Complex Analysis
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Module - 3
Complex Integration theory
Lecture - 8
The first and Second Derivatives of Analytic Functions

Hello viewers, today we will see the higher order derivatives of analytic functions exist on the domain of analyticity of a analytic function. So, this will imply that the differentiation of analytic function, we will see that this will imply that that is also an analytic function. So I will start with a reminder of the third version of Cauchy's theorem which tells you that if you have a positively oriented simple closed curve, and then f is analytic on and inside this simple closed curve.

And if you have yet another simple closed curve contained in the inside of the first curve, then integrating f on the first curve is the same as the, same as integrating f on the second curve. So, that was the deformation theorem or Cauchy's theorem version three and we will put that to you today.

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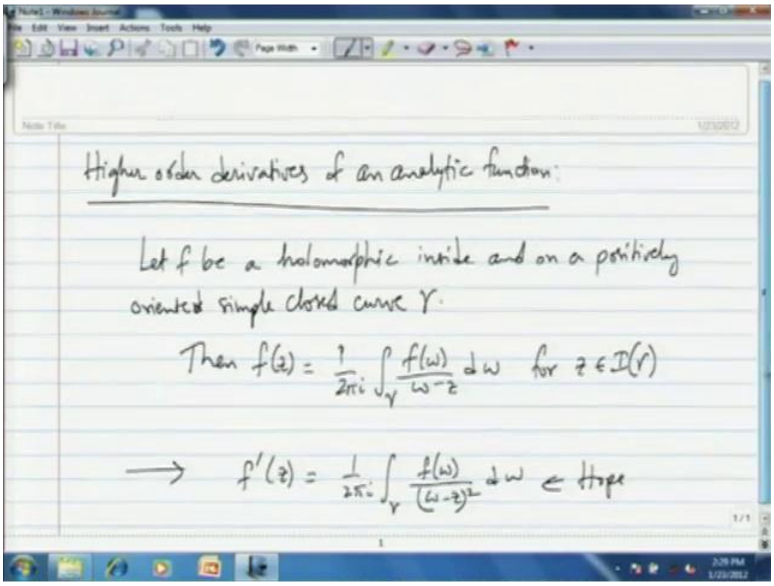

So, today we are going to see higher order derivatives of an analytic function. So there is the set up let f be Holomorphic function, so let f be a Holomorphic or analytic so I explained that Holomorphic is another word for analytic Holomorphic, inside and on a positively oriented simple closed curve gamma. So, then the conclusion is that when f of z is well this we know from Cauchy's integral formula, so that f of z is 1 by 2 pi i integral over gamma f of w by w minus z d w, for z belongs to i of gamma the interior of the simple closed curve gamma.

So, now intuition is that we will be able to differentiate under the integral, so hopefully we are able to differentiate under the integral sign then we hope to say that f prime of z is 1 by 2 pi i times the integral over gamma of well the differentiation with respect to w we hope to write this as w minus z square minus and the minus cancel d w. So, this we hope so, but we should justify the differentiation under the integral sign in order for us to be able to do this and that is the content of the of what follows. So, what I will do here is I will follow the textbook Presley and in here first I will try to justify the formula for or justify this Cauchy's integral formula for f prime which we wrote as hope here.

Then I will justify or produce a formula Cauchy's integral formula for f double prime of z then show that higher order derivatives of f exist or in other words all derivatives of all orders of f exist. And then I will give an proof based on induction to produce the formula for Cauchy's integral formula for all higher order derivatives.

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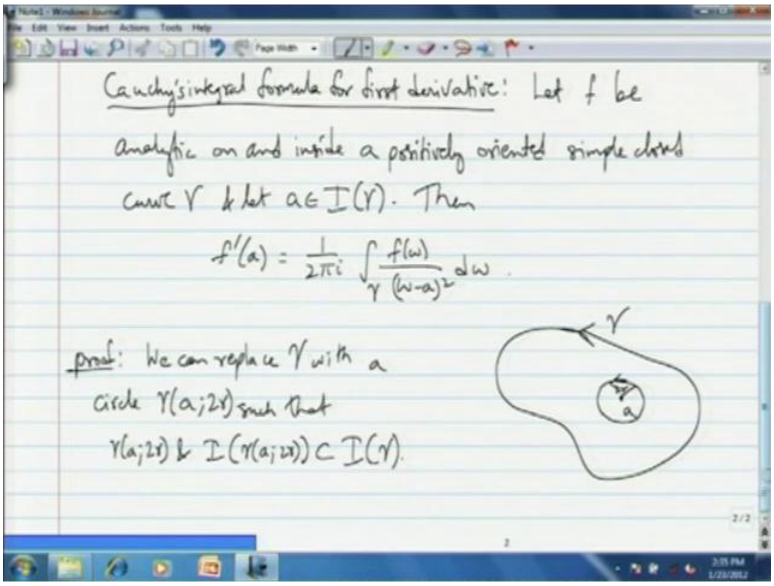

So, that the program, so here I start with the following I will start with the first derivative Cauchy's integral formula for first derivative. So, this about the justification of differentiation under this under the integral, so let here is the set up once again let f be analytic on and inside a positively oriented simple closed curve gamma and let a belongs to the inside of the simple closed curve gamma. Then f prime of a is 1 by 2 pi i integration of integration over gamma of f of w by w minus a square d w so that is here Cauchy's integral formula for the first derivative.

So, we will, so we will prove this here and notice that I will well I just said simple closed curve gamma where a belongs to the inside of the gamma. So, the idea of the proof is that I will take this simple closed curve gamma positively oriented and here, here is a I will replace gamma by a circle of radius r or 2 r around a where 2 such that you know this disc of radius 2 r close disc of radius 2 r around a is completely contained inside this curve gamma.

So, now notice that the integration on the boundary of this disc is the same as the integration on this curve gamma of the integral that is because we will use the Cauchy's theorem version number three. So, that is the idea and then we will be able to do something with this circle of radius 2 r o, here is the proof so we can replace we can replace gamma with a circle gamma a 2 r. So, recall gamma a 2 r is a circle centred at a and of radius 2 r and it is oriented in the positive sense and we get so we can replace that with that we will decide what r is shortly. So, such that at least we need the following of r such that this circle gamma a 2 r and it is interior and the inside of gamma a 2 r are contained in the inside of gamma, that we can do by Cauchy's theorem version three.

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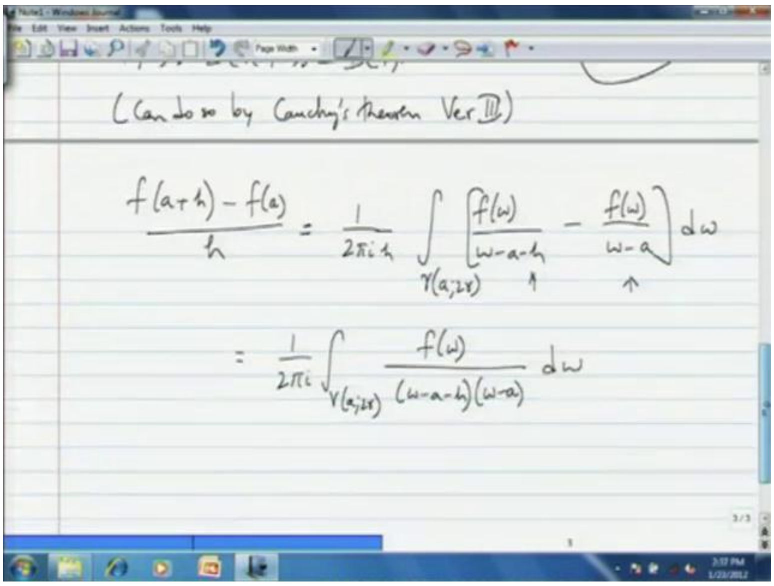

You can do so by Cauchy's theorem version three. Now, we will calculate this quotient f of a plus h minus f of a by h which in the limit will give you the differentiation of f. So, using the Cauchy's integral formula for f which we have already proved this is 2 pi i times 1 by 2 pi i times well there is this h in the denominator and then integral over this gamma a 2 r I am going to apply Cauchy's integral formula for f of a plus h that will give me f of w by w minus a minus h and then I am going to apply Cauchy's integral formula for f of a that will give me f of w by w minus a all this times d w.

So, I am using the Cauchy's integral formula twice and upon simplification this gives 1 by 2 pi i this h in the denominator cancels with the h I am going to obtain when I add these two expressions or subtract whichever way you consider it then w minus a minus h minus times w minus a once you make these two expression into one you get a h in the numerator that cancels the h outside then you get this d w.

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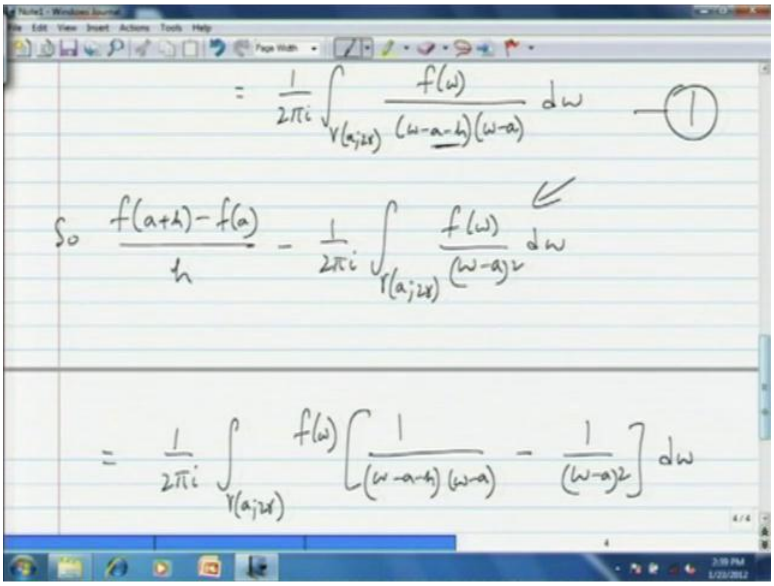

Now, the idea is hopefully this when this h is small when you take the limit as h goes to 0, the idea is the integrand will become w minus a whole square. So, we will justify that, so here is it is what we have so f of a plus h minus f of a by h minus 1 by 2 pi i integral over gamma a 2 r f of w by w minus a square d w. So, we obtained this expression 1 for the quotient f of a plus h minus f of a by h. So, what we are going to do is we are going to estimate the difference between this quotient and this, and this integral here this is different notice from the expression we got from 1.

So, that difference is equal to 1 by 2 pi i times the integration over gamma a 2 r of I am using 1 now then I get f of w times 1 by w minus a minus h times w minus a that is from your 1 then minus 1 by w minus a square that is from this differential that is the second expression here. So, I am subtracting the integrands and then I get this that is your difference here this difference here.

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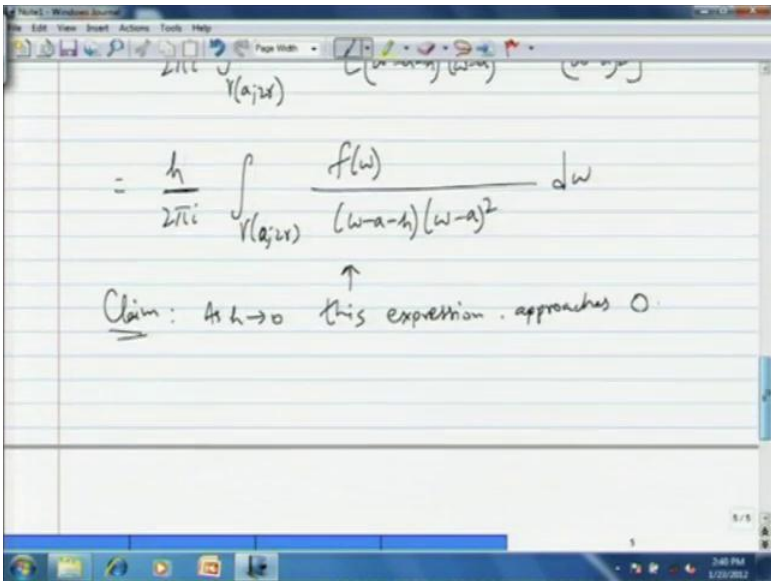

So, then that is 1 by 2 pi i after clearing the fraction I get integral gamma 0 2 r or a 2 r rather a 2 r f of w times h divided by w minus a minus h times w minus a square d w. So, I will transfer the h to the front it is free of the integrand, so I will just integration rather so I will move that to the front.

So, now the claim is so here is the claim now the claim is that as h approaches 0 this integrand this not just the integrand this expression, this expression here tends to 0 approaches 0. So, that is the idea and in order to prove that approaches 0 we will, we will use the estimation theorem that we talked about few sessions ago that is the useful tool when you want to estimate something.

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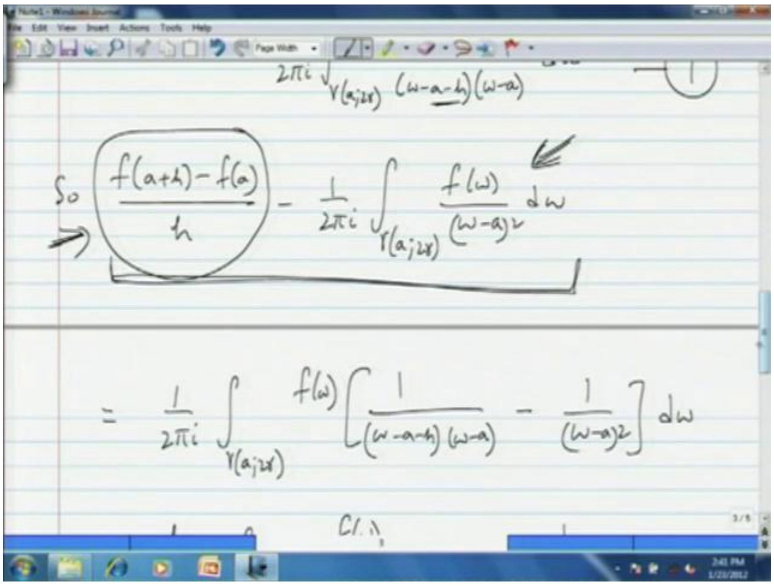

After showing that tends to 0 we would have shown that this difference in the limit here tends to 0 as h tends to 0 which means the differentiation this quotient here is your differentiation right of f then h tends to 0. So, we would have shown that f prime is equal to this integration or this integral is here as h tends to 0. So, well this integration is free of h so the differentiation of f would be equal to this and which is the Cauchy's integral formula for f pi, so we will do that we will show that this approaches 0 as h approaches 0.

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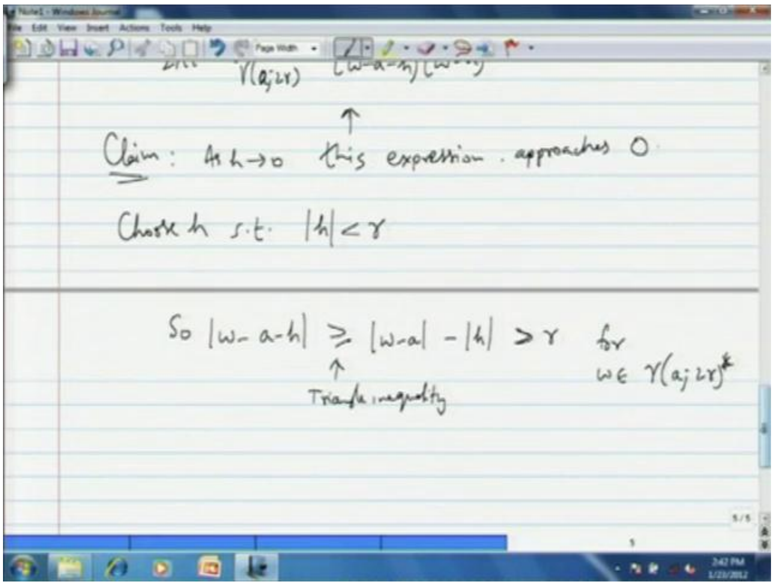

So, choose h so the idea is as follows you choose h such that the modulus of h is strictly less than r. So, the modulus of w minus a minus h is going to be greater than or equal to modulus of w minus a minus modulus of h that is why the triangle inequality, triangle inequality and then that is a strictly greater than r because well for w belongs to gamma a 2 r star the trace of gamma 2 r.

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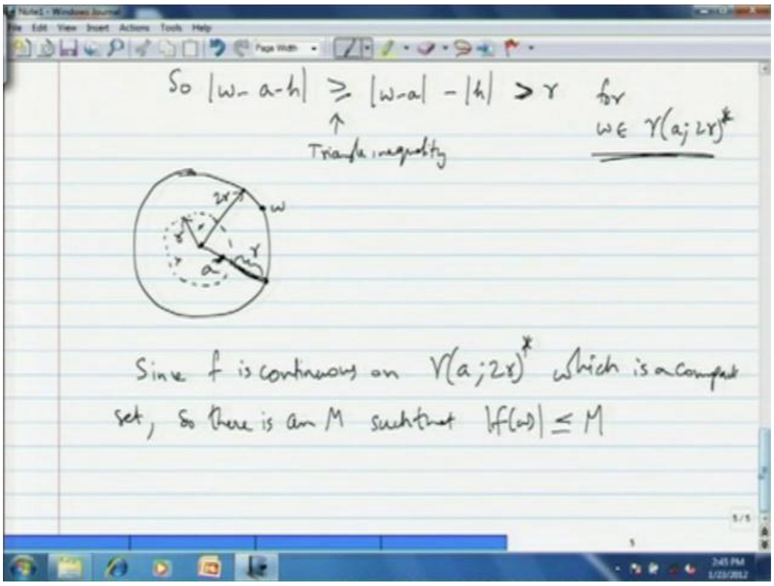

So, what is happening here is you have a and then you have a circle of radius 2 r so that is the circle of radius 2 r I will show the radius 2 r and as your w varies on this circle w can be any point on this circle. So, it is choose h so that the modulus of h is less than r so that well so a plus h is a vector which is somewhere inside so this is a circle of radius r the dotted circle is the circle of radius r your a plus h is some point within this disc. Anywhere it is you see that the shortest distance from a plus h to this circle of radius 2 r is at least r is at least r.

So, if you suppose that is u r a plus h then it is at least this distance that distance to that circle of radius 2 r is at least r. So, that is the idea so this is strictly greater than r and we will use this to say we will actually use this to estimate the denominator now we will also do something about the numerator since f is continuous I am talking about the denominator numerator in this expression of course. So, in the numerator since f is continuous on the trace of gamma a 2 r star or trace of gamma a 2 r which is a compact set it is a circle so it has to be a compact set which is compact set we know that a continuous function on a compact set assumes a maximum value.

So, actually the modulus of f assumes a maximum value so there exists there is an M such that the modulus of f of w is less than or equal to M. So, f is continuous and the modulus is a continuous function so the modulus of f is a continuous function on this compact set so it assumes a maximum volume and then the modulus of f of w is bounded.

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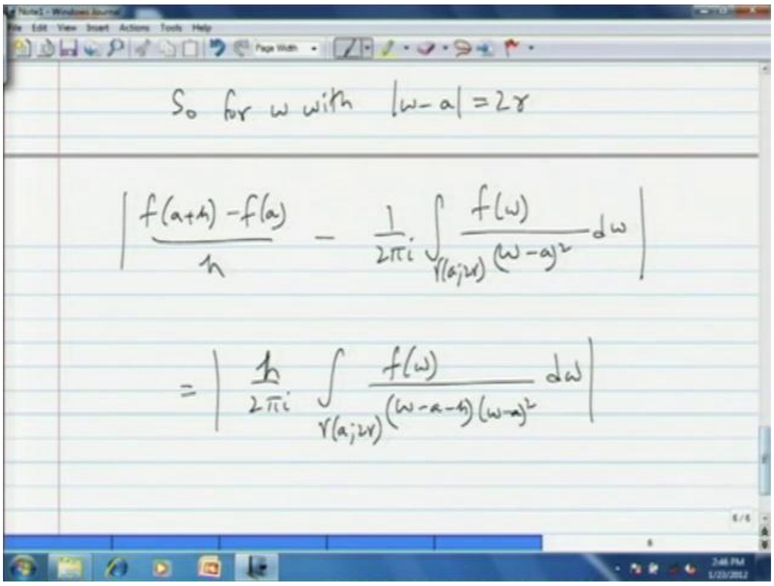

So, for w with modulus of w minus a equals 2 r that means for w on that circle of radius 2 r what we have is the modulus of f of a plus h minus f of a by h minus 1 by 2 pi i integral over gamma a 2 r f of w divided by w minus a square d w. The modulus of this quantity which is here which is underlying here is equal to the modulus of this quantity of the expression which they are handling, so is less than or equal to well firstly let me write the expression is equal to the modulus of 1 by 2 by i or h by 2 pi i times integral over gamma a 2 r of f of w by w minus a minus h times w minus a square d w.

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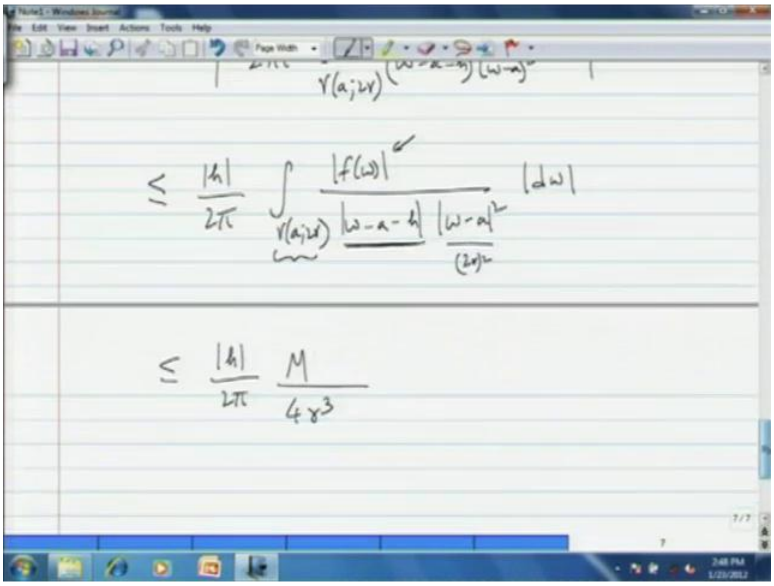

So, this is less than or equal to well modulus of h by 2 pi integral over gamma a 2 r the modulus of f of w divided by the modulus of w minus a minus h times the modulus of w minus a square times the modulus of d w, that is by the estimation theorem this is less than or equal to the modulus of h by 2 pi. We know that the modulus of f of w on gamma a 2 r is less than or equal to M because f is bounded on a compact set and then in the denominator we know that the modulus of w minus a minus h we showed that is at least r.

So, it is here that is at least r, so 1 by that is at most r and likewise w minus a in modulus is equal to r so this is equal to 2 r rather so this is 2 r square then this quantity is at least r so 1 by that is at most 1 by r. So, we get r times 4 r square 4 r cube here times, times this 2 pi times the radius h radius is 2 r that is the length of the curve.

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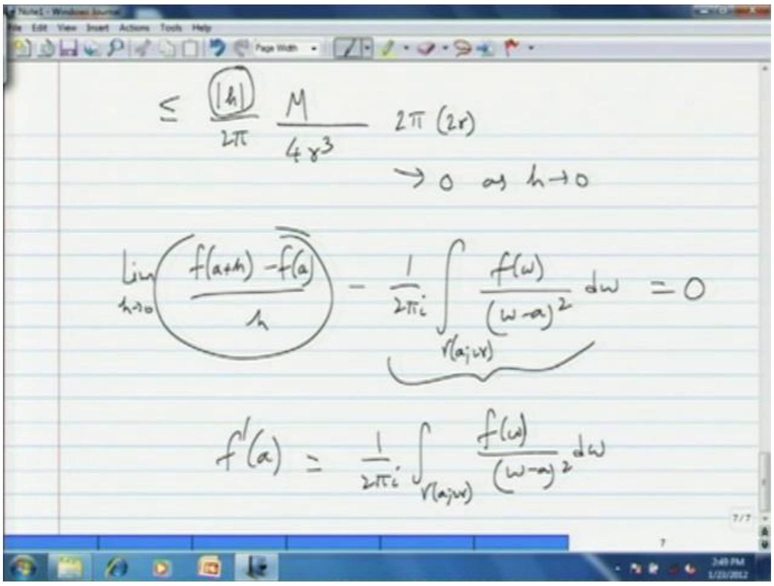

So, all in all this quantity, this quantity on the right hand side tends to 0 as h approaches 0 clearly the modulus of h in the denominator the other are all constants, so the modulus of h in the numerator causes this quantity to go to 0 notice that the quantity on left hand side then is equal to 0 in the limit as h goes to 0, so the limit as h goes to 0 of f of a plus h minus f of a by h minus 1 by 2 pi i integral gamma a 2 r f of w by w minus a square d w this is equal to 0 according to this statement, actually the modulus of this is equal to 0 of course, the only complex number with modulus 0 is 0 this has to be the case. So, since quantity is free of h we only take the limit for this quantity which is f prime of a that we declare is 1 by 2 pi i integral over gamma a 2 r of f of w by w minus a square d w.

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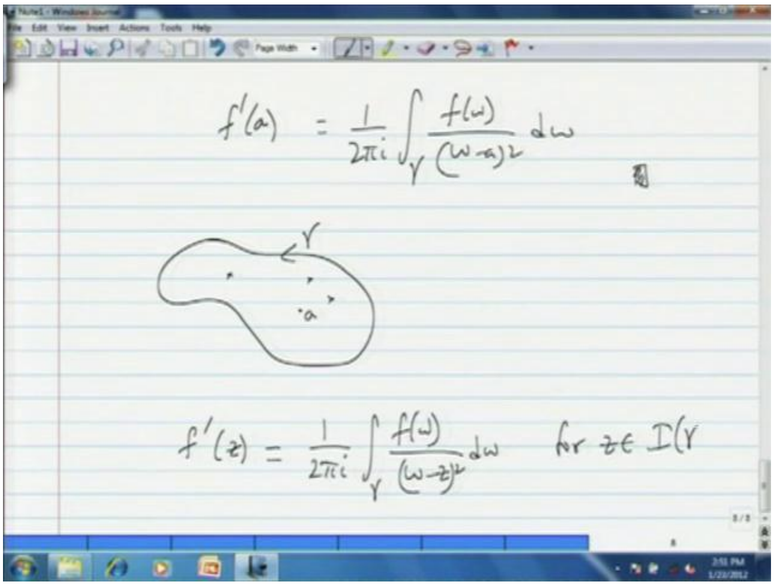

And by the deformation theorem like we remarked earlier that is equal to 1 by 2 pi the integration over gamma of f of w by w minus a square d w, that proves the Cauchy's integral formula for the first derivative so f prime a. Now, the situation works your f is analytic on and inside the simple closed curve gamma and a is any arbitrary point on the inside of the gamma and this formula holds for a so you can think of a varying on the inside of gamma so a can be any point so you have a variable point.

So, we can we usually write this formula as f prime of z just to remind you that it is a variable point inside the closed curve gamma simple closed curve gamma 2 pi i integral gamma f of w by w minus z square d w for z belongs to the inside of the simple closed curve gamma notice that gamma should be oriented in the positive sense of course, simple closed curve. So, that is your Cauchy's formula for the first derivative using this what we what we can show is that one can extend this formula for the second derivative as well.

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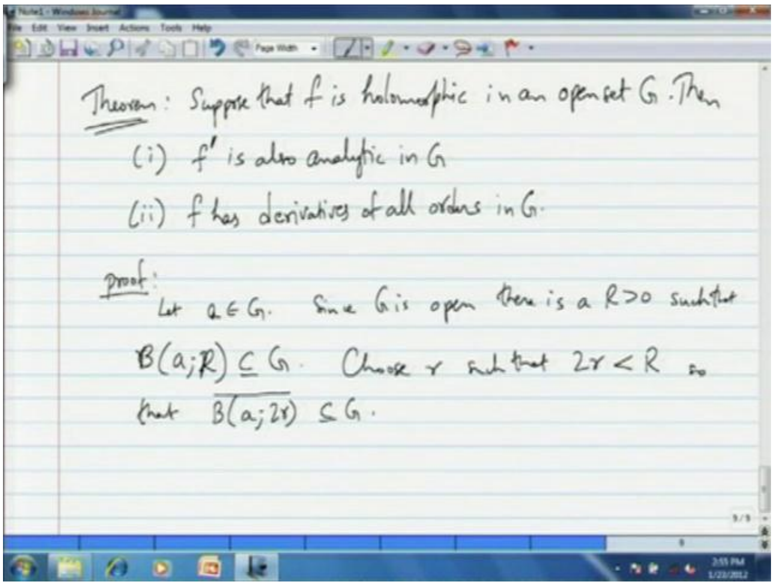

So, we will extend this and not only that we will show the following theorem. So, suppose that f is Holomorphic that is analytic in an open set G then the first statement is that f prime is also analytic in G from the first statement we can conclude that f has derivatives of all orders in G. If we show that f prime is analytic in G then we can apply the first statement to f prime which is analytic in G and show that f double prime also exists and then we can continue this and say that the derivatives of all orders of s exist.

So, the proof of second statement is direct using the first statement so we have to show you have to prove the first statement. So, let e belongs to G since G is open there is capital R positive such that the ball of radius r around a is contained in G that is the definition of open set. So, choose choose little r such that 2 r is strictly less than capital r, so that ball of radius 2 r is contained in G not only that the closure of the ball of radius 2 r is also contained in G.

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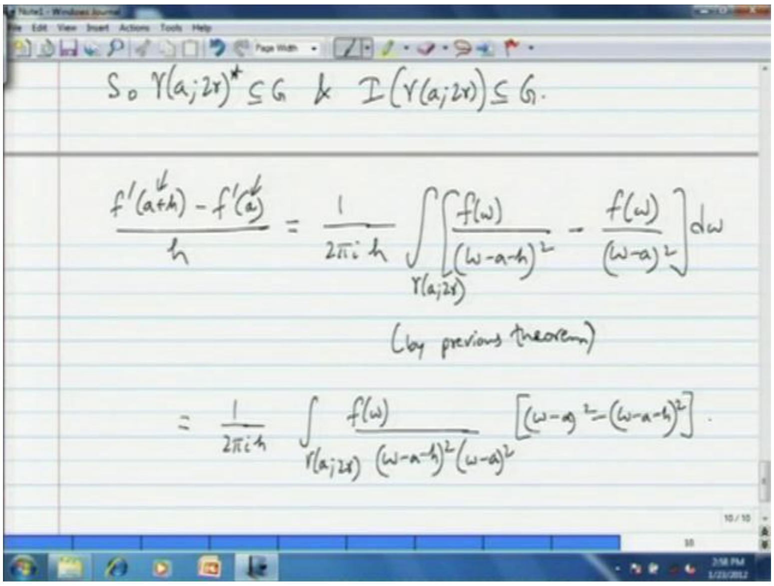

And the boundary of this which we typically known by gamma a 2 r is contained in G. so this is what we need so this implies that gamma a 2 r so gamma a 2 r star which is circle of radius 2 r centred at a however parameterised however, but oriented in the counter clockwise direction is contained G or its contained in G and inside of this is also contained G is also contained in G this is what we want.

So, now we will look at this quotient f prime of a plus h minus f prime of a divide by h like we did for the previous proof we will consider this quotient we will notice that this is 1 by 2 pi i times h this h I am in the denominator I am carrying it forward I leaves use the Cauchy's integral formula for the first derivative which was proved in the previous theorem on f prime of a plus h and on f prime of a.

So, that will give me integration over gamma a 2 r of f of w by w minus a minus h square minus f of w by w minus a square d w by previous theorem, notice that we restated it here shortly z can be any point in the interior of the simple closed curve gamma here z instead of z I have w minus a minus h once and then I have or rather a plus h once I apologise a plus h once and then I have a for the second instance of this formula. So, then let us see that this is 1 by 2 pi i h upon simplification this gives us integration over gamma a 2 r and then f of w divided by w minus a minus h square times w minus a square.

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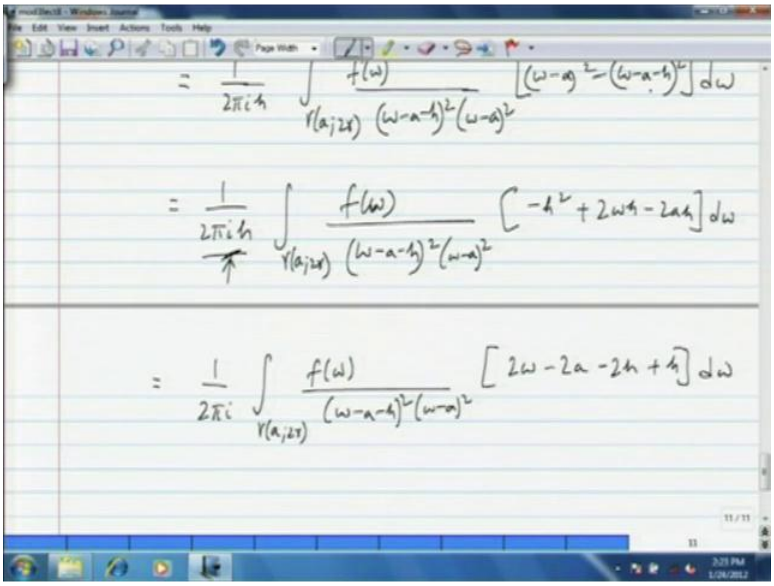

I am trying to I took the common denominator then I have w minus a square minus w minus a minus h square d w, there is some cancellation when you expand this squares so you get gamma a 2 r f of w divided by w minus a minus h square times w minus a square, so w square cancels with w square here a square cancels with a square so we are left with well 2 a w also cancels so we are left with minus h square from the second part and then we are also left with minus h square then we have minus 2 w h plus 2 a h d w. So, from here actually this should be 2 plus 2 w h minus 2 a h right so including that minus sign so what I will do is I can filter out or factor out h so that I can cancel with the h is in the front here, so I get 1 by 2 pi i times the integral over gamma a 2 r of f of w by w minus a minus h square w minus a square times 2 w minus 2 a minus h.

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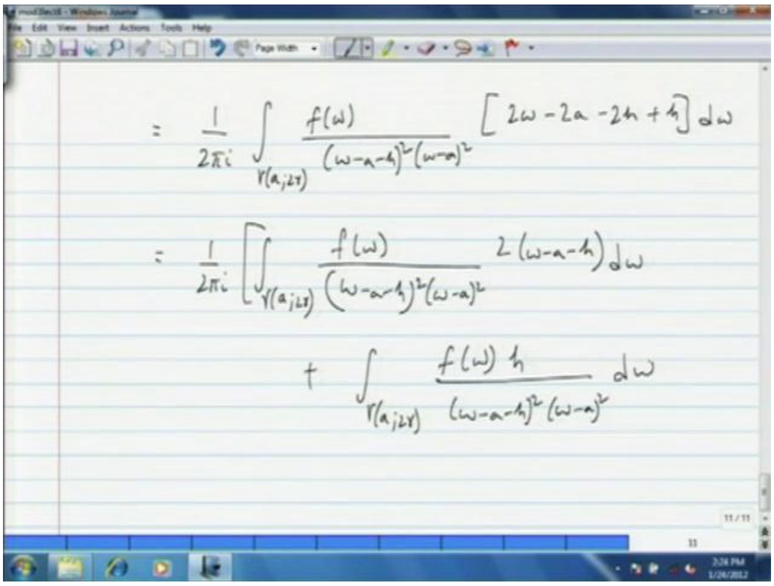

So, allow me to write this as minus 2 h plus h. So, that what I can do is, I can separate this integral into two things, so this is gamma a 2 r times f of w divided by w minus a minus h square times w minus a square times 2 times w minus a minus h then plus well, d w so I will separate the integrals. So this plus integral over gamma a 2 r f of w divided by so times h divided by w minus a minus h square times w minus a square d w.

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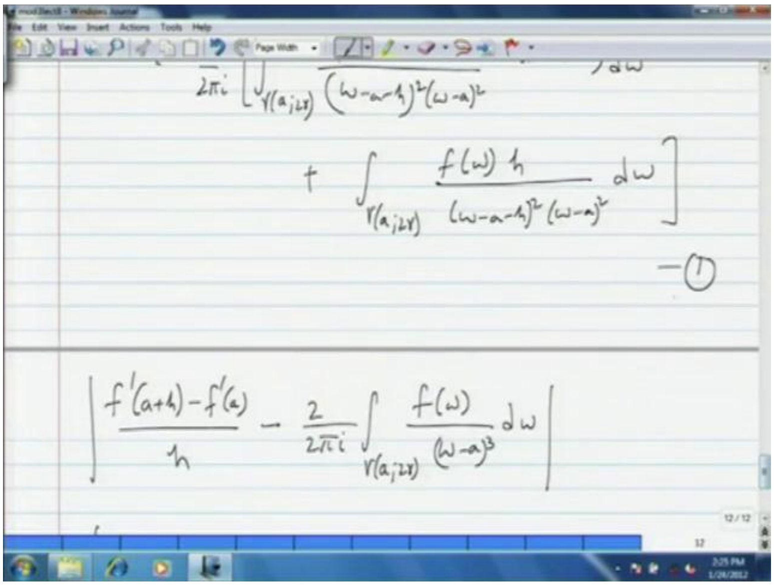

So, now we are estimating this quantity let us estimate f prime of a plus h minus f prime of a divided by h so this expression 1 is this expression here then let us subtract 2 by 2 pi i so 2 by 2 pi i integral over gamma a 2 r of f of w divided by w minus a cubed d w. So, this is the desired expression we wish to show that in the limit this expression is equal to this expression positive with the, with the plus sign rather so this is what we wish to show so let us estimate this difference so the modulus of this difference is equal to modulus of for the first expression.

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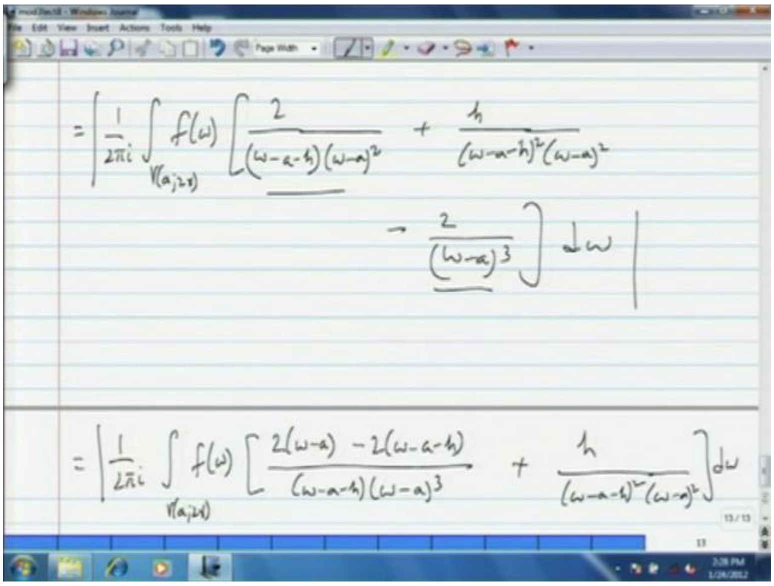

I will use this expression 1 and say that it is 1 by 2 pi times the integral over gamma a 2 r of f of w times 2 by what I have here is 2 by w minus a minus h times w minus a square plus h divided by w minus a minus h square times w minus a square time d w. So, actually what I have done is I have cancelled the w minus a minus h here with 1 factor here hence I have 2 by w minus a minus h times w minus a square. And then for the second integral here I have h divided by all that h divided by all that.

So, I have this expression and then you subtract 2 by subtract 2 by w minus a cube then times it by d w so all this is the integrand so and splitting it into 2 line lack of space, but that is your expression. So, this is in turn equal to 1 by 2 pi i in the modulus in the modulus integration over gamma a 2 r of f of w times so let me combine the first the third expressions here, so that I get 2 times so w minus a minus 2 times w minus a minus h when I take the common denominator w minus a minus h times w minus a cube then I will leave the third the second expression alone.

So, that will be h divided by w minus a minus h square times w minus a square d w.

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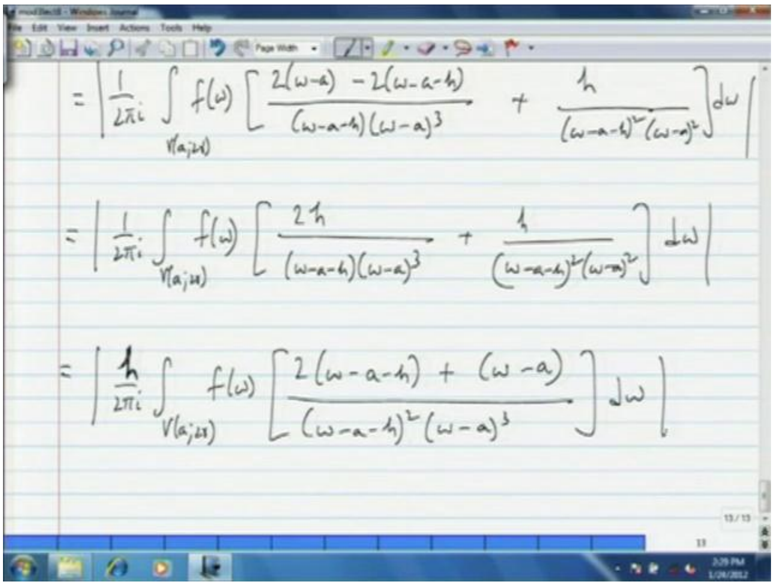

So, that what I get so all this within modulus this is equal to modulus of 1 by 2 pi i integral over gamma a 2 r f of w times well after cancellation what I have in the numerator is 2 h divided by w minus a minus h times w minus a cube plus h divided by w minus a minus h square times w minus a square d w within modulus.

So, this is in turn equal to the modulus of 1 by 2 pi i integral over gamma a 2 r f of w so I am carrying out the simplification in 2 steps. So, that was the first simplification this is the second simplification I am going to write this as 2 times well there is h common all throughout so I will push the h out of the integration into the numerator here and then 2 times w minus a minus h plus 1 times w minus a then I take the common denominator w minus a minus h square times w minus a cube d w.

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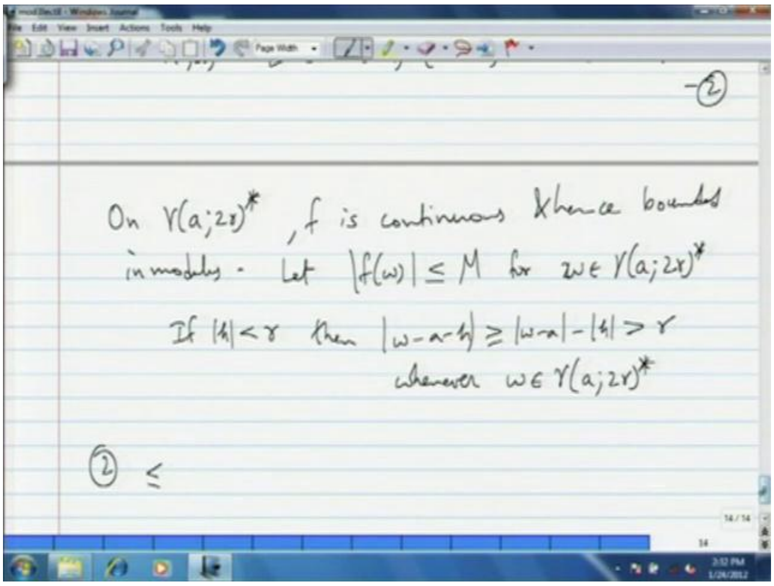

So, after all that ordeal we can use the estimation theorem now say that on gamma a 2 r star f is continuous f is a continuous function and hence bounded in modulus like the previous the proof of previous theorem hence bounded in modulus. Let f of z the modulus of rather w I am using w here f of w in modulus be less than or equal to M for w belonging to gamma a 2 r star, so continuous function on a compact set is bounded. so that I am using like in the proof of previous theorem if modulus of h is strictly less than r if you choose h such that modulus of h is strictly less than r then modulus of w minus a minus h is greater than or equal to modulus of w minus a minus modulus of h which is strictly greater than r whenever w is on stress of gamma a 2 r. So, that is very similar to the proof of previous theorem then what we can do is we can use all this to estimate what is within this modulus.

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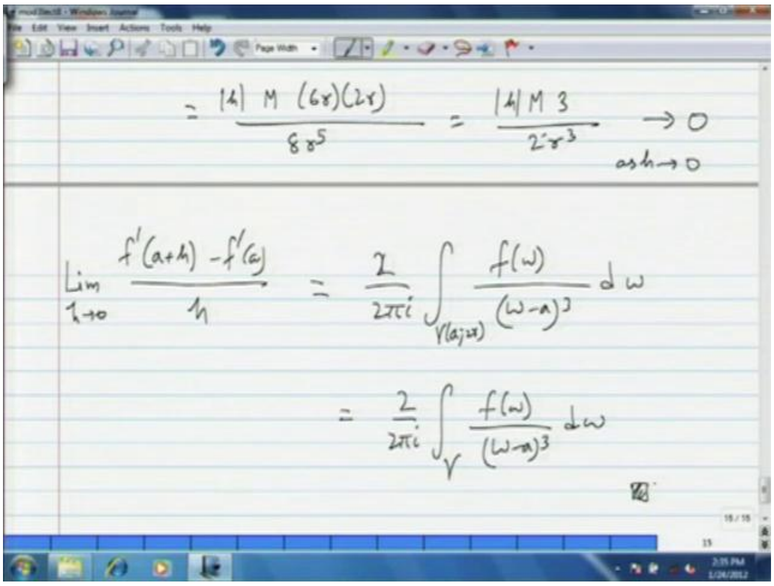

So, let me call this 2 so 2 the expression 2 that is the expression within the modulus is less than or equal to modulus of h divided by 2 pi, so that is what is in the front and then modulus of f of w is less than or equal to M so I have a M here. And then modulus of 2 times w minus a minus h plus w minus a so that is a definitely less than or equal to 2 times r 2 times r plus 2 times r.

So, this in modulus is less than or equal to 2 r definitely this in modulus is less than or equal to 2 r so I have a 2 times 2 times r plus 2 r for the numerator for the denominator I will use all these here, so what I have is r w minus a minus h square in the denominator is actually less than or equal to in modulus is less than or equal to r square here and then w minus a square well w is on the trace of gamma a 2 r so that is 2 r then times the length of the curve gamma a 2 r is 2 time 2 pi times 2 r.

So, all in all this expression what does it mean it come to it comes to 4 plus 2 6 r so this is modulus of h m by 2 pi is cancel so this is 6 r times 2 r divided by 8 r power 5 so that is what it comes. So, the summary is well this goes to 0 as h goes to 0 the modulus of h in the numerator whatever the other expressions are they are all constants so that goes to 0 that takes the expression to 0 as h goes to 0.

So, we conclude that f prime of a plus h minus f prime of a divided by h in the limit as h goes to 0 is equal to 2 by 2 pi i integration over gamma a 2 r h f of w by w minus a cube which in turn is equal to 2 by 2 pi i integral over gamma recall by deformation theorem these 2 integrals are equal Cauchy's theorem version three these 2 integrals are equal gamma was a simple closed curve such that f is analytic on an inside this simple closed curve so we have prove this theorem.

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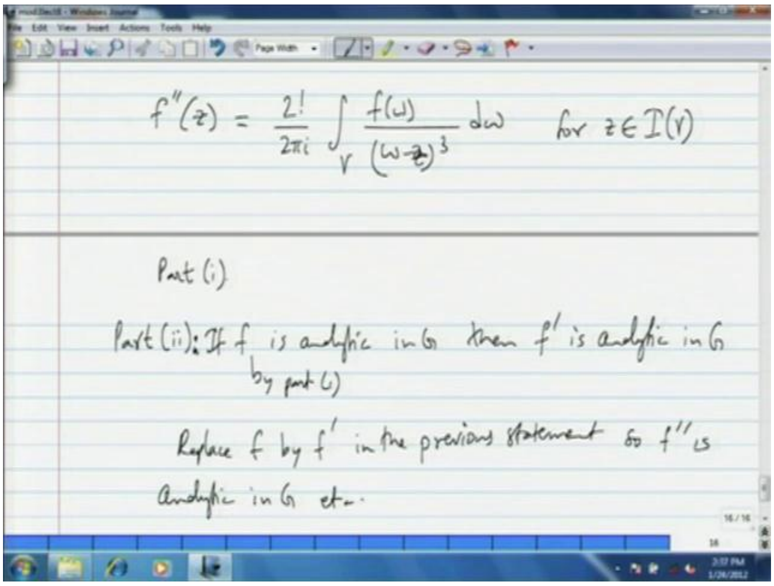

So, a notice that f double prime of now z so we can treat a as variable now solely because a was an arbitrary point in the statement of the theorem a was an arbitrary point so f double prime of z 2 by 2 pi i or 2 factorial by 2 pi i integration over gamma f of w by w minus a cube z d w for z belongs to any of for the interior of gamma. So, that is the proof of a part 1 part 1 now part 2 follows easily using part 1 f is analytic what we showed is that if f is analytic in g then f prime is analytic in G.

So, this is the proof of part 2 we showed by part 1 we showed that f is analytic in G then f prime is analytic in G, so replace f by f prime in the previous statement so f prime is analytic in G implies that f double prime is analytic in G so on and so forth so replace f by f prime so f prime is double prime is analytic G etcetera.

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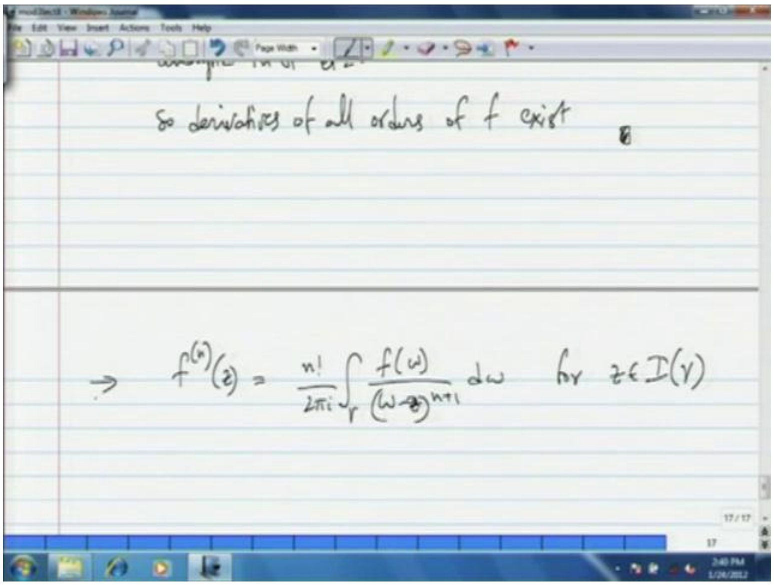

So, you get that so derivatives of all orders of f exist and that is the proof of part 2 of this so that is the, that is the end of 2 of this theorem. So, we have a proved the Cauchy's integral formula for the second derivative and we did that for first derivative we had a Cauchy's integral formula for the function itself and what we are going to do next is prove the Cauchy's integral formula for the n th derivative.

So, I will briefly give the formula for the n th derivative here so under appropriate conditions the n th derivative of f at the point z is equal to n factorial by 2 pi i times integral over gamma, integral over gamma f of w by w minus z rather power n plus 1 d w for z belongs to the interior of or inside of curve gamma. So, the appropriate conditions of course, are the that gamma is the simple closed curve and that f is analytic on and inside this simple closed curve.

So, given these conditions we have a Cauchy's integral formula for the n th derivative, so we can prove this using induction on n which we will see next time, but for now we have the Cauchy's integral formula up to the second derivative at least and when we see the general proof we would have sort of over redden our proofs anyway, but the previous theorem at least gives us that the derivatives of all orders of f of an analytic function f exists in an open set when f itself is analytic in the open set.

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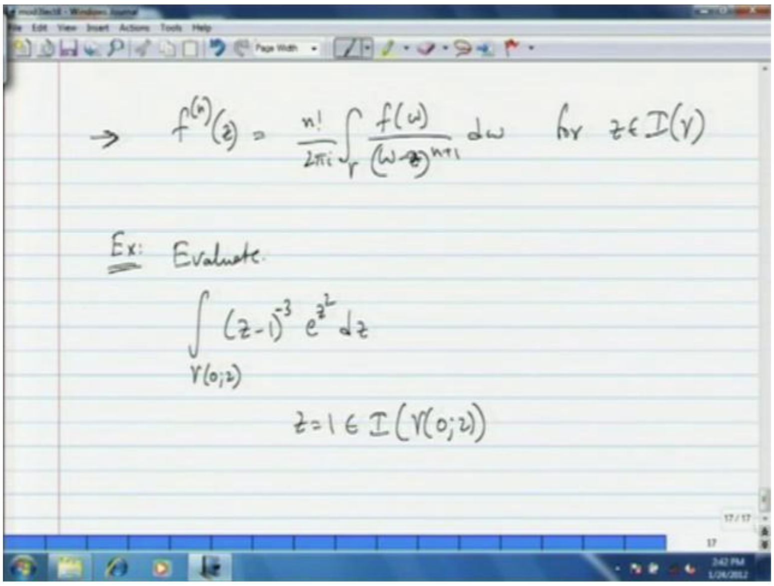

So, let us see an example of how to evaluate integrals using the Cauchy's integral formula. So, here is an example involving the second derivative we can have examples involving higher derivatives as well once we prove the formula for higher derivatives. So, here is an example evaluate the integration over gamma 0 2 recall what that is say circle of radius 2 centred at the origin and then centred at 0 and then its oriented in the counter clockwise direction or in the positive sets and then z minus 1 cube e power z square d z so evaluate this, so rather that is z minus 1 power minus 3 times 3 power z square d z.

So, notice that z equals 1 is a point which will appear in the denominator of integrand or z equals 1 is singularity of the of the integrand and this belongs to the inside of the curve gamma 0.

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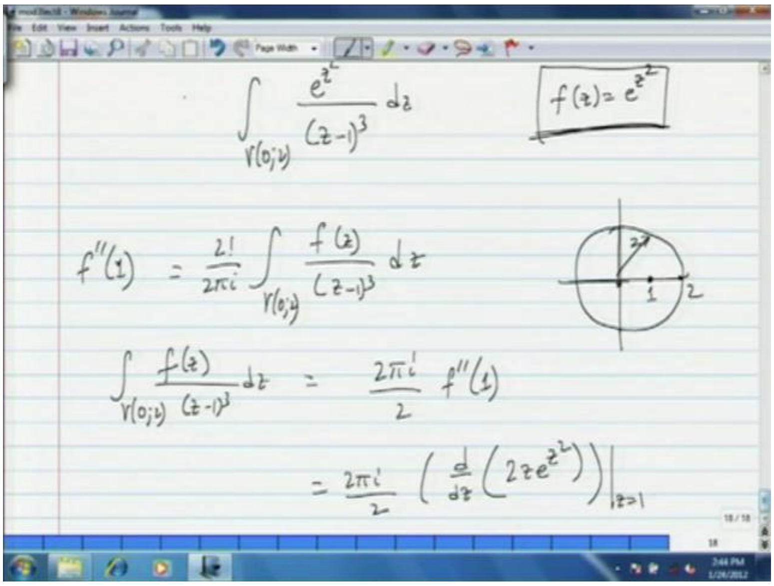

So, what we will do is we will treat this as so this integral over gamma 0 2 e power z square divided by z minus 1 cube d z so we will view this as follows we will take f of z equals e power z square so that this looks like f of z divided by z minus 1 cube the integration being over gamma 0 2 d z.

So, this is simple closed curve gamma 0 2 is a simple closed curve around the point 1 or it contains the inside of it contains point 1 so here is 1 and here is gamma 0 2 this is 1 and this is 2 circle of radius 2. So, by the Cauchy's integral formula so 1 by or 2 factorial by 2 pi i times this is equal to the second derivative of f at the point 1. So, this is a second derivative at the point 1 so what we can do is we can say integration over gamma 0 2 of f of z divided by z minus 1 cube d z is equal to 2 pi i by 2 while that is pi i times f double prime of 1.

So, we can easily compute the second derivative of this function f of z equals e power z square so this gives us 2 pi i by 2 times the second derivative well d square by or for the first derivative of e power z 2 z e power z square that is the first derivative of e power z square and then take its derivative again and all this evaluated at z equals 1.

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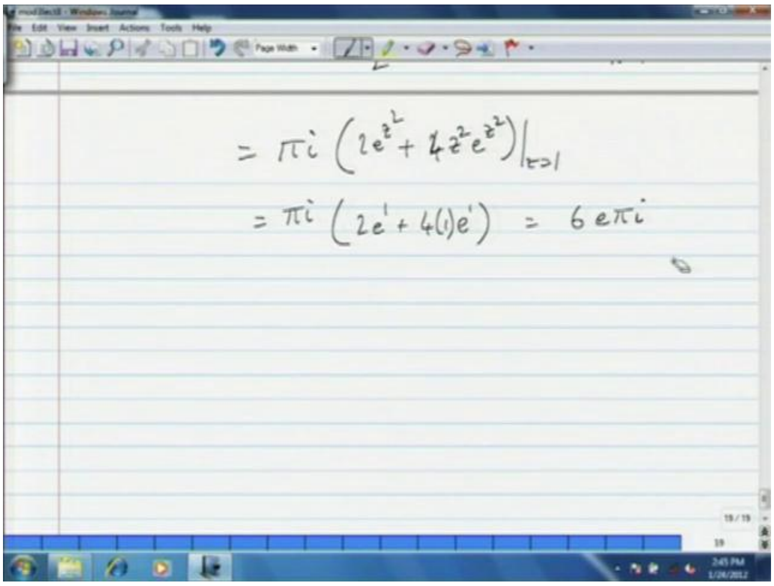

So, that is well that is equal to pi i times the differentiation of 2 z e power z square is 2 e power z square plus 2 z times 2 z 4 z square e power z square and all this evaluated at z equals 1. So that gives us pi i times 2 e power 1 plus 4 times 1 times e power 1, which is 6, so this gives us 6 e pi i that is the value of this integral.

So, we can evaluate integrals of this type using Cauchy's integral formula formulae, particularly if the denominator for example, here was z minus 1 power 4 or 5 or any higher integer; we would have used the n th derivative formula Cauchy's integral formula. So we will see a proof of that next time, and then we will also see an easy consequence of the theorem we proved towards the end of this session. This theorem here, sorry go back, this theorem has a easy consequence which is a partial converse of the Cauchy's theorem, and that is called the Morera's theorem; and we will see a proof of that next class.