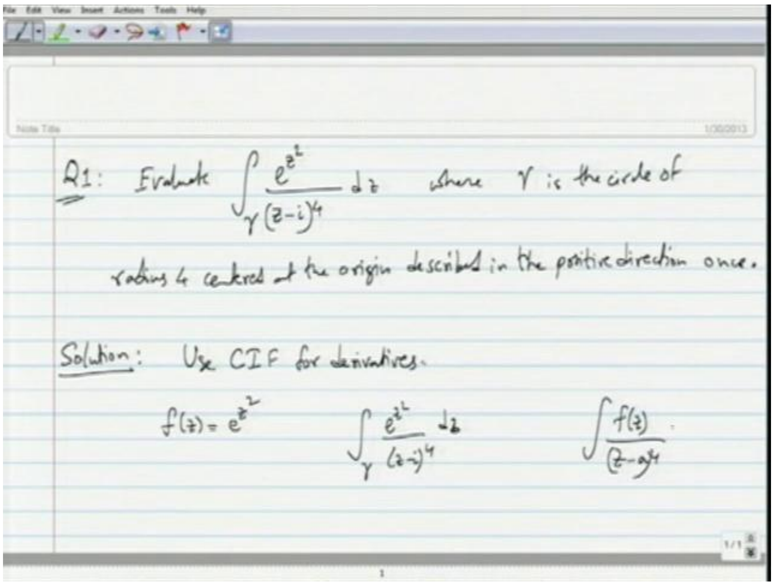
Complex Analysis
Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 3
Complex Integration Theory
Lecture - 10
Problems Solving Session

Hello viewers, in this session, we will solve some problems based on the theory covered so far. Like in the previous review problem session, please try to pause the video after the each question and try to solve it yourself before looking at the solution, which I will any way present here. So, let us starts with problems.

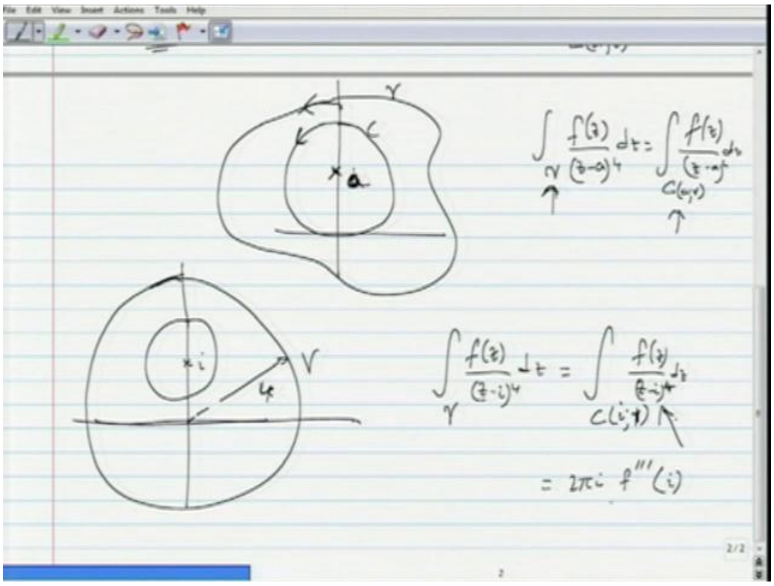
(Refer Slide Time: 00:40)



So, the first question is as follows. Question one - it is a very simple question. Let us try to look at a contour integral, so a very simple contour integral. So, evaluate integration over gamma e raise to z square by z minus i power 4 d z, where gamma is the circle. As the circle of radius 4 centred at the origin described in the positive direction that, is the counter clock wise direction once. So, that is the question. So, I will present the solution. So, this question is a simple exercise in Cauchy's integral formula for higher derivatives of analytical function. So, clearly if you try to use the parameterization of gamma to actually compute the contour integral directly it will be rather tedious. So, it is easier to use Cauchy's integral formula. That is the point of this exercise.

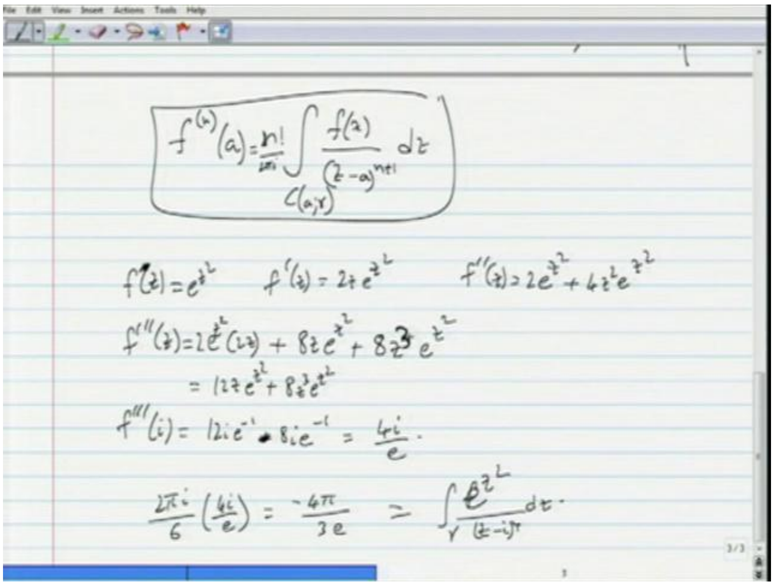
So, you should recognise the integral as an of a integral of a certain form. So, if you consider f of z equals the numerator e raise to z square, then this integral, integral gamma, integral over gamma e power z square by z minus i power 4 d z can be thought of as an integral of the form f of z by z minus a power 4 d z by the Cauchy's theorem version three. So, if you have or the deformation theorem of the integration over that kinds of curve gamma of f of z by z minus a power 4 d z is going to be the same as integration over the circle c a r f of z by z minus a power 4 d z.

(Refer Slide Time: 03:23)



So, you have the following situation you have i here and you have a circle radius 4 around a origin that is not to scale, but that is the picture. So, this is circle of radius 4. That is your gamma, it is not a circle centred at i but, that does not matter. By Cauchy's theorem version three we know that integration over gamma of f of z by z minus i power 4 is d z is equal to the integration over a circle of radius r circle of radius r around i, r small enough say 1, f of z by z minus i power 4 d z. Once again this by the Cauchy's integral formula for the derivatives is 2 pi i times the third derivative we have a fourth power in the denominator, so it is the third derivative of f at the point i divided by 3 factorial.

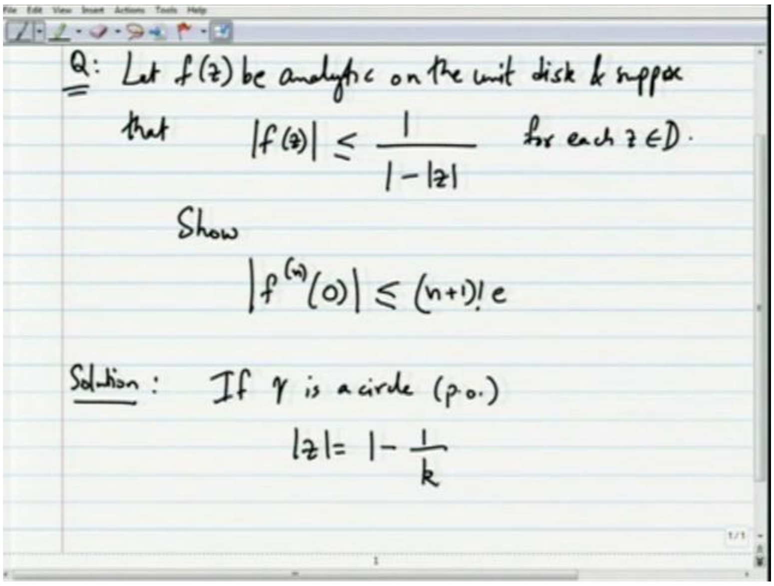
(Refer Slide Time: 04:26)



For higher derivative, we have a following formula integration c a r f of z. There are constant here there are constants 2 pi i. So, n factorial divided by 2 pi i times integral over c a r of f of z by z minus a power n plus 1 d z, that is the Cauchy's integral formula. So, here we apply with this with n equals 3 to get this. So, if you compute the third derivative of our f at i and substitute and substitute it here we are going o get value of this integral. So, the third derivative of f itself is easy to compute. Well, f is e power z square f prime of z is z square 2 z e power z square f double prime of z by the product rule is 2 e power z square plus 4 z square e power z square.

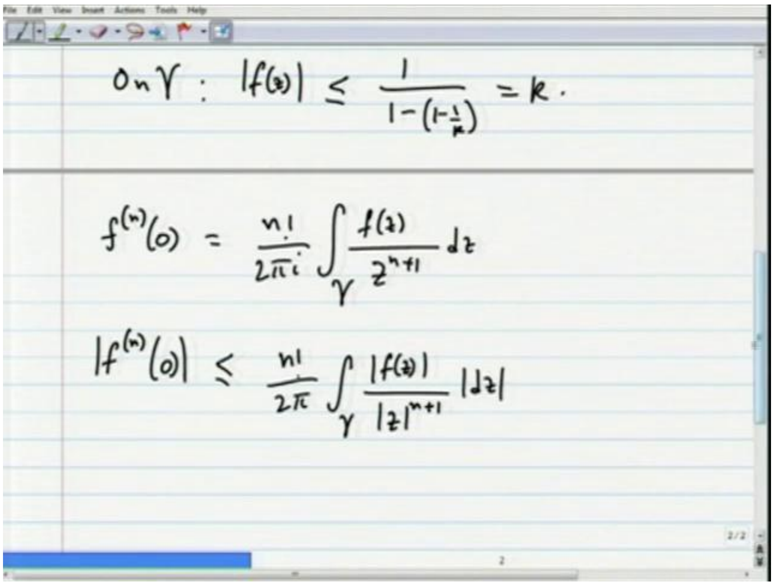
So, the third derivative of f is 2 e power z square times 2 z 8 z e power z square plus 8 z cube e power z square. So, 4 plus 8, 12 z e power z square plus 8 z cube e power z square. So, f triple prime at i is 12 i e power minus 1 plus 8 minus i. So, this is minus 8 i e power minus 1 which is 4 i by e. So, that is your f triple prime of i. Substituting that in here we get 2 pi i by 3 factorial 6 times 4 i by e which is minus 4 pi by 3. So, that is your integration over here gamma of f e power z square by z minus i power 4 d z. So, that is the solution add to this problem.

(Refer Slide Time: 06:59)



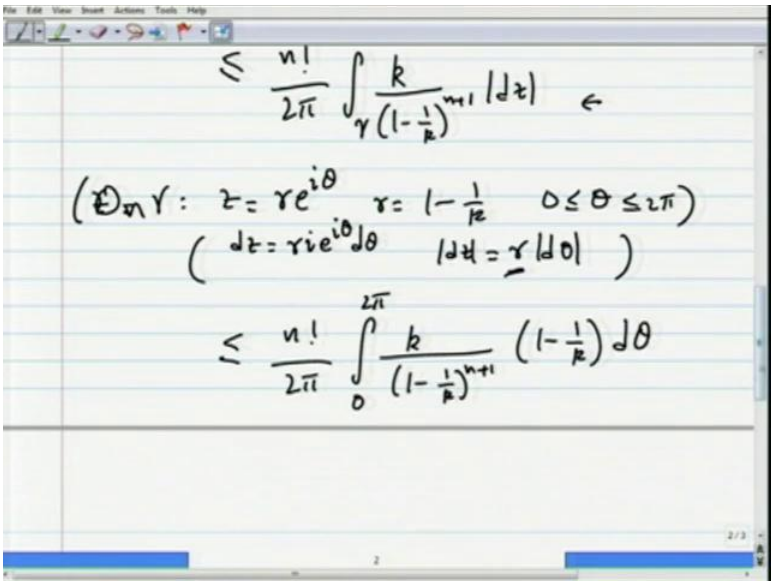
The next question is as follows. So, let f of z be an analytical function or be analytic on the unit disk. Suppose that the modulus of f of z is less than or equal to 1 by 1 minus modulus of z for each z belongs to D. So, under these circumstances try to show that the modulus of the n eth derivative of f at the point 0 is at most n plus 1 factorial times e? So, the key here is to use an appropriate contour in the unit disk so that we can we can bound the n eth derivative. So, more precisely we will choose an appropriate a circle of appropriate radius so that the n eth derivative at 0 can be bounded by the values of the function f on that circle.

(Refer Slide Time: 09:13)



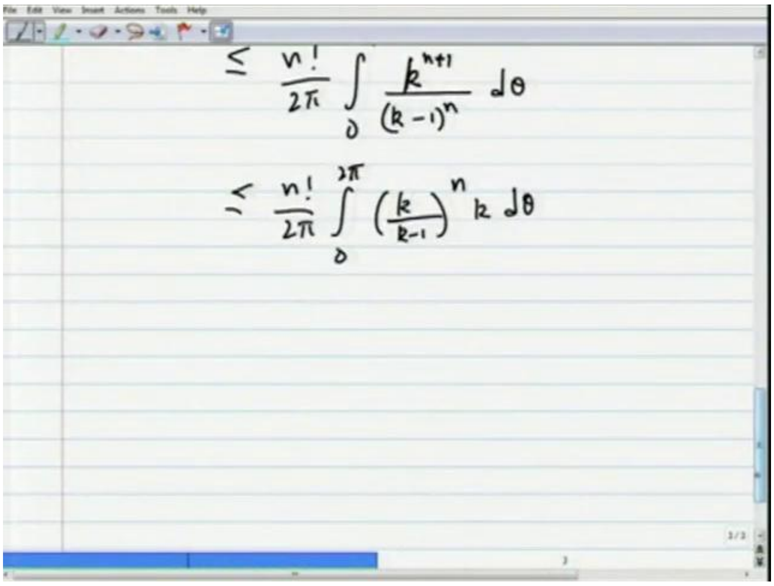
So, please pause here and try to see if you can come up with the solution. Here I will present the solution. So, as I said the idea is to choose the appropriate a circle. So, if gamma is circle oriented in the positive sense, so positively oriented mod z is equal to 1 minus one by k, let us decide what this k is little later let us just to see. So then, what we get is, the modulus of f of z on gamma on the circle gamma modulus of f of z, we know is less than or equal to 1 by 1 minus 1 minus 1 by k which gives us a k. Cauchy's integral formula for the n eth derivative at 0 is given by well, f n of 0 is a n factorial by 2 pi i times integral over a gamma this particular gamma will work, because f is analytic inside and on the circle of f of z by z power n plus 1 d z.

(Refer Slide Time: 10:33)



That is the Cauchy's integral formula for the derivative n eth derivative f at 0. So, the modulus of the n eth derivative of f at 0 is less than or equal to n factorial divided by 2 pi integration over a gamma of the modulus of f of z divided by modulus of z power n plus 1 mod d z. On gamma we have bounded f of z it is bounded by k will set this k shortly. This is less than or equal to n factorial divided by 2 pi times integration over gamma of k well divided by mode z is 1 minus 1 by k. So, I have 1 minus 1 by k power n plus 1 mod d z. So, on gamma z is equal to r e power i theta where r is equal to 1 minus 1 by k theta goes from 0 to 2 pi.

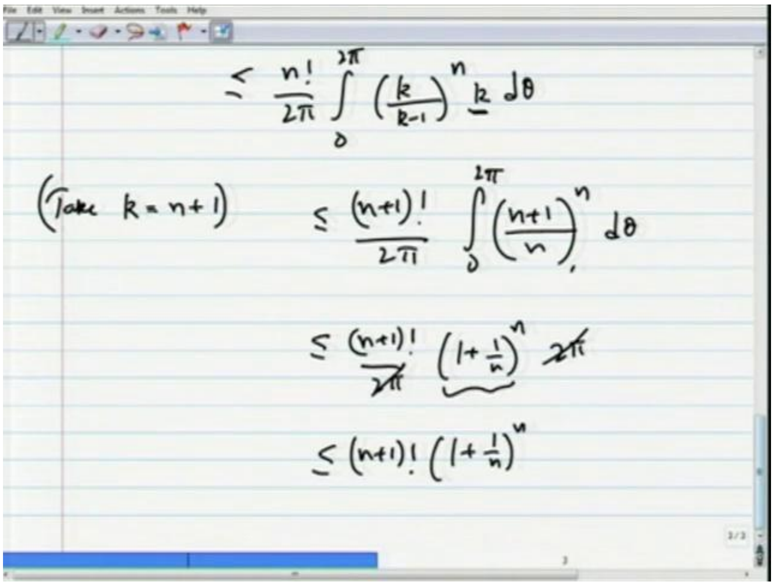
(Refer Slide Time: 12:16)



So, if that is the parameterization then d z will give us r i e power i theta d theta or mod d z will be r times mode d theta. So, this expression here is less than or equal to n factorial divided by 2 pi times integration over gamma. Now, we have converted everything n to theta so theta goes from 0 to 2 pi of k divided by 1 minus 1 by k power n plus 1 times 1 minus 1 by k which is r here and then modulus of d theta. Well, we can take that to be d theta. I mean it is not modulus it is the absolute value. So, then take that to be d theta itself.

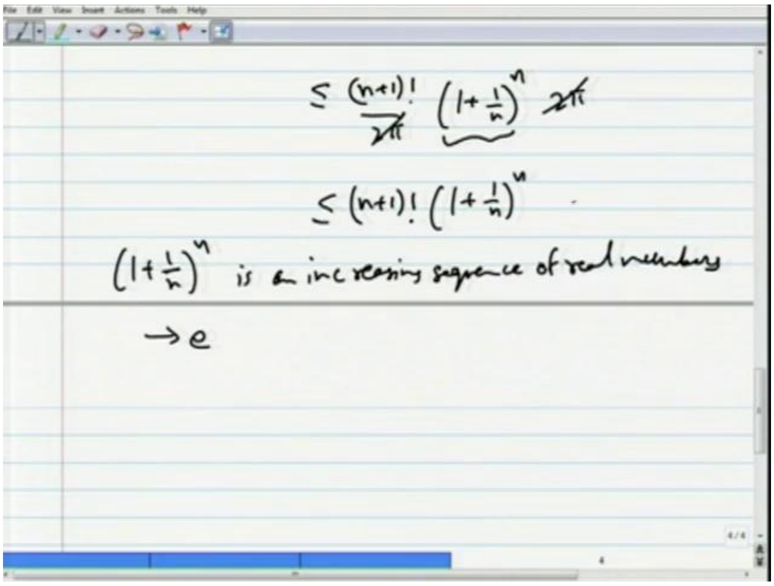
Then this we see is after after some cancelation we see that this is less than or equal to or equal to in fact n factorial divided by 2 pi integration from 0 to 2 pi k power n plus 1 factor here cancels the denominator. Something like this here one factor here cancels with one factor here. So, we get k power n going to the top so we have k power n plus 1 which is k power n here times k divided by k minus 1 raise to n and then d theta. Then this is less than or equal to n factorial divided by 2 pi. Well, now is the time to set k so or lets still decodes this. This looks like k by k minus 1 power n times k times d theta.

(Refer Slide Time: 13:31)



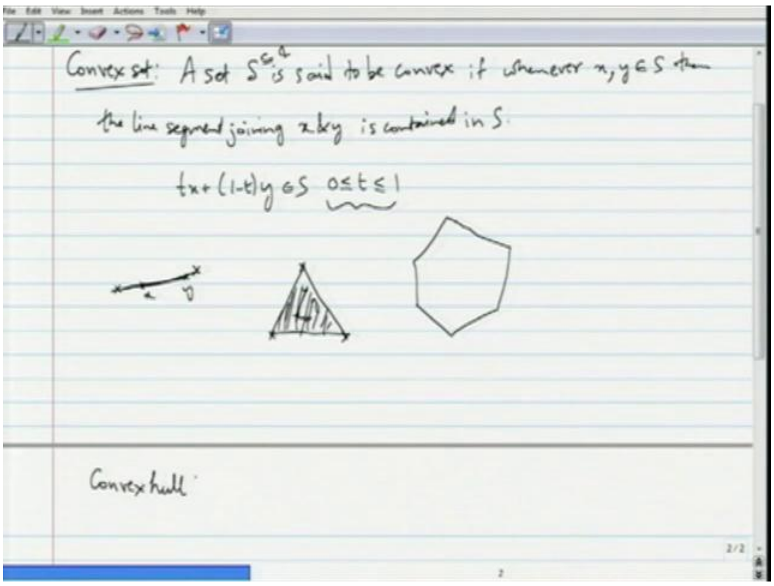
Since we want to show that this is bounded by this n eth derivative at 0 is bounded by n plus 1 factorial times e. So, in order to get that n n plus 1 factorial probably correct to take k equals n plus 1. So, that k and this n factorial give us n plus 1 factorial. So, this is take k equals n plus 1 n plus 1 rather. So, this is less or equal to n plus 1 factorial divided by 2 pi times integration from 0 to 2 pi of n plus 1 by n power n and then d theta. This is of course this is constant which is clear of the integration. So, we can actually put it outside the integration. This is less than or equal to n plus 1 factorial divided by 2 pi times 1 plus 1 by n power n then 2 pi.

(Refer Slide Time: 14:46)



The integration from 0 to 2 pi of d theta gives us the 2 pi. So, you can cancel these and get this is less than or equal to n plus 1 factorial times this factor 1 plus n by power n. We know that this, this, this is the sequence 1 plus 1 by n power n is an increasing sequence of, of real numbers. One proves in a first course in real analysis that this tends to e. The limit of this increase in sequence either one defines that to be e or e is define otherwise then one proves that converges to e. So, in any case this is an increasing sequence. So, before we go to the next problem we will introduce some terminology namely convex set and the convex hull of a set of points.

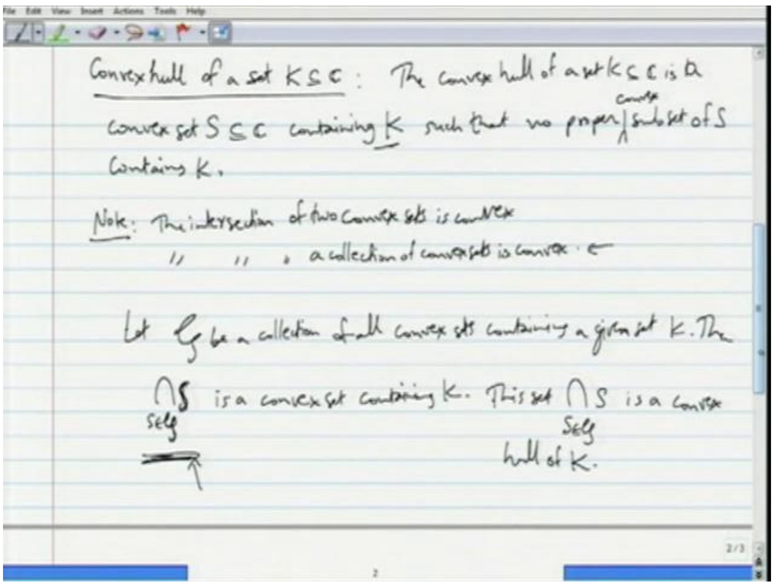
(Refer Slide Time: 15:45)



So, let us starts with convex set. So, a set is said to be convex it has the property that if x and y belong to the set then line segment joining x and y belong to the set. So, set S is said to be convex, for simplicity we will assume this as sub set of c. So, set contained in c is said to be convex if whenever x comma y belong to S then the line sigma joining x and y is contained in S. So, another way to say this is well one can always parameterize line segment by t x plus 1 minus t times y. So, t between 0 and 1 will parameterize the line segment joining x and y.

So, if this belongs to S this point belongs to S for t between 0 and 1 whenever x and y belong S then you say that s is a convex set. So, for example, convex sets could look like this. So, line segment itself is convex because if you take any two points on the line segment x and y the line segment joining them is a contained within this line segment. So, it is a convex set, so likewise if you take triangular piece like that and then the inside of this along with the boundary that is a convex set. So, you pick any two points x comma y inside the set they could be on the boundary for example, the line segment joining them is contained in the set.

(Refer Slide Time: 18:39)

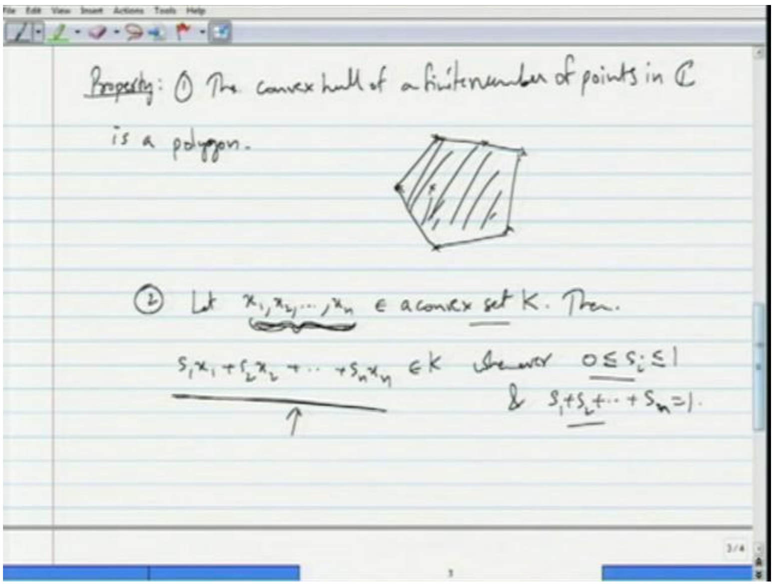


So, this is a convex. In general the inside of the polygon along with the boundary of polygon if you wish is a convex set. So, that is the convex set and then there is the concept of a convex hull. So, in order to consider is convex set which is, which contains a bunch of points we can construct a convex hull which has a an additional minimality property as well. So, here is convex hull. The convex hull of a set k contained in c is a convex set S contained in c containing in k. So, it contains k. Not only that it is such that no proper sub set of S contains k.

No proper convex set I should say no proper convex sub set of S contains k. So, such things are normally constructed as follows. Notice that, note the intersection of the intersection of two convex sets is convex. In general the intersection of a collection of convex sets is convex. So, since this happens what you do is you consider the set of all convex sets which contain a given k and you take their intersection. So, let G be collection of all convex sets containing a given set k. Then the intersection of all these, so intersection of S such that s belongs to g is a convex set by the above, above note is a convex set containing k.

By construction this is the convex hull of k because no proper sub set of this you contain k. Because if there is a there were a proper sub set of this which contains k then it would belong to the sub collection then it would appear in the intersection. Then so it would be the intersection or else super set of the intersection. So, that cannot occur. So, there is no proper set of this which contains the convex sub set of this which contains k. So, it is how we construct a convex hull. So, and is a so I will remark here that this set intersection S belongs to G, S is a convex hull of k. That is how we construct convex hulls.

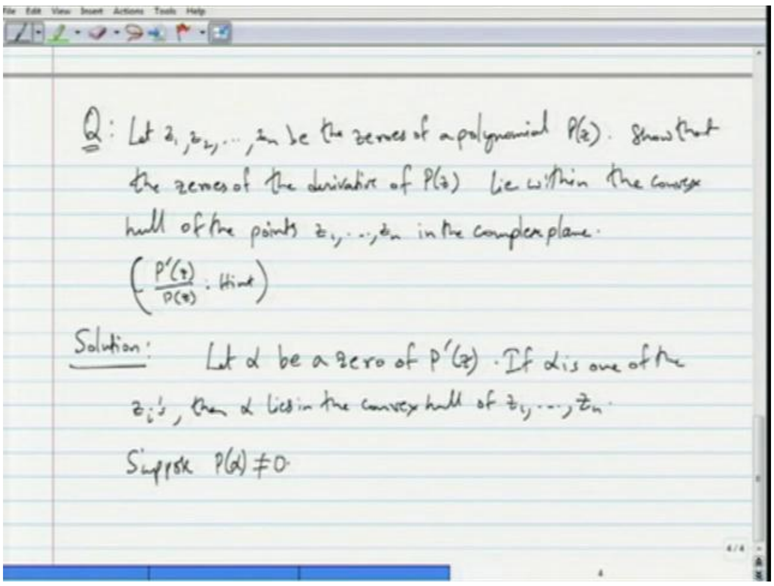
(Refer Slide Time: 22:12)



So, we will note one property of convex hull, it is as follows or couple of them. So, the convex hull of a finite number of points in C is a is polygon; so this intuitively clear. So, depending up on how many of them are collinear etcetera the geometry of the polygon differs. But eventually some or all of them are going to be the vertices of polygon. So, suppose some point is like this some point is like that then the this collinear etcetera. So, then eventually will get a polygon which will be the convex hull of bunch of this points. So, we will not prove this property but, this is true.

And then it is one can actually try to prove this property by taking coordinates etcetera. So, this is easy to do. So, you just have to use the definition of a convex set and of a convex hull to prove this. Then, then next property that I want to state is that, let x 1 x 2 so on until x n belong to convex set k. Then S 1 x 1 plus S 2 x 1 plus so on until S n x n belongs to k whenever 1 less than or equal to s i less than or equal to 1. The sum of s i's s 1 plus s 2 plus so on until S n is equal to 1. So, this is once again easy to prove using the definition of a convex set. So, one can show that all this point for any values of S like this is belongs to a convex set k.

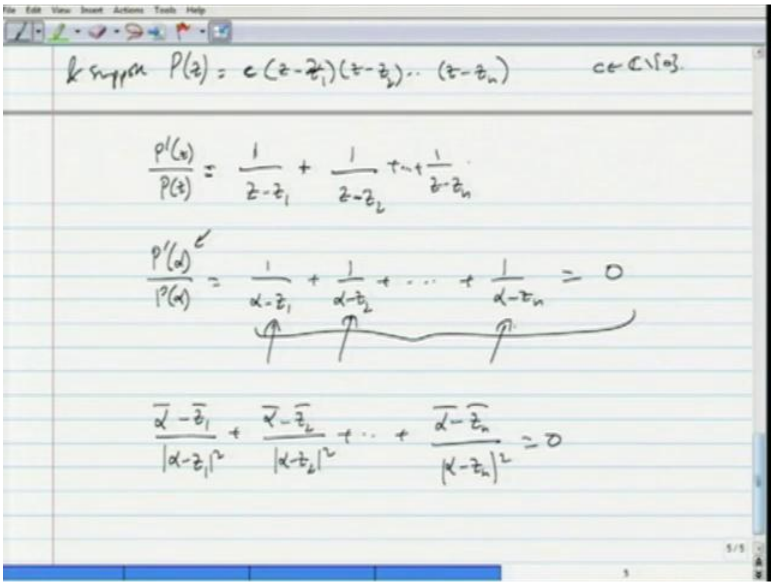
(Refer Slide Time: 25:16)



Whenever these points starts, when you start with these points on a convex set k, in particular if you consider the convex hull of these bunch of points it is a convex set. So, then this kind of combination where S i's are like this is also a point in that convex hull. So, with this we are ready to now state the next question. So, let z 1 z 1 so on until z n be zeroes be the zeroes of the polynomial P of z. Show that the zeroes of the derivative of p lie within the convex hull of these zeroes of the points z 1 through z n in the complex plane.

So, consider using the quotient P prime of z by P of z to prove the same is a hint use P prime by z by P of z. So, try to solve this problem and I will provide this solution here. The statement of this problem is called the Gauss Lucas theorem. So, the solution is as follows. Let alpha be is zero of P prime of z. If alpha is one of the z i’s then it sure lies in the convex hull because a point themselves lie in the convex hull of those points. So, if alpha is one of the z i’s then alpha lies in the convex hull of z 1 through z n. Now, suppose that alpha is a not 0. So, if not a suppose now that P of alpha alpha is not equal to 0. So, alpha is not one of the z i’s now.

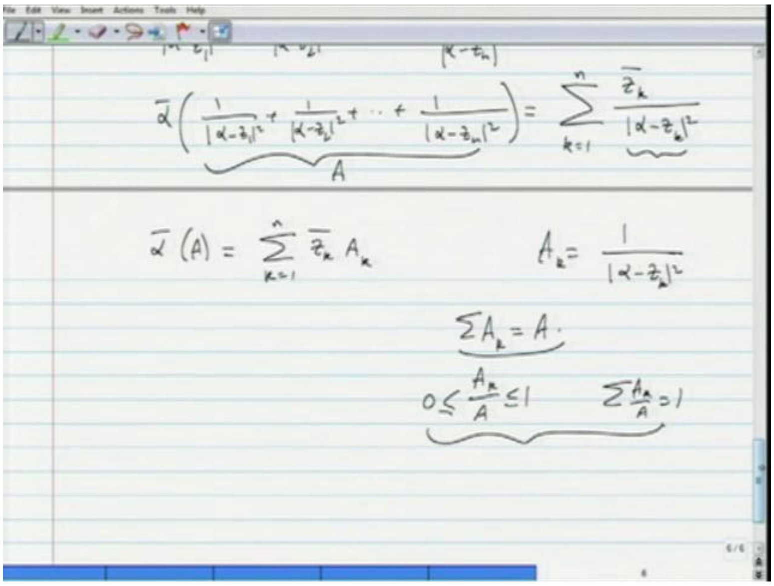
(Refer Slide Time: 28:01)



So, in this case let me first write P of z as some constant c times perhaps the constant c times z minus z 1 times z minus z 2 so on until the z minus z n. Because such a factorization is true for p of z because these are all the zeroes of p of z z 1 through z n are all the zeroes. So, perhaps there is multiplying constant c c belongs to c c not equal to 0, so belongs to c minus 0. Suppose P of z is like that then P prime of z is easy to calculate. P prime of z each time you take the derivative of one factor which is 1 and use the product rule. So, I can actually directly divide by P of z to get P prime of z by P of z is equal to 1 by z minus z 1.

So, the factor which is missing in P prime figures in the in that reciprocal like this. Then there are P prime is the sum of all these factors so it is plus 1 by z minus z 2 plus 1 by etcetera so on until plus 1 by z minus z n. So, then since P of alpha we are assuming is not 0 now. So, then P prime of alpha by P of alpha is equal to 1 by alpha minus z 1 plus 1 by alpha minus z 2 plus so on until 1 by alpha minus z n which is equal to 0 because the numerator P prime of alpha is equal to 0. Alpha is a 0 of P prime by assumption. So, so this is equal to 0 and then we will use this equation to do the following. So, we will use this particular equation. So, first multiply the numerator and denominator each of these terms with the conjugative of the denominator.

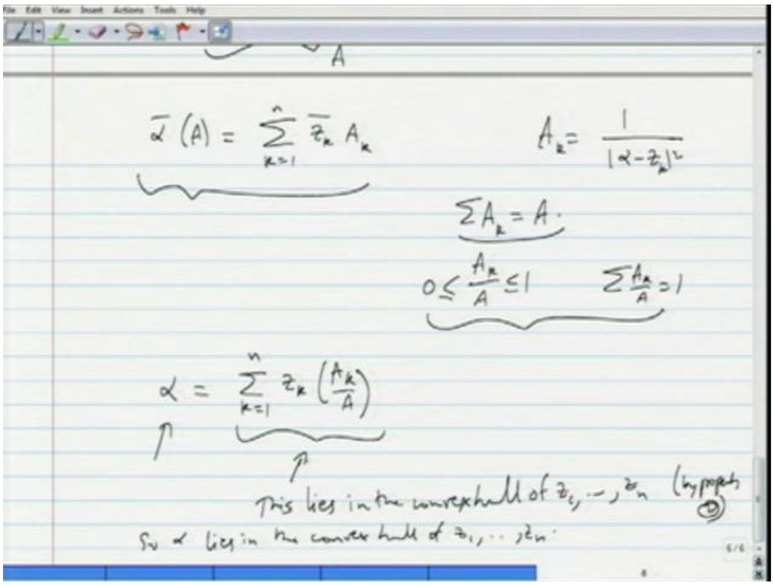
(Refer Slide Time: 30:46)



So, we get alpha bar minus z 1 bar divided by modulus alpha minus z 1 square plus alpha bar minus z 2 bar divided by modulus of alpha minus 2 square etcetera plus alpha bar minus z n bar divided by alpha minus z n modulus square is equal to 0. So, alpha bar times let me factor out an alpha bar. I get the 1 by modulus of alpha minus z 1 square plus 1 by alpha modulus of alpha minus z 2 square plus so on plus 1 by modulus of alpha minus z n squarer. This is equal to on the right hand side I will collect these terms z 1 bar divided by modulus of alpha minus z 1 bar etcetera. So, on the right hand side I get sigma i or k is equals 1 through n of z k bar divided by the modulus of alpha minus z 1 z k square.

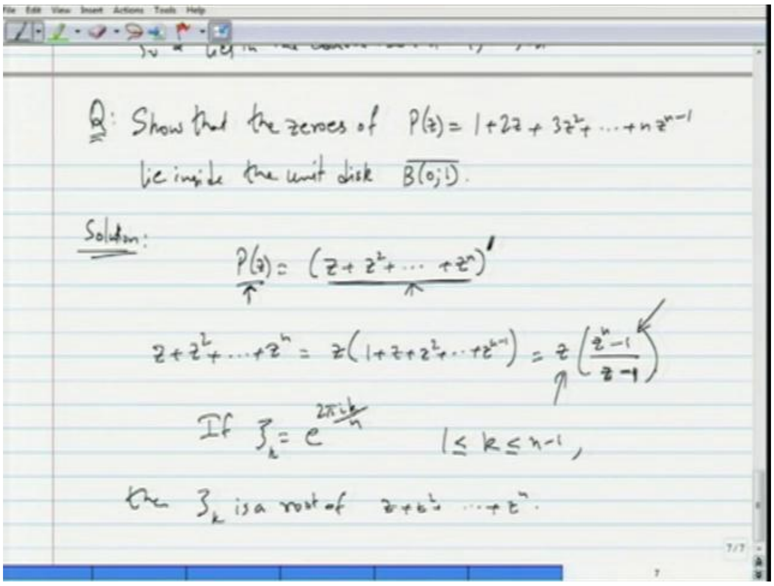
Now, notice that this is a real number. Let us call that A and the coefficient of z k bar is also a real number and so let us call that A k. so this is A k 1 by this is A k. So, alpha bar times A is equal to sigma k is equals 1 through n z k bar A k. where A k is equal to 1 by modulus of alpha minus z k square. Also notice that the sum of A k is actually equal to A. So, A k by a is such that A k by A is in between 0 and 1. These are all modulus of numbers complex numbers. So, they are positive or 0. Since the sum is equal to A sum of A k's equal to A, A k by A is in between 0 and 1. Sigma A k by A is equal to 1. So, this is where the second property of the convex hull that I mentioned is coming into picture.

(Refer Slide Time: 33:32)



Let me go back, so this S i's so these are the candidate A k by A's are candidate for S i's. So, alpha bar so I will conjugate on both sides of this equation A and A k are real numbers. So, the conjugation does not affect them. So, what I get is alpha is equal to sigma k is equals 1 through n of z k times A k by A. Now, we recognise that this is a combination like in the property two. So, this is a combination which lies in so this this lies in this complex number lies in the convex hull of z 1 through z k or z n rather by property two above. So, which is what we want, so alpha like in convex hull. So, alpha lies in the convex hull of z 1 through z n.

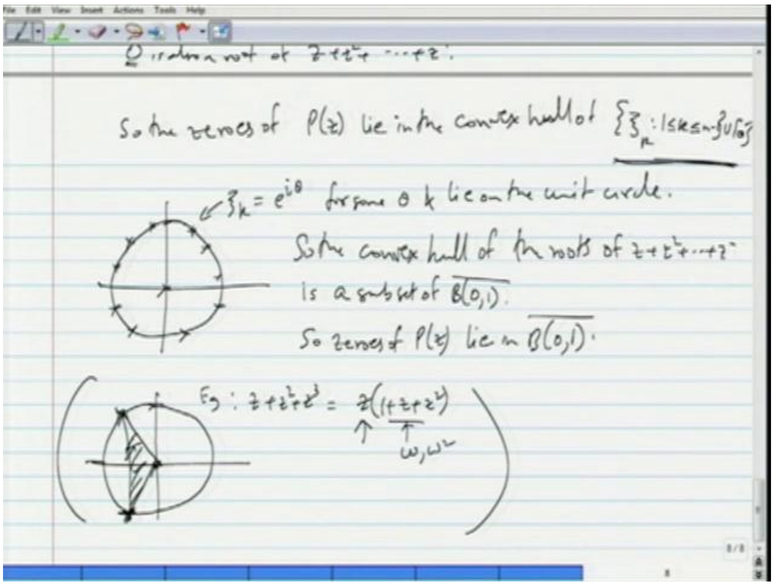
(Refer Slide Time: 34:59)



So, in either case whether alpha is 0 of P of z or not alpha lies in convex hull of z 1 through z n which proves the problem. This like I mentioned this is called the Gauss Lucas theorem. So, the next question is as follows. This is an application of the Gauss Lucas theorem. Show that the zeroes of the polynomial P of z is equal to 1 plus 2 z plus 3 z square plus so on until n z power n minus 1 lie inside the unit disc b 0 1 bar. So, the solution is as follows so try to see if you can work out the solution and then represent the solution here.

So, notice that P of z looks like the derivative of z plus z square plus so on until z power n. So, that prime means derivative. So, P of z can be thought of has the derivative of that polynomial and then we will use the Gauss Lucas theorem which says that the the zeroes of P of z lying on the convex hull of the zeroes of this polynomial. So, let us concentrate on this polynomial within the parenthesis. So, this polynomial can be factorised as z times 1 plus z plus z square so on until z power n minus 1which is z times in recognize that z power n minus 1 divided by z minus 1.

(Refer Slide Time: 37:45)

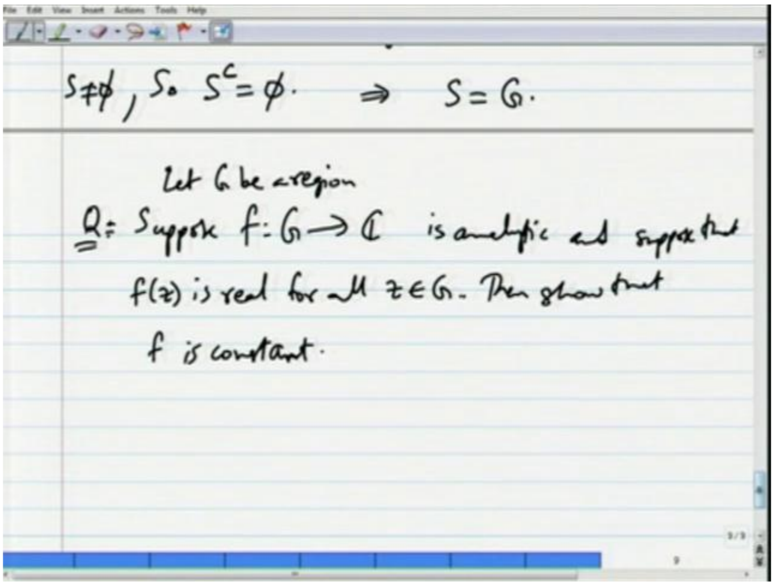


We recognize that the this z power n minus 1 the roots of this polynomial z power n minus 1 precisely the n through roots of unity and when we divided by z minus 1 we forget the root one itself. So, if zeta k is equal to e power 2 pi i k by n 1 less than or equal to k less than or equal to n minus 1 then zeta k is a root of this polynomial z plus z square plus so on until z power n. And notice that 0 itself is the root of this polynomial. So, 0 is also root of z plus z square plus so on until z power n. Now, the zeroes these are 1 minus 1 number. 0 is including 0 we have n roots an n th degree polynomial have at most (( )) roots.

So, we know all the n roots. These are all distinct (( )) the point. So, we now know all the roots of this. So, the zeroes of the given polynomial of P of z lie in the convex hull of the set zeta k which is e power 2 pi i k like I am mentioned here. So, this is zeta k such that one less than or equal to k less than or equal to n minus 1 union 0, but now what is the convex hull of these? Well, the roots of unity lie on the unit circle. So, they divided the unit circle to equal number of circular pieces. So, these zeta k lie they look like e power i theta for some theta and lie on the unit circle. 0 is in the centre.

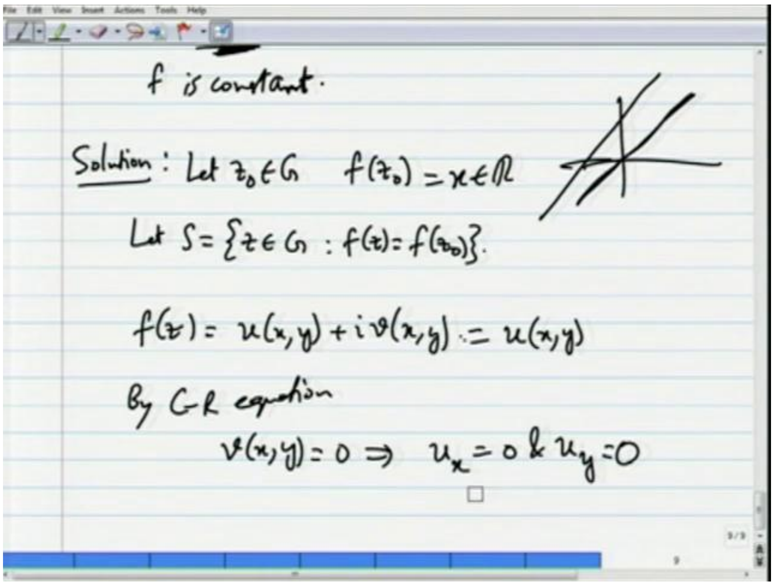
So, the convex hull of the roots of of this set the roots of z plus z square plus so on until z power n is the unit disc. Actually is a sub set of unit disc b bar 0, 1. So, if k equals 3 for example, we are looking at the thirds roots of unity and when you forget 1 itself when you have this this and 0 the roots of z plus z square when n is equal to 3 we have z plus z square plus z cube. So, this is an example z plus z square plus z cube when n is equal to 3 you get this is z times 1 times z plus z square. The roots of these this are omega and omega square. The cube roots of unity, which are here and here and then you have 0 itself which is the root here.

(Refer Slide Time: 41:37)



So, the convex hull is going to be this triangular region. So, this is an example. So, likewise you are going to get a sub set of b bar 0 b 0, 1 bar the closer of b 0, 1. That is the convex hull, so the zeroes of P of z definitely lie in b 0, 1 closer. So, that is the solution add to this problem. This kind of questions are standard. So, suppose f from G to C is analytic. I should start with let G b a region. So, G is a region and suppose f from G to C is analytic. And suppose that and f of z is a real for all z belongs to G. Then show that f is constant. So, a real value analytic function has to be a constant function. So, there are other versions or kinds of problems which look like this.

(Refer Slide Time: 43:07)

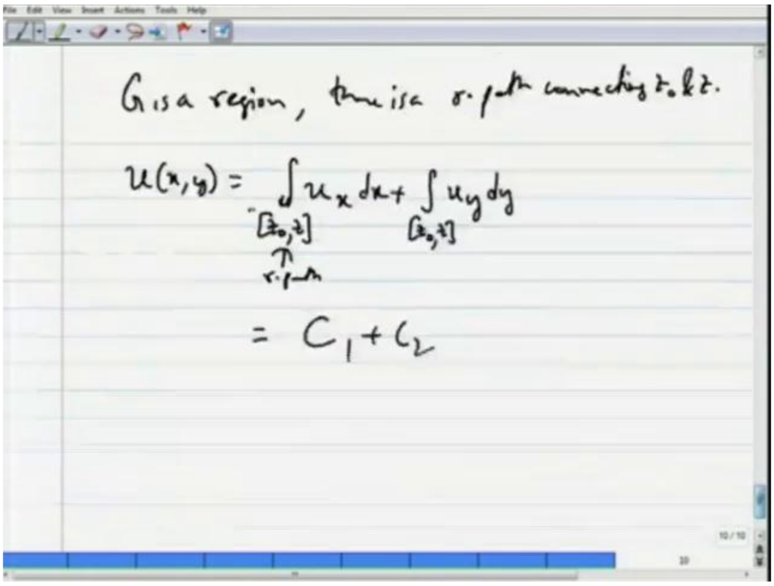


So, one can ask that show that if modulus of f of z is constant then f is constant. Or one can ask that if f of z is purely imaginary always then show that f is constant. And one can more generally ask that if the value f of z lie on a line passing through origin for example or on straight line then show that f is constant etcetera. So, all these will will follow from a theorem which we will do in the next few over the few next session called the open mapping theorem which says that an analytic function should take open set to open sets. So, or a non constant analytic function should take open sets to open sets. So, f were to be non constant it will take G which is the region it is an open set to an open set.

So, what that means is that well, so it cannot take in particular it cannot take c two the real axis which is not open in the complex plane. Or a part of real axis any part of real axis which is not an open set the complex plane. Likewise it cannot take the complex plane to a particular circle for that matter where the modulus is fixed. Now, we can use the Cauchy's Riemann equations directly to show that f is the constant. For now since we do not have the open mapping theorem at hand you can use the Cauchy's Riemann equations to show this problem. So, here is the solution. Let z naught belong to G. Let so f of z is equal to or f of z naught is the real number that is given f of z naught f of z is real.

Let S equals the set of all z belongs to G such that f of z is equal to f of z naught. We will show that S the whole set G. So, firstly let us look at f of z which is which looks like let me separate into its real and imaginary part. It looks like u of x y plus i times v of x y and u and v satisfy the Cauchy's Riemann equations. It is given that f of z is always real which means v of x y is identically the zero function, so on G. So, u of this is u of x y plus i times 0. By the Cauchy's Riemann equations we have, what do you have we have v of x y is 0 implies u x the partial of u with respect to x is 0 and the partial of u with respect to y which is minus the partial of v with respect to x that is also a 0.

(Refer Slide Time: 46:15)

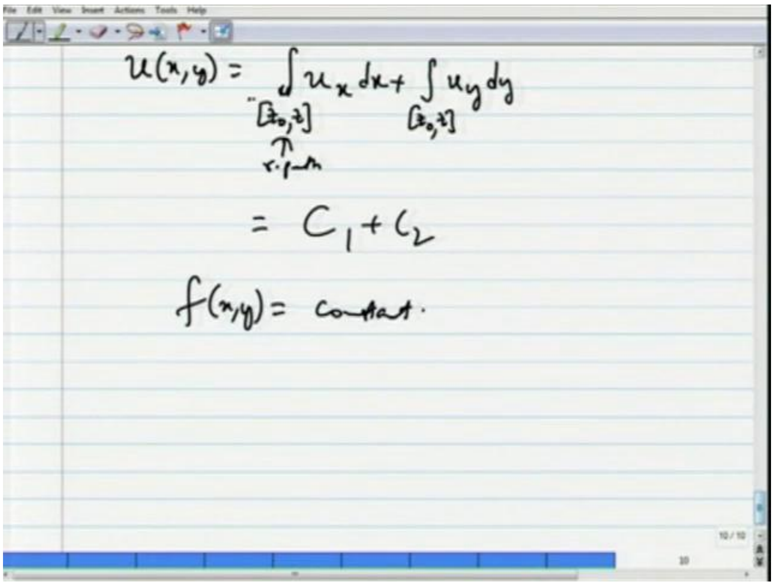


This is 0 because u x is equal to v y and v y is 0. So, both these are 0. Since G is a region there is a rectangular path connecting there is a r path connecting z naught and z. So, using that rectangular path you can try to recover u of x y by integrating along by integrating along the line segments parallel to x axis. So, on x axis line segments parallel to x axis. So, I will just not say that I will just say on z naught z this is to indicate the rectangular path connecting z naught and z. Then you can integrate with respect to y partially so d x. Then on the line segments connecting z naught and z on the rectangular path connecting z naught and z on the on the line segments which are parallel to y axis y axis rather.

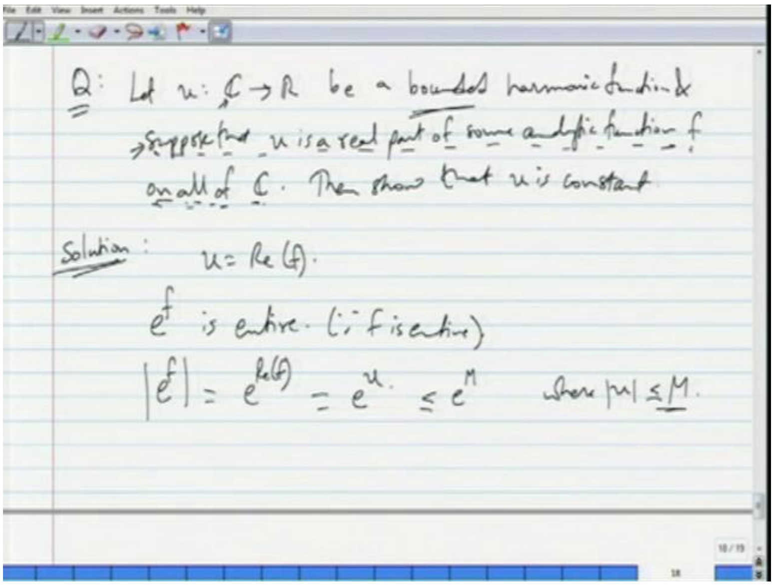
U can integrate with respect to u i with respect to y. But the these both both of these are zeroes so which means both are these are given to be 0 and u x any way is 0 on on vertical line segments. And then u y likewise 0 on horizontal line segments. So, actually identically 0 in the whole region. So, this gives you the constant of integration that is all.

So, two constant of integration, so u of x y is constant. So, likewise so, well that that already shows that f of x y is a constant, the real constant. So, that is solution to this problem alright so then next problem is as follows.

(Refer Slide Time: 48:09)



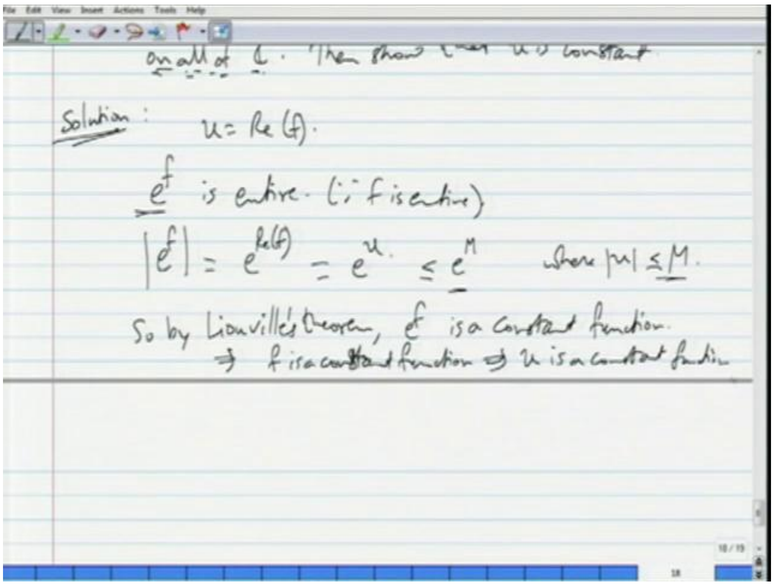
(Refer Slide Time: 48:27)



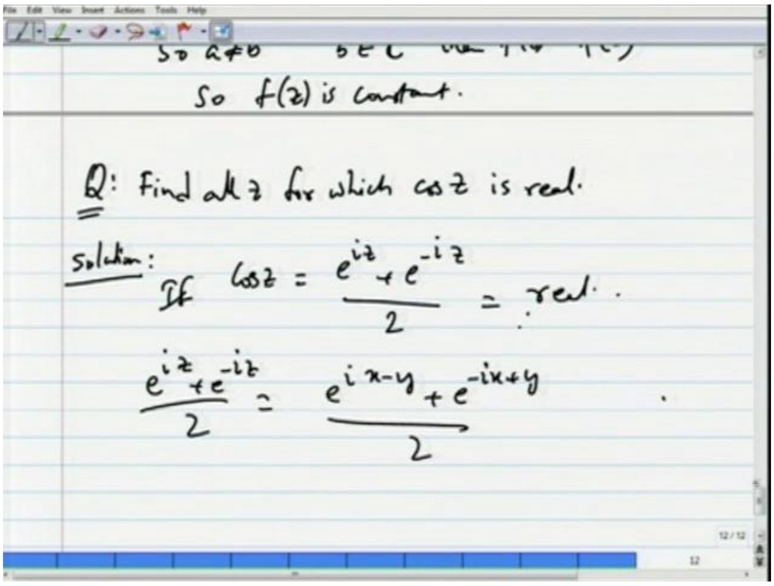
So, let u from C to r be a bounded harmonic function. Hence suppose that u is a real part of some analytic function f on all of C. Then show that u is bounded, u is constant rather. So, what that means is show that u further constant function. So, u is the real part of f for every complex number z. So, try to solve this problem and I will present the solution here. So, let me first remark that it can be shown that this supposition is unnecessary. What I mean by that is this is the additional, but since the we did not shown that in the theory we will assume this in addition to u being a bounded harmonium function.

Since u is the real part of f for every z belongs to C what you can do is you can consider e arise to f e arise to f is analytic. Or in fact it is entire because f is entire, since f is entire. This f here in the problem is entire. So, the modulus of e arise to f is going to be e raise to the real part of f at every point z and that is e raise to u. So, since it is given u is a bounded so this is less than or equal to e raise to m where u is bounded. So, the modulus of u or the absolute value of u is less than or equal to m. It is bounded function so there is such m.

(Refer Slide Time: 51:18)

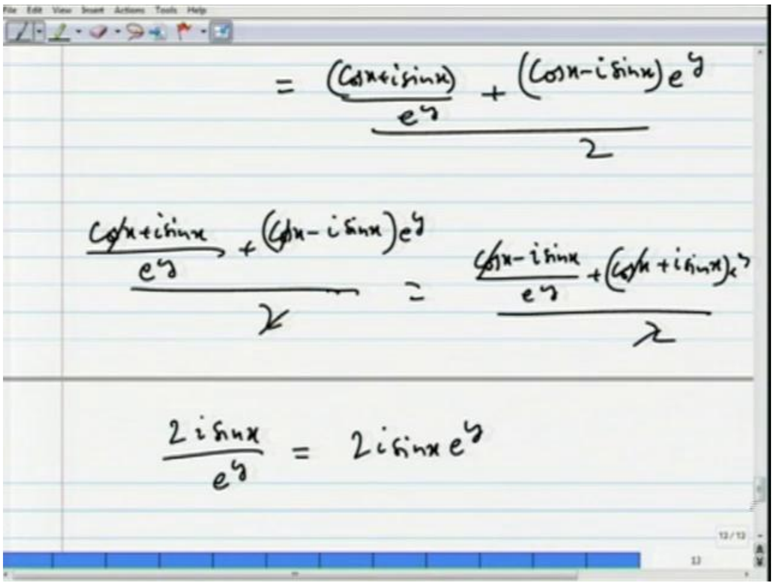


(Refer Slide Time: 52:05)



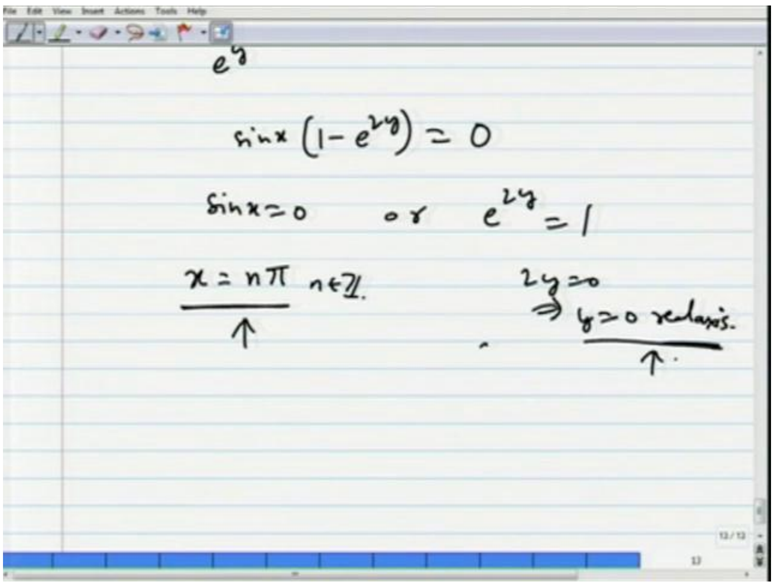
So e raise to f is bounded entire function. So, by Liouville's theorem we know that e power f is a constant function. That immediately implies that f is the constant function and which in term shows that the real part of f u is the constant function. So, that is also an easy exercise. So here are few more problems. So, here is the question. Find all z for which cosine z is real. So, the solution well, you just have to look at the expression for cosine z this is e power i z plus e power minus i z by 2 and then this is real number. So, we have to use the fact that real number is equal to its conjugate. So, we will use that fact, let us first see what this is. e power i z plus e power minus i z by 2, this is equal to e power i x minus y plus e raise to minus i x plus y divided by 2.

(Refer Slide Time: 52:56)



This is equal to well, this is cosine x plus i sine x divided by e power y plus cosine x minus i sine x times e power y divided by 2. So, this number should be equal to its conjugate for the above condition. So, so this number cosine x plus i sine x divided by e power y plus cosine x minus i sine x times e power y divided by 2 is equal to its conjugate. The conjugate of this is going to be cosine x minus i sine x divided by e power y plus cosine x plus i sine x times e power y divided by 2. So, up on simplification well these cosine x's cancel these cosine x's is cancel. Then you are left with 2 cancel of course, so you are left with 2 i sin x divided by e power y is equal to 2 i sine x e power y.

(Refer Slide Time: 54:16)



So, you end up with the equation sine x times 1 minus e power 2 y is equal to 0. So, well either sin x can be 0. So, this is 0 this product is 0 if sine x is 0 or if e power 2 y is equal to 1. Sine x is equal to 0 has the solution x equals n pi where n is an integer. Then e power 2 y is 1 implies 2 y has to be 0 which implies y is 0, which is the real axis. We know that cosine is real on the real axis. And whenever x is n pi even then your co cosine z is a real. So, these two give you the set of all points where cosine z is real and we will end this problem session with this problem.