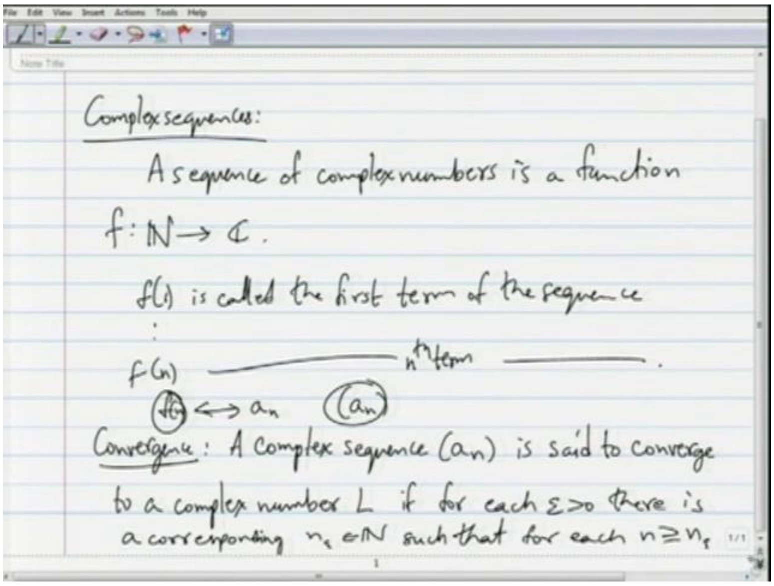
Complex Analysis
Prof. Dr. P A S Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 4
Further Properties of Analytic Functions
Lecture - 1
Introduction to Complex Power Series

Hello viewers, in this session we will start complex power series. So, we will see complex series which will help us to explore the properties, the local properties of analytic functions even further. So, to begin with we will see power series, complex power series as an example of analytic functions, and later we will show that every analytic function can be expressed as a power series around its point of analyticity. So we will make that clear in due course.

So, firstly to begin with, I will give a refresher on complex sequences and a series. So, the definitions and the some of the preliminary results in complex sequences and series are very similar to the results, in the corresponding results in real analysis or functions of one real variable real functions of one real variable.

(Refer Slide Time: 01:39)



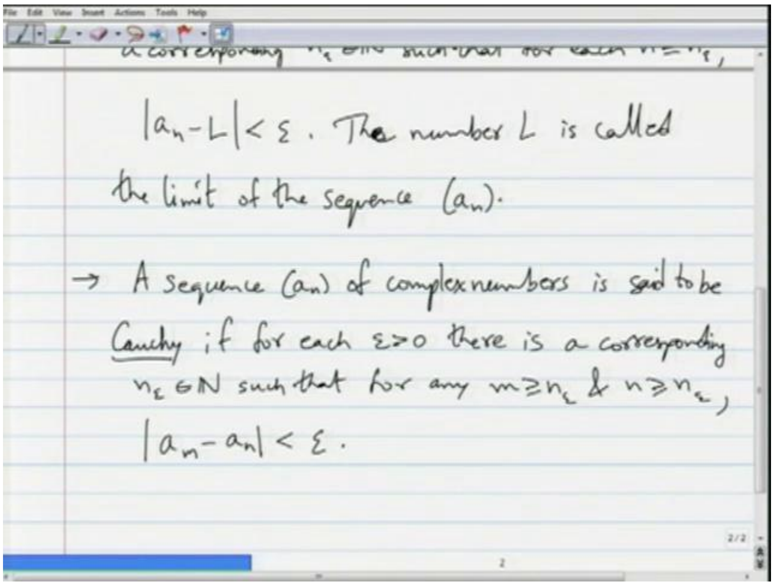
Let me first define complex sequences, so complex sequences, so what are complex sequences? A sequence like in the case of a real numbers, a sequence of complex numbers is a function f from the set of natural numbers to the set of complex numbers.

So, what that means is that assigned to each natural number or to each counting number there is a complex number.

So, there is a first complex number, a second complex number etcetera that we can talk of via this function f, so that is a complex sequence. So, f of 1 is usually called is called the first term of the sequence etcetera, generally speaking f of n is called the n eth term of the sequence. So, we can also talk about convergence of the sequence, like we do for real sequences, so convergence, so a sequence, a complex sequence in this case. So, a complex sequence, a complex sequence, I am switching to the notation a n, so to say what this a n is, this f can also be denoted by, so f of n is also denoted by a subscript n sometimes the n-th term of the series and usually when one writes a n in soft parentheses like this that means the sequence f okay.

So, that means this particular sequence f whose n-th term is given by a n a subscript n. So, a complex sequence a n is said to converge to a complex number L, if for each epsilon positive there is a corresponding n naught or n epsilon let me say in the counting numbers such that, for each n bigger than are equal to this particular n epsilon so again we will go back okay.

(Refer Slide Time: 04:51)



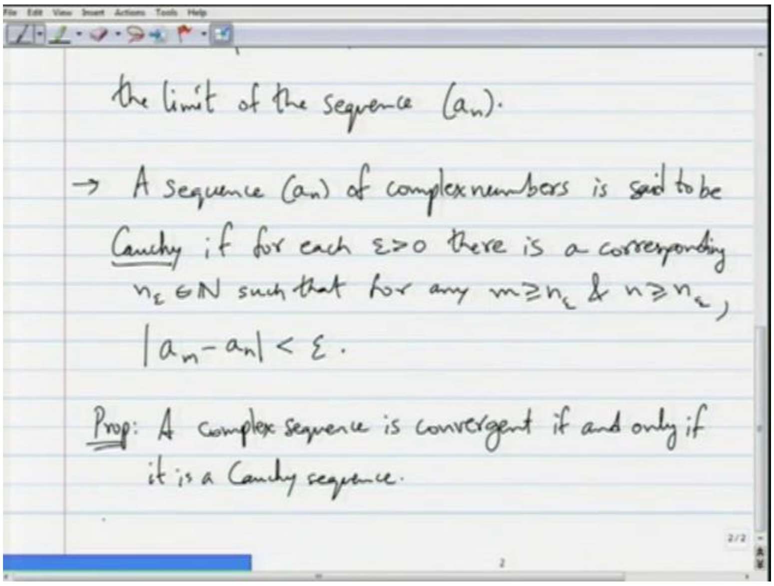
So, what happens is, the absolute value the modulus of a n minus L is strictly less than that particular given epsilon, so the number, number L is called the limit of the sequence a n. So, this definition is very similar to the definition of convergence of a sequence of real number to a limit L okay.

So, the only thing to notice here is that, in case of real numbers we consider the absolute value of a n minus L, where a n is the nth term of the series and we wanted that to be less than epsilon for every n greater than or equal to a particular capital N depending on epsilon. In this case that absolute value is replaced by the modulus of the complex number a n minus m, so apart from that particular difference pretty much the definition is the same. So, we can use the modulus, in modulus of a complex number in place of the absolute value for real numbers, so for this that is the definition of convergence.

Next, we would like to talk about a Cauchy sequence, so like in the case of real sequences we have we can call a complex sequence Cauchy, if a similar condition holds, so here is the definition a sequence, so this is the next point. A sequence a n of complex numbers, is said to be Cauchy excuse me, is said to be Cauchy. If for each epsilon greater than 0 there is a corresponding N epsilon belongs to the set of natural numbers such that for any m bigger than n epsilon and n bigger than n epsilon.

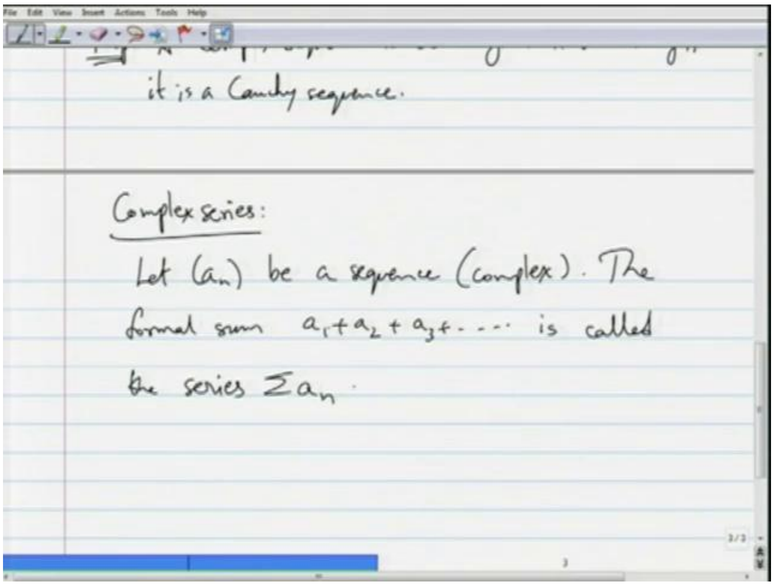
The modulus of the complex number a m minus a n, the difference, the modulus of the difference like that is strictly less than epsilon. So, the condition for a sequence to be called Cauchy is very similar to the condition for real sequences to be called Cauchy, except that once again the absolute value in the case of real numbers is replaced by the modulus in the case of complex sequence.

(Refer Slide Time: 08:23)



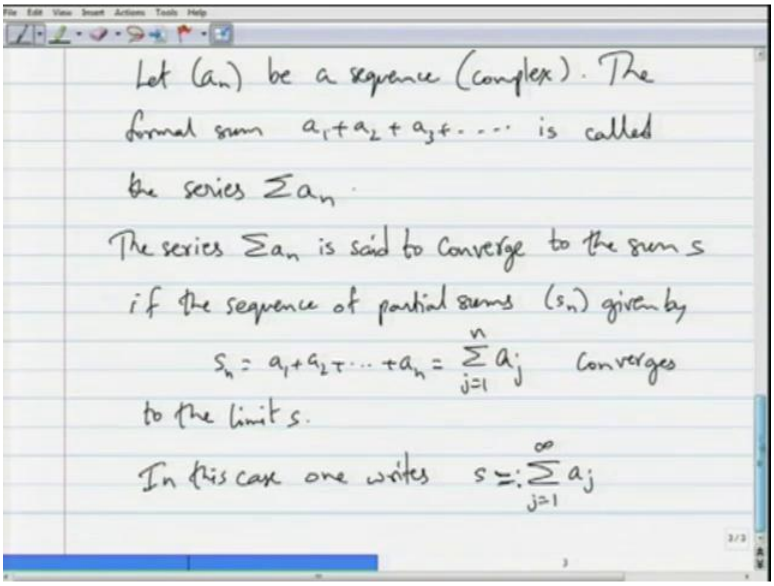
Once again why is this Cauchy condition useful? Once again I will state that preposition here, here is a preposition. A complex sequence is convergent to some number L, if and only if it is a Cauchy sequence, so this preposition which I am stating without proof, proof is very similar to the real case. This preposition tells us that, complex Cauchy sequences can be judged to be convergent without actually knowing what the limit is, so just by considering the difference, the modulus of difference of terms eventually, you can you can conclude whether the sequence is convergent are not. So, that is the use of this Cauchy condition, so like you know from a real analysis and then we proceed to define complex series this line.

(Refer Slide Time: 09:35)



So, consider a complex sequence, so let a n be a, a sequence complex sequence and the formal sum a 1 or a 1 a 1 plus a 2 plus so on, a 3 plus so on, is called the series sigma a n. So, that formal sum where you add all the terms in that sequence is called a series, a complex series, all right this is very similar to once again a real series.

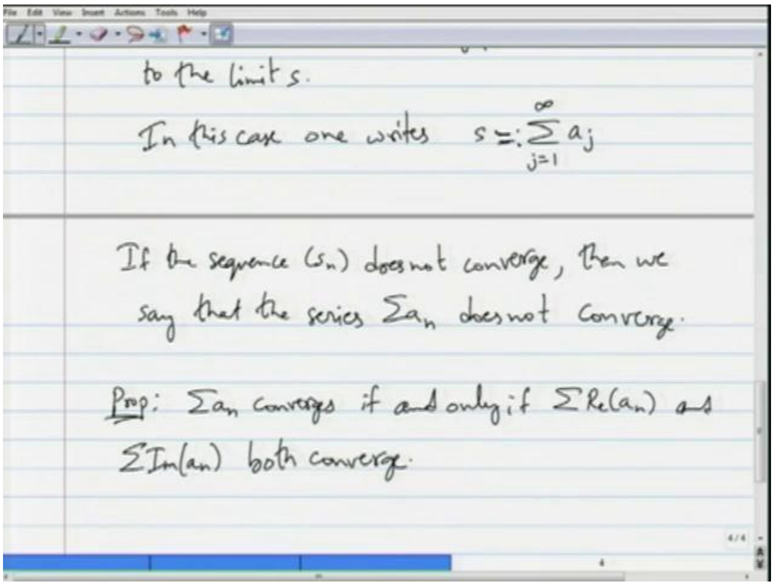
(Refer Slide Time: 10:43)



The series we have the notation convergence of series the series sigma a n is said to be con are said to converge to the sum S, if the sequence of partial sums s n given by, so you consider the partial sums s n equals a 1 plus a 2 plus so on, until a n or more formally sigma j equal 0 through are 1 through n of a j. So, the sequence of partial sums like that these s n's converges to the limit s. So, consider the sequence of partial sums and if this sequence converges to a limit s you say that, the series sigma a n itself converges to the sum s okay.

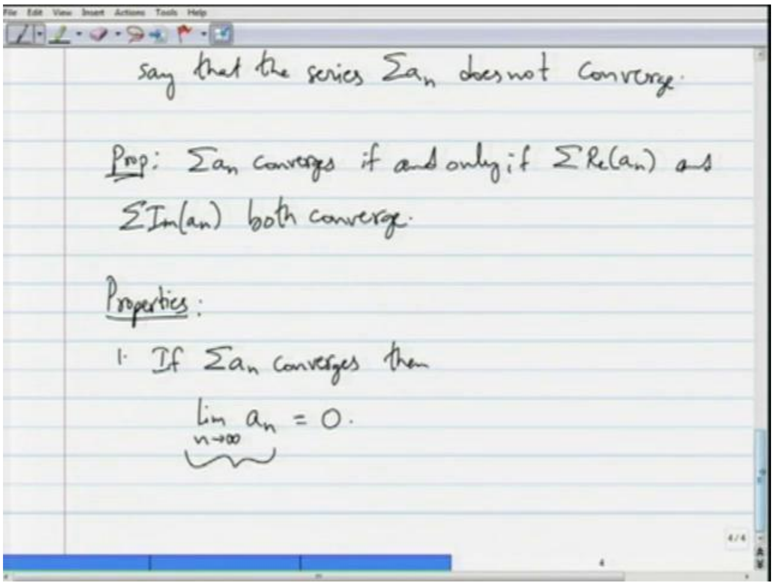
In this case, one writes s is equal to sigma n or I will use j, j equals 1 through infinity a j, so that is the notation, so in this case one, one defines are, one writes s that some s like that, so that is a notation for this sum s.

(Refer Slide Time: 12:47)



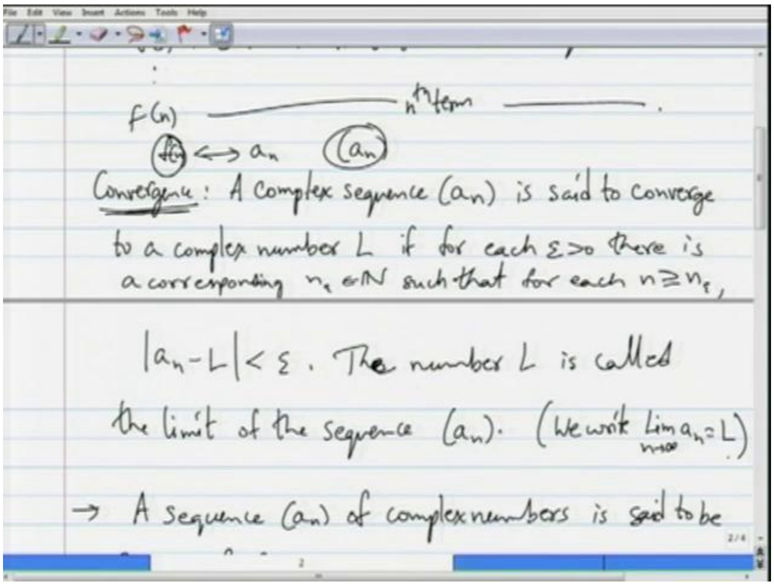
If the sequence s n does the of partial sums does not converge, then we say that the series sigma a n also does not converge and one can by splitting a complex number a and n to it is real and imaginary parts. One can conclude the following sigma a n converges if and only if the sigma of real parts of a n. The real part of a n and sigma of imaginary part of a n-th converge, that is just by splitting a complex number a n. The n-th term in the sequence again into its real and imaginary parts and then that that is used in the partial sums sequence and from there you can conclude this preposition. So, this preposition the proof of this preposition itself is not very hard it is just splitting a complex number into its real and imaginary parts okay.

(Refer Slide Time: 14:36)



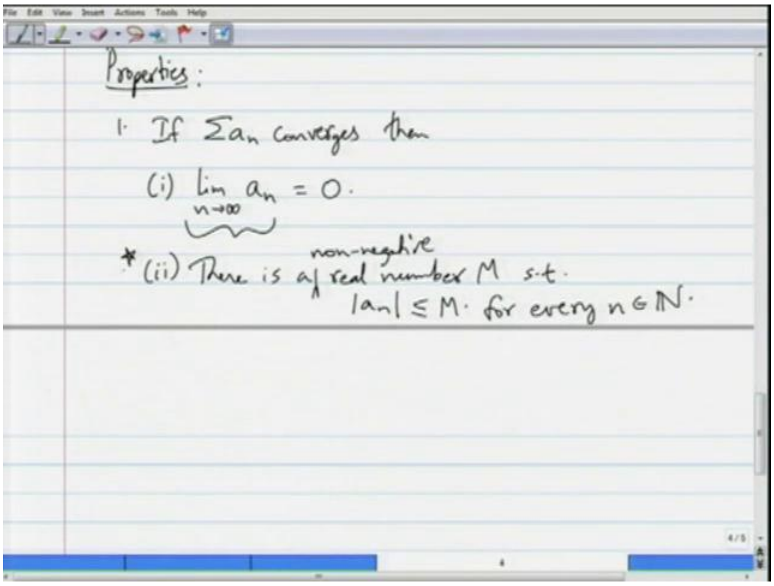
So, we will now see some properties of series complex series. So, firstly if sigma a n converges, then the limit as n tends to infinity of the n-th term in the sequence from which we are constructing these series. So, limit as n goes to infinity of a n is has to be 0. So, although I did not introduced this notation, limit n goes to infinity a n, it is what it means in the case of real sequences. So, notice the definition for convergence, let me go back, so notice the definition for convergence.

(Refer Slide Time: 15:44)



In this case, we write we normally write we write limit as n goes to infinity of a n is equal to L, this complex number L. So, that is the notation like in the case of real sequences. So, the viewer might be familiar with it from real sequences and we are using a similar notation for complex numbers, limit as n goes to infinity of a n is has to be 0, if sigma a n, if the series sigma a n converges.

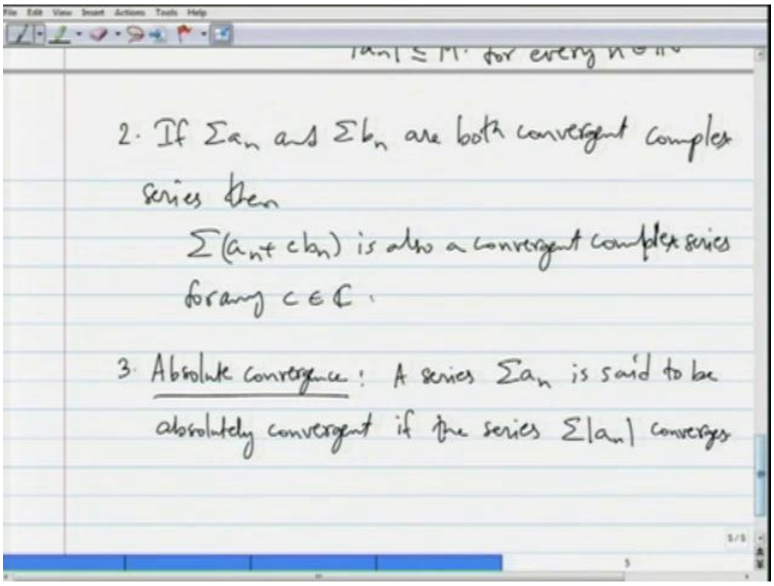
(Refer Slide Time: 16:21)



So, that is one that, I will call this as point 1 and the second of these is that, there is a number m. So, I should say this is there is a real number m or even further I can say there is a positive real number m such that, well maybe I should say there is a non negative because it could be 0, there is a non negative real number m such that, the modulus of a n is less than or equal to m. So, whenever the series converges sigma a n the series sigma n converges, the modulus of the n-th term has to be less than or equal to m, for every for every n belongs to.

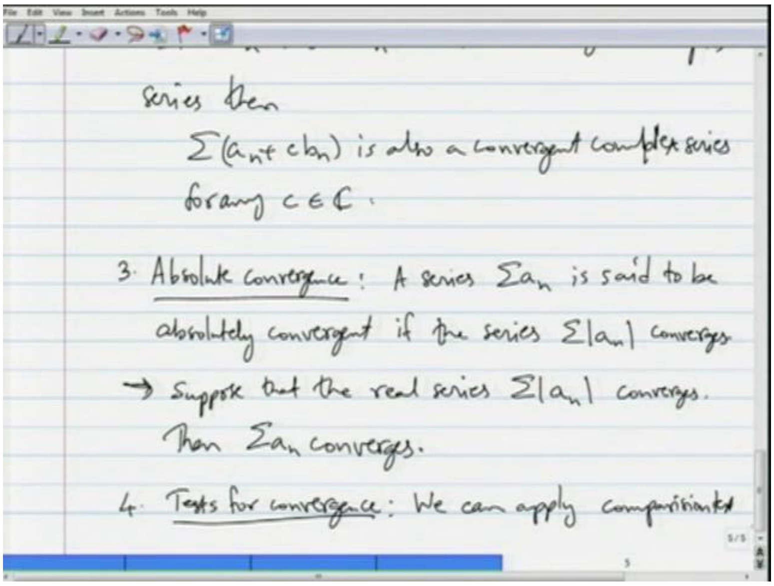
So, that has to hold for there is a fixed number m for which modulus of a n is less than or equal to m alright. So, this property is going to be used several times in the in this or the coming lecture, so this is important and then well. The proof of either of these is very similar to the corresponding proof for complex or sorry a real sequences and series, so I will omit the proof the proof is very simple. So, the viewer can treat it as an excise.

(Refer Slide Time: 18:22)



So, that's property one and then property two, if sigma a n and sigma b n are both convergent complex series, then sigma a n plus b n or more generally, sigma a n plus c times b n is also a convergent series, complex series for any c belongs to C. So, for any fixed for any c belongs to C for any complex number like that, the linear combination a n plus c b n is also a convergent a complex series whenever, sigma n and sigma b n are convergent. Once again the proofs of these properties are very easy, absolute convergences, a series sigma a n is said to be absolutely to be convergent, if the series of absolutes values of the n-th term, namely sigma absolute a n converges okay.

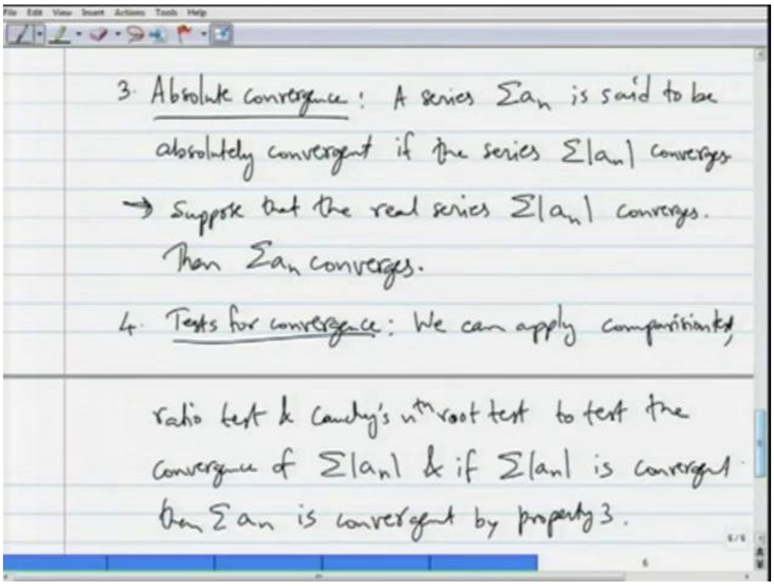
(Refer Slide Time: 20:24)



So and then by applying triangle in equality one can show that, suppose that one can show the following suppose that, the real series notice that the absolute value series, where you take the modulus of the n-th term. Consider the series sigma modulus a n that is a real number, real series, series of real numbers and suppose that real series converges, then you can conclude that then sigma a n converges okay.

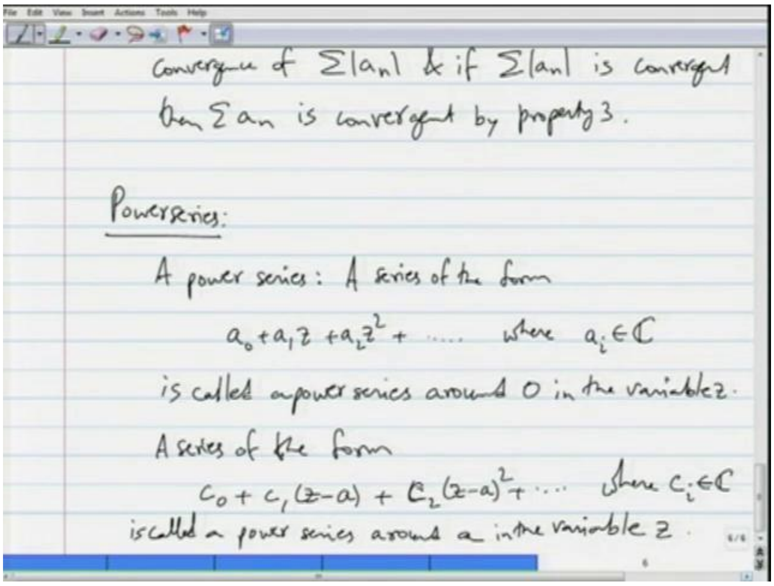
So, this property can be proved using a triangle in equality, on the partial sums and notice that this is also summed up sometimes, as if a series is absolutely convergent then it is convergent. So, absolute convergence means the convergence of the absolute values series are the modulus series, in the case of complex numbers alright. So, now the fourth property is the tests for convergence hold are about tests for convergences firstly.

(Refer Slide Time: 22:06)



So, we can apply a comparison test, ratio test and row test Cauchy's and row test. To test the convergence of sigma absolute n or sigma modulus n and if sigma a n is convergent, sigma absolute a n is convergent, then sigma a n is convergent by the above, by property three by what we have just said. So, since sigma absolute a n is a real series, we can use the usual comparison test, ratio test and the n-th row test and then accordingly, we can conclude if sigma a n is convergent. So, notice that in property three, notice that if sigma a n is convergent, it does not imply that sigma absolute a n is convergent, you of example is already in the case of real numbers. So, that that very same example applies for complex numbers as well. So, sigma minus 1 power n by n for example, is such a series which is convergent, but not absolutely convergent alright.

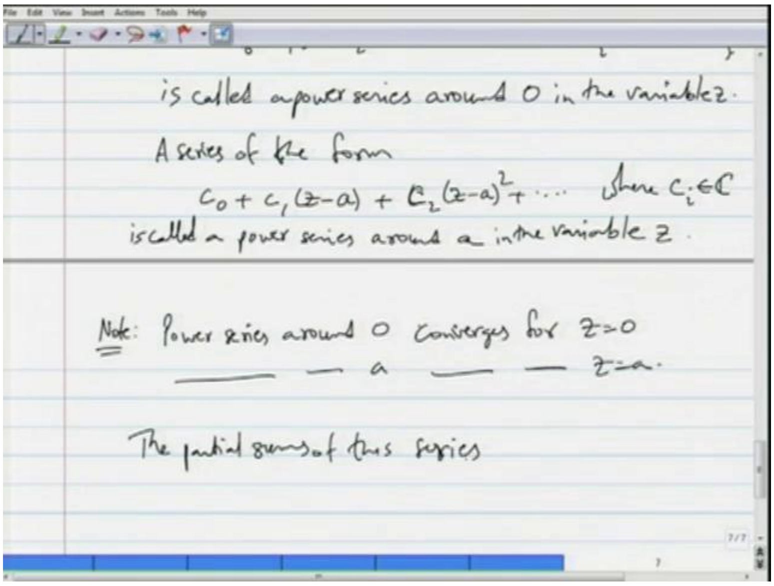
(Refer Slide Time: 23:56)



So, now we can proceed actually to define power series. So, power series, so that was a really crash course is short course on, what complex sequences and series are? But since the viewer is already familiar with real sequences and series, so one can immediately extrapolate what the corresponding things are for complex sequences and series. So, the power series, for complex numbers though take an interesting turn, one can see some important properties like, why the radius of convergence should be a particular number? In the context of complex series more clearly than in the case of real series, I will elaborate this at an appropriate movement. So, for now I am going to define power series.

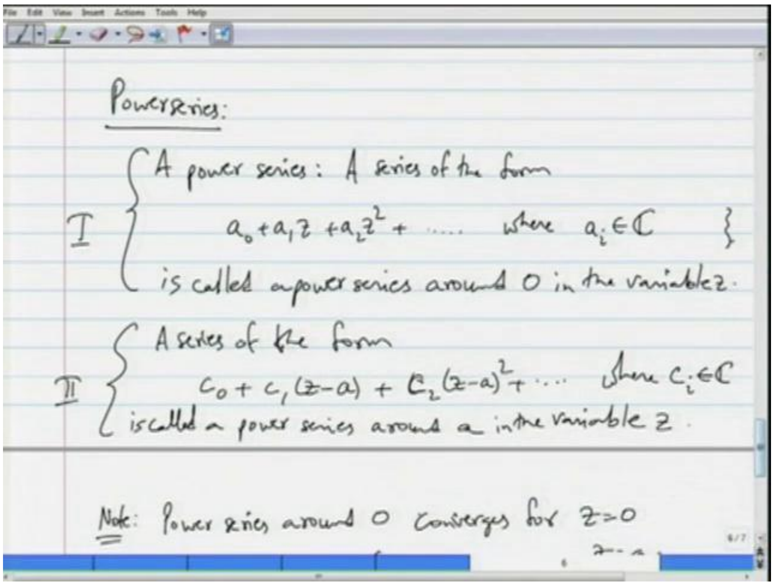
A series of the form is called a power series around 0, in the variable z. So, z is a variable and that is called a power series around 0. So, if you are wondering why it is called a power series around 0? So, like in the case of real numbers a series of the form more generally, a series of the form, so let me use c naught here, c naught plus c 1, z minus a plus c 2 z minus a square plus so on, where c i’s belong to complex numbers is called a power series around a, around the complex number a, in the variable z. So, that is complex power series and we will see that under appropriate conditions okay, sometimes the series converge.

(Refer Slide Time: 26:37)



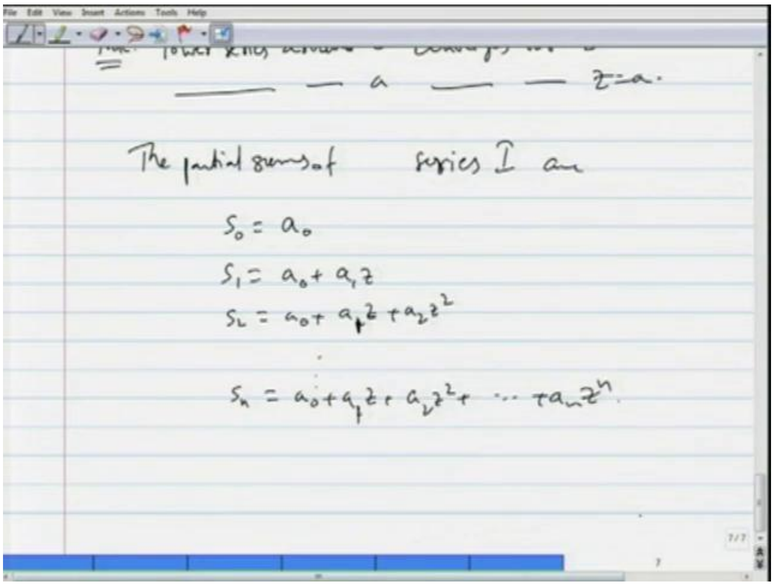
Firstly note that the first series, if you have power series around 0. So, power series around 0 for example, or more generally power series around a converges for one number at least namely 0, converges for z equals 0 or power series around a converges for z equals a. So, the set of points where this series converges is at least is at least non empty. So, but more is true depending on the coefficient a naught or c naught in either of these cases, so we will see what happens. So firstly, the notice that the partial sums, so the partial sums of this series.

(Refer Slide Time: 27:56)



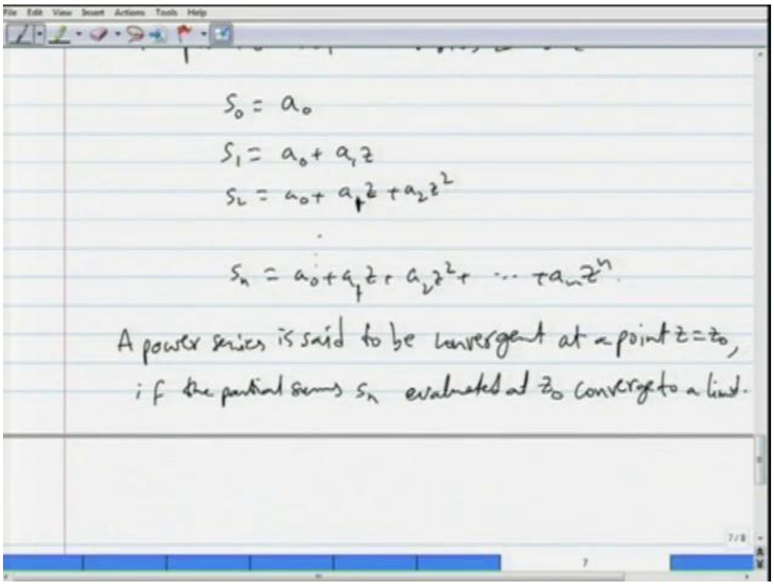
So, first let me call this series of type one and type two, I have to keep going back to these two types. So, I will talk about type one and type one series, power series and similar statements holds for type two series. So, here I am going to talk about this type one series the partial sums of series one.

(Refer Slide Time: 28:28)



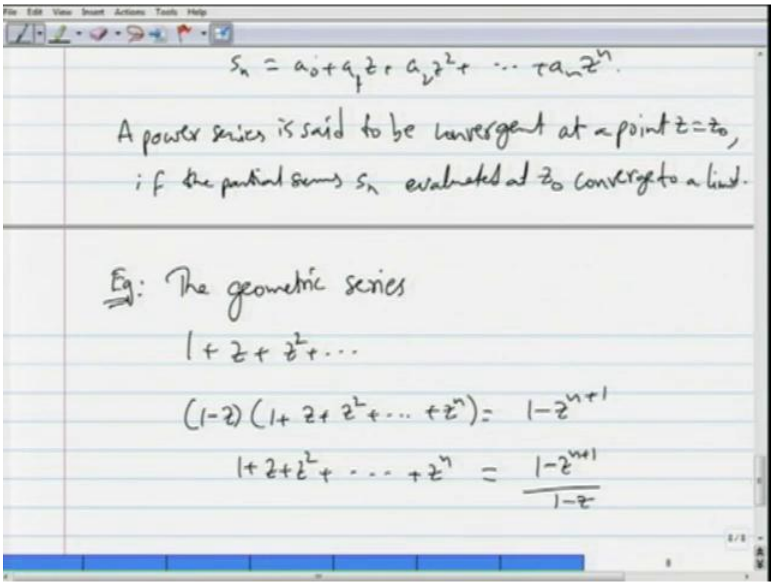
What are they? They are s naught equals a naught, S 1 equals a naught plus a 1 z, S 2 equals, I will just go until S 2 a 1 z plus a 2 z square. So, actually they are all the partial sums are all polynomials in z and the next best thing you can do to polynomials is sort of the infinite version of the polynomials, which are these power series a 1 z plus a 2 z square so on, until a n z power n.

(Refer Slide Time: 29:19)



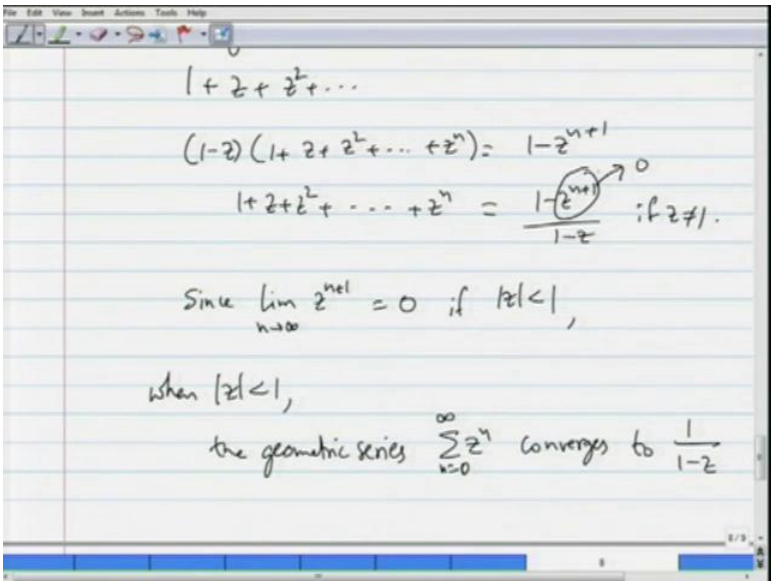
A power series is said to be is said to be convergent, at a point z naught, at a point z equals z naught, if the partial sums s n evaluated at z naught converge to a limit. So, if for a fixed z naught these partial sums converge, then we say that the power series converges at that particular z naught. So, that is the convergence and then, we will now see examples ok.

(Refer Slide Time: 30:26)



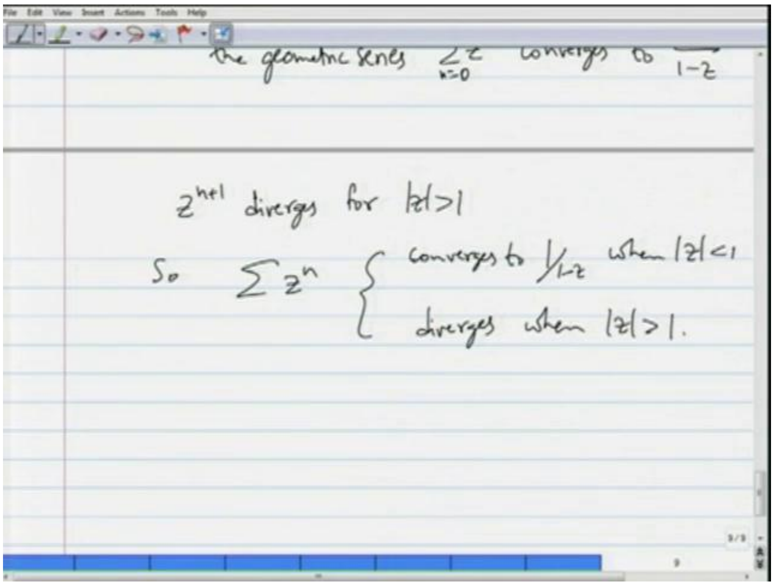
So, like in the case of real power series, we have the following examples, the first one is the geometric series. So, what is the geometric series? This is 1 plus z plus z square plus so on. So, we know by simple arithmetic that 1 minus z times 1 plus z plus z square plus so on, until z power n is 1 minus z power n plus one. So, 1 plus z plus z square plus, plus z power n which is the partial, n-th partial sum of the a geometric series is 1 minus z power n plus 1 by 1 minus z.

(Refer Slide Time: 31:20)



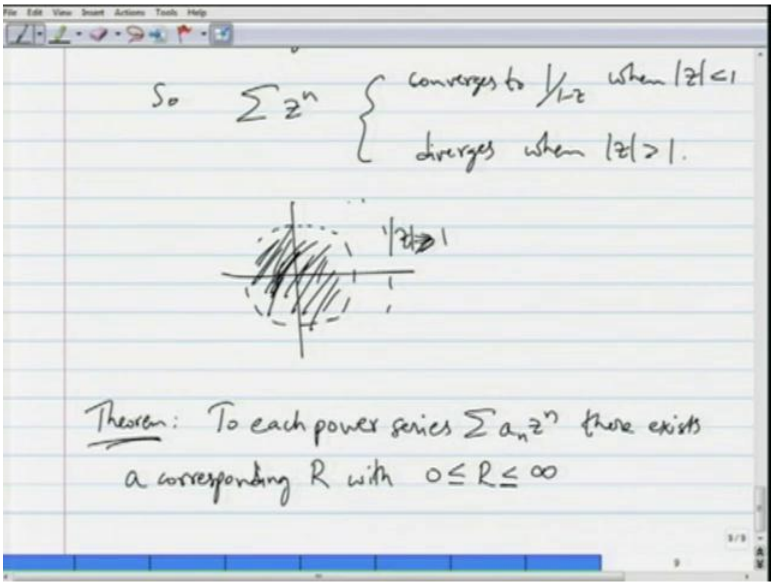
If of course, we need that z cannot be equal to 1 and so since, the limit on the right hand side of n goes to infinity of z power n plus 1 is 0, if mod z is less than 1. What we can conclude is that, when mod z is less than 1. When the absolute are the modulus of z is less than 1, the series the geometric series sigma z power n, n equals 1 through infinity or 0 through infinity in this case converges and it converges to 1 by 1 minus z okay. So, this term here up here, this term is and tends to 0, so this geometric series converges to 1 by 1 minus z, when the modulus the of z is less than 1.

(Refer Slide Time: 32:40)



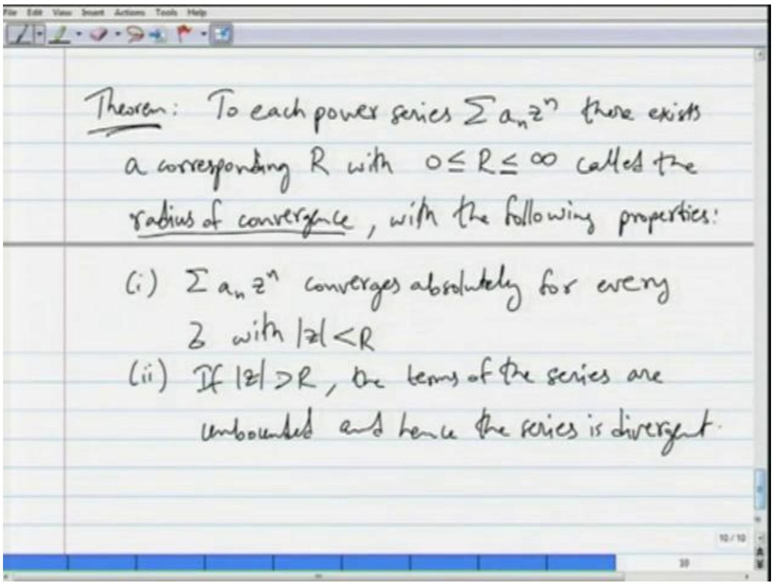
So, also sigma or z power n plus 1 diverges mod z greater than 1 and we know that. So, in summary, sigma z power n has two kinds of behavior, it converges to mod converges to 1 by 1 minus z, when mod z less than 1 and it diverges when mod z is greater than 1. So, we will not very worry about it is behavior, at the point mod z is equal to 1, I on the circle on the unit circle in the complex plane, but what is important is that the geometric series behaves in the following manner, there is this disk, there is this unit disk.

(Refer Slide Time: 33:36)



If there is this unit disk inside of which it converges and outside of which it diverges. So, everywhere outside mod z outside the unit circles mod z strictly greater than 1 it diverges. So, there seem to be a disk of convergence, in this case and we will see that this is typical of any power series. What I mean by that is, we will see that there is a certain round disk in the complex plane, in which given series converges. So, that is the preposition that I am going to present. So, here is a theorem to each power series sigma a n, z power n there exists a corresponding R with 0 less than are equal to r less than are equal to infinity, what I.

(Refer Slide Time: 35:19)

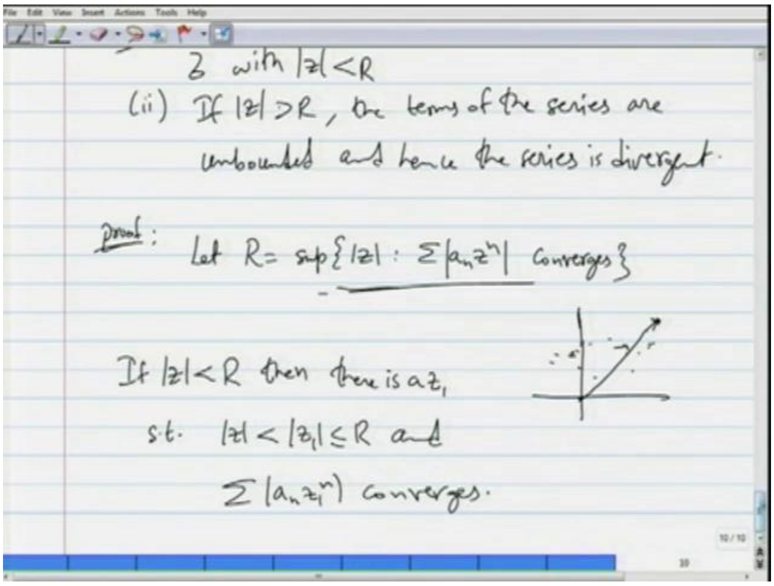


So, we will allow infinity to be 1 value of R, what that means is that R is unbounded called the radius of convergence, with the following properties. One sigma a n z power n, converges absolutely for every z with modulus of z strictly less than R and if mod z minus, if mod z is greater than R there is a different behavior. If mod z is greater than R the terms of the series diverge of the series a n z power n diverge are unbounded are unbounded and hence the series is divergent okay.

So, this theorem, so what it is stating is that there is the behavior exhibited by the geometric series is typical, its telling that not necessarily the unit disk, but there is a disk centered at 0, for the power series centered at 0. In which inside of which the power series converges absolutely and outside of which the power series diverges and on the circle itself, on the circle of radius R itself, the behavior of sigma a n z power n is not is not told by the this theorem. So, what happens to sigma a n z power n is not predicted by this theorem.

So, what is import is there is this number R, the existence of this number R to be small are it could be large, which is given in this bound. So, R can be anywhere between 0 and infinity. So, like I have already commented earlier R can be 0 and so we have seen that the power series of type one or type two converge atleast for one point namely, the center of convergence itself namely 0, in the case of type one and a in the case of type two. So, it could be 0, I mean that could be the set of convergent points for the power series and it could be as large as infinity. So, we will see some examples of where the radius of convergence is infinity okay.

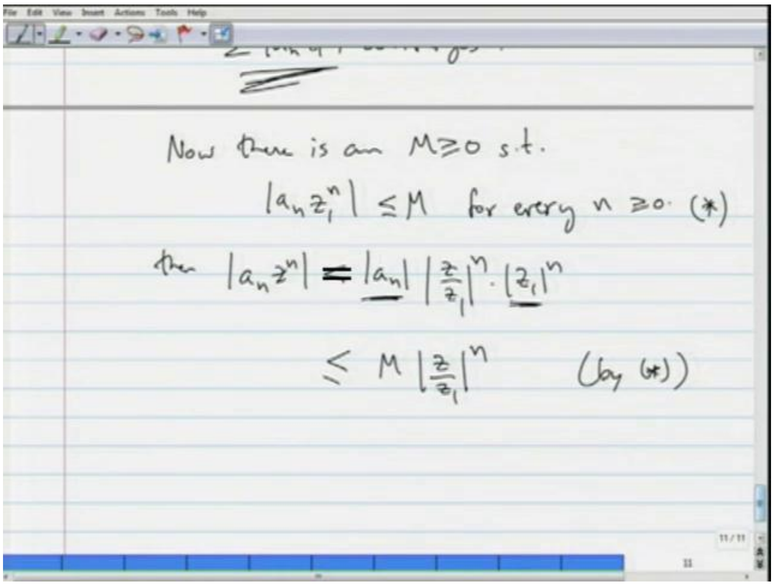
(Refer Slide Time: 38:49)



So, first we will prove this theorem proof. So, this strategy is to, let R be the supremum of the modulus of all such z's such that, sigma a n z power n in absolute value. So, the absolute value series sigma a n z power n converges. So, geometrically speaking, so if we know a bunch of points where around 0, where this series converges absolutely, you are picking the z, such a z which is the supremum of all, I mean such a z whose modulus is the greatest. If one exists or if you know if there are many of these points then we pick the supremum of this set, of this set of modulus of such numbers and that is the candidate for our radius of convergence according to the theorem.

So, if the modulus of z is strictly less than R then, since R is the supreme of this particular set above, then there is a z 1, such that modulus of z is less than modulus of z 1 less than are equal to r and by definition of R. There is this z 1 such that this a n z 1 power n converges absolutely. So, the series sigma absolute a n z 1 power n converges and z 1 there is such a z 1 between mod z and r by the definition of r itself okay. So, since R is the supremum of such mod z's for this for this series, here we can say that.

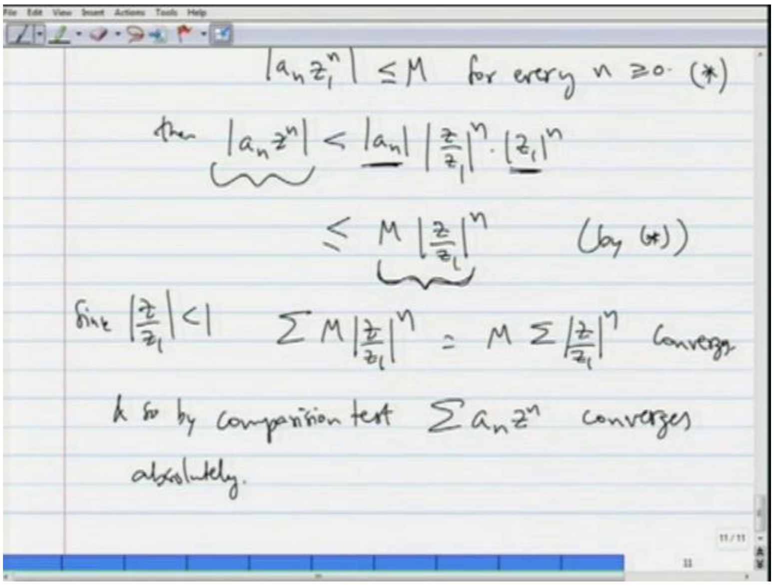
(Refer Slide Time: 41:17)



Now, there is an m greater than or equal to 0, such that the n-th term of this series is less than or equal to m, for every integer n every positive integer n or for every n greater than are equal to 0. So, this comes from the property that I have listed earlier for series. So, let me go back, allow me to go back here and point to this property here okay.

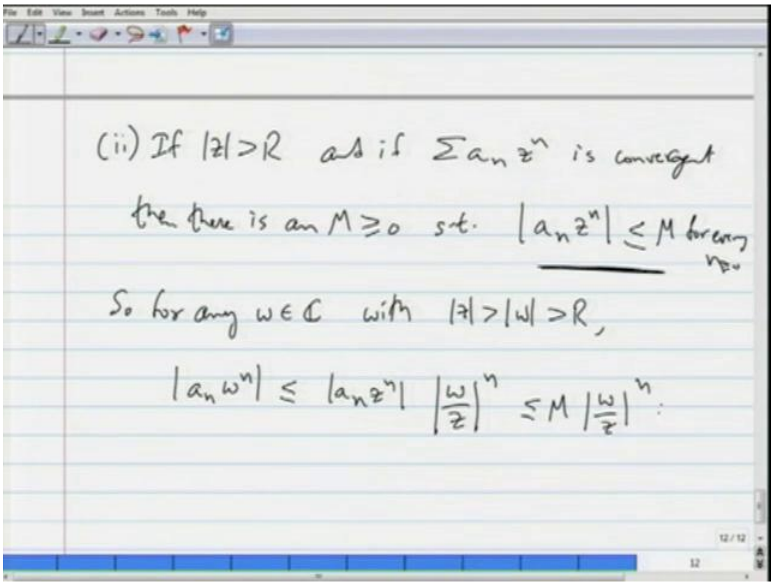
So, if sigma a n converges there is a this property says that there is non-negative real number m, such that the modulus of a n is less than are equal to m for every n belongs to m. So, I am using that property for sequences or series rather. So, the n-th term is less than are equal to m for every n positive or n greater than are equal to 0. What happens is that, the modulus of a n z power n is less than are equal to is less than are strictly less than modulus of a n times the modulus of z by z 1 power n whole raise to n times z 1 power n, in modulus. I will club these a n and this modulus of a n and modulus of z 1 power n and use this star, here to say that this is less than are equal to m times modulus of z by z 1 power n and since by star this is by star since modulus of z is less than modulus of z 1 modulus of z by z 1 is less than 1.

(Refer Slide Time: 43:26)



So, this becomes a geometric series, this becomes the n-th term of a geometric series. So, since this is less than one sigma m times modulus of z by z 1 power n is equal to m times, sigma modulus of z by z 1 power n converges. So, by compression test, this real a number is lesser than this real number in compression. So, by compression test, we can conclude that sigma a n z power n converges absolutely right this is less than this. So, this converges absolutely.

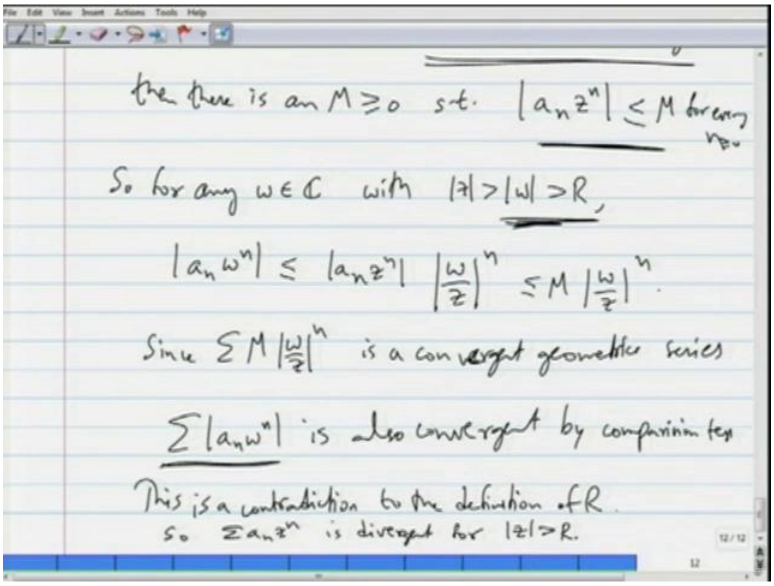
(Refer Slide Time: 44:20)



So, we have proved part one of this theorem. Part two of this theorem asserts that if, mod z is greater than R and suppose and if sigma a n z power n is convergent for is convergent for that particular z with mod z greater than R. Then, there is what we can say is that, there is an m greater than are equal to 0 such that, the modulus of a n z power n is strictly less or less than are equal to m for every n greater than are equal to 0, that is once again by the previous property.

So, for any complex number w with modulus of z greater than modulus of w greater than R, the modulus of a n w power n will be less than are equal to the modulus of a n. So, I will include the z power n here times. The modulus of w by z power n, which is less than are equal to by the above estimate, this is less than are equal to m times the modulus of w by z power n. So, it is a similar estimate to what we have done before in the previous case.

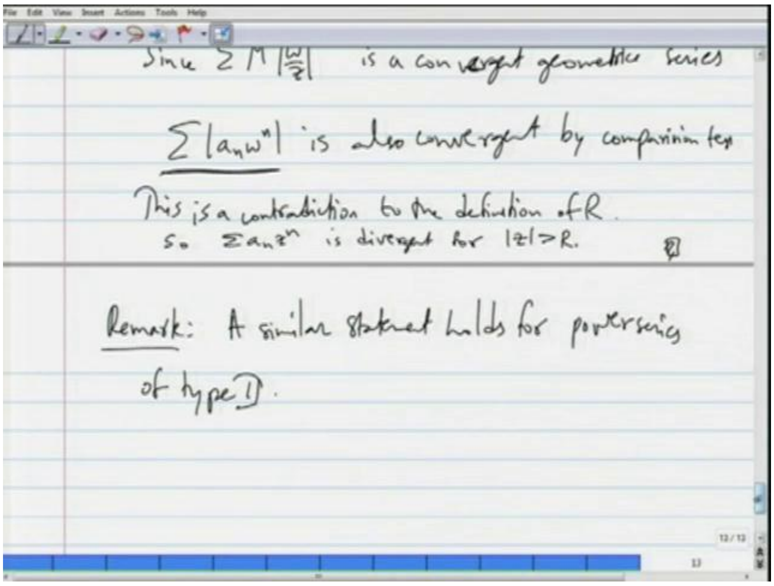
(Refer Slide Time: 46:08)



Since, sigma mw by z power n is a convergent geometric series, sigma absolute a n w power n is also convergent by compression test, but notice that the modulus of w is greater than R. We are saying that there is a number w, whose modulus is greater than R and sigma a n w power n is a absolutely convergent. So, this is a contradiction to the definition of R, R is the, what is R? R is the supremum of all such modulus z you know for which sigma a n z power z is absolutely convergent okay.

So, here sigma a n w power n is absolutely, convergent an modulus of w is greater than the supremum of all such things. So, this is a contradiction to the definition of our, so this cannot happen. So what cannot happen that, this is convergent cannot happen, so sigma a n z power n is a divergent for mod z greater than R okay.

(Refer Slide Time: 48:01)



So, that concludes the proof of this theorem and I will remark that, a similar statement holds for power series of type two. What I mean that is, there is a disk of radius R around the point, a inside of which the power series converges, b the power series of type two converges and outside of which the power series of two diverges. So, that is that is a statement in the case of power series of type two. So, we will conclude this session here.