Complex Analysis
Prof. Dr. P.A.S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 3
Complex Integration Theory
Lecture - 9
Morera’s Theorem and Higher Order
Derivatives of Analytic Functions

Hello viewers, recall last time we proved this theorem, which said that if f is analytic in an open set G, then so is its derivative. Using this theorem we are able to conclude that, f has infinitely many derivatives, if f is an analytic function on open set G .

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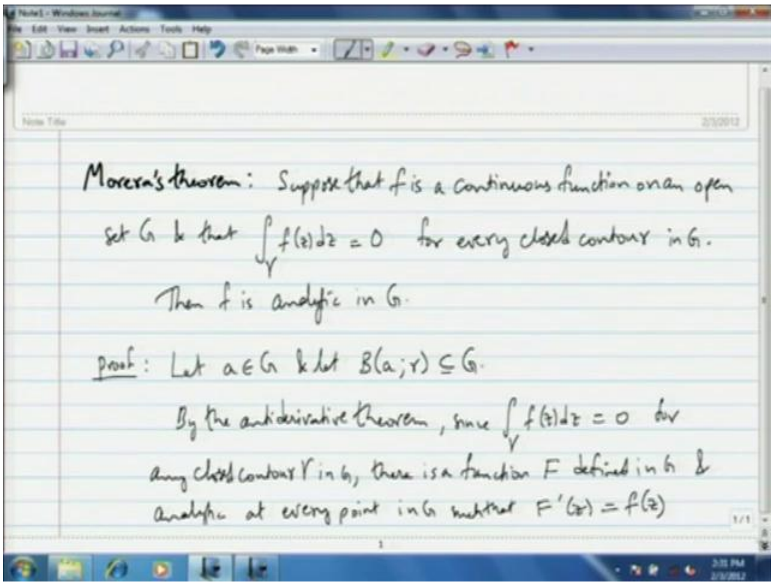

So, that we are going to put to use in the following consequence of that theorem, so which is often called as Morera’s theorem. So, this Morera’s theorem is a partial converse to your Cauchy’s theorem. So, it is states the following, suppose that f is a continuous function on an open set G and that, and suppose that the integration f of z d z on a contour gamma is 0 for every closed contour. So, in particular for every simple closed contour in G. If G, if gamma is any closed contour in G, then assume that the integration the contour integration of f on gamma is 0.

Then, the conclusion f is analytic in G, so the idea is essentially when when you have the integration over any closed contour is 0, use the anti derivative theorem to say that f is, f has a primitive or an anti derivative in an open set G. Use that to say that little f is the derivative of sum capital F in G. Since, capital F is analytic by the anti derivative theorem, little f which is derivative of capital F is also analytic by the previous theorem.

So, here is the proof I will write that down, let a belongs to G and let a bowl of radius r B a contained in G. Well actually, so by the anti-derivative theorem, since the integration of f on any closed contour is 0, for any closed contour gamma in G. There is a function there is a function capital F defined in G and and analytic on G, analytic at every point in G such that the derivative of F is little f.

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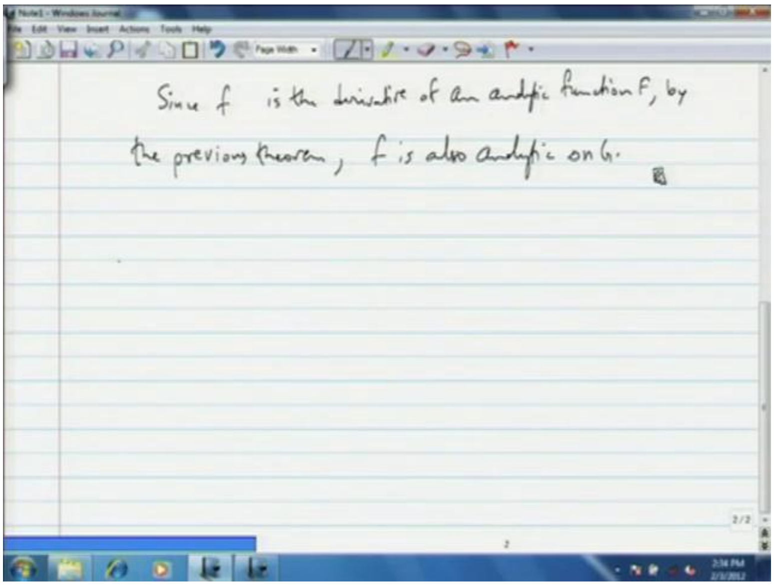

Since little f is the derivative of an analytic function capital F, by the previous theorem we conclude that, f is also analytic, right? We we just showed that in the previous session we showed that, if I mean the derivative of analytic function is also analytic. So, we can conclude from that that little f is also analytic on G at every point on G, so that is the proof of Morera’s theorem.

Let us now look at Cauchy’s formula for higher derivatives of an analytic function. So so far we have showed that there is an integral formula or an expression in terms of integration, contour integration of derivatives of the first derivative and the second derivatives of an analytic function. So, here we wish to show that such a formula exists for any higher order derivative.

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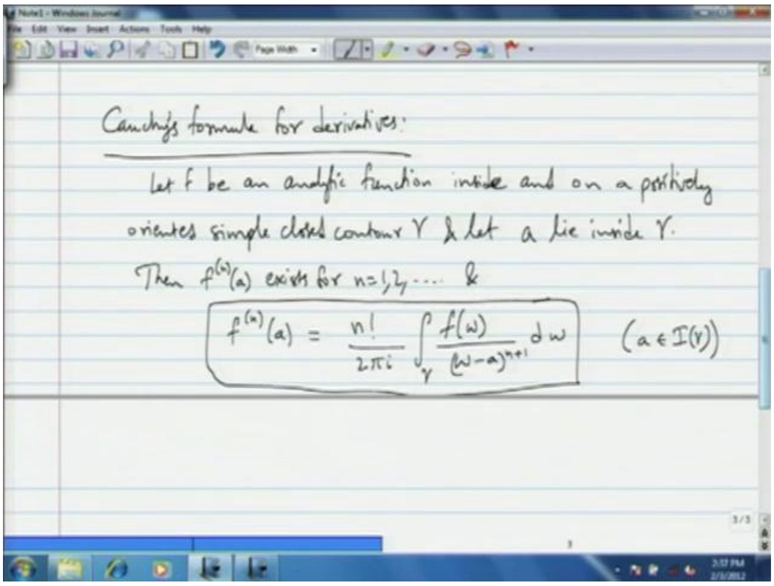

So, here is Cauchy's formula, I already introduced this formula earlier, so here it is. Here is the proof of it, so Cauchy's formula for derivatives. So, let f be an analytic function, inside and on a positively oriented, simple closed contour gamma. So, recall that if f is analytic inside and on positively oriented simple closed contour, then it actually analytic in an open set containing the inside of the simple closed contour and the trace of the simple contour itself, so that is what this means.

So, also and let a lie inside gamma, so there is a point a in the inside of gamma. Then, f n of a exists for n equals 1 2 etcetera, any natural number and the n th derivative of f at a is given by the formula n factorial by 2 pi i times the integration over gamma of f of w by w minus a raise to n plus 1 d w, where a is belongs to the inside of gamma. Already said that it lies inside of the gamma and that is your integral formula.

Now, the proof of this theorem sort of overrides the previous proofs, so we could have done this directly, but by giving a proofs separately for the first derivative and the second derivative, we sort of c the the workings of this theorem already or of the workings of the proofs of this theorem already.

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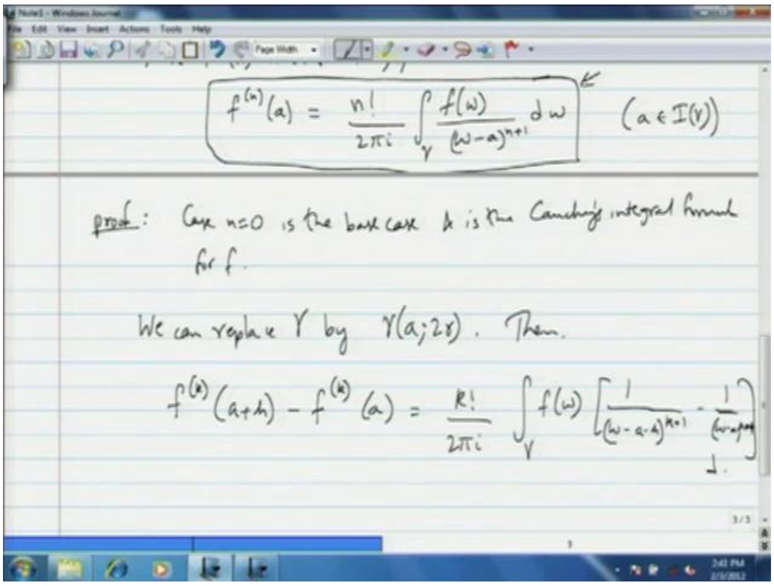

So, here is a proof. So, I should say this is really a sketch of the proofs. So here case n equals 0 is the base case, and is the Cauchy's integral formula. So, for n equals 0 we already have a proof via the Cauchy's integral formula for f. Now we can without loss of generality, like we discussed on the proof of earlier theorems, we can replace gamma by a gamma a 2 r where gamma by gamma a 2 r . By gamma a 2 r we mean a circle of radius 2 r centered at a, so we have done that in the previous theorem, so we will do that here as well. By the deformation theorem or Cauchy's theorem version 3, the integration on gamma the contour integration on gamma of of this expressions is equal to the contour integration on gamma a 2 r.

Of course, here we are assuming that r is sufficiently small, so that gamma a 2 r the closer of a gamma a 2 r lies completely inside of a the simple closed curve gamma. Then, we estimate the k plus 1 th derivative of f at a plus h minus f of k plus 1 th derivative of f at a. So, we calculate this expression, by the way we already showed that f has all derivatives derivatives of all orders. So, firstly assume that assume that the above mentioned formula or this formula in the box works for up to k.

So, we are going to prove by using mathematical induction, the case n equal 0 is your Cauchy's integral formula and assume that the statement is true for up to k. Then this equals k factorial divided by 2 pi i times the integration over f of w, over gamma over f of w times 1 by w minus a minus h of k plus 1 minus 1 by… Apologize, this should be please make the note, this should be the k th derivative, the k th derivative. Then f of w times 1 by w minus a minus h power k plus 1 minus 1 by w minus a power k plus 1, that is k plus 1 d w. So, let us look at this expression within the square brackets.

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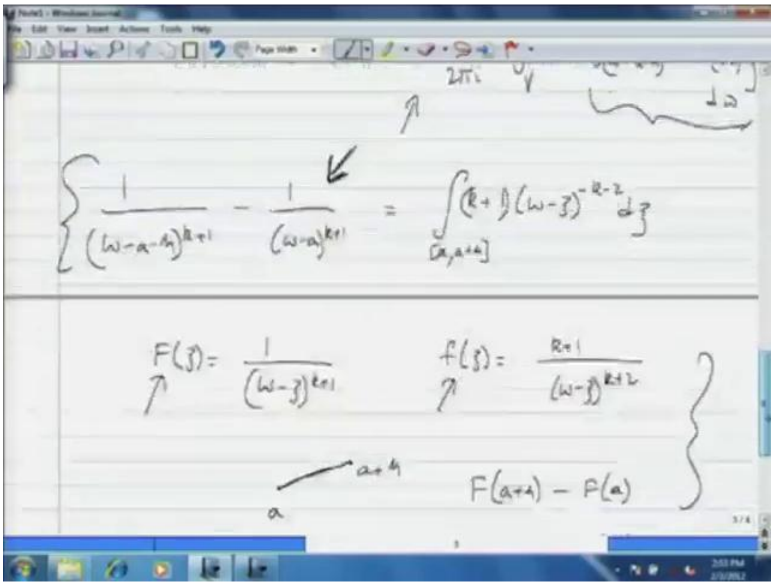

So, 1 by w minus a minus h power k plus 1 minus 1 by w minus a power k plus 1. So, this expression can be thought of as, let me put this in parenthesis. So, this can be thought of as a k plus 1 times times w minus zeta power minus k minus 2 integrated. So, k plus 1 times this integrated over the straight line connecting a and a plus h, so that is the notation for the straight connecting a and a plus h d zeta.

So, recall the fundamental theorem of calculus, this is the the use of fundamental theorem calculus in the opposite direction. So, recall that if if I take the function capital F of zeta equals 1 by w minus zeta power k plus 1, then the derivative of capital F of zeta is little f of zeta equals k plus 1 divided by w minus zeta power k plus 2. This this f is continuous on the required region here, on the inside of this curve, a simple closed curve and this capital F is analytic.

Since, f little f is a derivative of the function capital F, the fundamental theorem of calculus tells you that, when you consider the straight line connecting a and a plus h, then the integration, the contour integration of little f on this contour is equal to the value of capital F at a plus h minus the value of capital F at a, which gives us the expressions here on the left hand side. We are essentially using the fundamental theorem of calculus and then let us substitute it in this in this expression here, okay?

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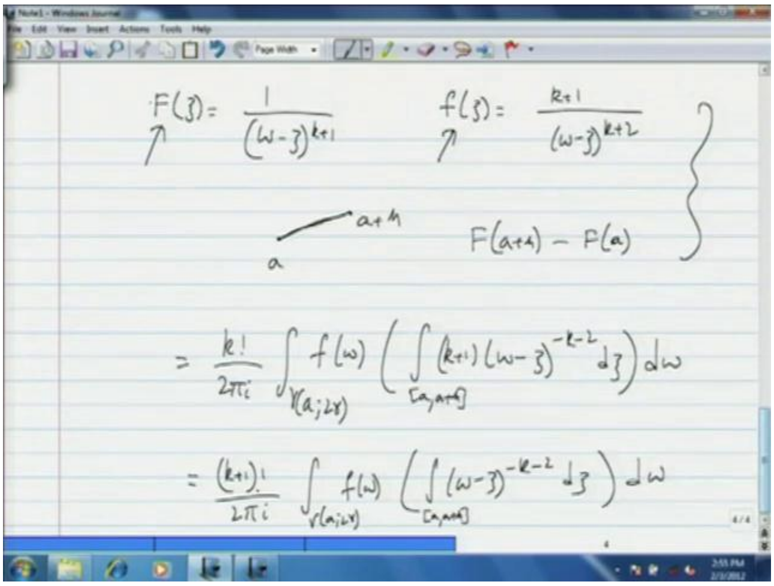

So we get, that is equal to k factorial divided by 2 pi i times the integration on gamma a to r, recall I said that, I can replace the gamma with gamma a 2 r, times of the integration of f of w, times the integration over the straight line joining a comma a plus h. Please note that is the notation for a straight line joining a and a plus h once again. Then k plus 1 I have k plus 1 times w minus zeta raise to minus k minus 2 d zeta and then d w.

So, I can use the k plus 1 to make it k plus 1 factorial, so this gives me k plus 1 factorial divided by 2 pi i and then the rest of the expression. So, integral gamma a to r f of w times the integration of w minus zeta raise to minus k minus 2 d zeta on the straight line joining a and a plus h and then d w.

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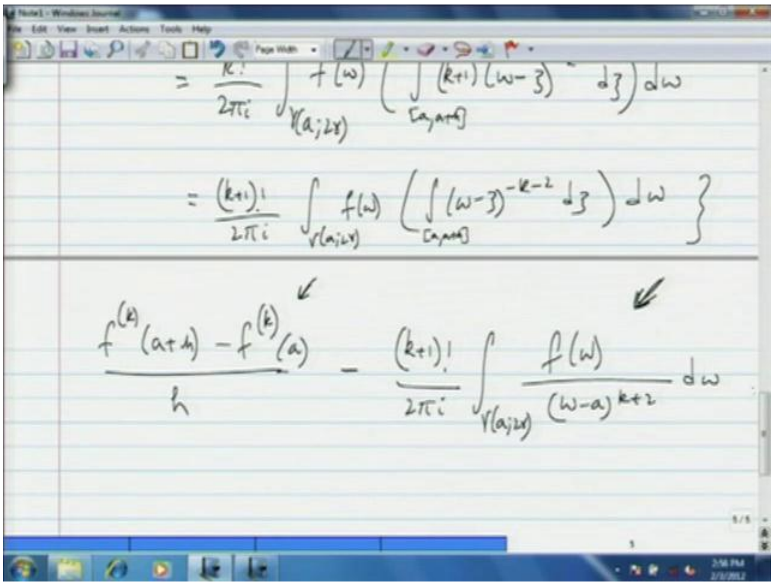

So, that is your f of k at a plus h, minus f of k are the k th derivative of f at a. Then, when we divided this by h, then we let us try to estimate this quantity minus k plus 1 factorial times or divided by 2 pi i times the integration over gamma a 2 r of f of w divided by w minus a raise to k plus 2 d w.

This is the desired expression, we want to show that the k plus 1 th derivative of f at the point a has this form. Then, we we have some, we have constructed some expressions which is equal to the numerator of this expression here. Then we will now estimate this difference, so this difference and we will show that this can be arbitrary small. We conclude that the k plus 1 th derivative of f, not only exists and at a not only exist, but it is equal to this other expressions that we claim, it is equal to…

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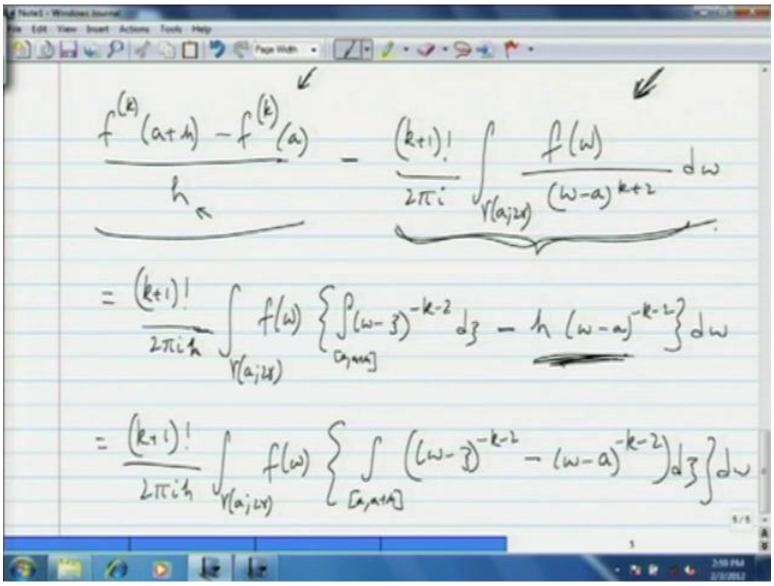

So, this difference let us see is equal to k plus 1 factorial divided by 2 pi i, as above like like it is here, but, there is a h in the denominator here, so I use that h and then I have integration over a gamma a 2 r of f of w times there is the other stuff here. So, what I will do is, I will recall that a here this is w the integration of w minus zeta raise to minus k minus 2 d zeta on a a plus h, on the contour a a plus h. Then, I have this second expression here, which I will decode a integrand. So, this is minus, there is a h here in the denominator. So, this gathers h times w minus a raise to minus k minus 2 d w.

So, this part comes from here and then the rest is from the first expression. Now, this is equal to k plus 1 factorial divided by 2 pi i h, integration over gamma a 2 r of f of w times. Now, I can write h times h times w minus a raise to minus k minus 2 as follows; I will write integration from a to a plus h times w minus zeta per minus k minus 2 d w, I will postponed the d w or d zeta. Then minus w minus a raise to minus k minus 2, so only because the integration of w minus a power minus k minus 2, which is constant with respect to zeta over the straight line joining a and a plus h is w minus a power minus k minus 2 times h.

So, I am using the fact to include this underline stuff here in the earlier expression into the integrand, into the inner integrand. Then write a d zeta like that. So, that is that and then times d w.

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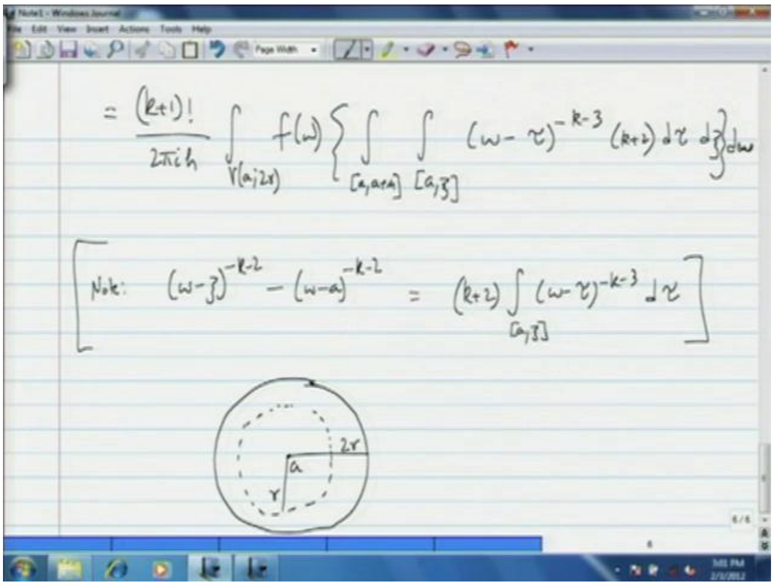

Now, I will once again use the fundamental theorem of calculus, the complex for complex analytic functions and write the inner most integrand as a further integration. So, this is f of w times the integration over the straight line joining a and a plus h, and then I will interpret this inner most integrand, as the integration over the line joining a and zeta. That is a and zeta, of the w minus yet another variable tau power minus k minus 3. Then I have a constant k plus 2 and then d tau d zeta and then d w.

So once again notice that, I am using the fundamental theorem of calculus, so here is a, note I am using w minus zeta power minus k minus 2 minus w minus a power minus k minus 2 is equal to k plus 2 times integration over straight line joining in a is zeta of w minus tau power minus k minus 3 d tau. That is by the fundamental theorem of calculus, like I explained earlier. So, with all this now we are ready to make an estimate, so here is the picture. So, here is a here is your point a, and here is the circle of radius 2 r centered a t, so this is 2 r. Now, look at the circle of radius r, this is the circle of radius r centered at a, assume. Okay?

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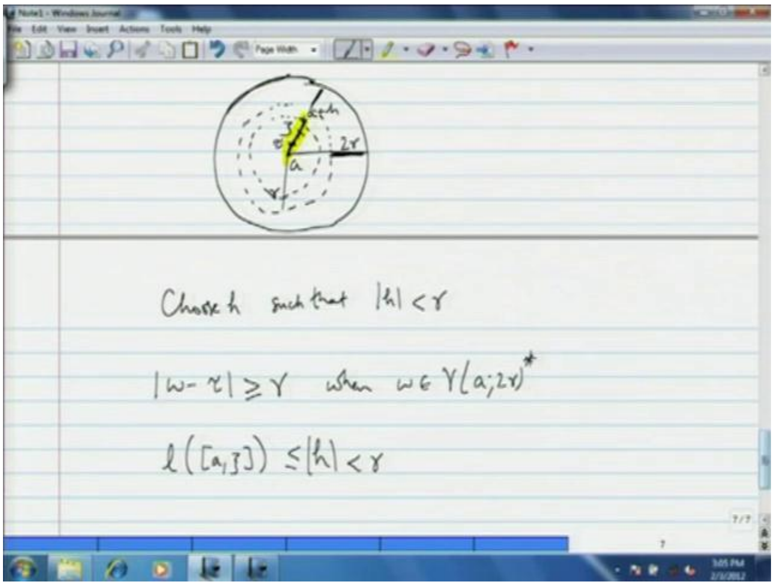

Now, choose choose h such that the modules of h is strictly less than r. So, that you are a plus h lies somewhere, a plus h could be anywhere inside this circle of radius r centered at a. So, let us suppose that is the point a plus h, so here is the cross mark, the point a plus h. Then we will draw a circle there, again get another circle, which is inside the circle of radius r centered at a. Then, here is the straight line connecting a and a plus h. So, may be gives a highlighted to show that and then a point zeta is a variable point on this straight line connecting a and a plus h. Then the parameter tau is any point on the line connecting a and zeta, so tau is a point on the line connecting a and zeta.

So, that is a the setup and given this setup it is clear that the modules of w minus tau, tau lie well within the circle of radius, modules of h around a. So, w minus tau is at least in modules, is at least r when w belongs to the trace of w belongs to the trace of gamma a 2 r star or star means trace. So, it is clear because this distance is at least maintained. So, that distance is at least or this distance is at least maintained, which is r already for any point w on this circle, the outer most circle. Also the length of straight line joining a and zeta is clearly less than or equal to h or modules of h, which is strictly less than r.

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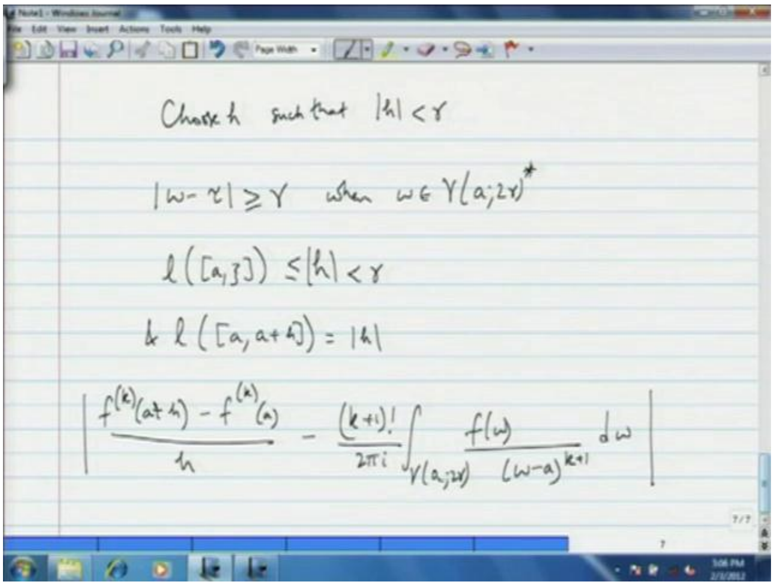

What is also important for this estimation is that, the length of the straight line joining and a plus h is equal to the modules of h of course. So, we will use this three to now estimate this expression write here. So, so the modules of the k th derivative or here k th derivative of f at a plus h minus the k th derivative of f at a divided by h minus k plus 1 factorial divided by 2 pi i times the integration of gamma a 2 r of f of w times or divided by w minus a power k plus 1 d w in modules. The modules of this is by the estimation theorem, this is less than or equal to, well let me go back to this expression. It is less than or equal to k plus 1 factorial by 2 pi times mod h. So, k plus 1 factorial by 2 pi times mod h times integration over gamma a 2 r of of all of this.

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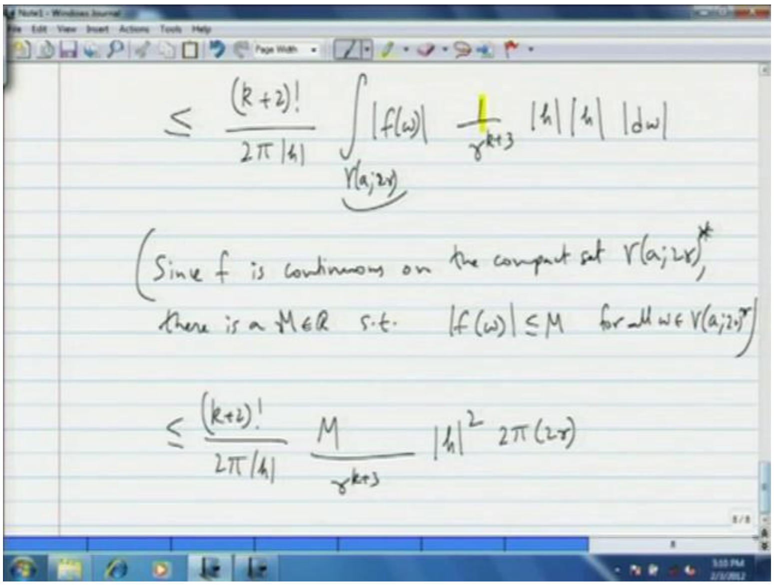

So, modules of f of w, of the modules of everything inside integration and modules of d w at the end, let me put that later. So, notice that w minus tau we said is is at least r, write here w minus tau is at least r. So, 1 by w minus tau power k plus 3, which appears now in the inner most here is at most 1 by r raise to k plus 3. So, I have a 1 by r raise to, excuse me, so 1 by r raise to k plus 3. Also there is k plus 2, which appears k plus 2. I will use the k plus 2 to write the expression in the front as k plus 2 factorial. Then, there is the length of this curve a from a to all other straight line joining a to zeta, which is modules of zeta minus a.

We commented that the modules of zeta minus a is less than or equal to or strictly less than r, or less than or equal to h. So, we have this is less than mod h and likewise we have the length of the straight line joining a and a plus h. Notice that notice that, this r power k plus 3 is independent of the parameter tau or the parameter zeta, which appear in integration. Then the the straight line joining a and plus h is at most modules of h in length. So, we have two modules of h there and finally, we have your d w in modules, so modules of d w. Since, f is continuous on the compact set gamma a to r, f is bounded on that set.

So, there is a M, so there is there is real number M, such that the modules of f of w is strictly or is less than or equal to M for all, w belongs to gamma a 2 r star. I guess, I should have said star here, which is the compact set because of that I will use this fatal. Put this in parenthesis and say that this is less than or equal to expression above, is less than or equal to k plus 2 factorial divided by 2 pi modules of h times M divided by r power k plus 3 times modules of h square times 2 pi times 2 r, which is the length of the curve gamma a 2 r star.

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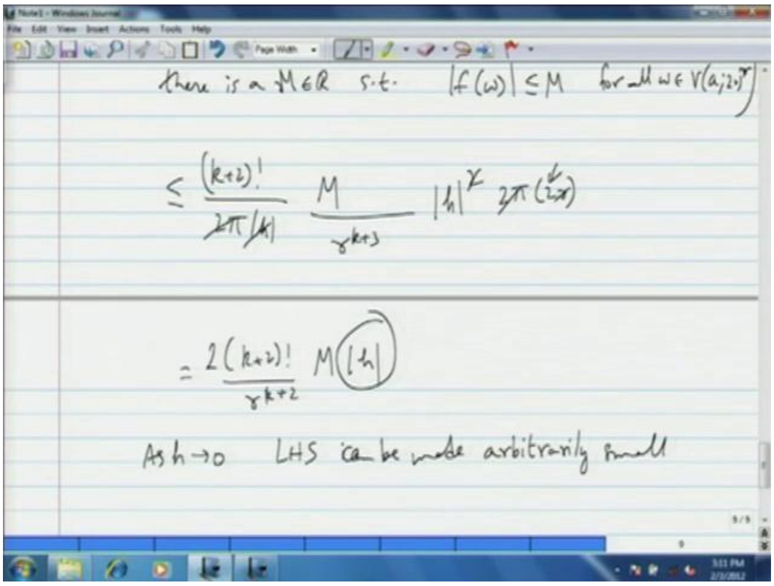

So, after cancelation it is clear that, this is less than or equal to or this is actually equal to k plus 2 factorial divided by r power k plus 3. h cancels with 1 factor h here in modules and 2 pi cancels 2 pi. Then actually 1 r cancels here with 1 factor, I guess, I have r power k plus 2 times M and then modules of h. Then there is 2 as well from here, so that is what this difference in modules is, less than or equal to and from this final expression. It is clear that as h goes to 0, and the L H S can be made arbitrary. L H S of this inequality can be made arbitrary small or it it tends to 0, because there is the factor modules of h here, all the rest are constant, so that tends to 0.

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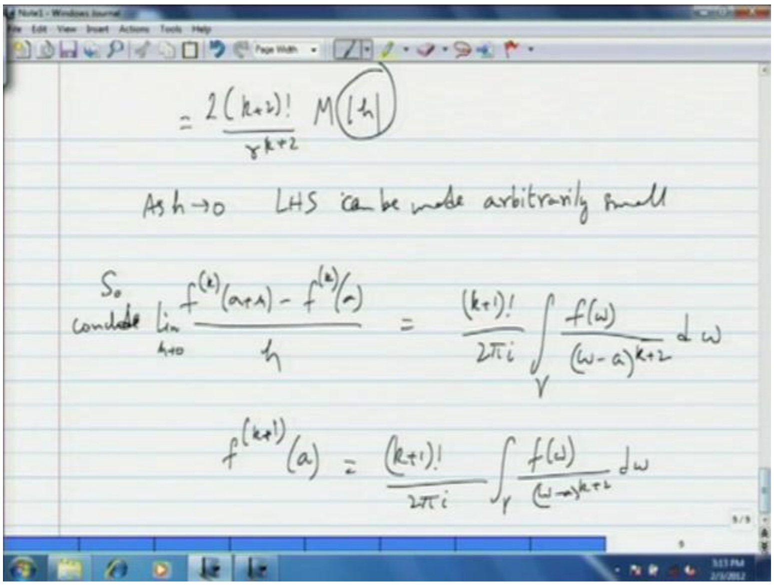

So, we conclude that, f k of a plus h minus f k of a divided by h is equal to, so conclude that this is equal to, this in limit as h goes to 0. This in the limit is equal to k plus 1 factorial divided by 2 pi i times the integration over gamma a to r, but gamma a to r the integration on that is equal to the integration on gamma of f of w divided by w minus a power k plus 2 d w. So, the left hand side is nothing but you are in the k plus 1 th derivative. At the point a of f and that is equal to above and that proves the induction step for induction for proof by induction, you can fill in the rest of the steps and and say that…

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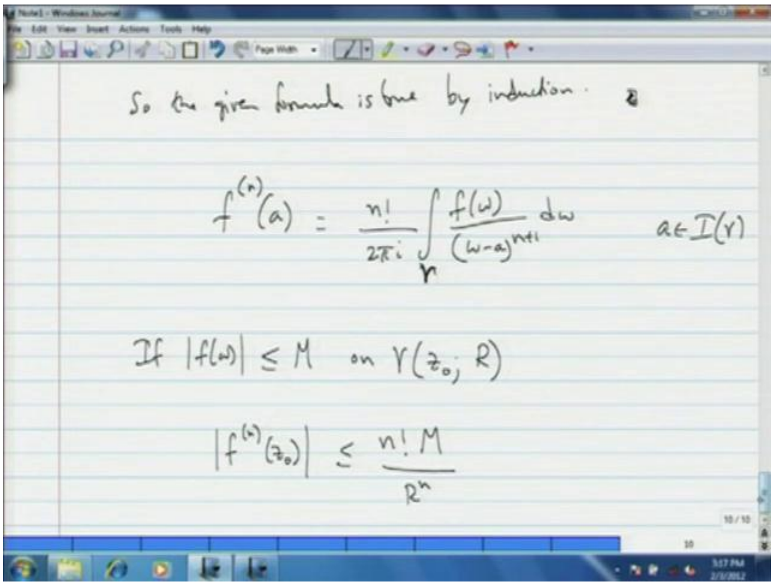

So, the given formula is true by induction, by principle of mathematical induction. So, that completes the proof of Cauchy’s integral formula for higher derivatives. So, let us write that down in order to prove certain inequality. The n th derivative give under a suitable assumptions. The n th derivative of f at a is equal to n factorial divided by 2 pi i times the integration over the simple closed curve gamma of f of w by w minus a power n plus 1 d w where a belongs belongs to the inside of the simple closed curve gamma.

So, the following is commonly called the Cauchy’s inequality for the n th derivative. So, if if a modules of f of w is less than or equal to m, let me use the notation I have already used in gamma z naught r, so this is a circle of radius r, so this is a contour whose traces circle of radius r centered at z naught. So, assume that the modules of f of w is less than or equal to M on that circle.

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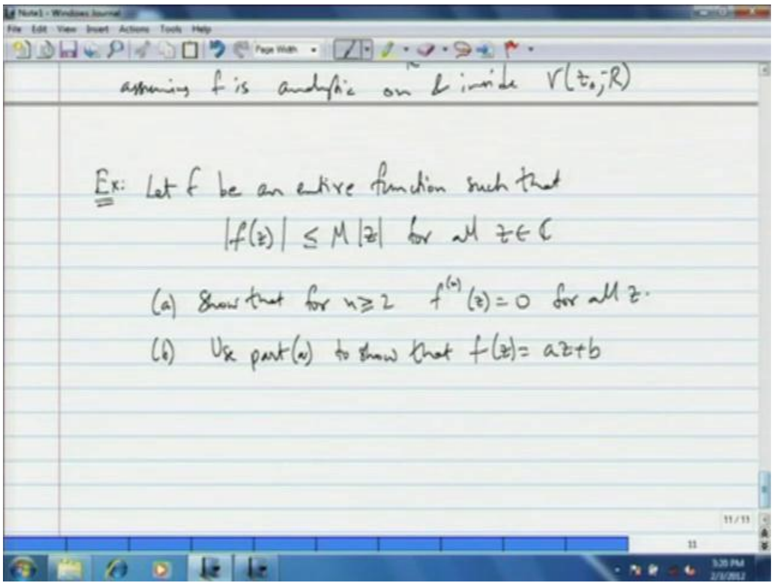

Then your modules of f n at z naught is less than or equal to n factorial times M divided by r power n provided of course, assuming that, assuming f is analytic on and inside gamma z naught r. That is your a Cauchy's inequality or Cauchy's estimate. That is the easy consequence of the above formula, by using the estimation theorem. Next, let us look at the following example and it says that let the f be and entire function such that, the modules of f of z is less than or equal to M times the modules of z, for all z in the complex plane.

So, firstly a show that for n greater than or equal to 2 the derivative, the n th derivative of f at z is equal to 0 for all z and part b asks you to use part a to show that f of z is a z plus b. So, notice that here we have a condition which is, you could say a slightly weaker then, a hypothesis of levels theorem. So, under this weaker condition we are able to show that f of z is actually equal to a linear function in z.

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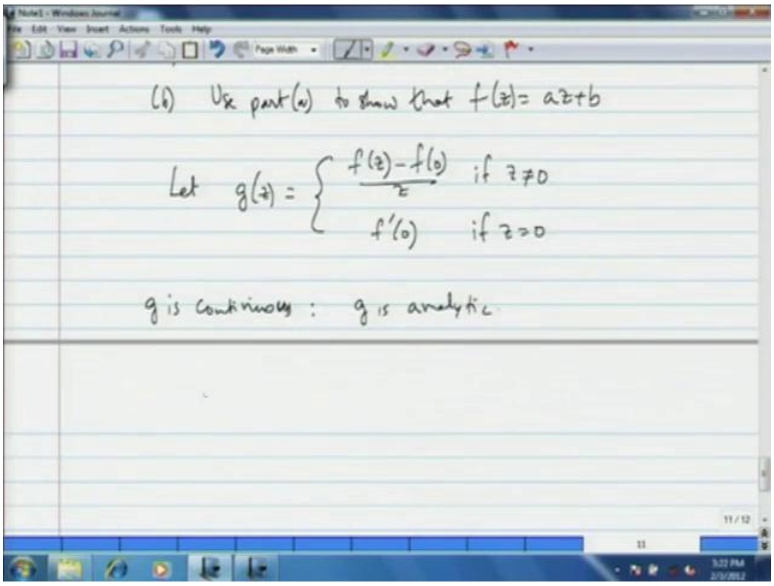

In order to in order to solve this exercise, we will let g of z be a function, will define a function g of z equals f of z minus f of 0 divided by z if z is not equal to 0. Since, f is entire function definitely its derivative at 0 exist, so then this is a f prime of 0. If z is equal to 0 and since the limit of the first expression is as z goes to 0 is equal to f prime of 0, g is continuous. Not only that g is actually g is analytic, so let me conclude that g is continuous. One can verify that g is actually analytic at 0 and hence analytic everywhere g is analytic.

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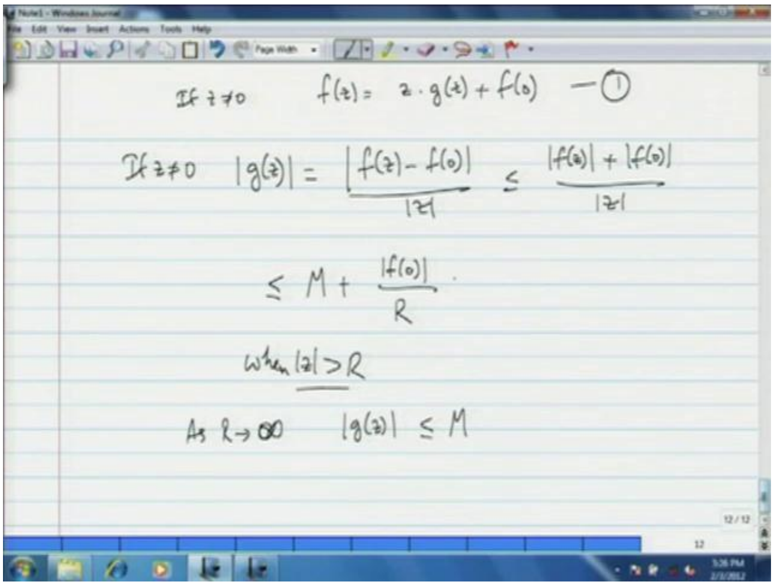

And also from the first expression, if z is not equal to 0, then your f of z is z times g of z plus f of 0. We will store this as equation one. Now, notice that g of z modules of g of z is equal to the modules of g of z, well z is not equal to 0 if z is not equal to 0 modules of g of z is equal to modules of f of z minus f of 0 by modules of z. This is less than or equal to modules of f of z plus modules of f of 0 by modules of z by triangle inequality and that intern is less than or equal to M plus modules of f of 0.

So, when when the modules of z is strictly less than or strictly greater than r the outside of a circle of radius r centered at 0. This is intern less than or equal to, well the first portion we know is less than or equal to M plus modules of f of 0 divided by modules of z is greater than r. So, this is divided by r and we can see that as R tends to zero or infinity modules of g of z is a less than or equal to M.

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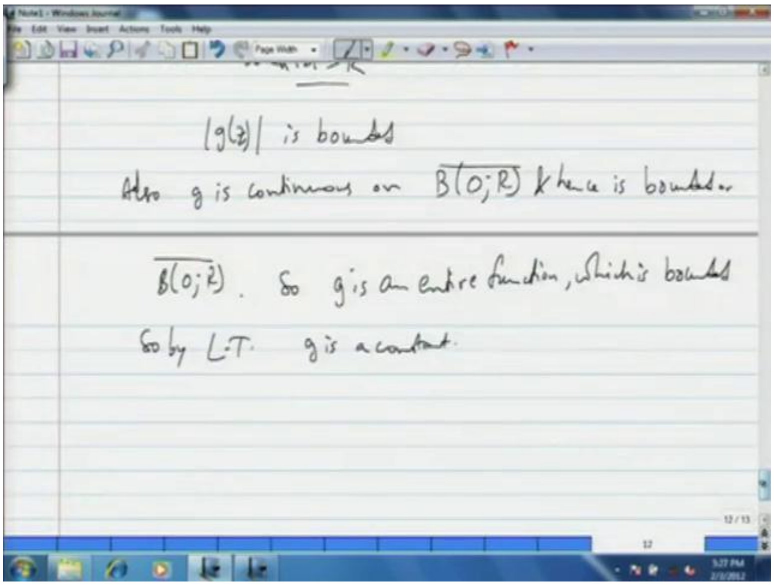

What I want to say is, modules of g of z is actually is bounded or g of z we say is bounded. So, by also g is analytic, g is continuous on b 0 R bar. Hence is bounded on b 0 R bar as well, so on outside of a circle radius R, it is bounded by the above estimate and on the inside of a circle of radius R it is also bounded by continuity of g, excuse me. So, g is an entire function which is bounded, so we conclude, so by levels theorem g is a constant.

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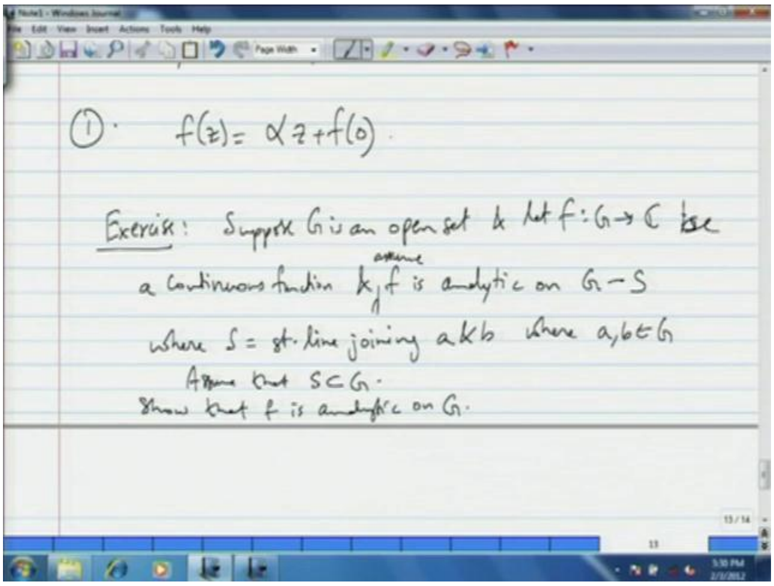

So g is a constant function, so we get f of z by from 1 using 1 we get f of z is some constant alpha, alpha times alpha times z. Let me go back to 1, z plus f of 0. This is what we want, so that shows the problem. So, there is another exercise that I will assign to the viewer in this connection, so using Morera's theorem, you can show the following. Suppose, G is an open set and let f from G to C is a be a continuous function and f is analytic or and assume f is analytic on G minus a set S, where S is the straight line joining a and b, where a comma b belong to G. So, a and b are two points in the set g and s is a straight line joining a and b.

Also, assume that assume that s is completely contained in G, there is a there there can be an open set with two points, such that the straight lines joining this two does not line the open set, so assume that the straight line is completely contained in G. So, under this assumptions show that show that at f is analytic on G, so if f is analytic on G minus S, so the straight line as then f has to be actually analytic G. So, use the Morera's theorem to show this exercise.

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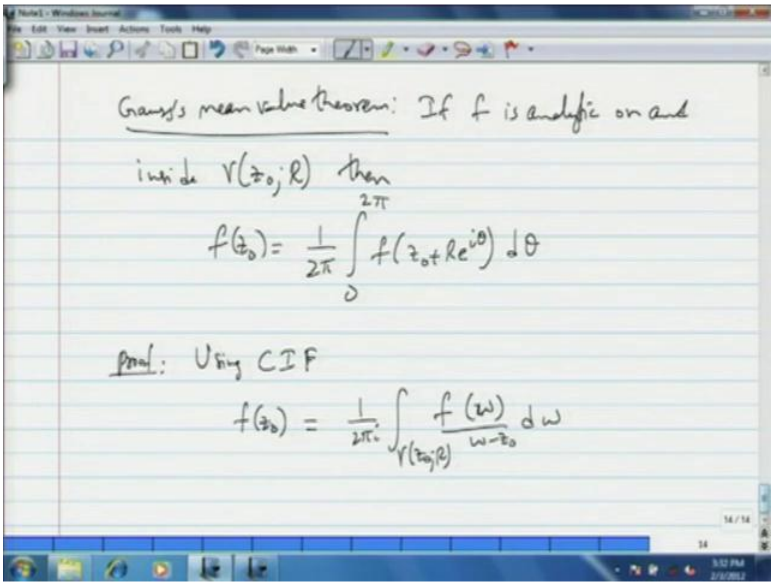

Now, we will sort of end end with the following result known as gauss mean value theorem, for completeness. I will code it, code this here, but it is a easy consequence of Cauchy's integral formula. So, if f is analytic on and inside gamma z naught R, then f of z naught is 1 by 2 pi integration from 0 to 2 pi of f of z naught plus R e power i theta d theta. So, this is an easy consequence of Cauchy's mean value theorem, so using or else sorry Cauchy's integral formula. So, using Cauchy's integral formula we know that f of z naught under the above a hypothesis f of z naught is 1 by 2 pi i times integration over gamma z naught r of f of f of w by w minus z naught t w.

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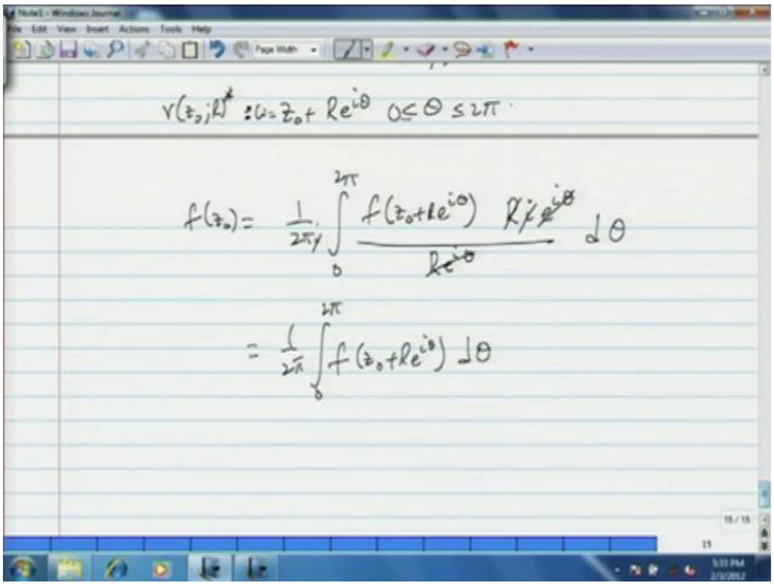

So, parameterized the circle of radius r as gamma z naught r, star can be parameterized, what is this? This is set of of all points z naught plus R e power i theta theta between 0 and 2 pi, so that is your parameterization. So, f of z naught is 1 by 2 pi i times integration from 0 to 2 pi of f of w, so this is w equals that, so f of w is f of z naught plus e power i theta. Then d w is going to be R i e power i theta and w minus g naught is simply R e power i theta. Then we have a d theta, from the derivative, this is your i cancels R e power i theta cancels and then i cancels o. This i you have 1 by 2 pi integration from 0 to 2 pi of f of z naught plus r e power i theta d theta, simply evaluation using the Cauchy's integral formula. So, this can be thought of as the value of f at a point z naught.

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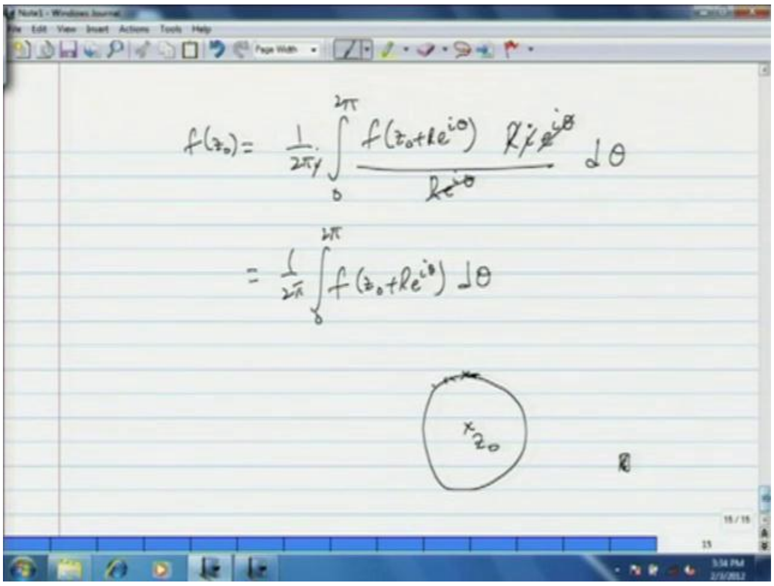

So, the value of f at a point z naught is actually equal to is is sort of mean of the values on a circle, so the integration can be thought of as a continuous sum. Then the length of that curve is R and then or the length of the curve is 2 pi times R. So, 1 by 2 pi times the integration is equal to f of z naught. So, that is gauss mean value theorem and that sort of completes our discussion of an integration theory to some extent. There is, there are more local properties an analytic function that need to be explored, which we will see in the next module.