Complex Analysis
Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 3
Complex Integration theory
Lecture - 6
Cauchy's Theorem Part – III

Hello viewers, in this session we will discuss yet another version of Cauchy's theorem, like we said last time. So, far we have Cauchy's theorem on a rectangle and then Cauchy's theorem for a disc. So, today we are going to see that if we have a two simple closed curves one inside other. I will I will make those terms more concrete. So, if one is in inside the other and both are oriented in the positive sense, then then the integration of f of an analytic function f, on one of them is equal to the integration over the other. So, I will I will make the statement more precise.

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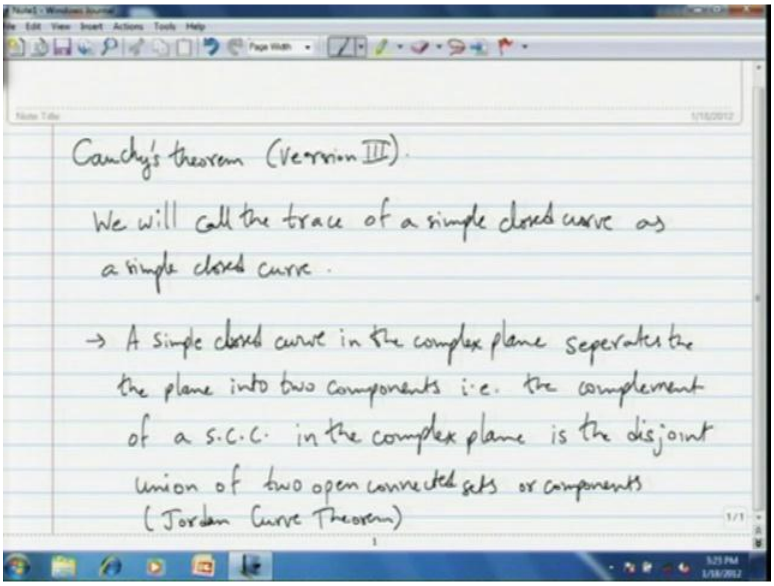

So, here is Cauchy's theorem version 3. From here on in this course, the curves that we are going to consider are always going to be contours unless mentioned otherwise. So, please take a note of this; so in order to make some preparation for stating this theorem, let us start with some recollection of properties of simple closed curve. So, we will call the trace of a simple closed curve as a simple closed curve. So, what that means is, I will constantly keep confusing the curve with its trace, when there is no much ambiguity.

Having said this, so the first fact we will need is or the property of a simple closed curve, that we will need is that a simple closed curve separates or its trace in the complex plane separates the plane separates the plane into two components two components I e the complement of a simple closed curve. I will say SCC for short a simple closed curve in the complex plane, is the disjoint union of two open connected sets or components. So, open connected sets are also called maximal open connected sets are also called components. Since, they are disjoined these are called components and this statement is called is called the Jordan curve theorem.

So, for simple minded curves this is reasonably easy to believe. So, for example, if you consider a circle in the complex plane, then it is clear to see that complement of the circle, you know is the disc inside the circle and the and the region outside the circle. But in general for a simple closed curve, this statements needs a proof and we we are going to assume that this statement is true for general simple closed curve.

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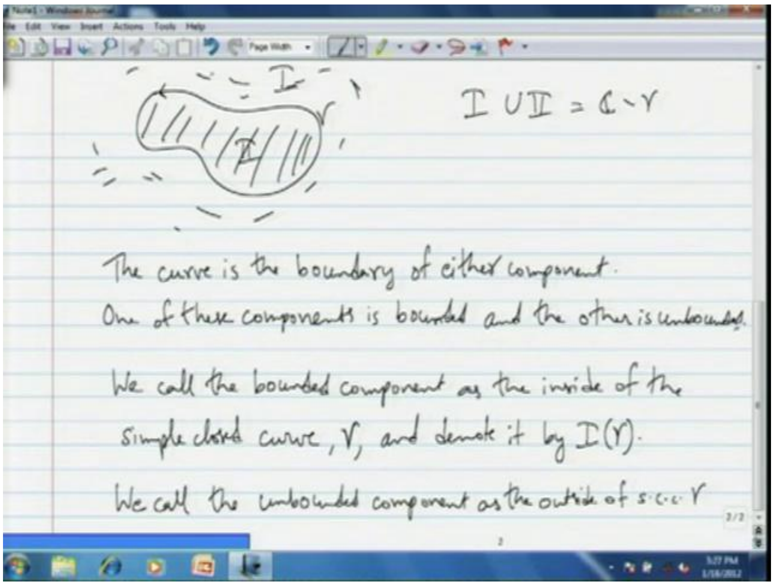

So, a simple minded example is like that. That is a simple closed curve or a trace of it. So, with some direction, let us say then the complement of this curve is that portion and then outside of that portion. So, this is region 1 region 2, so one union two the disjoint unit of this is the complement. So, if I call this gamma is C minus gamma. In this case where the curves simple closed curves separates, the complex plane into two components the curve is the boundary. What is also important is that the curve is the boundary, of either component.

One of these components is bounded. Since, the curve itself is bounded, one of the components it is intuitively easy to believe is bounded and the other is unbounded. And this this much we will take it on belief and we call the bounded component as the inside of the simple closed curve. Let us name the cure, let us call it gamma and denote it by I of gamma. So, I of gamma is a region in the complex plane. It is an open connected set you know, which the bounded component is corresponding to the simple closed curve gamma. Notice that I of gamma is defined only for a simple closed curve and not for a general curve, we call the unbounded component as the outside of m of the simple closed curve gamma.

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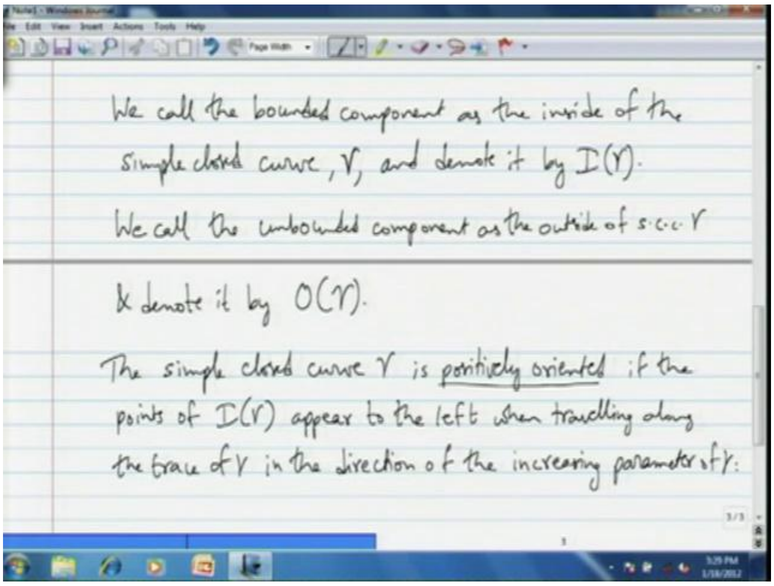

Denote it by O of gamma. Now, we want to talk of orientation of a simple closed curve. So, the simple closed curve gamma is positively oriented. So, we are trying to define positively oriented if the points of I of gamma appear to the left, when travelling along the trace of gamma in the direction of the increasing parameter of gamma. So, recall that we said, we gave a direction to gamma based on the increasing parameter value. So, if the direction of gamma is such that when one travels in the direction of gamma, the points on the inside for a simply closed curve fall on its left. Then we say that the curve, simple closed curve gamma is positively oriented.

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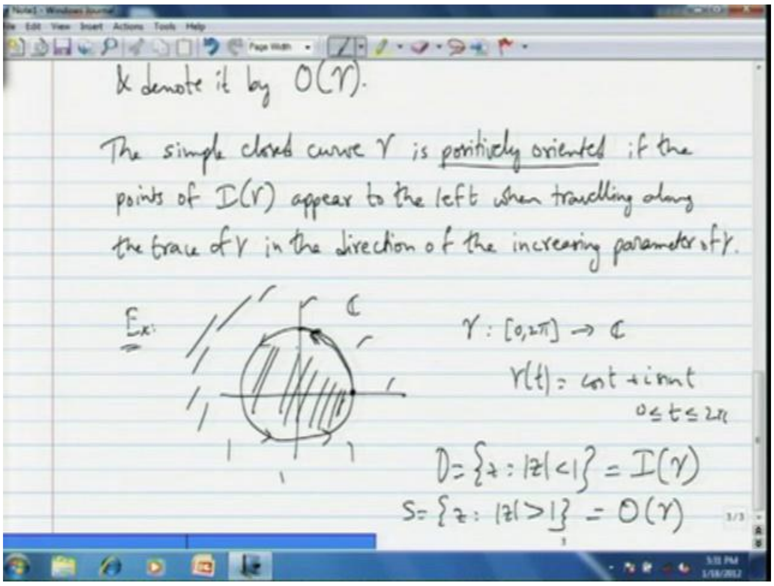

For example, if you consider a circle unit circle and then you consider the function cos t plus I sin t from 0 to pi. For example, to C gamma of t is cos t plus i sin t 0 less than or equal to t less than or equal to 2 pi. Then you start here and the trace in the complex plane, it goes in the counter clockwise direction to the observer from the top of the plane. Then this region unit disc D equals set of all z such that the modulus of z is strictly less than 1, it becomes your inside of this gamma. Gamma is a simple closed curve and D is the inside.

The set S is the set of al z such that the modulus of z is strictly greater than 1, that is everything which is outside becomes the outside of this curve. So and then we call this curve positively oriented. So, notice that if gamma is positively oriented, then we defined a curve minus gamma, which traces gamma in the opposite direction that is not positively oriented. We want to call that negatively oriented.

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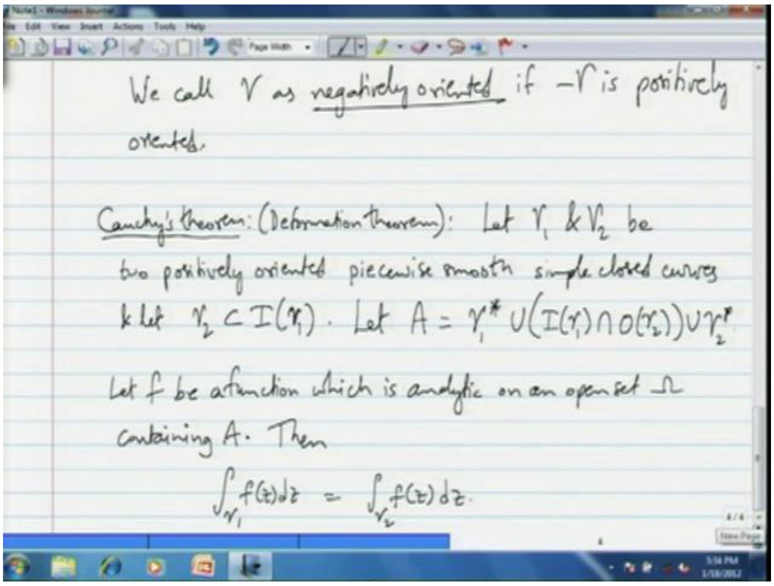

We call gamma as negatively oriented. Of course, it has to be simple closed curve, if minus gamma is positively oriented. So, that is positive and negative orientation for a simple closed curve. So, with these, with this terminology we are ready to state the third version of Cauchy's theorem. So, here is Cauchy's theorem, this version can also be called deformation theorem. So, this is, this can also be called as deformation theorem. Very simple version of what is usually called deformation theorem. So, let gamma 1 and gamma 2 be two simple, closed two positively oriented. It needs to be positively oriented simple closed curves.

I will also add piecewise smooth, since we are dealing with piecewise smooth curves only piecewise smooth simple closed curves. Let gamma 2 be contained in the inside of gamma 1. Also let A be note the set the trace of gamma 1 union, the inside of gamma 1 intersection the outside of gamma 2, union gamma 2 star union the trace of gamma 2. Let f be a function a complex function of course, which is analytic on an open set. Let us call that open set omega containing, containing this set A. Then the conclusion is that the contour integral on gamma 1 of f of z d z is equal to the contour integral on gamma 2 f of z d z.

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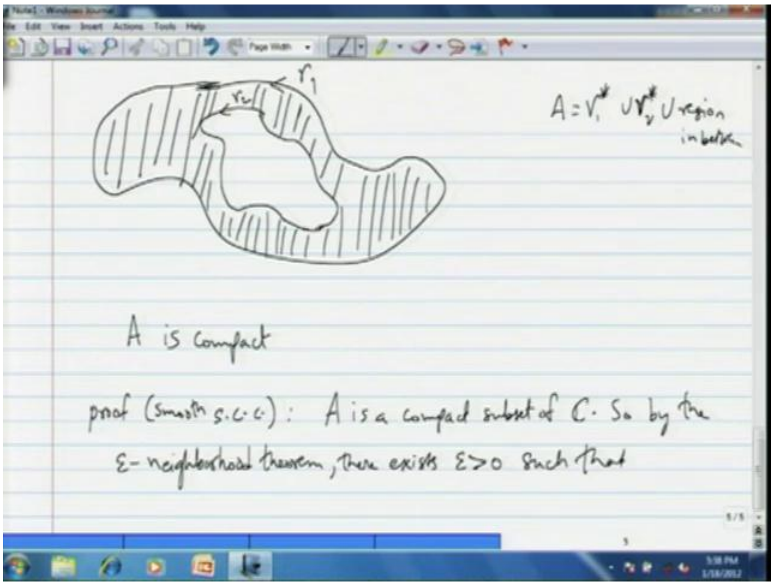

So, here is the schematic for this version of Cauchy's theorem. So, suppose you have some curve gamma 1 simple closed curve rather gamma 1 oriented in the positive sense. Then there is yet another curve gamma 2 not intersecting gamma 1 and also gamma 2 is completely contained inside gamma 1 and here is gamma 2. Both of them are oriented in the positive sense. So, A this A which appears in the statement of the theorem, is essentially your region in between. So, to say according to the schematic you and and it includes A includes the trace of gamma 1 and gamma 2 union region in between. In the region in between is clearly the intersection of the inside of gamma 1 and outside of gamma 2.

So, that is your set A. So, what is important is independent of the schematic? So, A is compact, it is a compact subset of the complex plane and that is a very significant. So, here is the proof, there are some details that I will skip, more or less this is proof. So I will assume that gamma 1 and gamma 2 are smooth smooth simple closed curves. The the proof for piecewise smooth simple closed curves can be given similarly, with some slight modification of the proof, that I am going to give. So, first what we will use is the fact that A is a compact set.

So, above A is a compact subset of C, so by the epsilon neighbourhood theorem, there exists an epsilon positive. Recall the epsilon neighbourhood theorem, it tells that if a compact set is contained in an open set, then there is an epsilon neighbourhood of this compact set, which is contained in the open set. So, here we have a situation where a is contained in the domain of analyticity of the function f, which is an open set omega. Then we can use the epsilon neighbourhood theorem.

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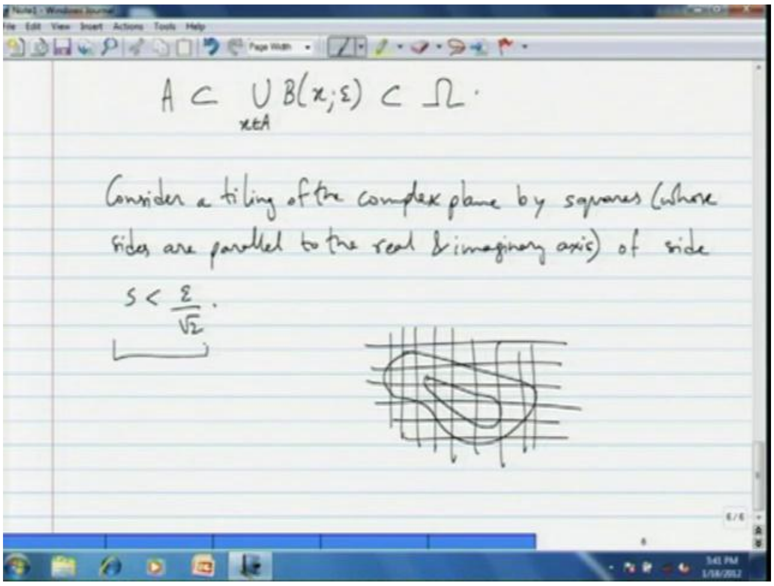

A is compact subset of C, so there exist an epsilon positive such that A is contained A is contained in the union of the epsilon balls, where the union is over all points contained in A. This is in turn contained in the set under question the open set under question omega. So, now what we will do is we will consider the intersection of a tiling of the complex plane, with with the set A, this compact set A. So, tiling is essentially imagine that the complex plane is divided into square tiles, whose adjust are parallel to the x and y axis or the real and imaginary axis.

So, so imaginary grid, so consider a tiling of the complex plane by squares of side by squares, whose sides are parallel t the coordinate axis or the real and imaginary axis of side. So, the sides have a side length S, which is strictly less than epsilon by square root 2. You will see why this is important? So, what you have is here is the compact set a schematically and then you have a tiled, the whole of the complex plane.

Using squares whose side length is S, which is strictly less than epsilon over root 2. It does not matter how small as you pick, as long as of course, it should be positive it is the length. So, you consider such a tiling consider its intersection with A.

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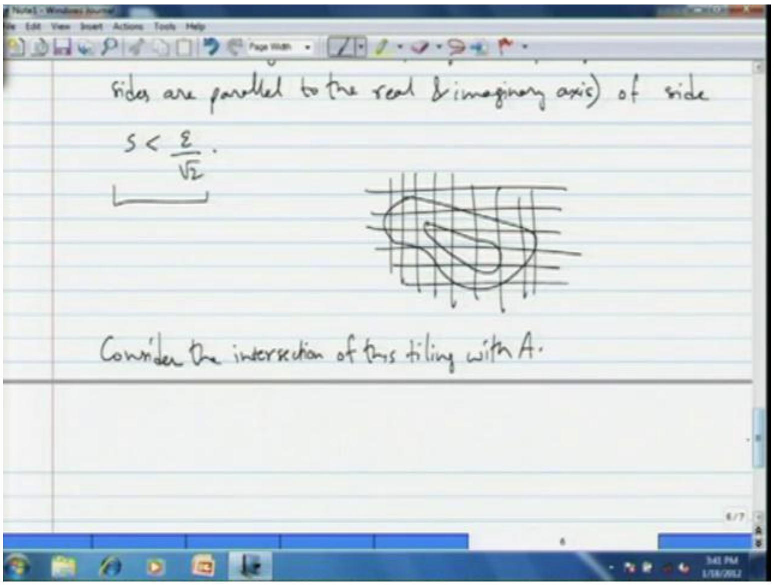

Consider the intersection of this tiling with A. We are going to use the structure of this tiling to prove the theorem. So, the structure is essentially, there are squares and then there are portions of the squares, when when the curve the boundary of A intersects the square tile.

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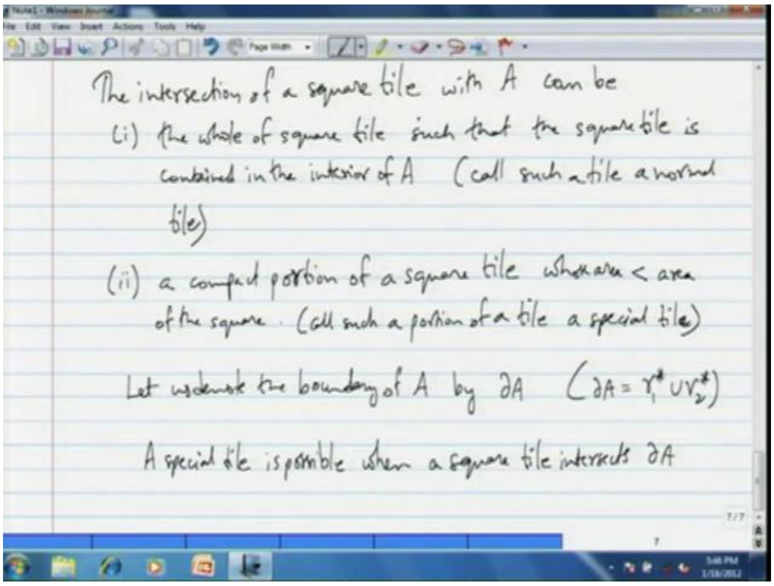

So, the intersection of a square type with A, it can be 1 the whole of square tile such that square tile is is contained in the interior of A. The containment of square tile in the interior of A is important; the whole of the square tile could intersect A with it intersecting the boundary as well. So, we will separate that one case. So, the whole of the square tile such that it is contained in the interior, calls such a tile a normal tile to... It could be portion a compact portion of the square tile, of a square tile whose area is strictly less than area of the square and call such portion of a tile, a special tile.

Now, a special tile could really be disconnected in the sense, it might have few portions or yeah not just a one compact piece. So, we will allow for that and we will still call all those portions for a given square tile as a special tile. But notice that number of pieces which A can split this square tile into has to be finite, because we are dealing with two compacts intersection of two compacts one is A and other is the square tile. Now, let let us denote the boundary of A by dou a notice that the boundary of A is nothing but gamma 1 star union gamma 2 star.

That is the boundary of A. Now, special tile is possible when when a square tile intersects the boundary of A. It is only possible when a square tile intersects boundary of A. A square tile is either completely in the exterior of A or it is in the interior of A or it could be that it intersects the boundary of A. We want to call a square tiles still a normal tile, when the intersection with the boundary does not take up any area of the square tile itself.

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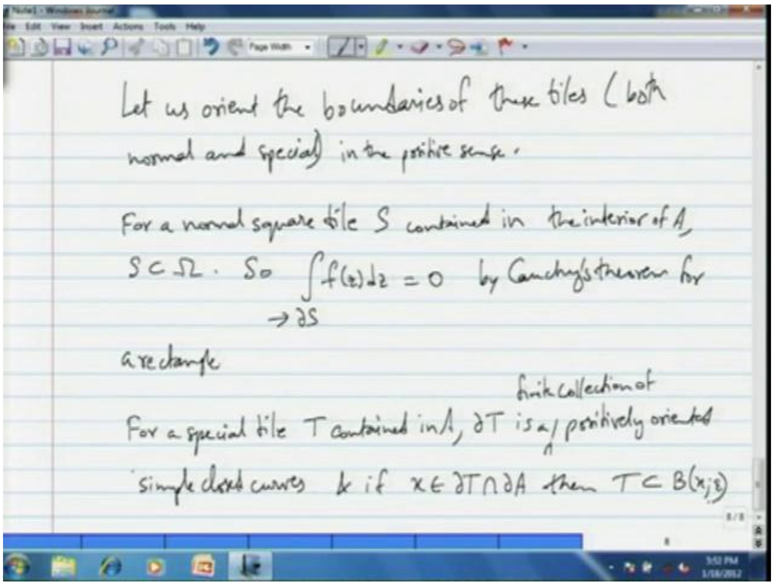

Now, let us orient the boundaries of these tiles both normal and special in the positive sense. So, for a normal type its easy you just consider the counter clockwise orientation of the boundary of the square, but for a special tile I just mentioned that the number of the number of pieces or the number of components of the boundary could be more than 1. Nevertheless each of them we will orient in the counter clockwise sense. So, after having done that we notice that for a square tile, for a normal square tile. Let us call it capital S contained in A in the interior of A S is clearly contained; of course, in omega because itself is completely contained in omega.

So, the integration on the boundary of S oriented in the counter clockwise sense of f of z d z is clearly 0 by Cauchy's theorem for a rectangle. So, here the, that symbol stands for boundary of S oriented in the positive sense. So, once the integral is 0 actually the orientation does not matter, but we want to keep the positive orientation for reasons, which will be evident shortly. So, the boundary of on the boundary of S in the contour integral of f is 0 by Cauchy's theorem for a rectangle and for a special tile for a special tile t.

Let us call it t, so actually t is possibly a union of finitely many compact pieces. Then for a special tile t contained in a the boundary of t, is a finite collection of positively oriented simple closed curves. If x is a point, which belongs to the boundary of t intersection boundary of A. Notice that, I said that a special tile is possible when a square tile intersects the boundary of A. So, there is such a point x, which is contained in the intersection of boundary of t, with the boundary of A. Then that special tile is completely contained in b x epsilon by the arrangement that on the side of any square tile is at most epsilon by square root 2.

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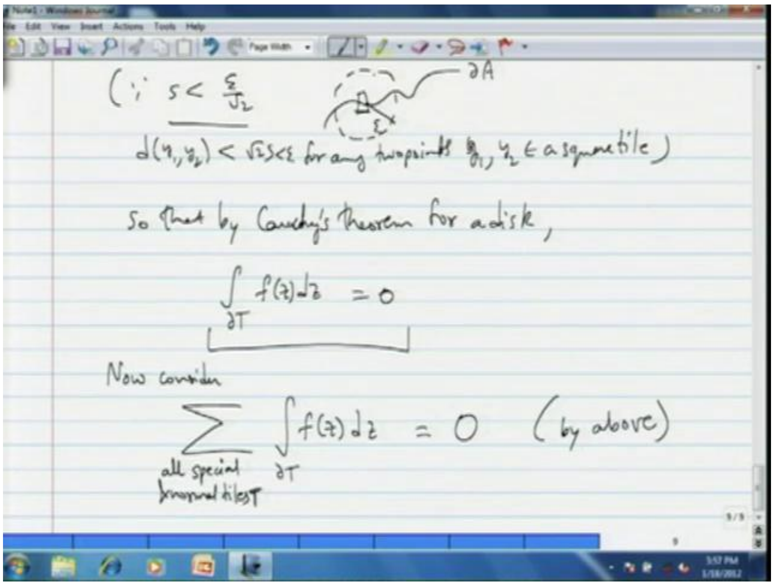

So, since S is strictly less than since S is strictly less than epsilon by square root 2, what you have is you take x to be x to be a point in the intersection and then this is your epsilon ball. So, this is our epsilon. So, since S is strictly less than epsilon by root 2 any two points contained in this square t, have to be or at most square root 2 S apart. So, they are they are at most epsilon apart. So, any two points the distance between any two points y 1 comma y 2 is at most a root 2 S for any two points. S 1 for y 1 comma y 2 belonging to a square tile. By arrangement S is strictly less than epsilon by root 2, so this is strictly less than epsilon.

So, if you consider an epsilon ball, which is a around a point, which is contained in this square tile of course, your whole square is contained in that epsilon ball. So, that schematic should convince you. So, your t is contained in this b x epsilon. So, as a consequence, so by, so that by Cauchy's theorem for a disc for a disc that was version two we saw last time for a disc the integration on the boundary of t oriented in the in the positive sense of f of z d z is equal to 0. Actually this integration is a possibly a finite sum of few integrals, which are on the boundaries component boundaries of boundary of t.

Now, we consider so whether you take a normal tile or whether you take a special tile, the integration on the boundary the contour integration on the boundary oriented in the positive sense of f is equal to 0 by two different versions of Cauchy's theorem. Now, you consider the summation of the contour integral on the boundary of a tile t f of f of z d z, where the summation runs over all square tiles all special and normal tiles t. So, you consider the collection of all normal and special tiles, these have to be finite because you are considering the intersection of a compact set A with the square tiles.

So, there are only finitely many tiles like this. So, you consider the finite sum of these integrals over the boundary of t oriented in the counter clockwise sense in the positive sense. So, and then by by either of above you are adding bunch of zeroes, so this is 0 by by above by the Cauchy's theorem version 1 and Cauchy's theorem version 1 for normal time; cauchy's theorem version 2 for a disc for the special tiles. Now, this is important, now this equation is important, we will analyse the LHS.

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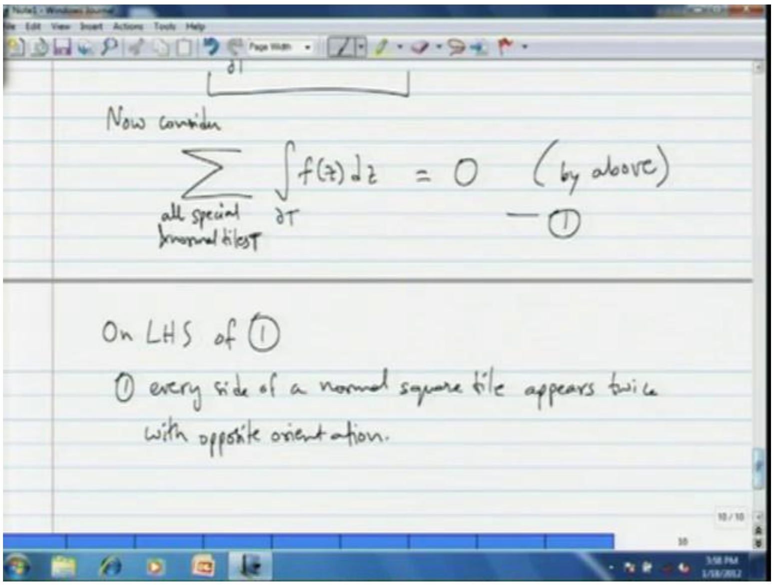

So, on LHS on left hand side of 1, let us call this 1, we will make the following observations, every side of normal tile normal square tile appears twice with opposite orientation.

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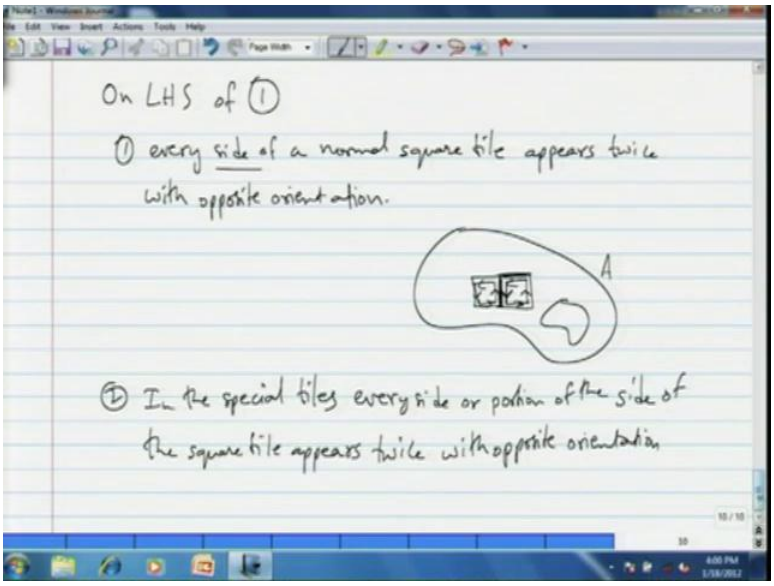

So, that is clear because if you have a square tile normal square tile, which is completely contained in the interior of the region A, so maybe that is the other curve gamma 2. Then there is an adjoining square, which shares that side. If you consider one side of of a normal tile, any side of the normal tile it is completely contained in the interior of A. So, so that side is common to yet another square, which which could be possibly special tile, but that side is shared by some other tile.

So, each of these sides, which are completely contained in the interior are shared by two tiles. Then one of them will have the orientation in the in one direction and the other will have orientation on that side in opposite direction, because both these styles are oriented in the positive sense. So, when you when you consider a side of a normal square tile, then that appears twice in the integration, one for this style and one for the other tile. So, that is one observation and the second observation is that, in special tiles every side or portion of the side of the square tile appears twice again with opposite orientation.

So, if like I mentioned in the earlier case, in observation one, if you have side of special tile does not matter normal or special. If you have side, which is completely contained in the interior of A, then the integration of f occurs twice on that side, with opposite orientations in equation one.

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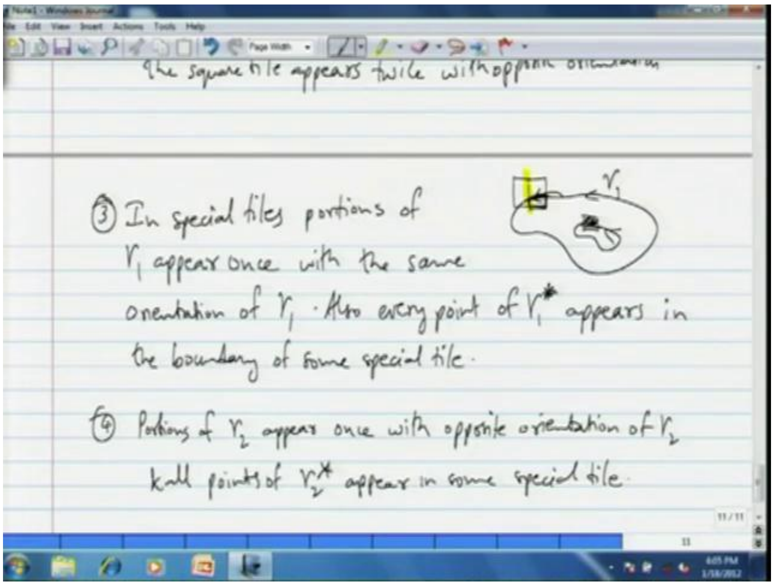

Now, if you have a portion of a square tile. So, for example, square tile whose portion, this is the portion I will probably is a highlight to show that portion. So, only a portion of it is is present in the interior of A or is present inside gamma 1 and in that event, that is also shared with an adjacent tile. So, that portion of the side of the square is shared with the adjacent square. So, and its exactly that portion which is shared and not more. So, the second observation holds. Now, there is a a third observation that I make. So, in special tiles portions of gamma 1 appear and only once with the same orientation as gamma 1 of gamma 1.

Not only that also every point of gamma 1 appears in appears as the boundary or appears in the boundary of some special tile, either of these are clear. So, in the picture I drew here you notice that here is gamma 1 oriented in that positive sense. So, that appears only once in a special tile, because I mean there is no adjoining square tile containing the the boundary as gamma 1. So, here is the special tile in this example picture or in the schematic gamma 1 only appears for this one. Then there is no other square tile sharing that portion of gamma 1.

Also since we are tiling all of the complex plane, it is clear that gamma each point of gamma 1 has to appear in some square tile, it has to fall in some square tile. Four yes, also notice that when you orient these these special tiles in the positive sense, that tallies with the orientation of gamma 1 and there is a fourth observation that portion of gamma

2 appear, once with opposite orientation of gamma 2 and all portions or all points of gamma 2 star. So, here also I, as I said you know this should be gamma 1 star actually in the third observation. So, all points of gamma 2 star appear in some special tile.

So, portion of gamma 2 portions of gamma 2 appear once with opposite orientation of gamma 2, in special tiles actually. So, if you have tile intersecting gamma 2, which is the inner curve, then you see that here is the portion hashed portion and that intersects gamma 2. When you orient the special tile in the positive sense, it picks up the opposite orientation to gamma 2. Gamma 2 is oriented in this sense. So, this is the fourth observation and these four observations lead us to conclude the following.

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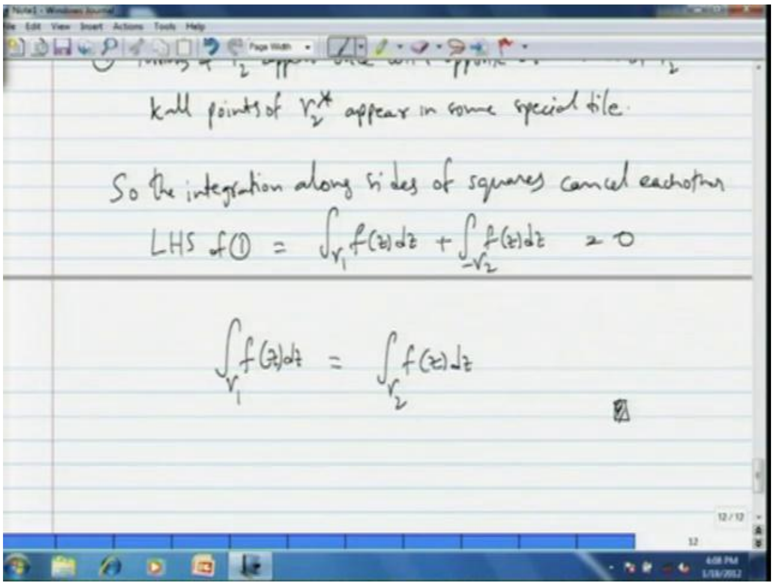

So, the integration along sides of squares, whether partial or full sides it does not matter. The integration along sides of squares cancelled each other when the integration on the LHS you know includes a portion of or complete side of square. Then it does, so twice with opposite orientation, so, they cancel each other. When you have these curves gamma 1 or gamma 2 or the trace of them actually appearing in the integration the portions of gamma 1 or with positive orientation and the portions of gamma 2 are with negative orientation. We also notice that each point of gamma 1 star and gamma 25 star participate in the integration on the LHS of the equation 1.

So, with these observations we see that the LHS boils down to LHS of 1 is actually equal to the integration on gamma 1 after rearrangement, because portions of gamma 1 could be you know here and there, in the integration. So, after rearrangement this is equal to the integration on gamma 1 of f of z d z plus the integration on the negative of gamma 2 f of z d z and the right hand side of course, is 0. When you have all these sides of squares the integration cancels. So, as a result we have that the the contour integral on gamma 1 of f of z d z is equal to contour integral on gamma 2 of f of z d z and that is proof for the smooth version at least. That completes the proof of Causchy's theorem.

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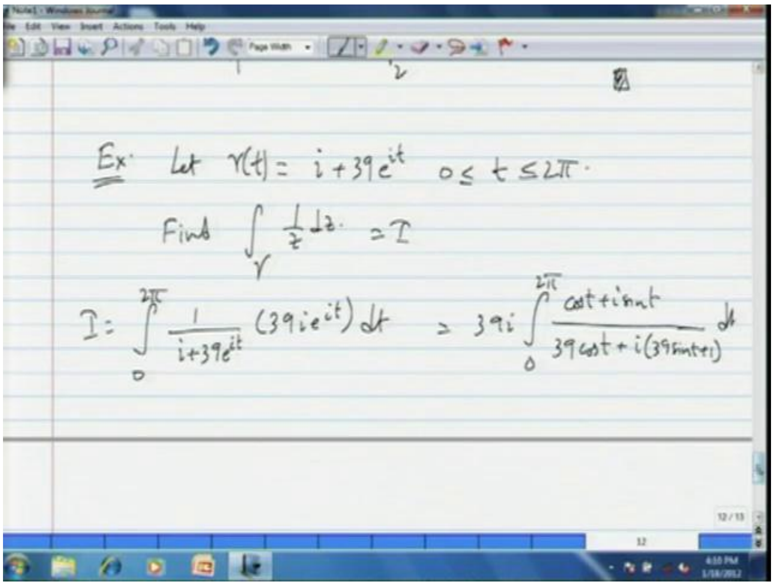

We will see quick applications of this version of Causchy's theorem, it is very useful. So, for example, let gamma of t be a circle of radius 39 does not matter, a very large radius centred at i 0 less than or equal to t less than or equal to 2 pi. Let gamma be that simple closed curve, then find the integration the contour integration on gamma of 1 by z d z. Now, if you start parameterising or if you start evaluating this integral by using the parameterisation, it becomes very complicated. So, you have 0 to 2 pi this integral I.

Let me call it I, I is equal to integration from 0 to pi of 1 by i plus 39 e power i t e times 39 i power i t d t, which is d z. So, 1 by d z d z and you see that this integral is not easy to tackle. You have 0 to 2 pi cosine t plus i sine t divided by 39 cosine t plus i times 39 sine t plus 1 d t.

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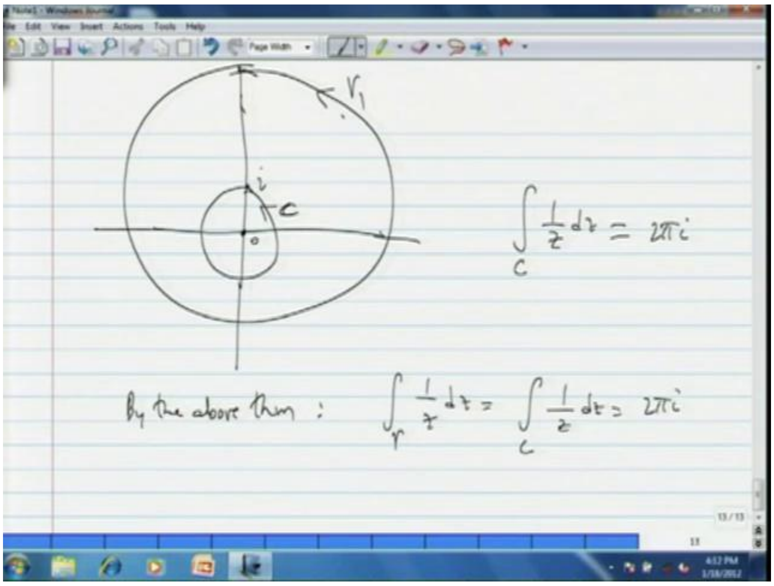

So, it looks rather daunting, but if you use the deformation theorem or Cauchy's theorem the version we just prove, you can see that here is i and you take a large circle centred at i and if the interior contains the point 0. So, clearly the interior of this curve gamma contains 0. Then you look at the units circle and we already know by the fundamental integral that 1 by z d z on the unit circle, so gamma 0 1. Let us call it, let us have a notation for unit circle, let us just call it C rather oriented in the positive sense.

So, we know that if that is the unit circle 1 by z d z on c gives us a 2 pi i this we computed that is the fundamental integral. So, by Cauchy's theorem, we see that this circle acts like gamma one, and this C acts like gamma 2 and C is completely contained in the interior of gamma 1. So, by the above theorem by above version of Cauchy's theorem, we immediately see that the integration on gamma 1 of 1 by z d z is equal to the integration on C or actually gamma integration on gamma of 1 by z d z is the same as the integration on C of one by z d z which is 2 pi i. So, you do not have to compute that integral. So, that is an application of this.

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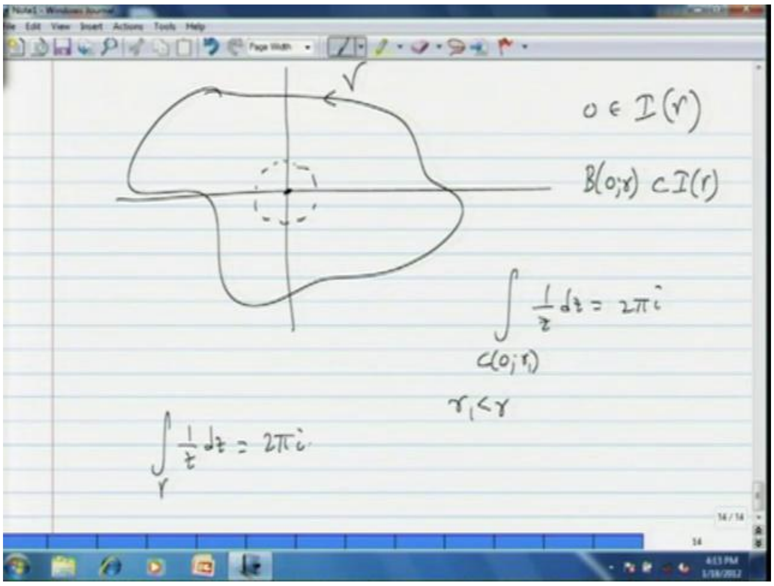

So, you could have really any complicated curve which contains 0 on the interior, 0 in the interior and as long as well... If zero is in the interior of that simply closed curve, and if the simple closed curve is oriented in the positive sense, then there is a ball of some radius around 0, which is contained in the interior. So, if 0 belongs to the interior of gamma, it is gamma, then B 0 r is contained in the interior gamma because interior is an open set. We know by the fundamental integral that C 0 r 1, r 1 strictly less than r.

So it is a circle of radius r 1 around r 0 of 1 by z d z is 2 pi i by the fundamental integral, that we calculated. So, from this we can conclude that the integration on gamma, which is oriented in the positive sense of 1 by z d z is also 2 pi i by using the Cauchy's theorem which we just saw. So, this is simple application and this version of Cauchy's theorem is very useful in deriving Cauchy's integral formula, which we will see in next time.