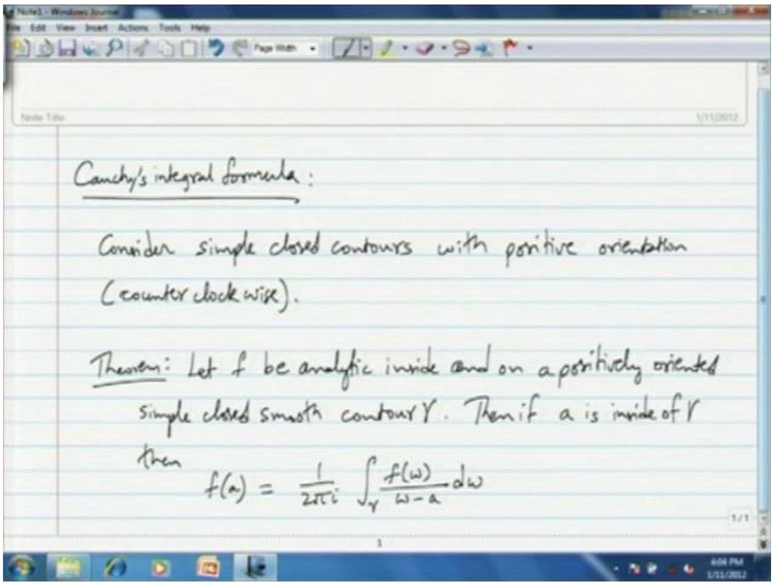
Complex Analysis
Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 3
Complex Integration Theory
Lecture - 7
Cauchy's Integral Formula and its Consequences

Hello viewers, so we saw the various version of Cauchy's theorem; and in this session, we will see will start some applications of Cauchy's theorem. So, as I mentioned earlier Cauchy's theorem is central to the analysis of complex analytic functions. So, today using Cauchy's theorem, we will first derive the Cauchy's integral formula. So, here is Cauchy's integral formula.

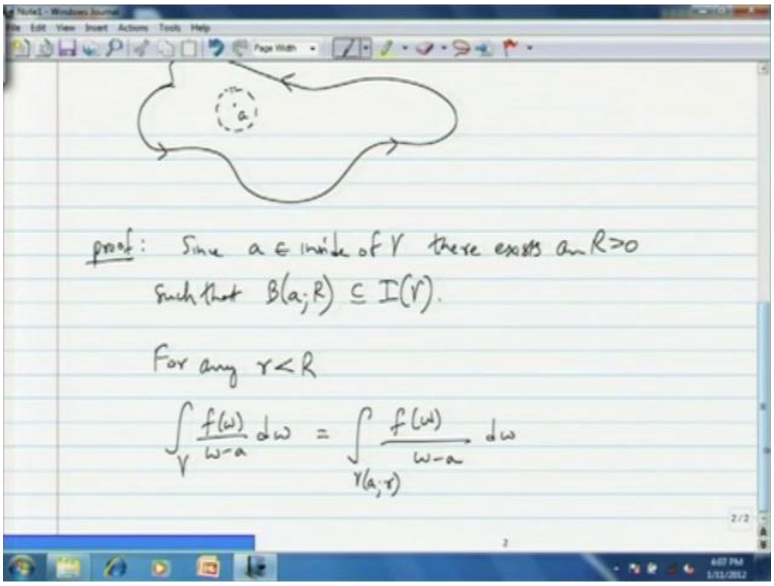
(Refer Slide Time: 00:54)



So, first for this purpose and from now on, we will assume a contour will be positively oriented and we consider simple closed contours. So, let me say that in order, so consider simple closed curves contours and then with positive orientation or clock wise or counter clock wise rather orientation. So, counter clock wise with respect to the points on the inside of the simple closed contour like we have defined earlier. So, the orientation matters now, it did not matter for Cauchy's theorem, but it matters for what we are going to do. So, here is the statement of the theorem which gives the Cauchy's integral formula.

So, let f be an analytic function f be analytic inside and on a positively oriented simple closed smooth contour gamma. Then if a is inside of gamma, then f of a is equal to 1 by 2 pi i, we get a formula for f of a it is in terms of the integral of f contour integral f this way. So, it is the contour integral of f of w by w minus a d w, where w is the variable of integration. So, 1 by 2 pi i times that contour integral gives you the value of f of a, where a is the point inside this simple closed curve. So, the requirement that the contour gamma is smooth can be draft, we can assume a piece wise smooth contour, but for simplicity I will assume a smooth contour.

(Refer Slide Time: 03:55)

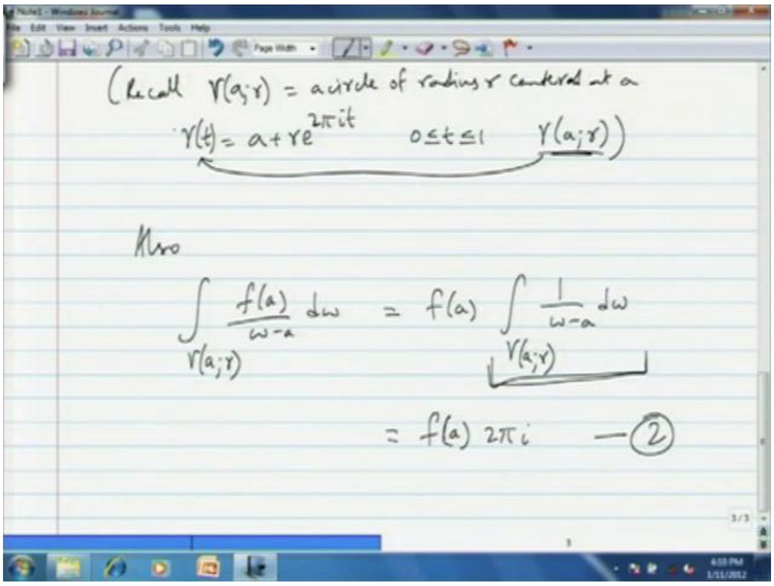


So, the intuit picture we keep in mind is the following; so, here is schismatic, so, suppose this is a simple closed contour oriented in counter clock wise direction if a is point inside, then the value of f well f should be analytic inside and on the curve which means its analytic in a set in an open set containing this curve gamma and the inside. So, if f analytic there then the value of f at the point a is given by the integral on the right hand side of the statement of the theorem.

So, here proof of a this theorem; so, the proof well for the proof what we do is since, a is in the inside, inside of a belongs to belongs to the inside of gamma clearly there is a disc about a, which is contained in the inside. So, there exists an R positive such that at B a R is contained in the inside of the curve gamma, recall that is the notation for the inside of the curve gamma. And then that is because once again recall that the inside of a curve is the is an open set. So, if you have a point in then then there is a bolls surrounding it an open boll of some radius surrounding it which is contained in set.

So, for any r strictly less than R then, so, here is a like I pictured above there is a open boll of radius capital R around a. So, for any r strictly less than R what we can do is we can take the integral f of w by w minus a d w This is the integral which appears in the formula statement of the theorem and this is equal to the integral over gamma or a gamma a r recall that notation that notation stands for gamma a r stands for circle of radius r centred at a. So, recall this gamma a r stands for is a is a circle of radius r centred at a.

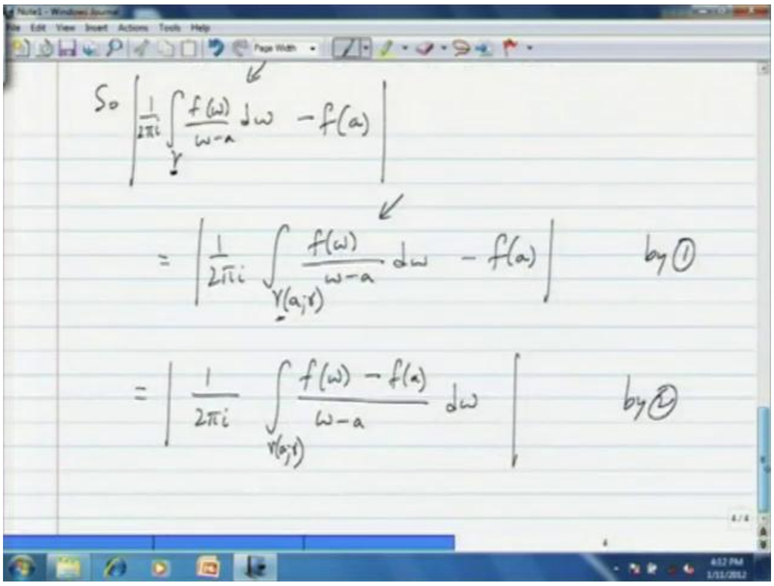
(Refer Slide Time: 06:47)



And, more precisely this is given by gamma of t is a plus r e power 2 pi i t, where t is in between 0 and 1, if you want to be precise this is your gamma of a r gamma of a r. So, please note there is some there is some ambiguity because I am using gamma here and on the right hand side, but here I am using the notation gamma of a r to be to be precise. And then this equality here follows from a version of Cauchy’s theorem we saw earlier. So, this is true by a Cauchy’s theorem for simple closed curve for a simple closed contour.

So, and then or more precisely that is by the deformation theorem, which we called Cauchy’s theorem for a simple closed contour and then by that. So, we will preserve this equality we call this one and also the integration over gamma a r of f of a by w minus a d w this is equal to f of a times the contour integral over gamma a r, the circle of radius r about a 1 by w minus a d w. Essentially because f of a is constant and this is equal to f of a times 2 pi i, because recall that this is the fundamental integral which we computed a few sessions ago. So, that the value of that integral is a 2 pi i. So, this we will store as equation two. So, the integration f of a by w minus a on the circle a is 2 pi i times f of a.

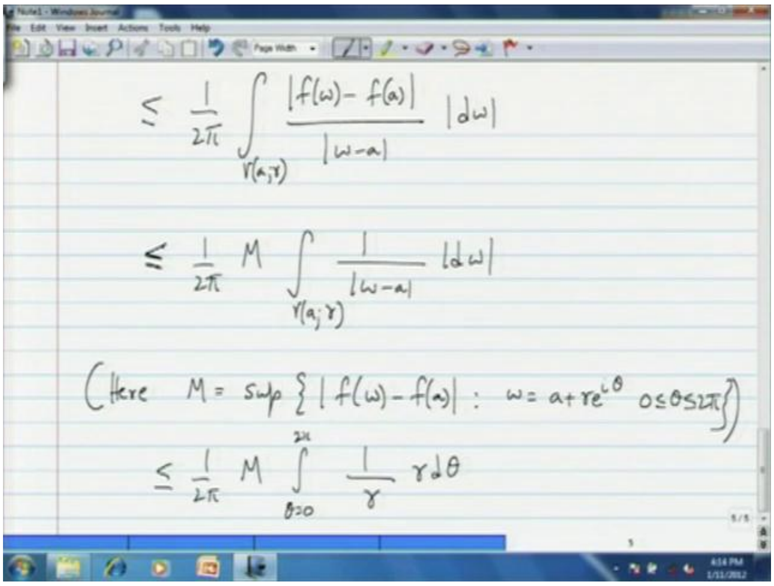
(Refer Slide Time: 09:34)



So now, we will estimate the difference 1 by 2 pi i times f of w by w minus a d w over gamma over simple closed curve gamma, which is given minus f of a. So, let us estimate this difference this in absolute value is a equal to 1 by 2 pi i in absolute value integration contour integration over gamma a r f of w by w minus a d w minus f of a. What I have done is converted the integral in the first expression to this integral in the second expression by using our equation one. So, notice that I have change this contour integral on gamma to contour integral on gamma a r. So, that gives us, well then that is equal to the modulus of 1 by 2 pi i integration of f of w by w minus a gamma a r minus f of a d w.

So, what I am doing is I am using this is by two, I am using the fact that f of a let me go back to here, I am using the fact that f of a is 1 by 2 pi i times the left hand side here. So, and I am using the left hand side to bring the integrands together. So, I get 1 by 2 pi i times f of w minus f of a by w minus a. like that. So, then by the estimation theorem we had earlier this is less than or equal to 1 by 2 pi times the integration over the circle gamma a r of the modulus of f of w minus f of a by modulus of w minus a and modulus d w.

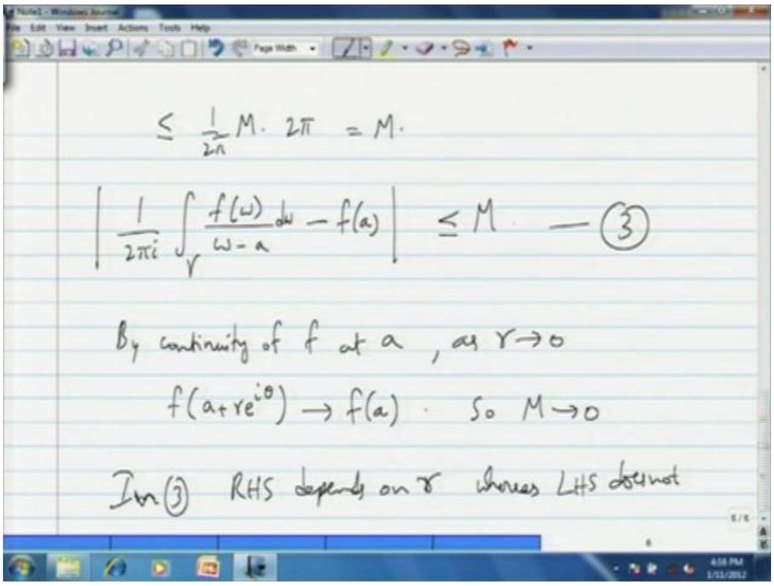
(Refer Slide Time: 11:42)



So, that is by the estimation theorem and this is equal to 1 by 2 pi times M, let me say what m is m is the maximum value or the supremum of modulus of f of w minus f of a on the circle. I will write the down; times the integral or gamma a r. So, this has to be less than or equal to this has this is less than or equal to 1 by modulus w minus a mode d w here. So, I will writing parenthesis here, M is the supremum of the set of modulus of f of w minus f f a, such that w belongs to well w is equal to a plus r e power i theta 0 less than or equal to theta less than or equal to 2 pi.

So, it is the value it is the supremum of the modulus of these differences, where w is on a circle of radius r around a. So, M is that and and so from this estimate we get this is less than or equal to 1 by 2 pi well let me parameterise now. So, this gives you gamma a r when I parameterise I get modulus of w minus a is simply r and then modulus of d w will give me r d theta, where theta now ranges from 0 to 2 pi. So, this is this is 0 2 pi, I am parameterizing using using r and theta.

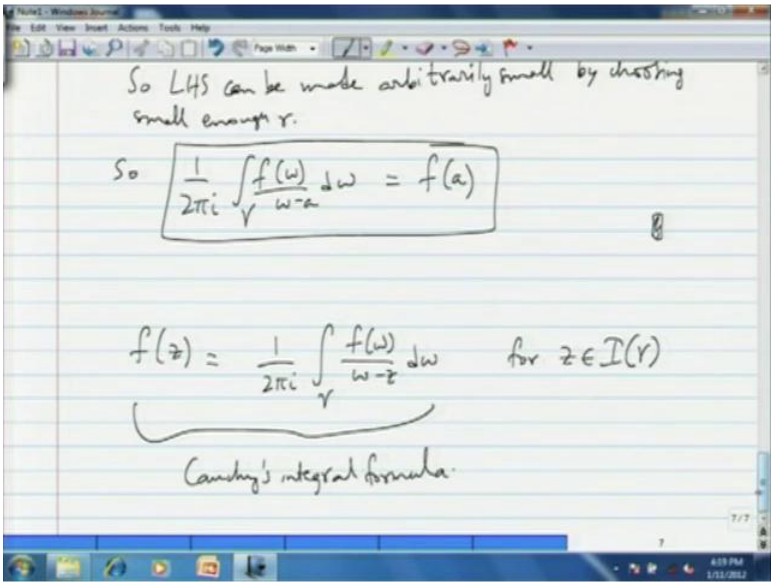
(Refer Slide Time: 14:10)



So, r cancels we get this is less than or equal to 1 by 2 pi times M times 2 pi which is M. So, all in on all what we get is 1 by 2 pi i times the integration over gamma of f of w by w minus a minus d w minus f of a. This in absolute value is less than or equal to M, where M is this supremum that I mentioned earlier. So now, by continuity of f at a f is analytic so, it is definitely continuous at the complex number a. So, by continuity of f at a this supremum as as r tends to 0 f of a plus r e power i theta matter what tends to f of a. So, this supremum M of the modulus of differences.

So, notice M is this the supremum of the modulus of these differences this tends to 0 as r tends to 0 and in this let me call this estimate three in three, in three R h s depends on r little r because the supremum does, where as LHS does not. So, what you can concluded is that you can take the limit as r goes to 0 and you can you can see that the left hand side is arbitrarily small.

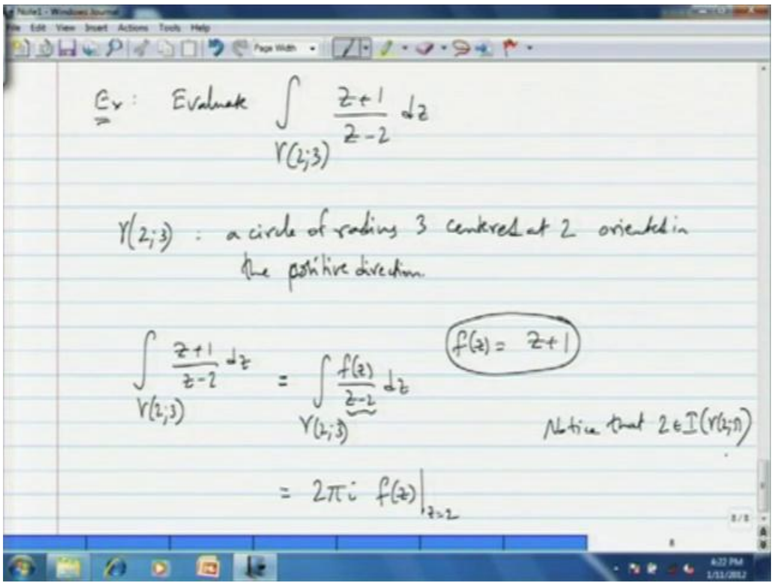
(Refer Slide Time: 16:19)



So, LHS can be made arbitrarily small by choosing small enough r little r. So, in conclusion we can say that since, the left hand side is arbitrarily small. So, the left hand side has to be equal to 0. So, so 1 by 2 pi i times the integral or a gamma of f of w by w minus a d w has to equal f of a and which is what you want. So, that is the proof of this theorem.

So, what one can what one can look at this as is as follows; one can vary this point a in the inside of this curve simple closed curve gamma and think of the integral representation as giving a formula in terms of integration. So, what I mean is we will instead of saying a we will replace it by variable z and say that f of z is 1 by 2 pi i times integral over gamma f of w by w minus z d w for z belonging to the interior or the inside of the simple closed curve gamma. So, you can imagine the z varying on on the inside of this gamma curve gamma and still this or or this integral on the right hand side gives a formula an integral formula. And so, this is refer to as Cauchy's integral formula and let us see some examples using this Cauchy's integral formula.

(Refer Slide Time: 18:39)

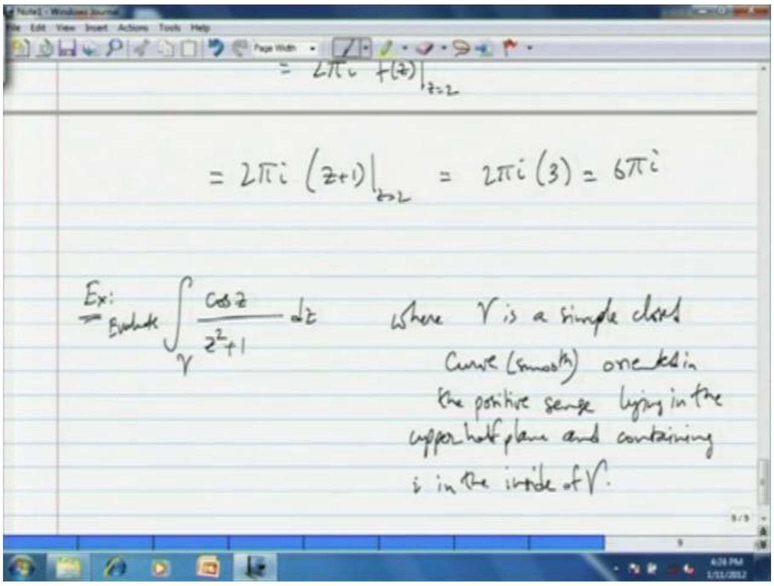


So, here is an example evaluate integral gamma 3 2 z plus 1 by z minus 2 d z. So, by gamma 3 2 once again I mean a circle of radius 3 centred at I apologise that should be gamma 2 3. So, gamma 2 3 is a circle of radius 3 centred head the point 2 at the complex number 2 and oriented in the counter clock wise in the positive direction. So, recall that how we parameterize this circle is immaterial because contour integrals are are free of this parameterization or they are independent how you parameterize the circle.

So, Cauchy's integral formula allows you to think of this as z plus 1 by z minus 2. So, you consider the analytic function z plus 1 and then this integral can be represented or can be thought of as integration of f of z by z minus 2 d z. So, considering appropriate analytic function is is important clearly that function z plus 1 is an entire function recall an entire function is analytic on all of complex plane. So, this is an entire function so, it is definitely analytic on gamma on the inside of gamma.

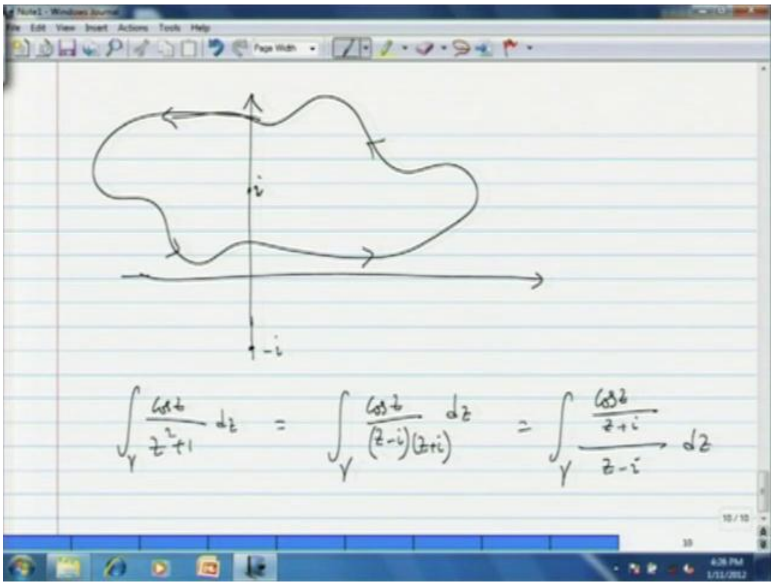
And so, this formula above let me go back this formula above tells you that the value of this integral is 2 pi i times that that constant 2 pi i multiplies to the left hand side now, times the value of the function f of z at the point z is equals 2. Notice that notice that z minus 2 this point 2, which is which is singularity in denominator notice that 2 belongs to the inside of the curve gamma 2 3, that is why we are using this formula.

(Refer Slide Time: 21:54)



And then this is 2 pi i times the value of z plus 1 at the point z equals 2. So, this is 2 pi i times 2 plus 1 3, which is 6 pi r. So, then that is one example. Let us see another kind of computation of this sort. So, consider or evaluate evaluate integration contour integration cos z divided by z square plus 1 d z over gamma, where gamma is a simple closed curve, smooth let us say. So, simples close curve smooth oriented in the positive sense lying in the upper half plane and containing i in the inside of gamma that is a weg description but, that is enough to evaluate the side integral.

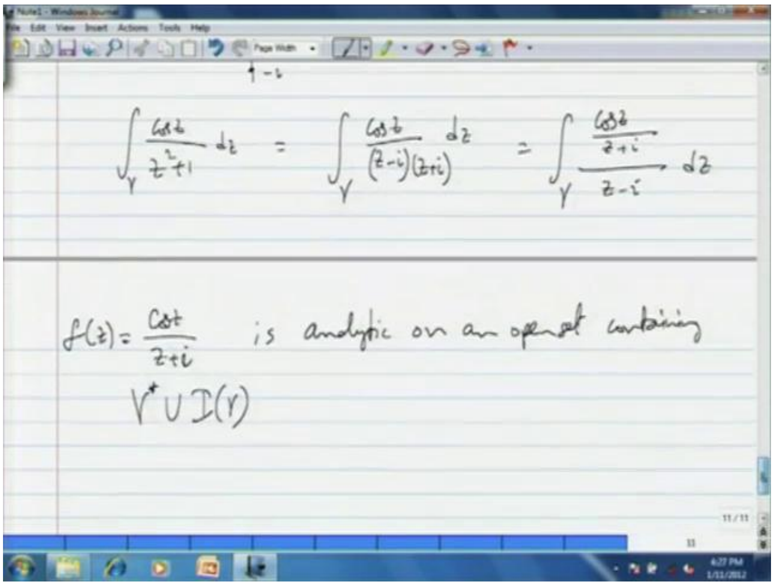
(Refer Slide Time: 24:04)



So, I could have taken a more concrete curve gamma, but this is a more general curve gamma. So, what this is telling you that here is the upper half plane for positive values of the imaginary part and then what is important is i is inside in the curve. So, you have a simple closed curve orientated in the positive sense and it is around i is what is important. So, i is in the inside of gamma.

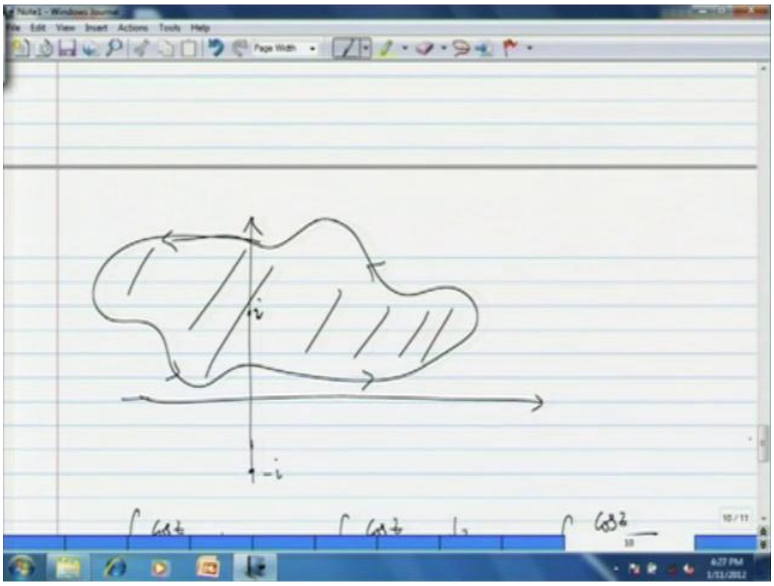
Now, we want to evaluate cos z by z square plus 1 d z. So, by Cauchy's integral formula... So, well before I use the Cauchy's integral formula I apologise, let me look at the integral let me look at integrand cos z by z square plus 1 d z this can be written as the integration of cos z by z minus i times z plus i d z. So, you see that there are two values of z, which make the denominator 0, one of them is i and the another one is minus i. And since, minus i lies in the lower half plane we do not have to worry about that cos z over, so, I will write this as cos z over z plus i divided by z minus i d z.

(Refer Slide Time: 25:57).

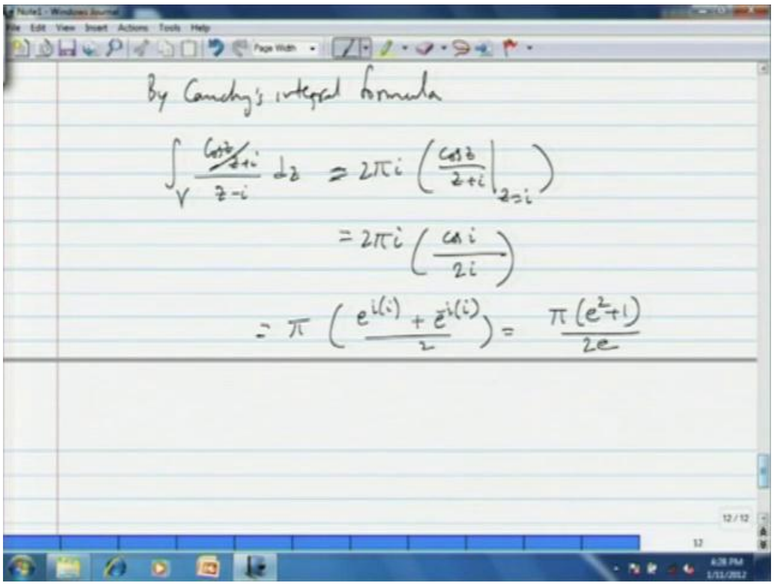


So, that cos z by z plus i if you choose that to be your f of z f of z equals this is analytic on an open set containing gamma union the inside of gamma. So, actually that should be the trace of gamma, gamma star union inside of gamma, so, is gamma star. So, here is the trace of gamma and on the inside cos z by z plus one z plus i is analytic.

(Refer Slide Time: 26:34)

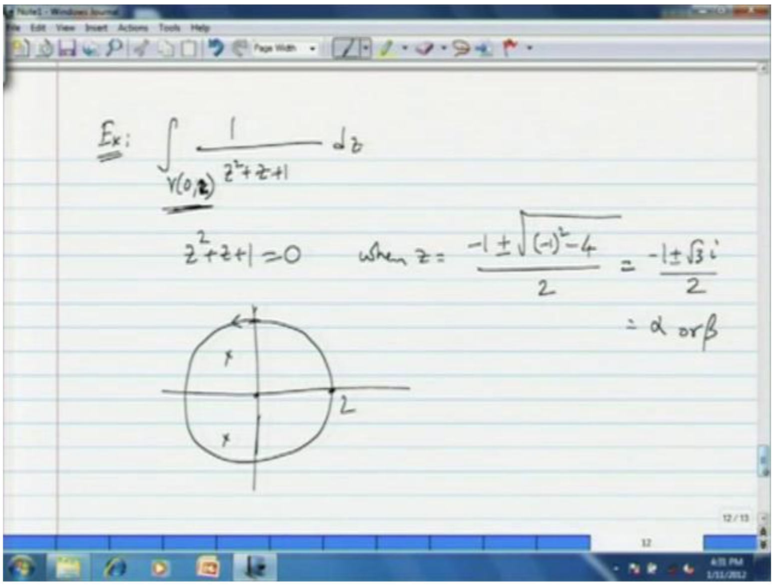


(Refer Slide Time: 26:41)



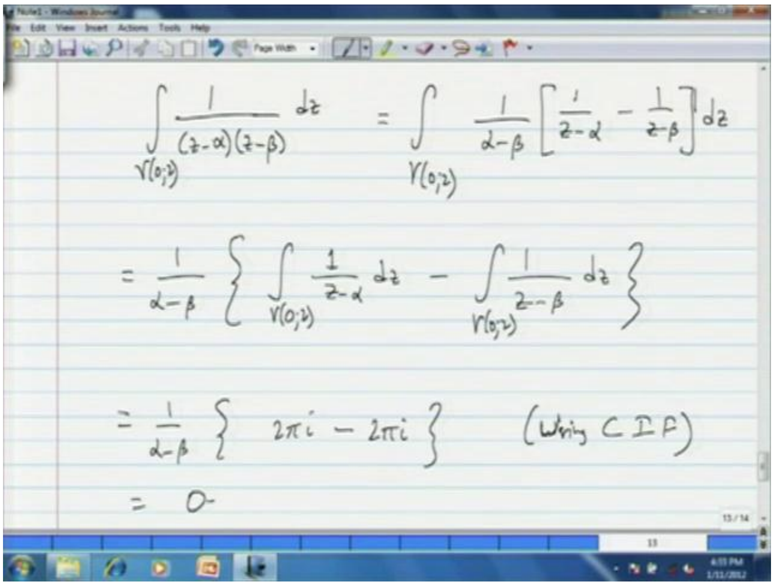
So, by by Cauchy's integral formula now; So, since i's inside of gamma this cos z by z plus i by z minus i d z is 2 pi i times, the value of the function cosine z by z plus i evaluated at z equals i, that is your value integral. So, is 2 pi i times cosine i by 2 i. This can be simplified cosine i, well first 2 i cancel 2 pi i for a pi and then cosine i is recall e power i times i plus e power i minus i times i divided by 2. So, that gives you pi times e squared plus 1 over 2 e so, that is here. So, that is what the integral evaluates two. So, this is another example of how to use the Cauchy's integral formula to compute integral.

(Refer Slide Time: 28:10)



So, gamma 0 2 1 by z squared plus z plus 1 d z so, evaluate this integral. Well this is gamma 0 2, notice that well first let me see where the denominator is 0 z squared plus z plus 1 is equal to 0, when z is minus 1 plus or minus square root of minus 1 squared minus 4 divided by 2. So, that gives me minus 1 plus or minus square root 3 i by both of these are the cube roots of unity, the non unit cube roots of unity. So, what is that means is if you take 0 and circle of radius 2. So, this is circle of radius 2 this is 2, which is your gamma 0 2. Then definitely both these numbers minus 1 plus root 3 i by 2 and minus 1 minus root 3 i by 2 both of them lie inside this contour, this simple close contour.

(Refer Slide Time: 30:03)

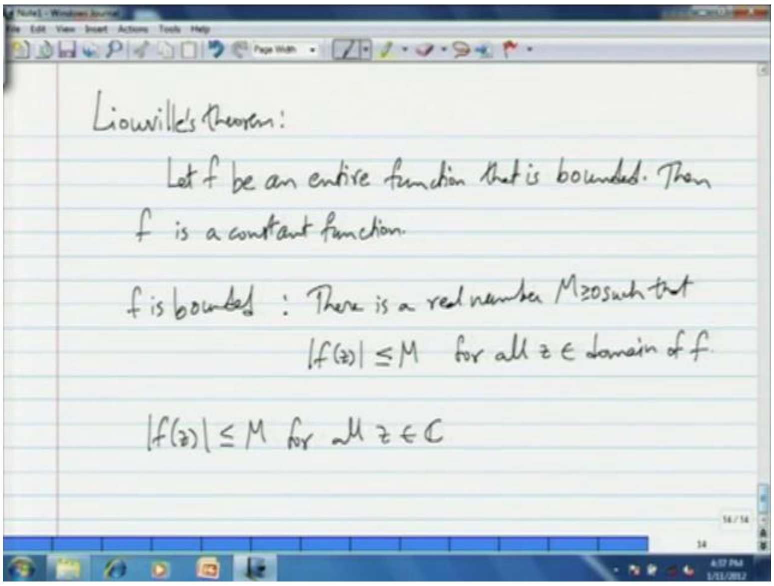


So, then we cannot evaluate this this integral directly by using Cauchy’s integral formula, but what we can do is we will call these two roots alpha and beta and what we can do is use partial fractions to separate the factors in the denominator. So, allow me to factorise this zee minus alpha times z minus beta over gamma 0 2 d z and now, I will separate this into partial fractions the integral into partial fractions. So, this is integrations over gamma 0 2 of 1 by alpha minus beta. So, I am using partial fraction this is 1 by z minus alpha minus 1 by z minus beta and then d z.

So, this gives me 1 by alpha minus beta times. So, let me use a flower parenthesis integration over gamma 0 2 1 by z minus alpha d z minus integration over gamma 0 2, I am separating the integrants as well 1 minus z by beta d z. So, since alpha and beta are both in the interior of on the inside of the curve gamma, gamma 0 2, what I can do is use Cauchy’s integral formula on either of these integrands. So, this gives me 1 by alpha minus beta times 2 pi i times, well here the analytic function is 1. So, that is gives me just 2 pi i or you can think of it as as using the the fundamental integral.

So, either way 2 pi i minus 2 pi i. So, using Cauchy’s integral formula so, we are using Cauchy’s integral formula, this gives us as 0. So, we can break the given integrand using partial fraction and then use the Cauchy’s integral formula to evaluate this part. The viewer is encourage to to do more problems of this certain evaluation of integrals. So, next let us look at the following application of Cauchy’s theorem.

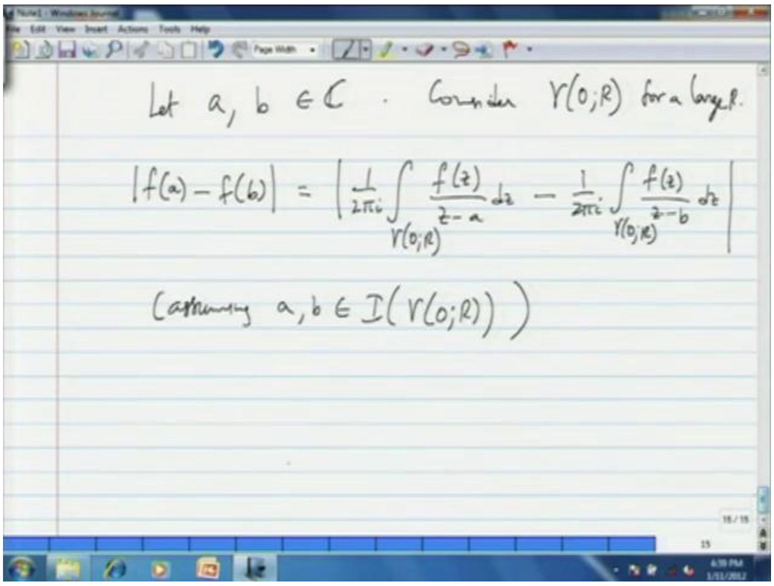
(Refer Slide Time: 32:46)



So, this is Liouville's theorem, Liouville's theorem assets that, if f is a bounded entire function then it is constant. So, let me stated f be an entire function that is bounded then f is a constant function. So, recall what an entire function is it is an analytic function its analytic on all of the complex plane. So, and a bounded function recall is what a bounded function is the bounded function of modulus f of z is less than or equal to M for all z belongs to C.

So, let me write the down recall f is bounded means that modulus of f of z or let me say there is an M there is a real number M capital M such that it has to be greater than 0 of course, greater than or equal to 0 such that greater than or equal to 0 such that modulus of f of z is less than or equal to M for all z belongs to C in this case C. So, normally speaking z belongs to domain of f that is what f is bounded means. So, in this case since the domain is all of the complex plane modulus is f of z is less than or equal to a fix real number M for all z belongs to C. So, a modulus of f of z is less than or equal to M for all z belongs to C.

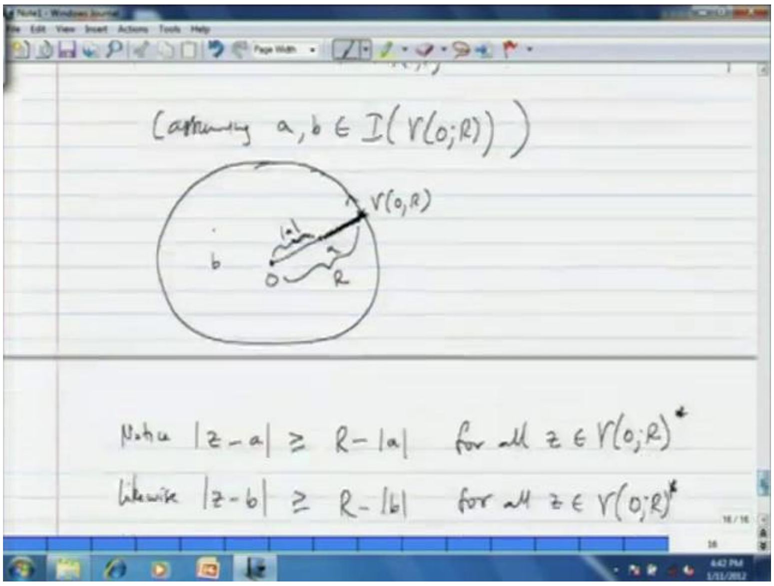
(Refer Slide Time: 35:18)



And using this what we can do is let let us start by fixing a comma b complex numbers a and b. So now, these are these are two fix complex numbers we will prove that the value of f at a is equal to the value of f at b, by proving that modulus of f of a minus f of b can be made arbitrarily small. So, it is a standard way of proving this. So, will estimate the modulus of f of a minus f of b. So, using Cauchy's integral formula this is equal to well what is f of a, in order to use Cauchy's integral formula I need contour. So, let me take a large circle so, let me take consider consider gamma 0 R recall what that means.

It is a circle of radius r centred at 0 for large R for some for large R, how large we will see later we wanted to be as large as we want to make f of a minus f of b in the modulus smaller. So, I make precise in a moment. So, here a modulus of f of a minus f of b is equal to by the Cauchy's integral formula 1 by 2 pi i times the integration over gamma 0 R f of z by z minus a d z minus 1 by 2 pi i times the integration over gamma 0 R f of z by z minus b d z. Assuming that assuming a b belong to interior of gamma interior of this gamma 0 R, by choosing large enough R of course, we can assume that this fixed a and b lie inside of this gamma 0 R. So, here I need to close the parenthesis.

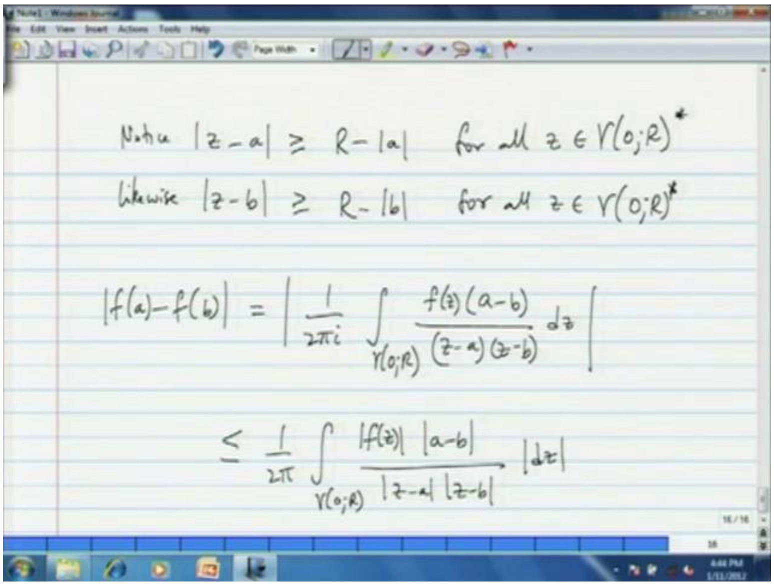
(Refer Slide Time: 37:52)



So here is the picture here is a here is a here is 0 firstly, here is a here is b let us say you choose a large enough circle such that both centred at 0, such that both a and b lie in the inside and what is also important here is that here is your gamma 0 R. So, if you consider this line connecting 0 and a and extended all the way until R, you notice that the distance between a and this point here this cross mark point here gives you the minimum distance between a and any point on the circle.

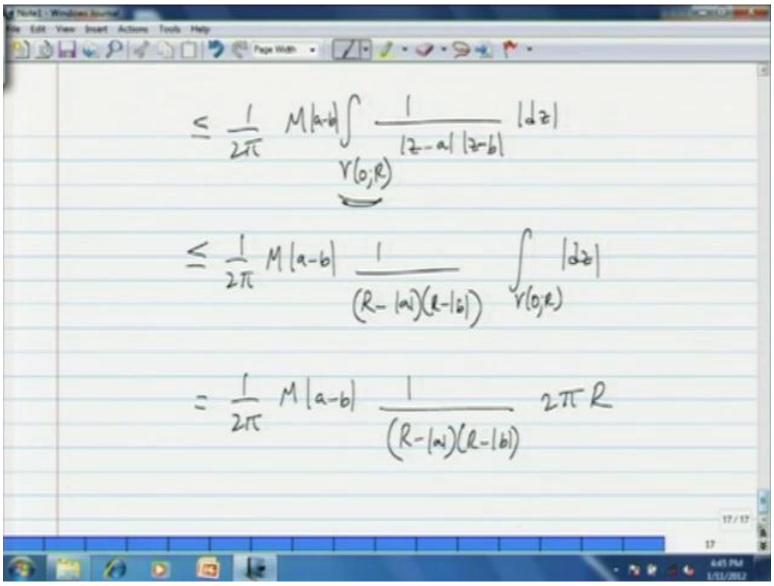
So, what I want to say is that notice that the modulus of since i am using z i guess, modulus of z minus a is a greater than or equal to R minus the modulus of a. So, from the picture here is the modulus of a and then this is the distance R minus modulus of a refers to this piece, which I am over writing (( )) and then the modulus of z minus a is greater than or equal to R minus mode a for all z on belonging to gamma 0 R. So, this technically should be this stress of gamma 0 R, but gamma 0 R i confuse it constantly with the trace. So, for all point of circle that is the minimum distance, likewise likewise modulus of z minus b is greater than or equal to r minus modulus of b for all z belongs to the trace of gamma 0 R. So, we will keep this aside will need this estimate in a moment.

(Refer Slide Time: 40:41)



And, then let us go back to our estimate of modulus of f of a minus f of b and then using Cauchy's integral formula we got that. So, modulus of f of a minus f of b is less or is equal to the modulus 1 buy 2 pi i times the integration on gamma 0 R, let me club the integrands now. So, I get f of z times a minus b by z minus a times z minus b d z. So, I am I am adding these two integrands or subtracting we have to say and then I get this expression. The the common denominator z minus a time z minus b etcetera and then I get a minus b in the numerator. So, this is by the estimation theorem less than or equal to 1 by 2 pi times the integration on gamma 0 R of the modulus of f of z times the modulus a minus b by modulus of z minus a times the modulus of z minus b times modulus of d z.

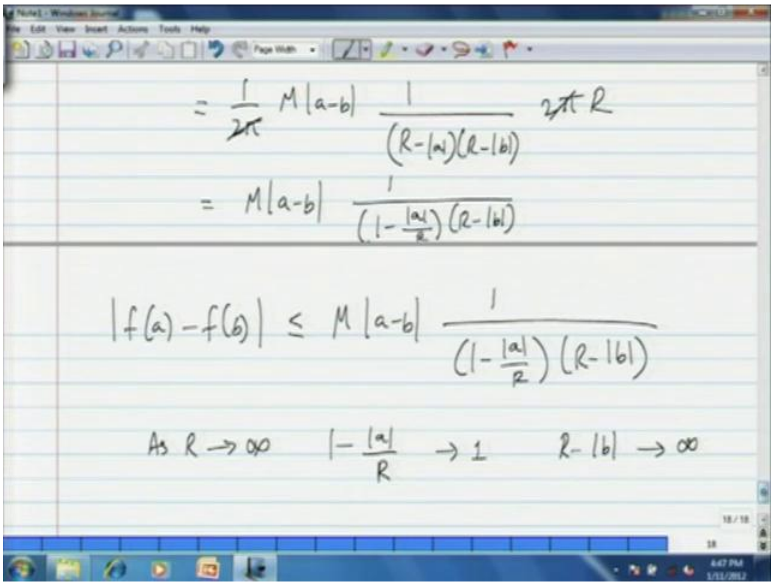
(Refer Slide Time: 42:10)



So this is recall that f is bounded function which means the modulus f of z is less than or equal to some fixed number M. So, whether z belongs to gamma 0 R or not this is less than or equal to 1 by 2 pi times M times integration on gamma 0 R, well a minus b is also a fixed number. So, get 1 by mode z minus a mode z minus b and we also saw that the modulus z minus a for z belonging to gamma 0 R is greater than or equal to r minus modulus of a, which means one over mode z minus a is less than r equal to r minus 1, 1 over r minus mode a.

So, this is less than or equal to 1 by 2 pi times m times the modulus of a minus b times 1 divided by r minus modulus of a times r minus modulus plus of b times the integration gamma 0 R of modulus of d z, which is just a length or the perimeter of the circle of radius capital R. So, this is equal to 1 by 2 pi M times modulus f of a minus b times 1 by R minus modulus of a times r minus modulus of b times 2 pi R 2 pi capital R.

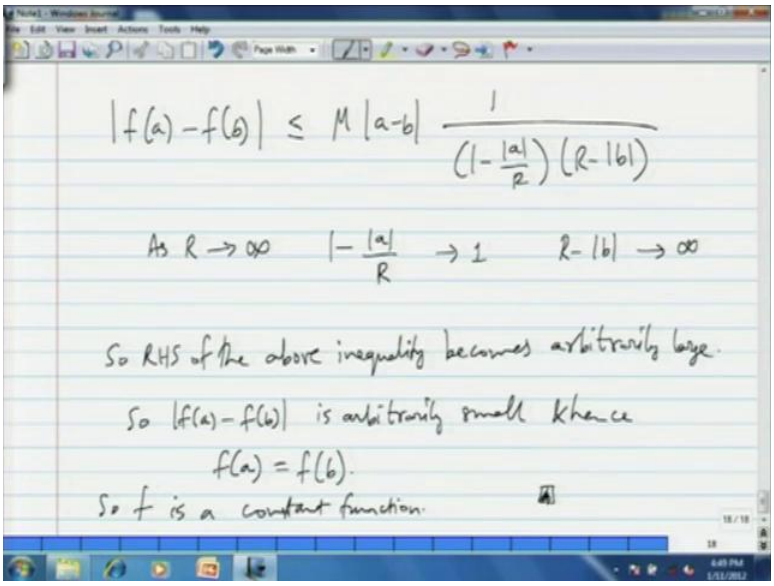
(Refer Slide Time: 43:34)



So, then we can cancel the two pi's and see that this gives us M times modulus of a minus b, which are all fixed numbers times 1 by divided the R into one of these factors 1 minus modulus of a by R times R minus modulus of b. So now, all in all what we have is modulus of f of a minus f of b is less than or equal to this M times modulus of a minus b times 1 by 1 minus modulus of a by R times R minus modulus of b. So, the left hand side does not depended on R on the right hand side depends on capital R. So, we are free to play with R.

So, as R becomes large and large R tends to infinity 1 minus modulus of a over R, recall R a and b fixed numbers. So, this tends to 1 and R minus modulus of b also tends to infinity becomes arbitrarily large. So, the denominator in the right hand side expression of the inequality becomes arbitrarily large.

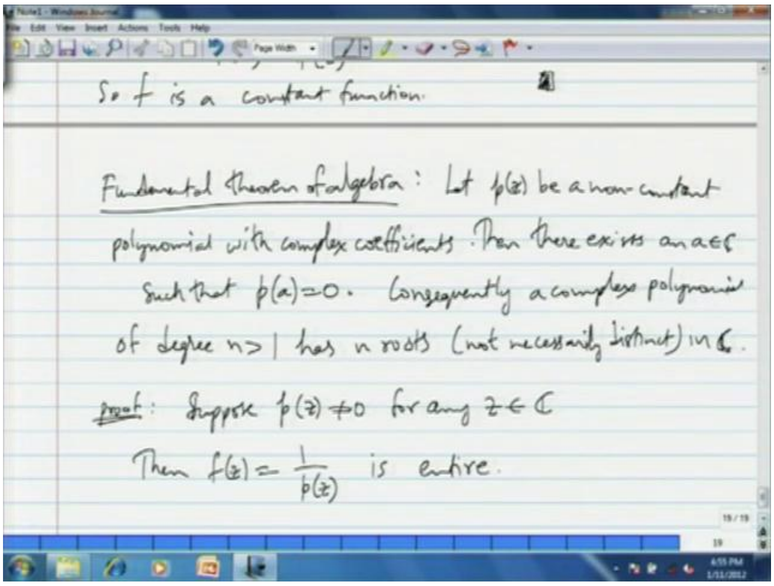
(Refer Slide Time: 45:04)



So, RHS of the above inequality becomes arbitrarily large and so, that is how large we want R. So, however much want modulus of f of a minus f of b to be smaller, we will make R that much larger. So, that gives us that, modulus of f of a minus f of b is arbitrarily small. Since, the RHS tends to 0 as R tends to infinity. So, we conclude f of a the modulus has to be equal to 0, which means f of a has to equal, the only number with modulus 0 is 0. So, f of a have to be equal to f of b for the above any quality to hold for arbitrarily large R. So, since a and b where arbitrarily chosen from the complex plane we conclude that f is a constant. So, f is a constant function.

So, we prove this by taking two different point say a and b and showing that the value of f at those two different point is equal, at the prove of course, use the Cauchy’s integral formula and Liouville’s theorem is very useful. So, for example; it gives as a quick proof of the fundamental theorem of algebra. So, here is so here is the fundamental theorem of algebra which can be quickly reduced from levels theorem.

(Refer Slide Time: 47:08)

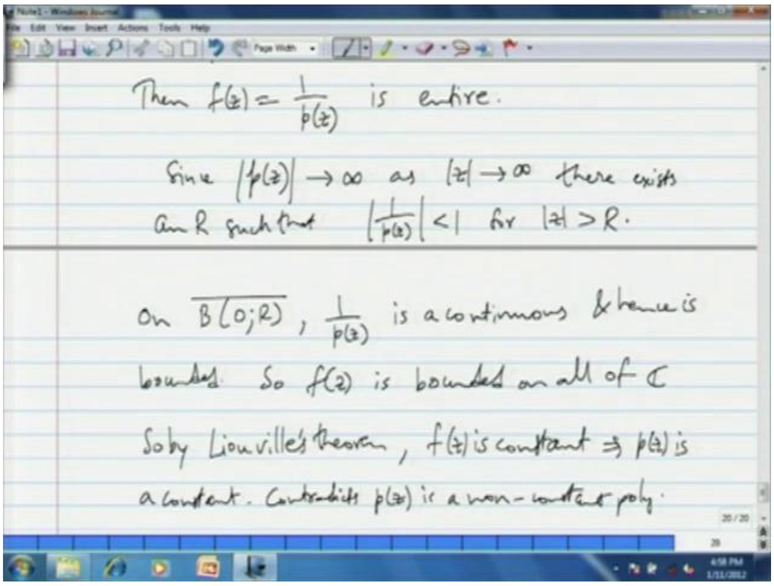


So, let p of z be a non constant polynomial with complex coefficients. Then there exists an a the complex number a such that f of a is 0 or p of a rather 0. So, there is a 0 of this polynomial non constant polynomial. So, applying this repeatedly we get that consequently a complex polynomial of degree n greater than 1 has n roots, not necessarily distinct in c. So, that the degree tells us for n greater than 1 for degree tells us the number of roots possibly non distinct ones of the polynomial.

So, that is for example for not true for polynomial real polynomial, because x squared plus 1 the polynomial x squared plus 1 in real numbers does not have any 0. So, here is the proof, but any complex polynomial any polynomial with complex coefficients has a root has 0 and then consequently applying this repeatedly it has n 0 where n is the degree it is a non constant polynomial. So, here is the proof let suppose suppose that it does not have a 0 let a and b, non constant polynomial and suppose p of z is not equal to 0 for any z belongs to c.

So, will assume to the contrary that p of z non 0 for any easy then of course, we can form the reciprocal 1 by p of z. Because, p is not 0 p of z is non 0 we can form the reciprocal function, this is entire because p is non 0 p is entire 1 by p is a entire by assumption. So, since this is entire and this will prove is bounded as well.

(Refer Slide Time: 50:32)

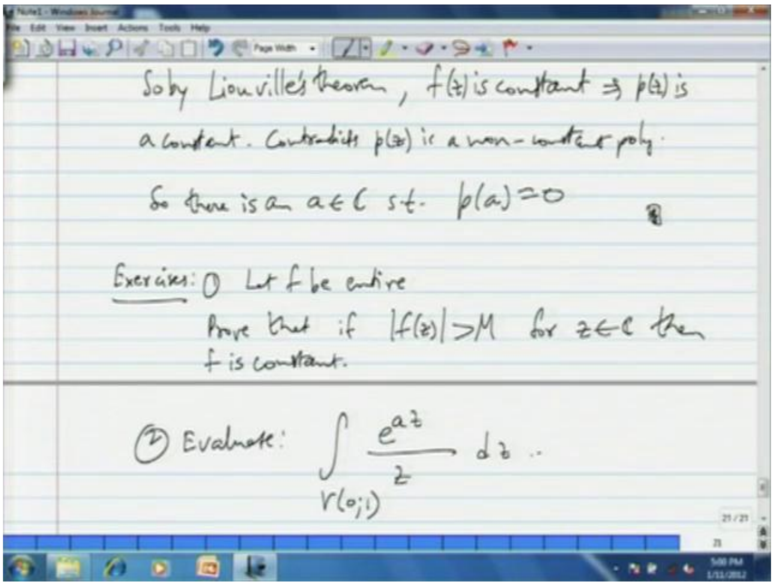


So, since the polynomial p of z non constant polynomial in modulus will tend to infinity as modulus of z tends to infinity, consider the leading term for example; the leading term has degree greater than or equal to 1. Since, it is the non constant polynomial, it will have a z with power greater than or equal to 1 and that tends to infinity and then in modulus and then that will dominate other terms.

So, this is true and then because this happens there exists an R for, for a large R there is an R such that 1 by p of z is strictly less than 1 for modulus of z greater than R, 1 by p of z has to tend to 0. So, 1 by p of z in modulus is strictly less than 1 for really large R really large modulus of z and then on b 0 R bar the closer of b 0 R itself 1 by p of z is a continuous function its analytic by assumption. So, this is continuous and hence, is bounded because a continuous function on compact set b 0 R bar is a compact set, continuous function on compact sets are bounded and, and so, one by p of z your f of z equals one by p of z is bounded on all of C.

It is bounded by some number capital M in b 0 R bar and it is bounded by 1 outside of b 0 R bar. So, the maximum of M and 1 will be the bound for f of z on all of C. So, f of z is bounded. So, by Liouville's theorem we get that f of z is constant f of z is 1 by p of z which implies p of z is a constant function, but by assumption p of z was a non constant polynomial. So, that is contradiction, contradicts p of z is a non constant polynomial. So, this cannot be possible which means that there is a certain a such that p of a 0.

(Refer Slide Time: 53:53)



So, there exists, there is an a belongs to C, such that p of a is equal to 0; that is the proof of the fundamental theorem of algebra. So, we will stop this session here, I will give couple of exercises to the viewer, here is exercise one; let f be entire prove that if the modulus of f of z is strictly greater than M in all of C, what that means is that for z belongs to C then f is constant f is a constant function.

That is the first exercise and the second exercise is to evaluate, evaluate on the circle of radius 1 centred at 0, the unit circle oriented in the counter clock wise direction. The value of this integral e power a z by z d z and use this to show use this to show, just by simple parameterization that integral 0 to pi e power a cosine theta cosine of a sin theta d theta is equal to pi.