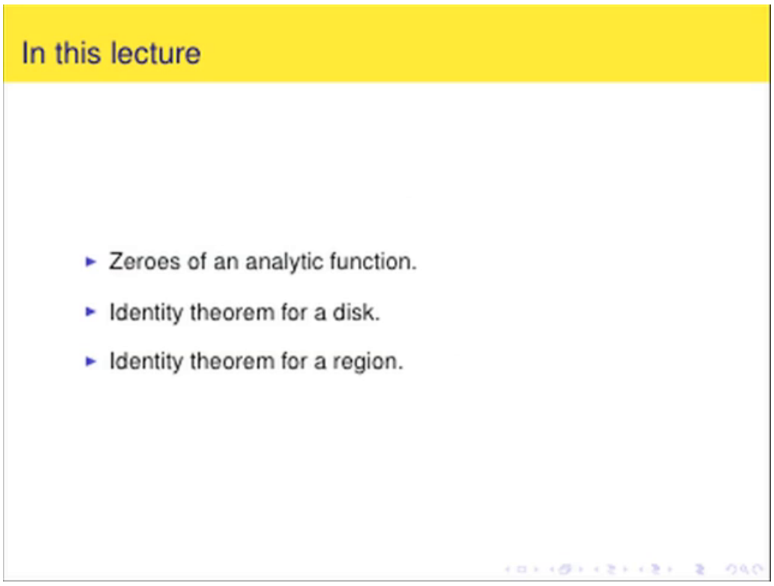

Hello viewers, in this session we are going to look at some examples concerning Taylor's theorem, and then we will analyse the properties of zeroes of an analytic function, that is the points, where an analytic function is 0. And we will see some theorem's concerning the zeroes of analytic function; so firstly some examples for Taylor's theorem.

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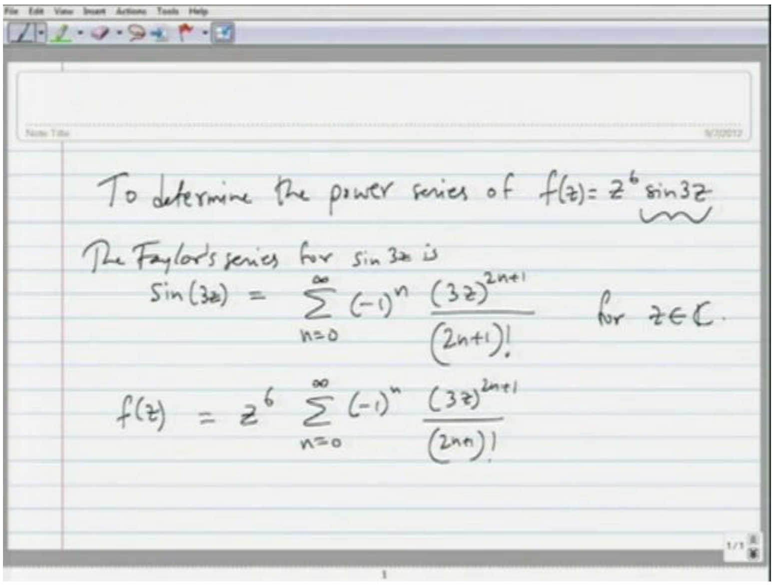

So, the first example is determining the power series of a function. So, to determine the power series of f of z equals z power 6 sin 3 z. So, what we can do is, we can actually use Taylor’s theorem and say that the power series of this function f of z will equal you know, whatever the Taylor’s theorem says, the coefficient c n are given by the n-th derivative of this function divided by n factorial etcetera, but to compute the derivatives of this function is tedious. So, instead what we will do is, we will compute the Taylor series for this function sin 3 z and then multiply by z power 6 that is easier than directly computing the Taylor series for f of z.

So, we know by one way or the other, that the Taylor series for sin 3 z well by using its derivatives. Let us say that the Taylor series so, the Taylor’s series for sin 3 z is sin 3 z equals sigma, n equals 0 through infinity of minus 1 power n d z raise to 2 n plus 1 divided by 2 n plus 1 factorial. So, the Taylor’s series sin 3 z, z being a complex number, tallies with the Taylor’s series in the real case for sin 3 x, where x is a real real number.

So, x replace with a complex number z here, works and the radius of convergence of this series is infinity. So, what this means is that for any for any z belongs to c. So, sin c z is this so, we can say that f of z, the Taylor’s series for f of z is nothing but z power 6 multiplied by this Taylor’s series n equals 0, through infinity minus 1 power n 3 z power 2 n plus 1 divided by 2 n plus 1 factorial.

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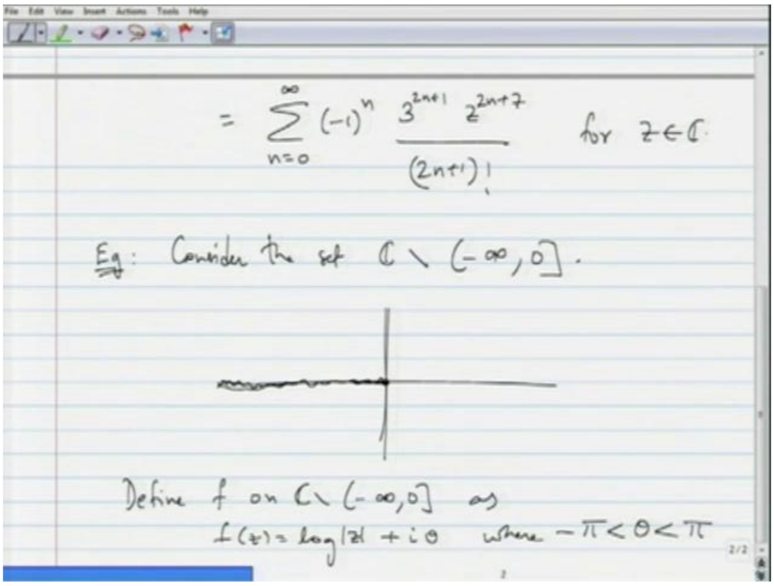

And then take the z power 6 inside and multiply, you can do that because you have a convergent power series and z power 6 is anyway an analytic function. So, this is n equals 0 through infinity minus 1 power n. So, this is 3 power 2 n plus 1 times z power 2 n plus 7 divided by 2 n plus 1 factorial, for any z belongs to c.

So, this is much easier than actually finding the derivatives of f and then figuring out its Taylor's series. So, this is the practical note as oppose to using the Taylor's theorem directly. Now, the next example is that of the logarithm a branch of a logarithm. So, here is the example so, consider the set complex plane minus the segment minus infinity is 0.

So, you know this is not really I mean this notation is kind of sloppy, but you understand what this is? This is the complex plane minus the negative real axis including the point 0. So, you have removed the negative real axis along with the point 0. So, you consider this set, so that is the branch cut and you can define a logarithm on this, as we know from before. So, f of z defined f on c minus minus infinity comma 0 as f of z is equal to log modulus of z plus i theta, where you considering theta to be between minus pi and pi.

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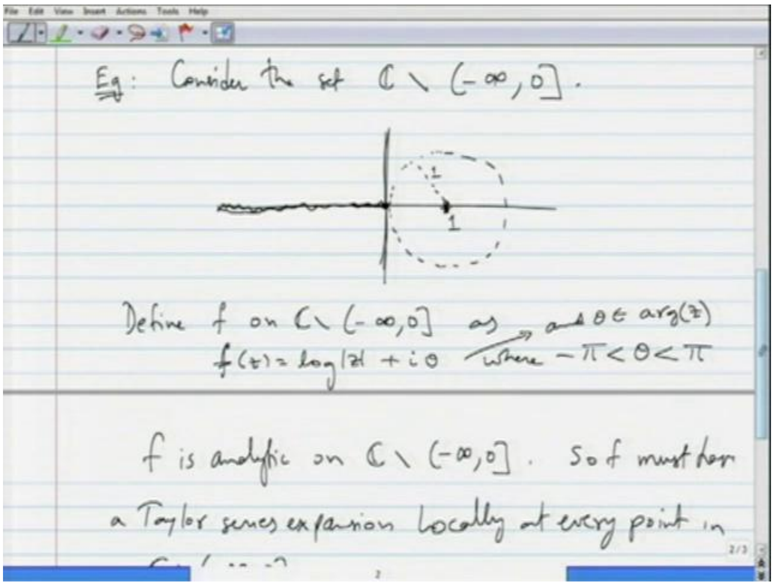

And theta here is the argument. So, I should say theta belongs to log mod z plus i theta and theta and theta is belongs to the argument, the set argument of z recall argument of a complex number is a set and the entries are at 2 pi apart from each other. So, if you considering theta to be in this region minus pi to pi, then you have a function here f of z equals log mod z plus i theta. So, this is this as we know there is a branch of logarithm. So, f is analytic on c minus, minus infinity comma 0 as we know from before.

So, f must have Taylor's series expansion locally, locally at every point in c minus, minus infinity minus 0 that is Taylor's theorem. So, in particular let us pick a point here so on. This complex plane on this picture, let us pick a point, let us say 1, 1 on the real line. Now, we can expect this Taylor's series for this ranch of logarithm to have a Taylor's series expansion in a disc about one, well how far can that disc stretch really well that can actually stretch until we hit a point, where the function is no longer analytic. So, we can expect this disc to have radius 1, this will have radius 1 and it centred at 1 it is an open disc remember.

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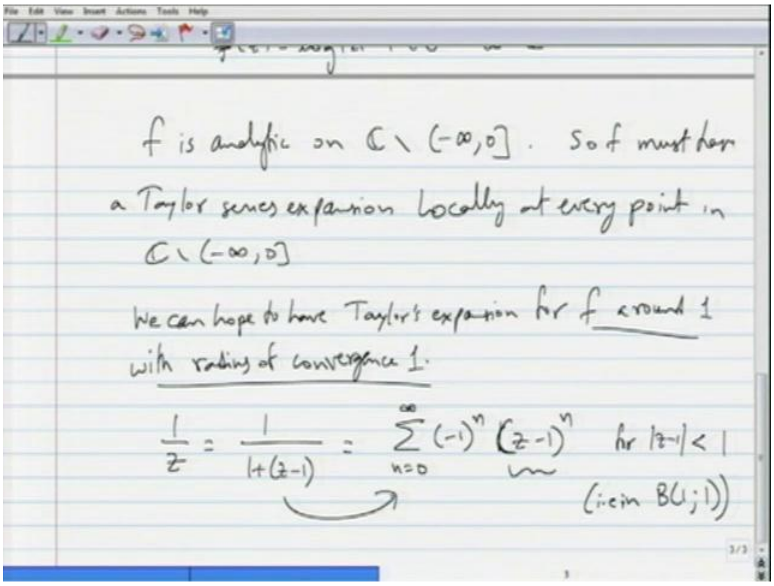

So, we can expand we can hope to have, we can hope to have Taylor’s, Taylor’s expansion for this for f, I will say f that branch of algorithm for f around around the point 1 around 1 with radius of convergence 1. And let us try to get the Taylor’s series expansion well instead of finding the derivatives etcetera directly once again we will resort to indirect methods.

So, first we know that, we know that 1 by z. The function 1 by z can be written as 1 by 1 plus z minus 1 and in this disc itself, in this disc under consideration or the disc our guess this is equal to sigma n equals 0, through infinity of minus 1 power n modulus or sorry z minus 1 raise to n. So, this function as this series expansion power series expansion it is the geometric series, as long as the modulus of z minus 1 quantity is strictly less than 1 this is i e in B 1, z i e z in B 1, 1. So, we will actually use the fact that this function 1 by z is the derivative of log z on o c minus on the domain given here c minus, minus infinity 0.

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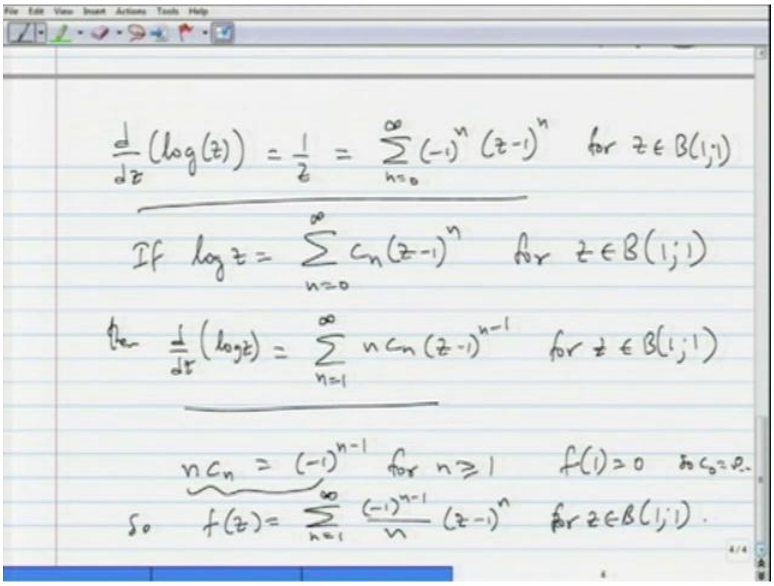

So, d by d z since log z is analytic, log z is analytic and its differentiation is 1 by z on the disc of interest at least affiliates its derivative equals 1 by z on all of its domain. So, this is equal to this is equal to sigma n equals 0, through infinity minus 1 power n z minus 1 power n for z belongs to B 1, 1. So, we also know that the power series if log z has a power series, if log z is equal to sigma c n z minus 1 power n n equals 0 through infinity for z belongs to B 1 1, then we know the d by d z of log z.

We saw that power series are analytic, and their differentiation is given by differentiating it term wise. So, this is n equals one through infinity of n c n z minus 1 power n minus 1 for the same z for z in B 1, 1. What this means is that, you know by comparison of this and this, what we get is that n. So, in case by shifting index, what we get is n c n is equal to n c n is equal to minus 1 power n minus 1 and not only that, this true for n greater than or equal to 1 and f of 1 is 0, log 1 is 0. So, c 0 is 0.

So, f of z is equal to sigma n equals 1 through infinity of minus 1 power n minus 1 divided by n because c is now, here c n is minus 1 power n minus 1 divided by n times z minus 1 raise to n for z belongs to B 1, 1. So, that the power series for this branch of logarithm. Once again, a little indirect way we use the power series expansion of the derivative of this branch of logarithm to obtain, the power series for the logarithm.

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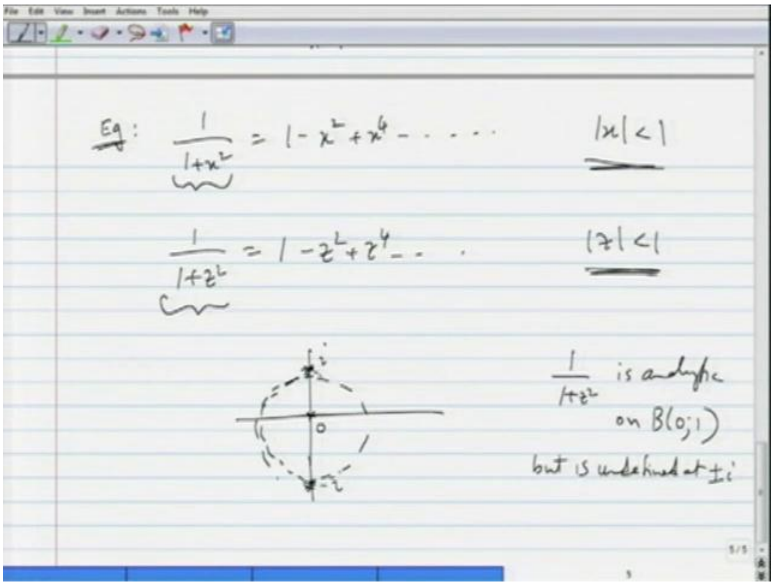

So, another sort of remark, really on Taylor's theorem; so the function when, we when we see the following function and its Taylor's expansion in the real case i, the function 1 by 1 plus x square as a function of real numbers, we see that its expansion is 1 minus x square plus x power 4 etcetera. So, and then the radius of convergence happens to be 1 it is not immediately clear. Why?

This function 1 by 1 plus x square should have a radius of convergence 1, when it comes to Taylor's series, in the real setting in the real number setting, but when we accent to the complex number setting, we see that 1 by 1 plus z square is 1 minus z square plus z power 4 etcetera with mod z less than 1 and we sort of see, why this 1 acts as a barrier for this power series around 0.

So, what I mean by that is around 0, you notice that 1 by 1 plus z square is analytic on B 0, 1, but is undefined at plus or minus i the numerator, the denominator actually becomes 0. So, i and minus i here actually act as a barrier for the expansion of power series around 0. So, this series really does not extend beyond this disc of radius 1. So, that is the geometric explanation of y, the radius of convergence of power series 1 by 1 plus z of 1 by 1 plus z square around 0 sort of has a radius of convergence 1, and you know it does not go beyond. So, this i and minus i which are not points of analyticity of 1 by 1 plus z square act as barriers. So, a similar phenomenon, we saw happens in the case of branch of logarithm.

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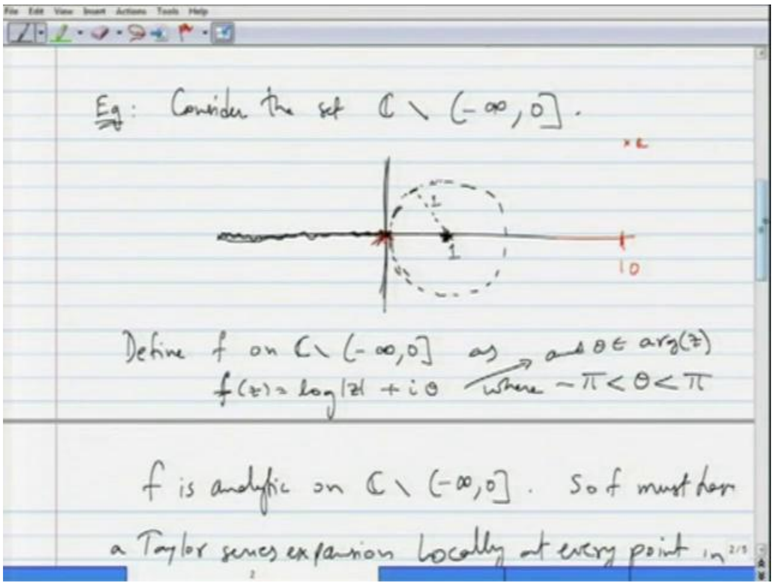

So, beyond, beyond this radius 1 we have points here, which are not points of analyticity of this function of this branch of logarithm. So, likewise if we going back to this example if we take let us say 10 on the real line or any complex number c for that matter. So, let us pick this example of 10 on the real line then, we can expect to have power series expansion for this function, this branch of logarithm of a with radius of convergence 10.

So, that sort of solves the quote and quote mystery of why this power series have a certain radius of convergence, and why that power series cannot be extended for a ever beyond point. So, these functions when they have certain barriers or points, where they are not analytic there power series expansion around certain points cannot be extended. So, that also we can note from Taylor's theorem and power series in the complex plane. So, next what we will do is we will use Taylor's theorem, to study the zeros of a complex analytic function.

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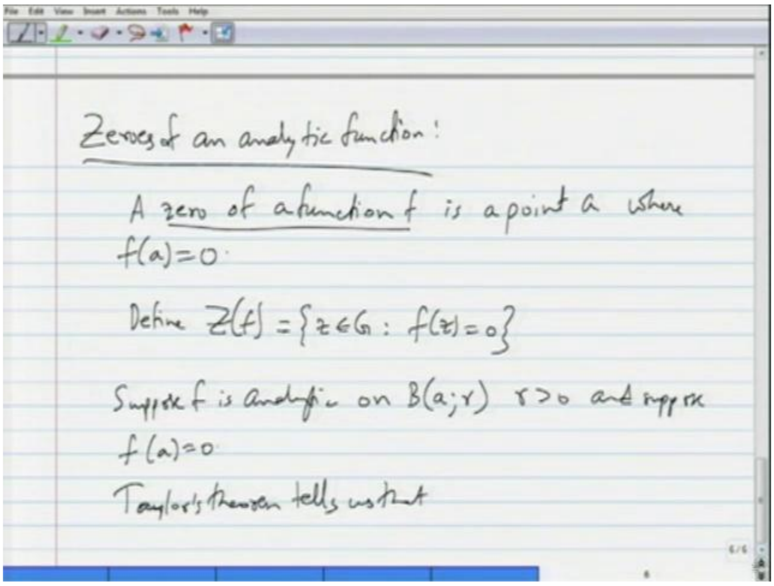

So, this is zeros of analytic function, of an analytic function. Firstly some notation a 0 of a function f like we know of a function, f is a point a where f of a is equal to 0 that we call zero of a function. So, let define z of f we will use this notation, z of f we will use this notation following the text book of Pressley, z of f is equal to z belongs to g such that, f of z is equal to 0 it is the set of all zeroes of the function f. So, Taylor's theorem gives us some important conclusions about the zeroes and we will see that. Suppose, f is analytic on a certain B a, r ball of radius r around a r strictly positive and suppose, that f of a is equal to 0 then Taylor's theorem tells us that.

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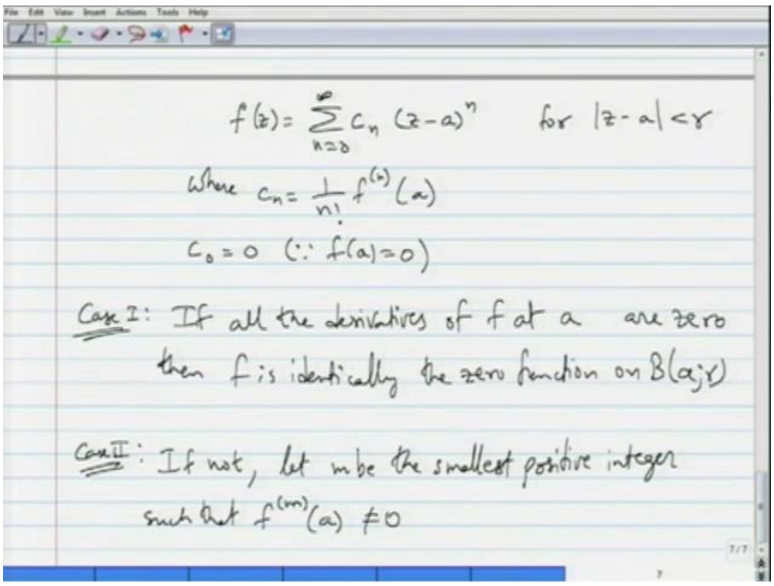

Firstly since f is analytic in the ball of radius r around a, f has a power series expansion about a with, with certain radius of convergence. So, c n z minus a power n, n equals 0 through infinity at least the radius of convergence is r or is at least r. So, modulus of z minus a strictly less than r, where we also know that c n is 1 by n factorial the n-the derivative of f a t, we also know that since f is 0 at a c 0 is 0. Since, f of a is 0 f of a is c 0. Now, there are two cases.

Case one if all the derivatives of f at a are 0. So, if all the derivatives of f at a are 0 then Taylor's theorem tells us that, then f is identically the 0 function, identically the 0 function on B a, r. Notice this is far from true for functions of real numbers, there are functions whose derivatives are all 0, but then the function itself is non zero.

So, since since analytic functions have local power series expansion so since, they are equal to a certain power series and the coefficients of this power series are nothing but the derivatives of f. If f is 0 at a point, and if all its derivatives are 0, then the function should be identically 0 on that disc of radius r. So, that happens for complex analytic functions. So, that is a that is a very important conclusion from Taylor's theorem, about the zeroes of a function and then case two.

Suppose, that is not true; so, if all the derivatives of f at a are I mean if not all of them are 0, then there is a smallest positive integer such that m such that the m-th derivative of f at a is non zero. So, if, if not if case one is not true, let m be the smallest positive integer such that, f m the m-th derivative of f at a is non-zero correspondingly that particular c m you know, in the Taylor's series expansion will be non zero.

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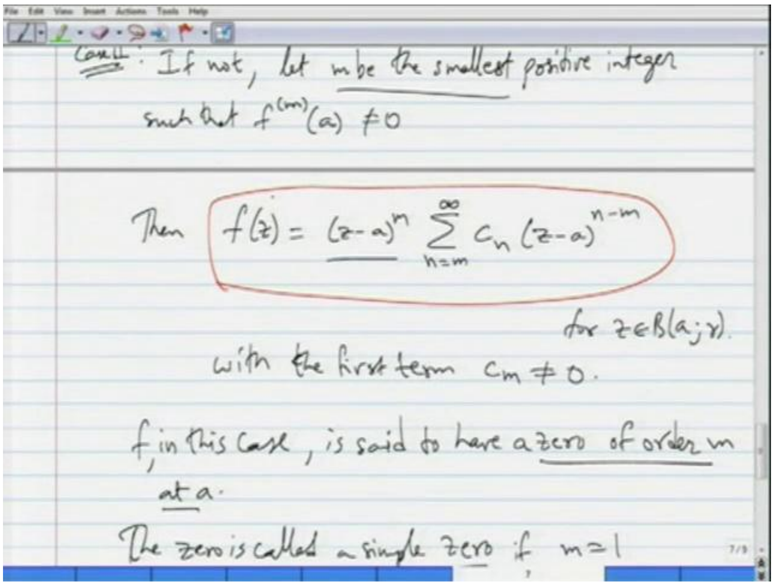

Then what we can say is that f of z by definition, m is the smallest positive integer. So, f of z looks like z minus a power m sigma n equals m through infinity of c n. Now, z minus a raise to n minus m, what I am doing is the expansion for f now, starts at n equals m because c n's are all 0 for n less than m. So, I will start the expansion from n equals m and when I start the expansion from n equals m. I know that i can factor out z minus a raise to m and when I do that, when i factor out z minus a raise to m, what I get is this, this form. So, this particular form so, this form and now also I should immediately note that, this is true for modulus for z belongs to B a, r did I use r i, guess I used r here.

So, now what is also important is that with this is true, with c m with the first term c m not equal to 0. So, that is the Taylor's series expansion, then f in this case, in case two, in this case is said to have a 0 of order m at a. So, the 0 is called a simple 0 if so, this 0 of order m at a and the 0 is called as simple 0, if m is equal to 1, which means, factoring out occurs for I mean m equals 1; that is the smallest integer for which the c n is non zero.

So, you see that from this thing circled in red, you see that f of z sort of behaviours like a polynomial. We know that polynomials, if they have a 0 at a point you can factor out that particular z minus a raise to a certain power m, where m is the order of 0 of that polynomial and then whatever is the remaining factor; that is non zero at that point a you can factor out a polynomial that way. So, likewise you can factor out a an analytic function locally, in that fashion. So, in that sense the analytic functions behave locally like polynomials in that sense. And then what want to say is here is I want to take an example.

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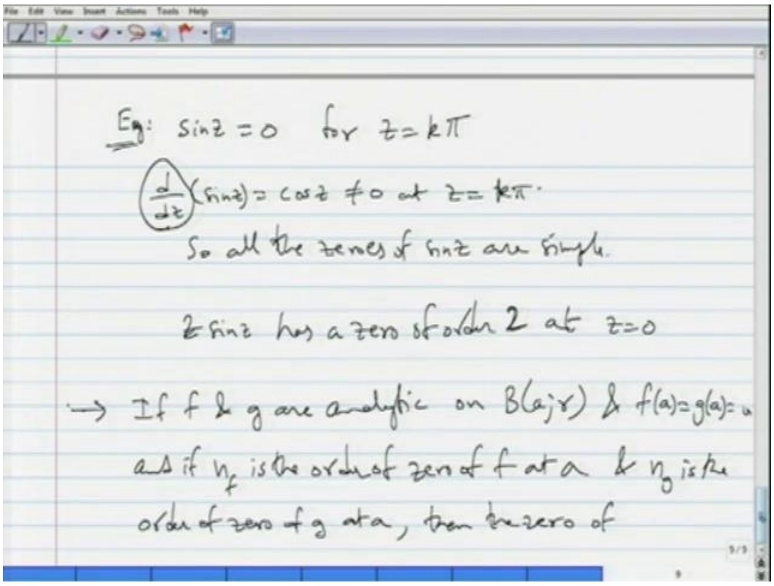

Here is an example sin z is equal to 0, we know for z equals k pi and only for these values of z. And we also know that the differentiation of sin z is cosine z, which is not equal to 0 at z equals k pi. So, we already see that the first derivative of the analytic function sin z is non zero at the zeroes of the analytic function.

So, here so all the zeros of sin z are simple because m. In this case m in this case is just one the first derivative is non zero. So, that happens for sin z so that is an example of an analytic function, which has simple zeros and it is likewise relatively easy to construct examples of analytic functions, which have double zeros or triple zeros. What I mean by that is they have zeros of the order 2, 3 etcetera. So, you just take z square sin z or you know z sin z as a 0 of order 2 at has a order 2 at 0 z has a 0 of order 2 at z equals 0.

So, I will actually state the following here is a fact, which you can prove easily or by just using what we have done using Taylor's theorem. So, if f and g are analytic on B a, r and f of a is equal to g of a is equal to 0 and if n f is the order of the 0 of f at a and n g is the order of 0 of g at a then, the 0 of f times g at a.

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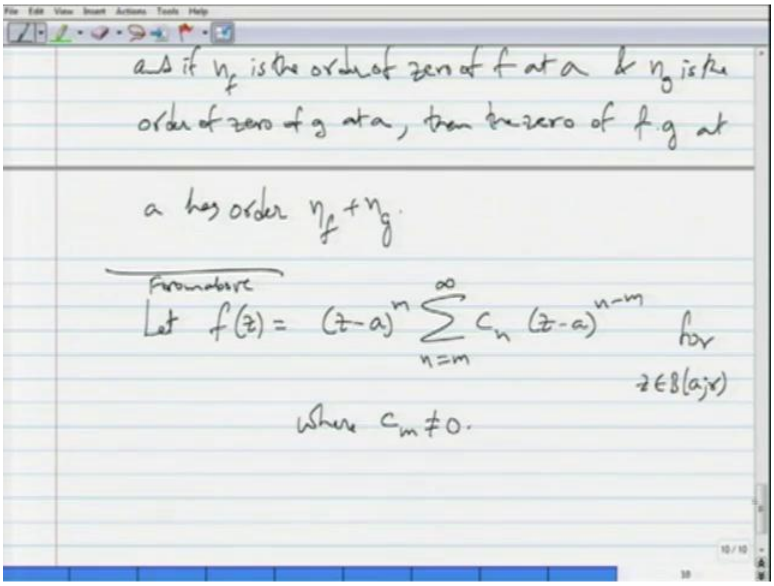

Firstly note that f times g will be 0 at a because f and g are both 0 at a. So, the 0 of f times g at a has order n f plus n g simply, it is the addition of the order. So, because of this fact, which can easily be obtained from this form of expansion and multiplication of power series, you can multiply power series in a certain fashion, and then you can immediately conclude this fact. So, from there I can say that z sin z has a 0 of order 2 at z equals 0 or for that matter. Now, I can construct a function having 0 of a certain order at a point.

So, that is a fact there and now I want to further analyse these thing, that I have circled in red what I want to conclude further is that, I will define so let f of z. So, f have a f of z equals so from above, from above that is easier from above the same set up f of z equal z minus a power m sigma n equals m, through infinity of c n z minus a raise to n minus m for z belongs to B a, r positive, where c m the first term is non zero.

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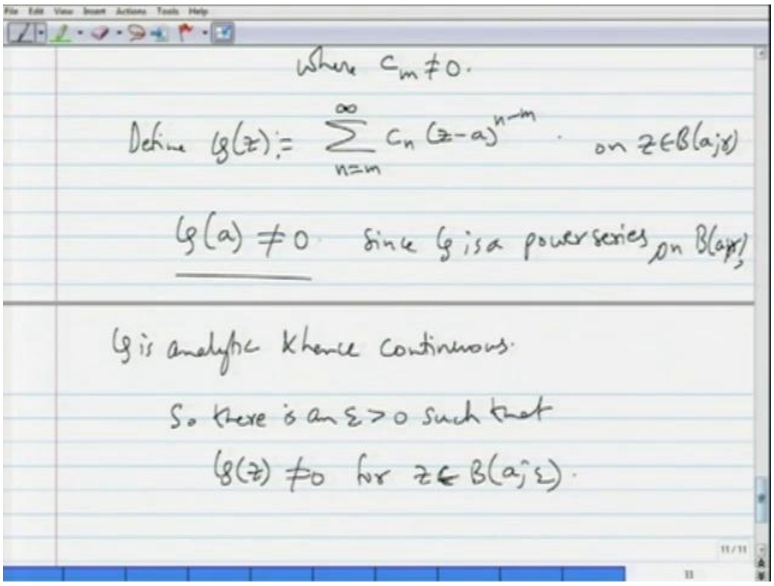

Now, what I will do is define I will take this part, which is non zero define phi of z is as z is equal to sigma n equals m through infinity of phi n z minus a raise to n minus m. So, first as noted above I can conclude that phi of a is non zero, right because c m is non zero. So, phi of a which is c m is non zero. And also note that phi is a power series so define this for set belongs to B a, r on the very same set. Since, phi is a power series function, we saw that power series are analytic since, phi is a power series in fact convergent power series is a convergent power series on B a, r phi is analytic by what we already know and hence continuous at least continuous .

So, that gives us an important conclusion that if phi of a is non zero. So, there is an epsilon positive phi of a is non zero means, that phi of z is non zero for a certain z in a neighbourhood around a. So, that is a property of a continuous function so, there is epsilon positive such that phi of z is non zero for z belongs to b a, epsilon. So, there is a whole neighbourhood around a, where phi is non zero. Now, this immediately tells us that f has a 0. Let us go back to this form once again this form here was in red.

So, f has a 0 of order m at a and whatever is remaining this, this is non zero at any point around a this function is definitely non zero around a, and we just concluded by continuity of this power series that this is also non-zero around a, around a neighbourhood of around a in a neighbourhood of a. So, what that gives is that as is that the zeros of f are actually isolated. What I mean by that is there is a small neighbourhood around the 0 such that, f is non zero in that neighbourhood.

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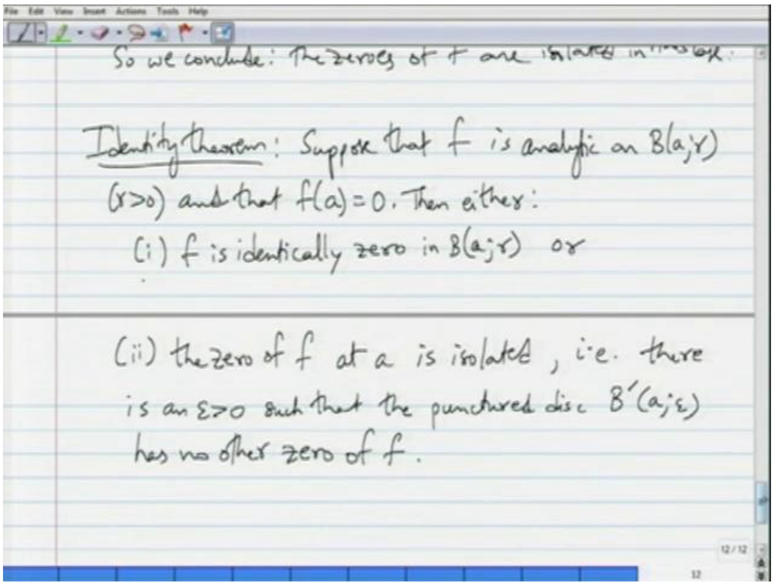

So, what we conclude is that. So, we conclude that the zeros of f are isolated in this case. What is this case? This is case two we are dealing with, this case two. So, we will sum these up in the following theorem, this we will call as identity theorem after the text book. Suppose, that f is analytic on B a, r, r positive and that f of a is equal to 0 then either of the two happens.

Case one f is identically 0 in B a, r or 2 that is this is the case two, we are working with 2, I will write below 2 the 0 of f at a is isolated that is there is epsilon positive. Such that, the punctured disc b prime of a epsilon recall what that is that is B a, epsilon minus the point a itself. So, B a, epsilon B prime a, epsilon has no other 0 of f.

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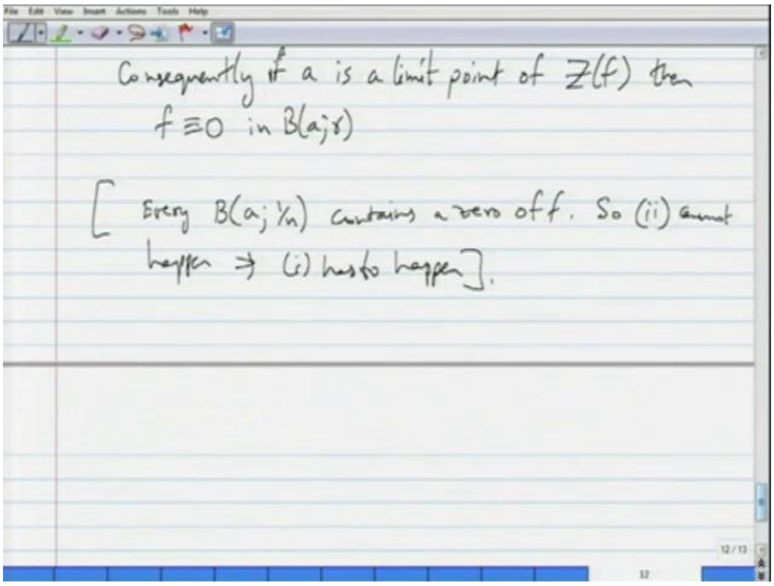

So, as a consequence of this, what we can conclude is that the consequently if a is a limit point of z of f then f is identically 0 in B a, r. So, I will briefly provide a proof of this consequence. What this says is that well if a is a limit point of z of f. So, what that means, is that every epsilon neighbourhood so, every B a, 1 by n lets say contains a 0 of f an element of z of f.

So, two cannot happen, two cannot happen which implies one has to happen. So, which means f is identically 0. So, next we will try to extend this theorem, this identity theorem for beyond a disc. So, obviously this is not true for the whole domain of analyticity. Well if the domain is disconnected I mean, if the region of analyticity of f is disconnected because f could be constant, 1 constant on 1 component and another constant on another component and so, you could have, you could have f not identically 0, but then you know all the derivatives are 0. So, here is actually a good extension of this theorem.

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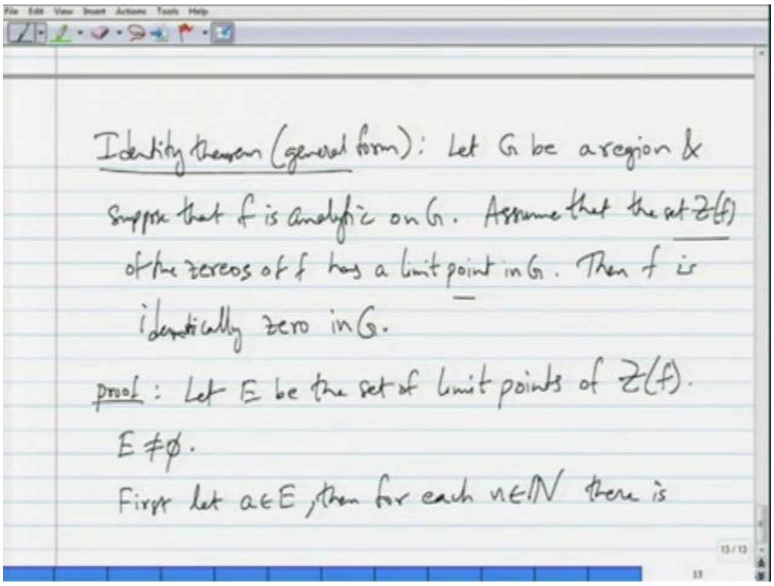

So, identity theorem, this is called identity theorem general form, it states the following. So, let G be a region so recall region is open connected set, the connectivity of G is essential here for this theorem. So, the proof as we can guess is topological hence. So, let G be a region and suppose that f is analytic on G. Assume that the set the set z of f i the zeros of f of the zeros of f has a limit point. So, there exists a limit point in G then f is identically 0 in G.

So, if there is a limit point for the 0 set of f in the region G then, f has to be identically 0. So, the proof is as follows, first let E be the set of limit points of the set z of f z of f, E is the limit point set of the set of zeroes of f, E is non empty is given it we assume that the set z of f has a limit point, which means, E is non empty so, that is given to us. So, first we will note that, first we will note that E is contained in z of f. So, first let a belong to E, we show that a is in z of f then for each n belongs to N.

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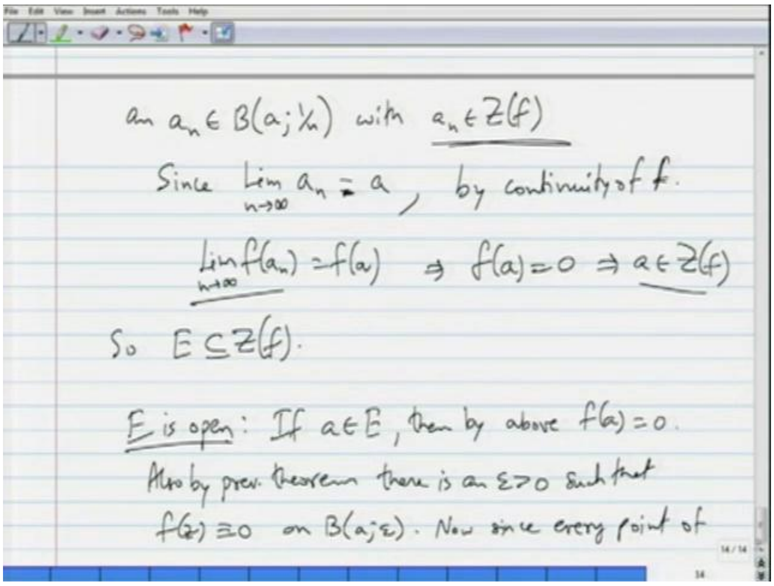

There is an n belongs to ball of radius 1 by n around a with with a n belongs to z of f. So, this sequence a n. Since, limit n goes to infinity of a n is equal to a by continuity of f. So, we I mean here is a note the limit definition of continuity is in order here. So, please note that if there is a, there is a sequence converging to a point and if a function is continuous at the point then the limit as a ,the limit of functional values on this sequence has to converge to the value of the function at that point, as long as the function is defined etcetera, etcetera at that point in a neighbourhood of the point.

So, if the function is continuous of course, it will be defined so, all that is true so, that is the sequential definition of continuity. So, by continuity of f the limit as n goes to infinity f of a n is equal to f of a, but we know that f of a n is 0 because a n belongs to z of f. So, this is a constant sequence 0 on the left hand side, this implies that f of a is actually equal to 0. So, this implies that a belongs to the set z of f. So, if we start with the point in the limit point, set of the set of zeroes, then that point itself is contained in a the set of zeros.

So, E is contained in z of f that is the first note. So, when what I am going to show is that this set E is both open and closed in the the the region G, what that will show is that is both open and closed and if G is connected there is only, there are only two sets which are both open and closed in G namely, the empty set the set g, but of course, is of non empty which will give us that E is the whole set g. So, the set of limit points of this zeros of f is the whole set G, which will mean that f is 0 identically on G so, that is the strategy.

So, E first I will show is open, if a belongs to E. I will show that there is a neighbourhood of a which is contained in E, then by what we have just done by above f of a is 0 because E is contained in z of f also by previous theorem, there is an epsilon positive such that, f of z is identically 0 on B a, epsilon. Noting this consequence consequently, if a is a limit point of z of f then f is identically 0 in B a, r. So, we are using that so since a is in E, we should have by the previous theorem that f of z is identically 0 on B a, epsilon. Now, since we know something about B a, epsilon epsilon positive.

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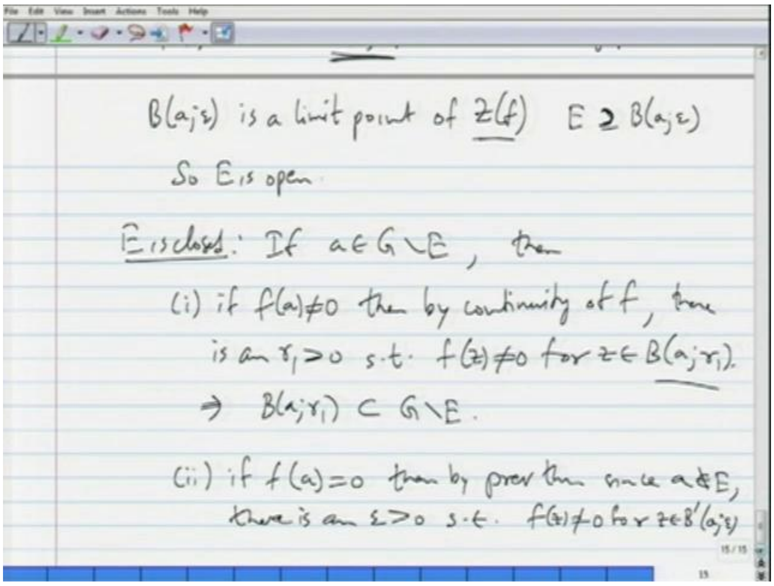

Now, since every point of B a, epsilon is a limit point of z of f. E contains B a, epsilon. So, f is identically 0 and this set is an open set. So, identically 0 on this open set and since every point of the open set is an limit point of z of f, E contains B a, epsilon. So, E is open likewise you can show that E is closed. What is the argument for E is closed, if a belong to the set G minus E, then we will do this in two cases.

Case one if f of a is non-zero then by continuity of a of f rather, there is an r 1 positive such that, f of a, f of z is non zero for z belongs to B a, r 1 and so, this implies B a, r 1 is not in the 0 set of f. So, it cannot be in the set E so B a, r 1 is contained in G minus a. Second if f of a is equal to 0 then by previous theorem. Since, a is not the limit point a does not belong to E. So, a is not the limit point there is there is an epsilon positive such that, B prime f of z is not equal to 0 for z in B prime of a epsilon, that is by the previous theorem once again.

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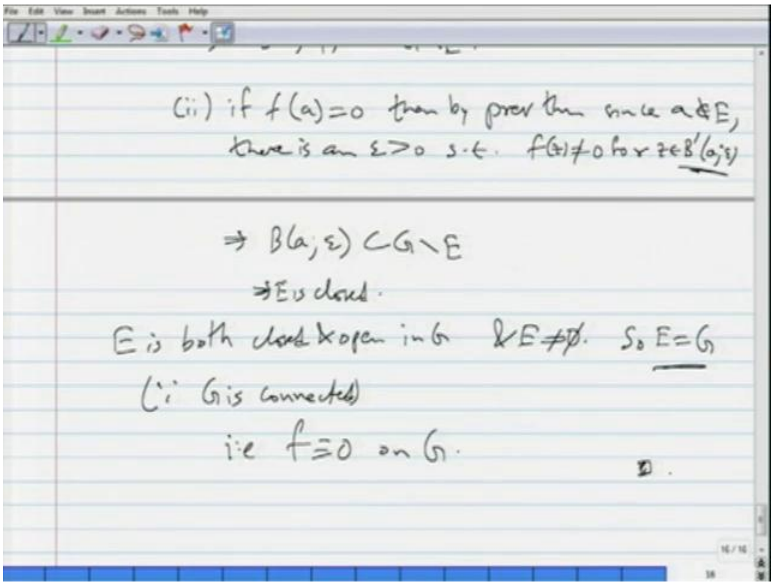

So, this tells us that, this implies B prime a, epsilon is disjoined from the 0 set and a is not in E. So, this implies B a, epsilon is contained in G minus E. So, in either case there is a neighbourhood around a which is contained in G minus E. If a is not in E so, this implies E is closed. So, E is both closed both closed and open in G and E is non empty. So, from previous topological considerations, we know that so E has to be equal to G. Since, G is converted that is the limit point set is the whole set G; that is f is identically 0 on G. So, that is the proof of this theorem. So, we will continue with this in the next session.