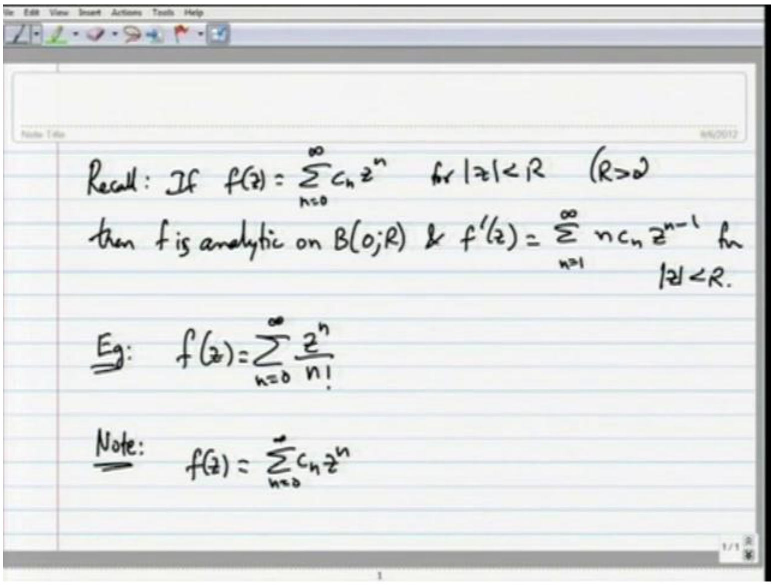
Complex Analysis

Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 4
Further Properties of Analytic Functions
Lecture - 3
Taylor's Theorem

Hello viewers, in the last session, we have seen that power series are analytic, and also that their differentiation is obtained by differentiating that term of the power series, and then summing up.

(Refer Slide Time: 00:41)

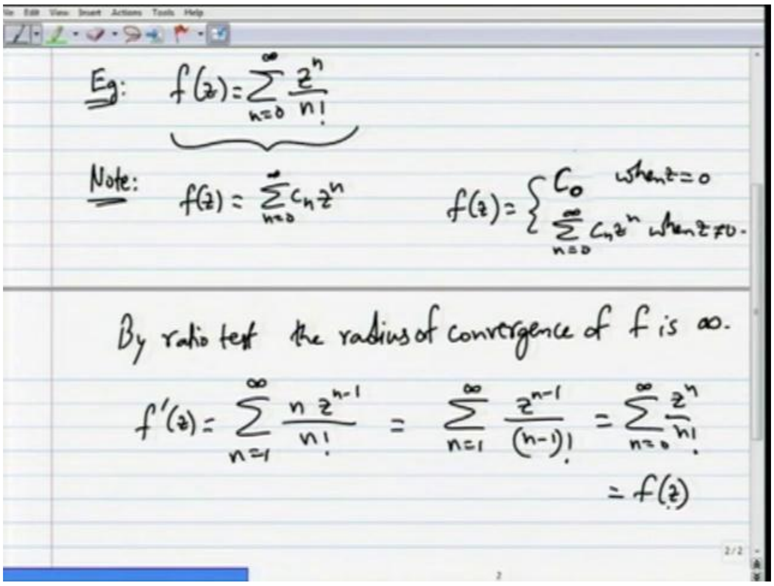


So, if f of z, so recall that if f of z is equal to sigma c n z power n, power series of type one, n equals 0 to infinity. Then for, for mod z strictly less than r, here we will assume r is strictly positive. Then we saw that then f is analytic on B (0;R), and f prime of z has the form n equals 0 through infinity n c n, so n equals 1 through infinity, I apologise. 1 through infinity n c n z power n minus 1 for r, for mod z less than r.

So, the differentiation of the power series is differentiating it term by term, and in summing up within the radius of convergence. So, that we have seen last time. So as an example, as an example, we will see the following if we consider the power series f of z equals z power n by n factorial sigma n equals 0 through infinity. Then we will show that this is the power series expansion for the exponential function e raise to z.

So, firstly there is one note, so within this example itself, I will make that note. There is a little ambiguity here, so when we write f of z, so when we write f of z equals sigma n equals 0 through infinity c n z power n for type one series. Then what we really mean is that, this is f of z.

(Refer Slide Time: 02:54)

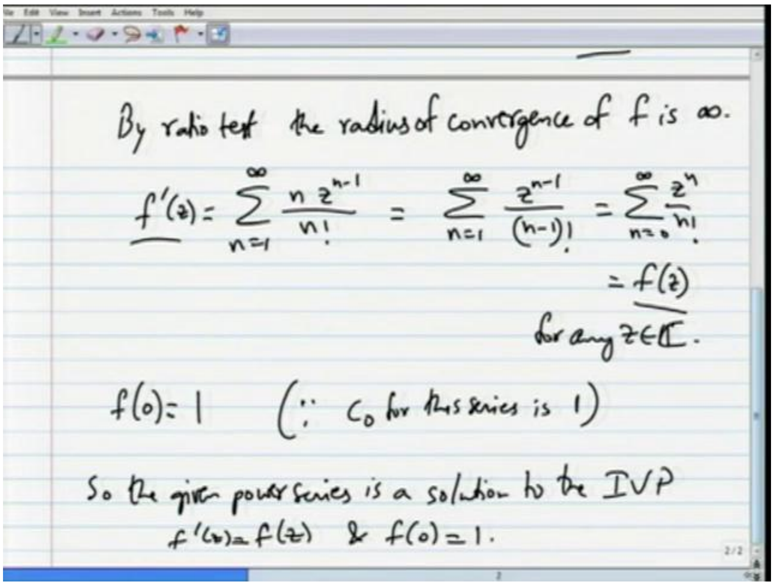


So, for type one series, this is a split function f of z is equal to C 0 when z equals 0, and f of z is that series when z is not equal to 0. Since 0 power 0 is not defined, we split the definition of f of z. Note for type two series that, f of z is defined to be C 0 when z equals a for type two series, and f of z is sigma c n z minus a power n when z is not equal to a. one needs to make distinction because at z equals 0, and n equals 0, you have the ambiguity of 0 raise to 0, and 0 raise to 0 is not defined, 0 raise to 0 is not defined.

So, there is that split form, and then, so having made that note let us look at this series. This series has the property that firstly the radius of convergence of f is by, by simple ratio test calculation, by ratio test the radius of convergence of f is infinity. Or you can use that Cauchy Hadamard formula either way one can get this. So, so we can differentiate this power series, so f prime of z for the same radius of convergence will looks like n equals 1 through infinity of the differentiation of z power n is n n z power n minus 1 divided by n factorial.

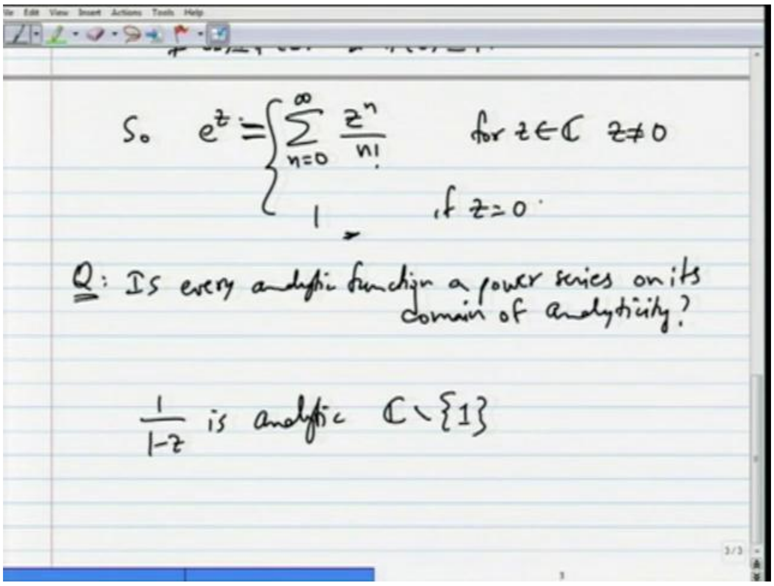
So, I will cancel the n in the numerator and the denominator. So, I will cancel n in the n factorial to the get n equals 1 through infinity, z power n minus 1 divided by n minus 1 factorial. By readjusting the index once again this is nothing, but n equals 0 through infinity z power n y n factorial which is your f of z to begin with.

(Refer Slide Time: 05:24)



So, we have shown that f prime of z is nothing, but f of z for mod z, or for any z belongs to r, or, not r, I apologise, to C, because the radius of convergence is infinite. Not only that, observe that f of 0 is 0, so or rather 1 f of 0 should actually equal 1, because C 0, since C 0 for this series, since C 0 for this series is 0, is, is 1 rather. 1 by 0 factorial which is 0 factorial is defined to be 1, so you have 1, f of z, f prime of z is equal to f of z. So, f, so the given power series is a solution to the initial value problem f prime of z equals f of z and f of 0 is equal to 1. So, since we know that, or by the existence and uniqueness of solutions to differential equations, we know that the solution to this i v p has to be unique.

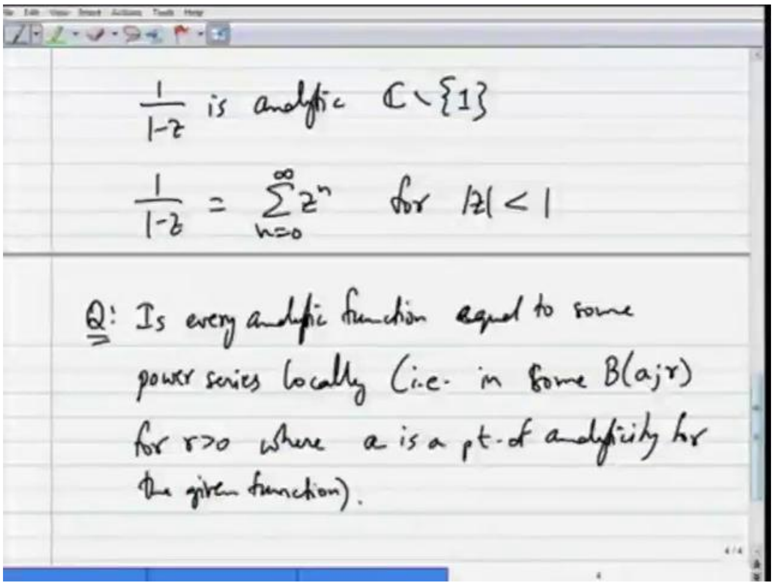
(Refer Slide Time: 07:12)



So, we know that e power z is also a solution, we defined in the solution to this i v p as e power z. So, e power z has to equal this power series sigma z power n by n factorial, n equals 0 through infinity. At z equals 0, we define this to be 1. So e power z is this power series has this power series expansion around z equals 0. For this is for a any z belongs to C, for z belongs to C, z naught equal to 0. So, I will split this, this is equal to 1, if z is equal to 0. So, that is the power series expansion of e power z, and it is valid all throughout the complex plane, all right.

So, we see that e power z has a power series expansion for any complex number, modulus of this small, small change at z equals 0. So, now a question can be as follows is every analytic function a power series or, so to say every analytic function be expressed as a power series. Well, we already know that the answer cannot be yes, or we know that the answer is no, because 1 by 1 minus z is analytic on. So, let me write the question the question is every analytic function a power series on its domain of analyticity that is the question.

(Refer Slide Time: 09:40)

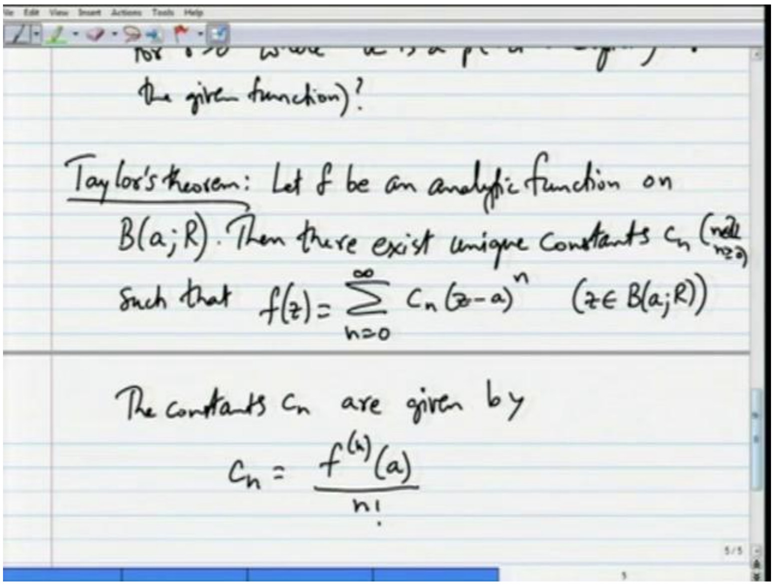


So, we know that 1 by 1 minus z is analytic on all of C except at one point at namely 1.So, on C minus 1, on the set C minus 1, 1 by 1 minus z is analytic. But, but 1 by 1 minus z is equal to one kind of power series which is the geometric series, z power n n equals 0 through infinity for, only for mod z less than 1. Beyond mod z less than 1, we know that we know at least that this power series is not correct for 1 by 1 minus z. So, 1 by 1 minus z does not equal this power series for mod z less than 1, mod z greater than 1. Sorry.

So, so the point is not all analytic functions are equal to a unique power series on the whole domain of analytic or their analyticity. So, that happens, so then one can ask the following modified question is every analytic function equal to some power series if not, if not on all of the domain of analyticity at least in a small neighbourhood around a point of analyticity. So, is every analytic function is equal to some power series locally i e in, in some b a r for r positive where, where a is a point of analyticity for f, for the given function. So, that is what locally means, locally means that there is a small disc around the point of analyticity of radius of positive radius such that you know certain property holds that is a local property and the property we are interested in here is whether every analytic function is locally an analytic power series rather.

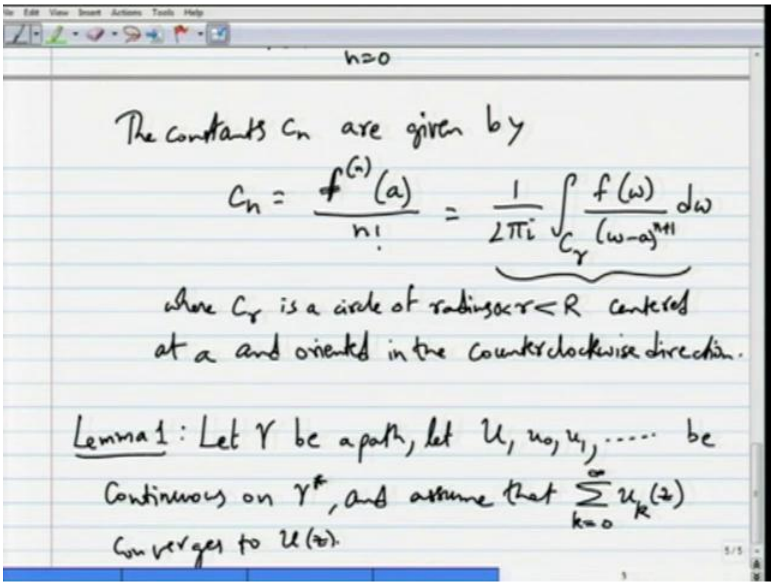
So, the answer to this question turns out to be yes and, and this is answered by Taylor's theorem, the Taylor's theorem tells us that every analytic function is locally a power series. So, let us see what is the Taylor's theorem says.

(Refer Slide Time: 12:27)



So, here is the statement of Taylor's theorem. So, let f be an analytic function on b a r, so since we are interested in expressing the analytic function as power series locally, although f could be analytic on a bigger set, we will only assume for the time being that its analytic on, on the disc itself b a r. Then, there exists or there exists unique constants C n such that, C n I should say n belongs to integers, n greater than or equal to 0, n belongs to integers and n greater than or equal to 0. There exists unique constants C n such that f of z is equal to sigma n equals 0 through infinity, c n z minus a power n z belongs to b a r, for any z belongs to b a r, f of z equals that power series on that disc.

(Refer Slide Time: 14:43)



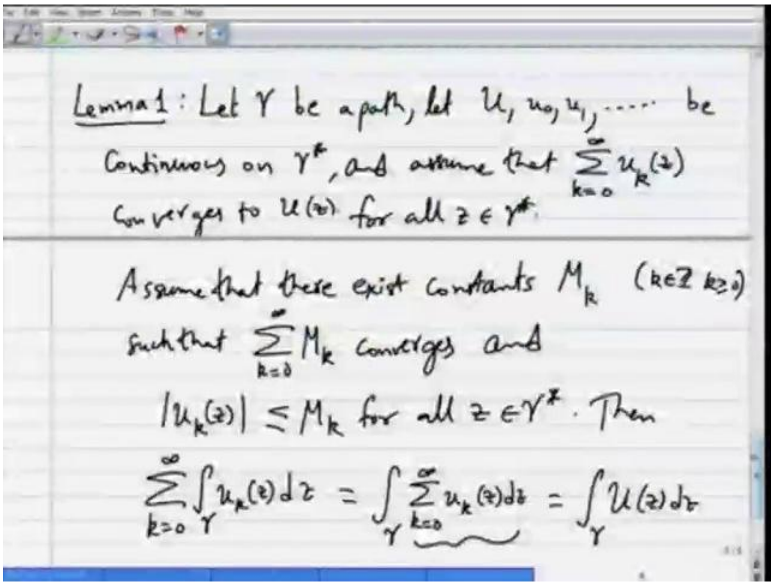
The constant C n furthermore, so the constants C n or given by C n is equal to the n eth derivative of the function f at a divided by n factorial, which is actually equal to 1 by 2 pi i the integration over C r, I will explain what C r is. It is a circle of radius little r around a of f of w by w minus a raise to n plus 1 d w, where C r is circle of radius r, radius r strictly less than capital R. So, I will say 0 strictly less than little r less than capital R, oriented centred at a, and oriented in the counter clockwise direction, in the counter clockwise direction, that is the positive direction recall.

So, firstly note that by, by what we have noted earlier by the theorems earlier we know that every analytic function is differentiable any number of times. So, the n eth derivative of an analytic function f exists in its domain of analyticity. So, this expression here f n the n the derivative of f at the point a is valid. Makes sense, and by Cauchy’s integral formula for the n th derivatives, we know that, we know that the n th derivative of f at a divided by n factorial is precisely the expression on the, on the right hand side. So, that is the statement of the Taylor’s theorem.

So, it not only tells us that the, that an analytic function is locally expressible as power series, but also that the, the constants C n in the power series are given by the n th derivative of f at a divided by n factorial. So, in order to prove this theorem, we will prove this theorem, in order to prove this we first need a couple of lemmas. So, here is the first lemma, the lemma. So, I am following the development here in from one of the text books introduction to complex analysis by Pressley. So, lemma 1 is since I did not define what uniform convergence is, I will have to I will have to, I need a result which will tell me when I can exchange integration and summation.

So, here is lemma 1, let gamma be a path, and let u, u naught, u 1. So, any u n for that matter, n belongs to non negative integers, so u be, be continuous on the trace of gamma in the complex plane, and assume that sigma k equals 0 through infinity of u k of z converges to u of z which is point wise convergence for all z belongs to gamma star trace of gamma in the complex plane.

(Refer Slide Time: 18:56)

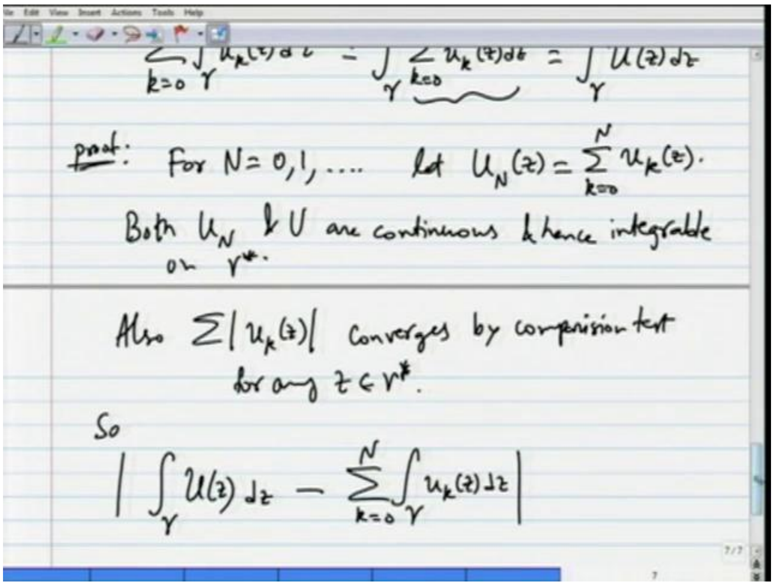


Now, assume that there exist constants M k, k belongs to non negative integers, k belongs to z, k greater than or equal to 0, such that sigma M k, k equals 0 through infinity converges and this M ks are related to u ks as follows u k of z in modulus is less than or equal to M k for all z belongs to the trace of gamma. So, for any z belongs to the trace of gamma the modulus of u k of z is less than or equal to M k. Then, then this sigma then the collusion is sigma k equals 0 through infinity of the integration contour integration of u k of z over the contour gamma is equal to the integration over the contour gamma over summation of u k of z d z k equals 0 through infinity.

So, we can exchange the integration and the summation. So, this is equal to integral over gamma of u of z d z by definition of summation. Since the summation converges to u of z, I am just replacing the summation by capital u z. So, we know that from our experience with functions of real numbers and series of, of real functions we know that it is not necessary that we can always exchange the integration and summation. So, under these circumstances this lemma gives us a sufficient condition under which we can exchange the integration and summation. Recall that the summation involves the limiting process, and if it were finite process, if you are taking finite sums you can always exchange the integration and summation because integration is a, is a linear operator.

So, but when you have infinite sums you have to be more careful there is a limiting process and it does not commute very well always with integration process. So, this lemma tells us that if you have a convergent series of functions term by term series of or point by point convergent series of functions. So, for each point z this series of functions converges to u of z, and at least for all z belongs to gamma star, and if these u k of z in modulus are less than or equal to some fixed constants M k, and sigma M k converges, then you can exchange the integration and summation for this u k of z over the contour gamma. That is the content of this lemma.

(Refer Slide Time: 22:52)

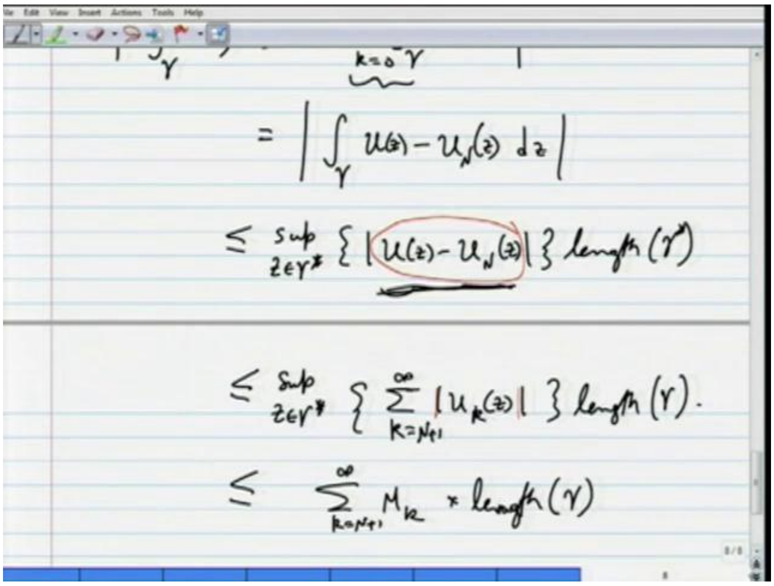


So, here is the proof. So, first we will take finite sums of these u k of z, so for N equals 0 1 etc. For any non negative capital N let U N of z capital U subscript of z be the finite sum sigma k equals 0 through capital N of u k of z, little u k of z. So, both U N capital U N and capital U are continuous of course, capital U N is continuous, because it is a sum finite sum of continuous functions, and capital U is given to be continuous. It is continuous by hypothesis.

So, capital U are continuous where hence integrable on gamma star on the set gamma star in the complex p. Now, also we know that sigma modulus of u k of z converges by comparison test that is clear, because sigma Mk converges, so the hypothesis of the lemma says sigma Mk converges, and term by term modulus of u k of z is less than or equal to Mk. So, definitely sigma u k converges for any z by comparison test for any z belongs to gamma star.

So, now what we will do is we estimate u of z using the u capital N of z. So, what we will do is that we will take the difference of u of z with u k of z but, what we will do is that we will take the difference of the integrals of these. So, and estimate them. So, contour integration of u of z d z over gamma minus sum from k equals 0 to capital N of the integration, the contour integration of u k of z d z over gamma in absolute value in, in the modulus.

(Refer Slide Time: 25:48)



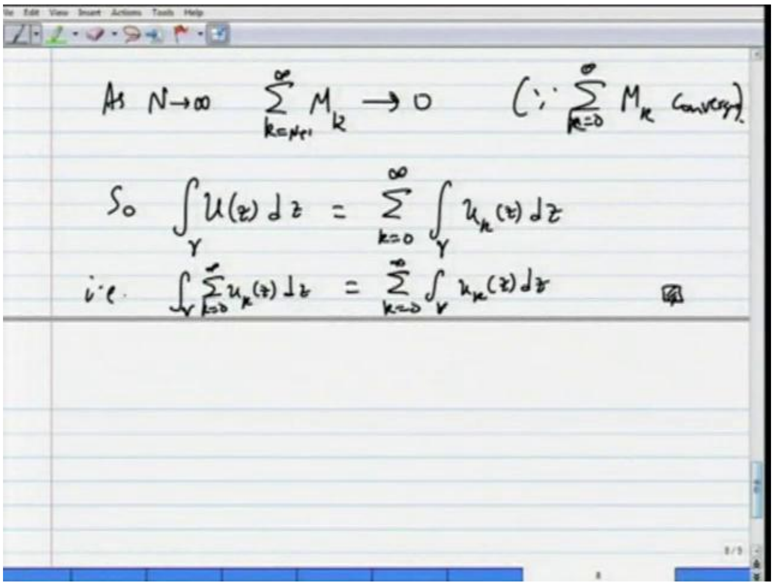
This is nothing, but the modulus of the contour integration of u of z minus u capital N of z d z. One and the same, you can exchange the integration and summation here, because your, you have a finite sum here k equals 0 through n.

So, this is less than or equal to, this is less than or equal to by one of the estimations we had on integration. This is less than or equal to the supremum over z belongs to gamma star of the modulus of the integrand u of z minus u n of z, times the length of gamma, the length of gamma star rather in complex, the length of the contour gamma. So, I will just say gamma, so this is less than or equal to, in turn this is less than or equal to, well the difference u of z minus u n of z is equal to, is equal to the tail of the series sigma u k of z.

So, this is less than or equal to the supremum over z belongs to gamma star of the modulus of the tail, and in which turn by infinite version of triangle inequality is less than or equal to k equals n plus 1 through this is the tail of the absolute value of series u k of z. so I am using an infinite version of the triangle inequality here. So, then we, we get that the earlier is less than or equal to. So, this is actually equal to notice that this thing within the, within the absolute value or the modulus is equal to this stale k equals n plus 1 through infinity. Sigma k equals n plus 1 through infinity of u k of z. And then, and then you shift the absolute value inside like this by triangle inequality, and then you get this times the length of gamma.

So, then this in turn less than or equal to the supremum well each of these u k of z in modulus, we know it is less than or equal to M k by hypothesis.

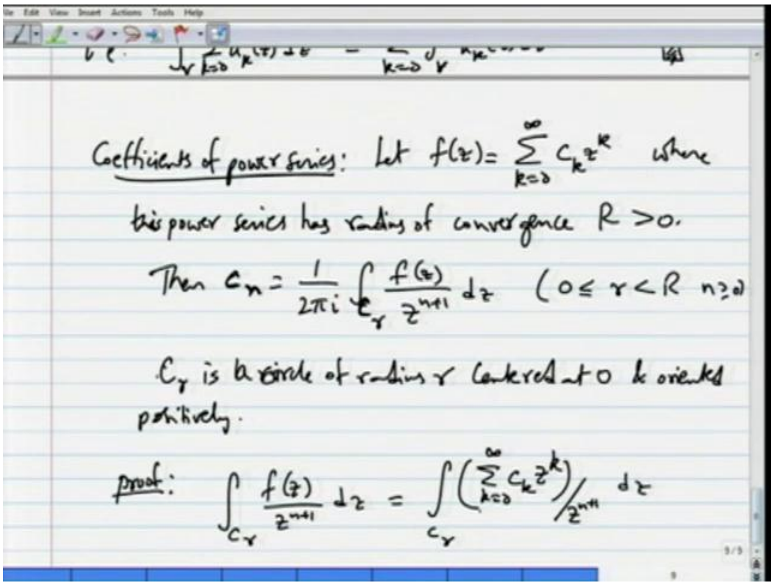
(Refer Slide Time: 29:00)



So, this is less than or equal to sigma k equals N plus 1 through infinity of M k times the length of gamma, and as, as N goes to infinity since M k, sigma M k converges as N goes to infinity sigma k equals N plus 1 through infinity M k tends to 0 since sigma M k converges. Right? If you have k equals 0 through infinity converges, if you have a convergent series the tail of the series we know tends to 0, so this, so this allows us to say that, so this estimate which we started of this as limit as N goes to infinity will be, will be tending to 0, will be tending to 0. So, the integration of u of z, the contour integration of u of z d z is equal to, so we have in modulus this is equal to zero. So, this is equal to sigma k equals 0 through infinity, limit as capital N goes to infinity, so we have infinity on the top of the summation, so its top bound of summation of the contour integral of u k of z d z.

So, this i e gamma sigma u k of z d z, because that is what capital U of z is, k equals 0 through infinity. This is equal to sigma k equals 0 through infinity, integral gamma, integral over gamma of u k of z d z. Or in short we can exchange the summation and integration under these circumstances, so it is a end of proof of this lemma and we will use this for the proof of Taylor's theorem. So, this is the first result that we need, and the next result that we will need to prove Taylor's theorem is as follows.

(Refer Slide Time: 31:23)

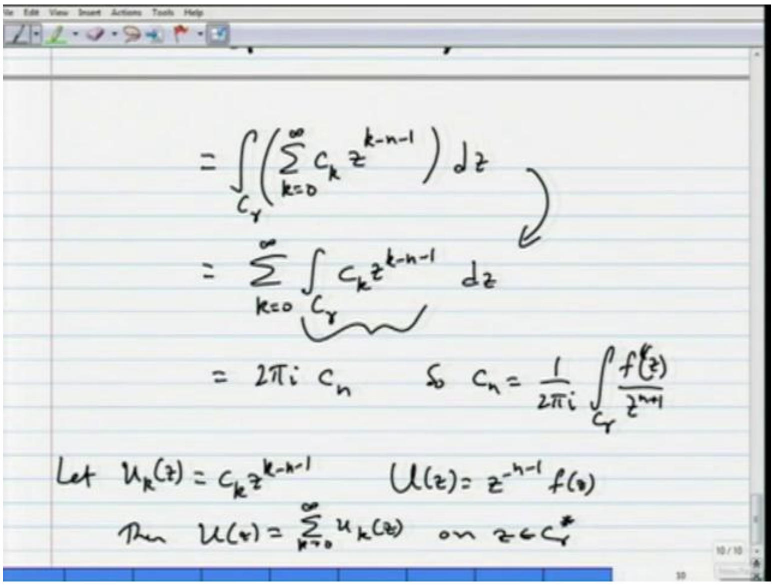


So, this is about the co-efficients of power series. So, let f of z equals sigma k equals 0 through infinity, c k z power k where the power series, where this power series has radius of convergence R strictly greater than 0, so positive radius of convergence. Then we know that capital little f is analytic, so its differentiable term by term or I mean its differentiation is the, differentiation of these terms and then summing it up.

So, there is also the fact that then the coefficients of this power series C n are given by 1 by 2 pi i integration over C r f of z by z power n plus 1 d z where 0 less than or equal to r strictly less than r n greater than or equal to 0, and and C r. So, I will write below and C r there is a flash of notation may be this is little c n and this is capital C r. So, and capital C r is a circle of radius r is the contour, the circle of radius r centred at 0, and oriented positively.

So, this tells us that if you take a power series which ahs positive radius of convergence, then its coefficients are, then its constants are unique. And there is a specific formula for that constant, and the, the constants are given by C n equals 1 by 2 pi i, the contour integration of f of z divided by z 4 n plus 1 over the contour C r, where r is C r is circle of radius little r strictly less than the radius of convergence, and its oriented positively. So, let us see the proof of this theorem of this little lemma, and it goes as follows the integration of f of z by z power n plus 1. The contour integration over C r of f of z divided by z power n plus 1 d z; this is equal to the integration over C r of clearly sigma k equals 0 through infinity of c k of z power k divided by z power n plus 1 d z.

(Refer Slide Time: 35:08)



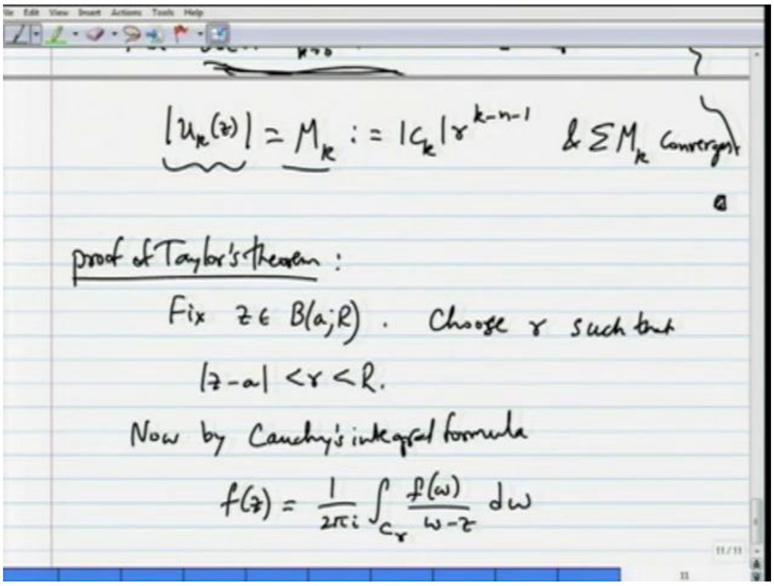
This in turn is the integration of C r sigma k equals 0 through infinity of c k. I will divide z power n plus 1 term by term, so I will get z power k minus n minus 1 d z. Then all this d z, and provided I can interchange the limit or, sorry the summation and integration.

So, I get sigma k equals, so let us for the time being assume that I can interchange them, I will show that I can interchange them by the earlier result. So, then this is equal to sigma k equals 0 through infinity of the integration contour integration over C r of C k z power k minus n minus 1 d z. C ks are constants, so this is equal to 2 pi i. So, there is only one survivor for this integral as k runs from 0 to infinity, when k equals n you have this integration for small enough r, this integration is actually equal to 2 pi i times C n.

So, this will give you 2 pi i times C n, then because of the survivor z power minus 1 when k equals n. The other functions z power k minus n minus 1 for k naught equal to n give 0 upon integration by the fundamental integral that we calculated earlier. Since C r is a simple closed curve, the integration upon C r of these functions gives us 0. So, this is 2 pi i times C n, and I should just now justify how I can exchange the integration and summation,

So, by the way this tells me that C n is actually equal to, so C n is equal to 1 by 2 pi i times the integration of f of z divided by z power n plus 1 over circle of radius r. So, firstly notice that if I set, let u k of z equals c k z power k minus n minus 1, and capital u of z is z power minus n minus 1 f of z. So, then I know that, then u of z is equal to sigma k equals 0 through infinity u k of z on at least z belongs to C r star. It is actually true on all of the, for all z belongs to the disc of convergence.

(Refer Slide Time: 38:31)



So, but at least true on the trace of C r, also the modulus of u k of z, and all these functions are continuous. This is clear because that is what f of z is. So, this is clear, and these functions are continuous, u k of z is equal to M k, what is M k? This is modulus of C k r power k minus n minus 1. That is the constant we are looking at, and sigma M k converges, because little r is strictly less than capital R which is the radius of convergence; so this, this shows that the hypotheses of the previous lemma are satisfied.

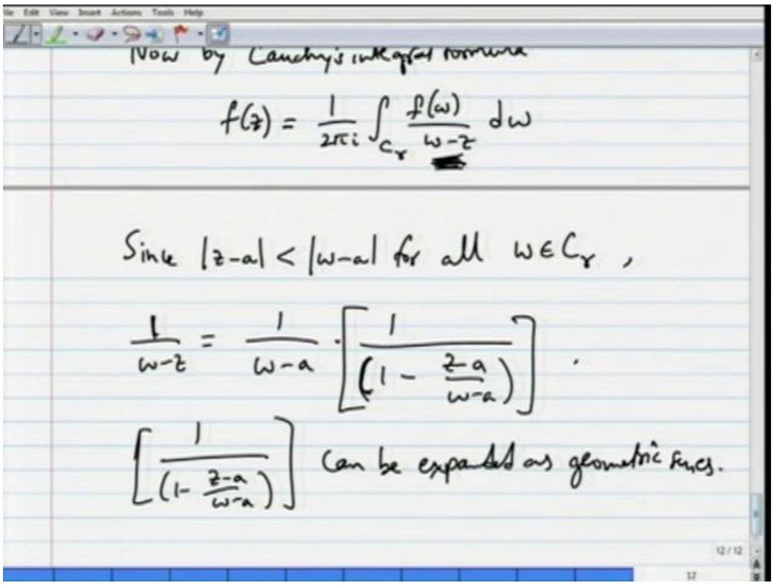
So, all the hypothesis are met, you, you have u k converging to sigma u k converging to u of z, and then u, and u ks are all continuous. Further you also have that, this modulus of u k of z is less than or equal to M k, actually it is equal to this M k and sigma M k converges. So, you can exchange the, the limit, sorry the, the summation, and the integration. So, you have this formula for C n the coefficients C ns in the power C s. So, that completes the proof of this second lemma and now we are ready to prove the Taylor's theorem.

So, we will start by fixing. Let, let me revisit the statement of the theorem. So, it says that function f is, is assumed to be analytic on B a R, R positive then f is locally a power series. And, and the constant C n are given by moreover the constant C n are given by the n the derivative of f at a divided by n factorial. So, there are several things to prove here, so firstly you fix a certain z belongs to B a R. what we have to show is that the power series actually converges to f of z. we have to show two functions are equal.

So, for z belongs to b a r we have to show that the values of the functions are equal. So, fix that, fix some z, and choose choose r little r such that, such that the modulus of z minus a is less than little r is less than capital R. So, we want a little r between modulus of z minus a and capital R now by Cauchy's integral formula, we know that f of z, the function f of z remember we are trying to reconcile the power series the value of the power series and the value of the f of z.

So, let me first look at the value of f at z, this is 1 by 2 pi i times the integration over C r, the contour integration over the contour c r, f of w divided by w minus z d z d w, so that much I know from Cauchy's integral formula.

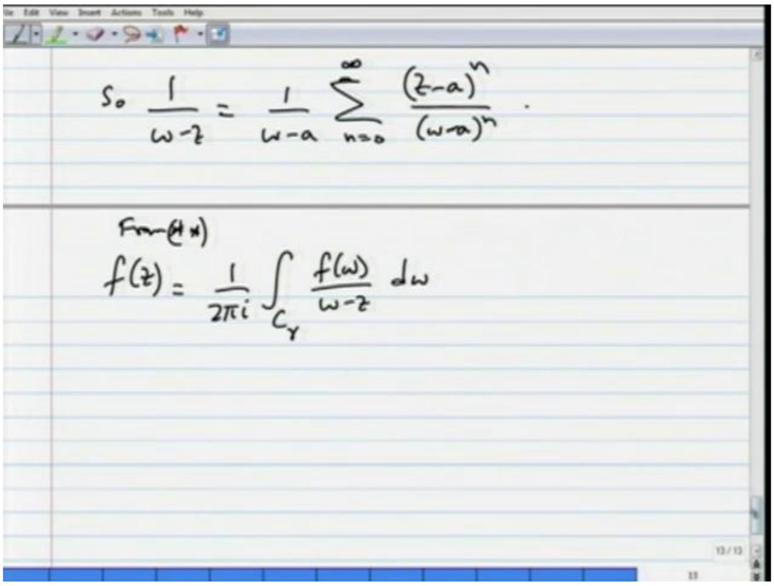
(Refer Slide Time: 42:19)



Now, since modulus of z minus a is less than modulus w minus a for all, w belongs to the circle of radius r centred at a. So, since z minus a, notice is strictly less than r, so the, what you can do is you can take 1 by w minus z which appears here in the integration.

So, in the integration the integrand is f of w divided by w minus z, I will concentrate on the denominator or namely, I will concentrate on 1 divided by w minus z. I will multiply f of w later, so 1 divided by w minus z can be written as, 1 minus w by a times times 1 by, this is a standard trick. We will write this as 1 minus z minus a by w minus a. And, since modulus of z minus a is less than modulus of w minus a, we can expand this piece within the square parenthesis; we can expand that as geometric series. I will write that 1 by 1 minus z minus a by w minus a can be expanded as geometric series.

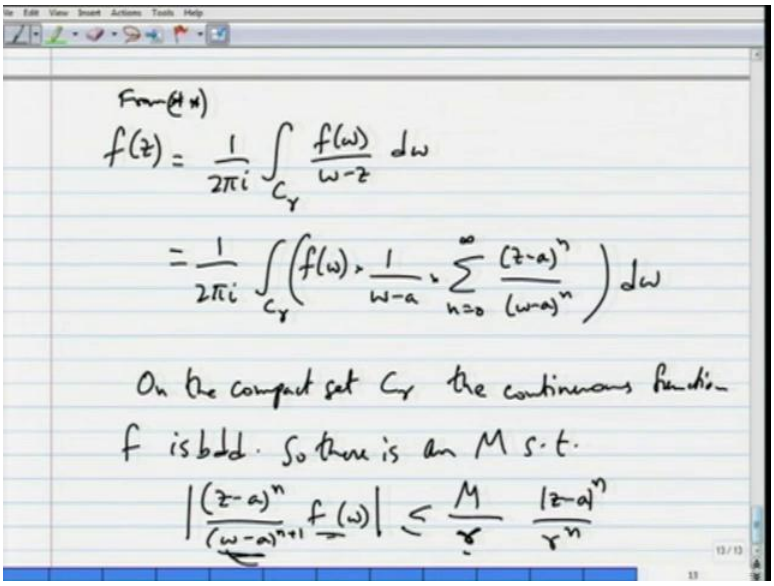
(Refer Slide Time: 43:56)



So, so 1 by w minus z, this expression here is equal to 1 by w minus a times that geometric series which is sigma, so sigma n equals 0 through infinity of z minus a by w minus a whole raise to 1 or z minus a power n divided by w minus a power n, so that much I can do. So, I have somehow got a power series within the value for f of z. Now, I will try to express f of z as power series itself.

So, f of z from, from that Cauchy's integral formula, I will say star, this is star, from star, did I use star earlier? From star or let me call it star star may be I have used star for star star. From star star f of z is equal to 1 by 2 pi i times integration over the contour C r of f of w divided by w minus z d w, and I have written w minus z in this fashion here.

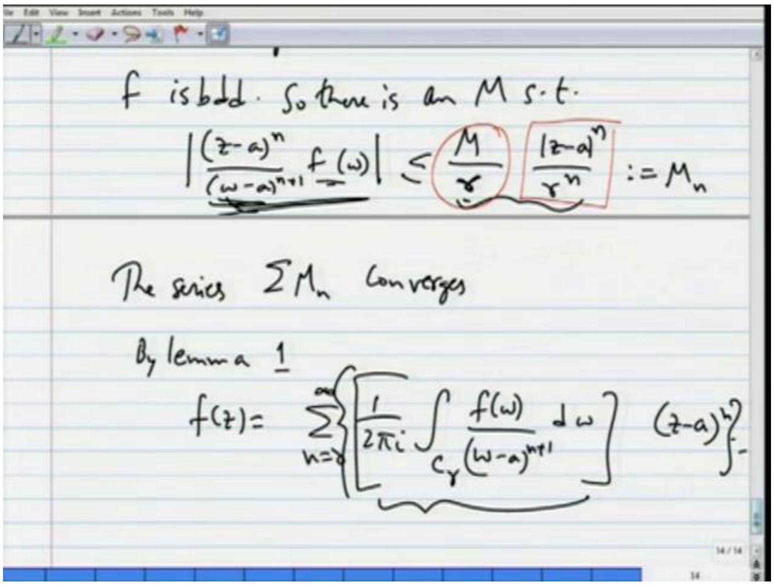
(Refer Slide Time: 43:25)



So, this is equal to 1 by 2 pi i integration over C r of f of w, I will keep the f of w a side times 1 by w minus a times the sigma n equals 0 through infinity of z minus a raise to n divided by w minus a raise to n, then all this d w. On the compact set C r, on the trace of C r, the continuous function f of w, f is bounded it is actually analytic.

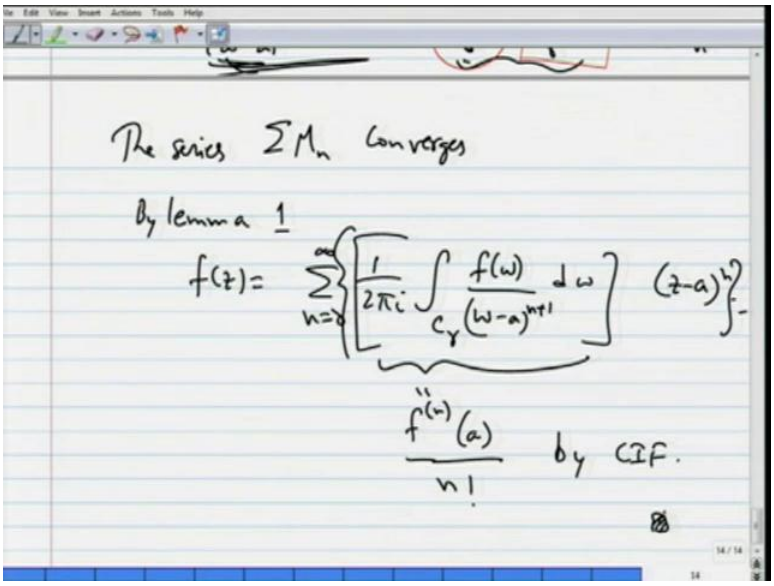
So, it is definitely continuous and it is bounded, so there is an M such that modulus of z minus a power n divided y modulus, or modulus of z minus a power n divided by w minus a power n plus 1 f of w is less than or equal to M divided by r. So, for this f of w I am using, and then w minus a in modulus z equal to r and then I will have, I, I took 1 r for modulus of w minus a, 1 modulus of w minus a. And then I take n r raise to n for the other remaining n, and then I have modulus of z minus a raise to n.

(Refer Slide Time: 47:17)



So let me call this constant, as so since we have fixed z all the, all of the right hand side here is a constant, let me call that M n. The series sigma M n converges, because what we have is a, is a constant, is a, it is a constant times geometric series, when we, when we sum up sigma M n, we have a constant times a geometric series, and that converges. Now, by lemma 1 therefore, we have, we have sigma M n converges, and this thing is less than or equal to M n for, for all n. And, so in here, I can exchange the integration, and summation, so by lemma 1 what I have is by lemma 1 f of z is actually equal to sigma n equals 0 to infinity 1 by 2 pi i times the integration over C r of f of w by w minus a raise to n plus 1 d w times z minus a raise to n.

(Refer Slide Time: 49:07)



So, I put this in parenthesis times z minus a power n, and everything is in sigma everything is in sigma, and this much from Cauchy’s integral formula we recognised that to be the n eth derivative of f at a divided by n factorial. So, by this is equal to this by Cauchy’s integral formula for higher derivatives. So, we proved all our assertions and that proves Taylor’s theorem. So, we will see, we will see some examples in the next session, I will stop here.