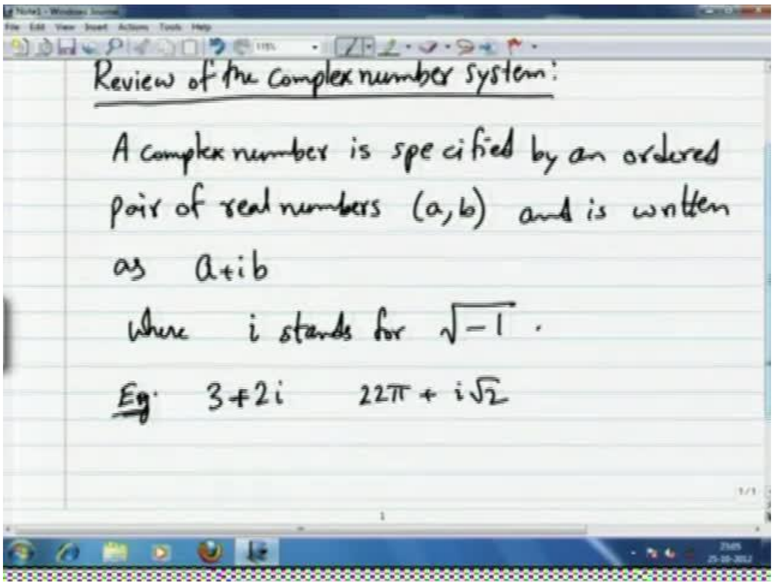
Complex Analysis
Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 1
The Arithmetic, Geometric and Topological Properties of the Complex Numbers
Lecture - 1
Introduction to Complex Numbers

Hello viewers, we will begin this course with a review of the complex number system and then we will proceed to study the geometry, and and then the topology of the complex plane from where we will start the study of complex functions which constitutes the study of complex analysis.

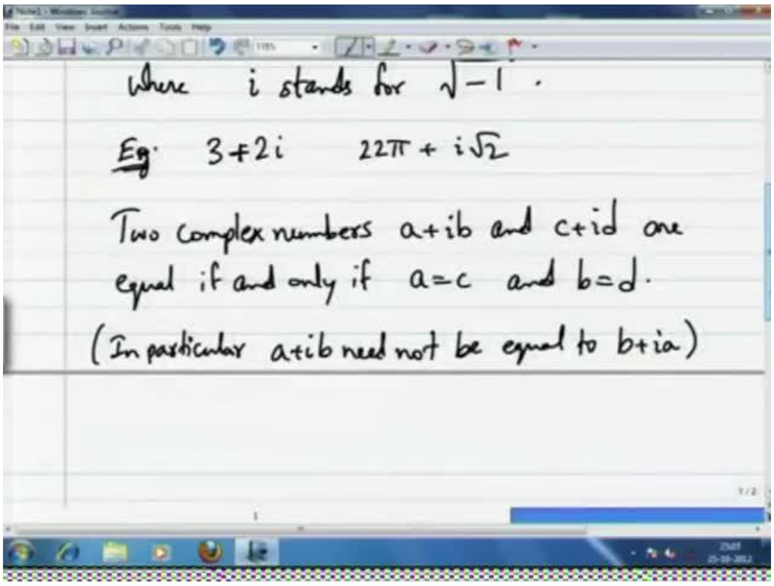
(Refer Slide Time: 00:54)



So, firstly let us review the complex number system. So, here is a review of the complex number system. So, in here what we are going to do is take an uncritical approach to the complex number system that is we are going to assume that the complex number system is given to us, and we will review the mechanics of how to multiply two complex numbers add them etcetera, and no subtract them and divide them etcetera. Then we will see some properties of modules and the conjugation of a complex number, etcetera.

So, let me first start with a complex number. So, a complex number is specified by a complex number is specified by and order pair of real numbers a comma b. So, an order pair a comma b gives rise to a complex number a plus i b. And is written as a plus i b, where a this i here stands for the square root of minus 1. So, i stands for the square root of minus 1. So, the complex number system provides for a an entity, which a acts as the square root of minus 1, which is not available in the real number system and that we denote by i, and given an ordered pair of real numbers we talk of a complex number a plus i b.

(Refer Slide Time: 03:30)

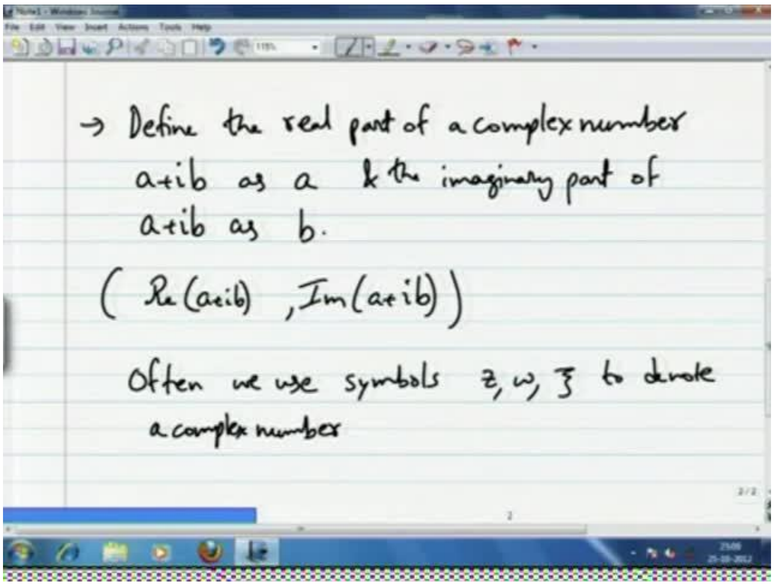


So, we can, I mean we can readily take any example like that 3 plus 2 i is a complex numbers and 22 pi plus i root 2 is also a complex number. So, such things we will call as complex numbers and two complex numbers are equal a plus i b. c plus i d are equal, if and only if a is equal to c and b is equal to d. So, and so in particular a plus i b need not be equal to a plus or the b plus i a, and that is why the order in which, this pair is given this pair a b is given is important. Hence, we said that a complex number is specified by an order pair, okay? So if a and b are not equal a plus i b is not going to be equal to e plus i a by this statement.

So, once again I am not going to give any justification. Here we are only doing the review of the complex number system. So, all I will pause here to mention that a good reference to critical approach, as to why this rules for complex number arithmetic arouse.

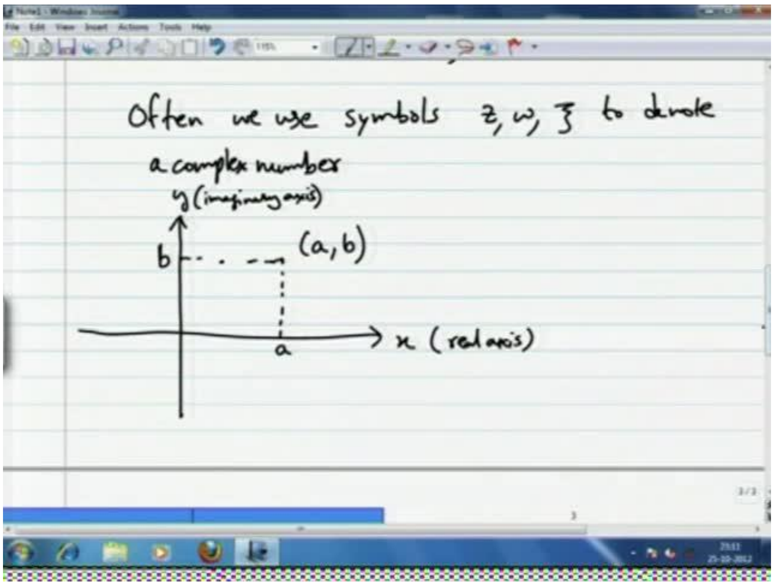
I will refer the viewer to book by Tristan Needham visualize complex analysis by Tristan Needham is a very good reference. So, there historical perspective is presented, where the evolution of this arithmetic of complex number is tressed and also a justification for y the complex number system has a reason is provided.

(Refer Slide Time: 05:56)



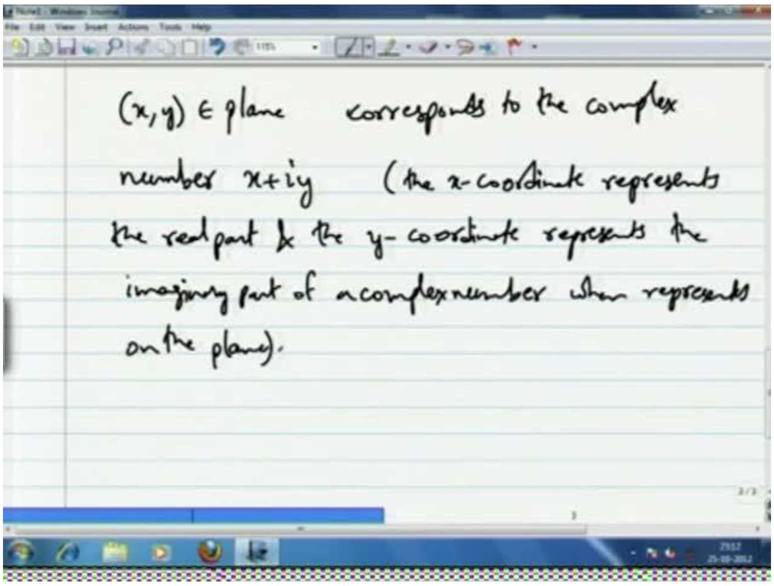
So, now we will owing to this earlier fact here. What we can do is that we can define the real part of a complex number a plus i b as a and the imaginary part of a plus i b as b. So, often they are abbreviated as a real part of a plus i b and a imaginary part i m of a plus i b. So, those are a and b, so this definitions are ambiguous because of the fact that a plus i b and c plus i d are equal, if and only if a is equal to c and b is equal to d. Often we use symbols like or alphabet z, w, zeta etcetera to denote complex number. So, that so just symbol and then what we can also do is, we can represent since a complex number is specified by pair of real numbers a comma b, we can write also represent the complex numbers on the Cartesian plane on the x y plane like that.

(Refer Slide Time: 07:52)



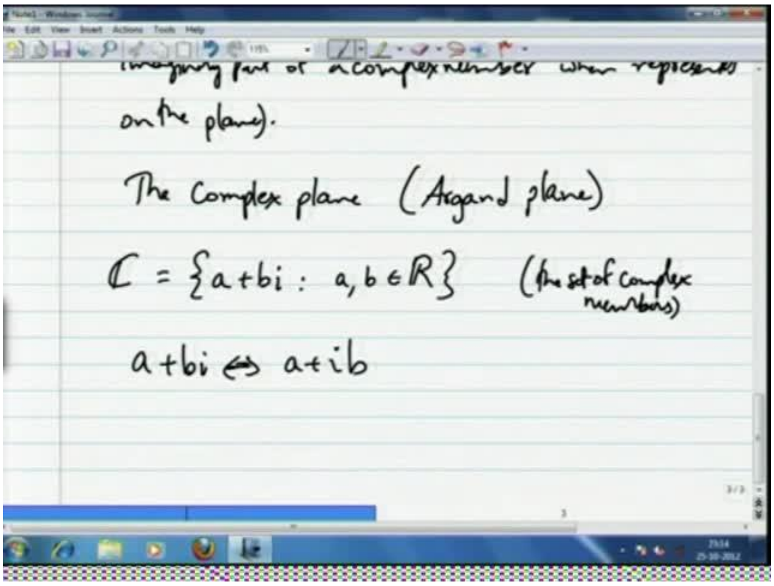
So, if this is the x y plane an order pair a b for the time being let us assume a and b are positive real numbers. So, the point a comma b falls in the first quadrant and this height is your b. Then this distance from the y axis is a, so this is the point a on the x axis and this is the point on the b y axis. So, a is the real part of a plus i b and b is the imaginary part. So, owing into this fact, this is called the real axis and this is called the imaginary axis. So the y co-ordinate gives the imaginary part.

(Refer Slide Time: 08:57)



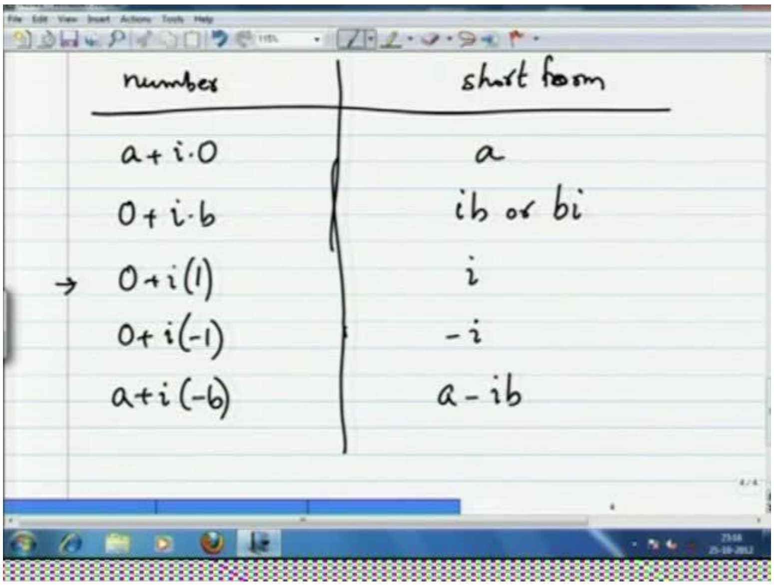
So, if x x comma y, so x comma y belongs to the plane corresponds to the complex number x plus i y. So, the x co-ordinate the x co-ordinate represents the real part. The y co-ordinate represents the imaginary part. The imaginary part of a complex number, when represented the on the plane and we will call this plane as the complex plane. So, we will call this plane as complex plane of an…

(Refer Slide Time: 10:17)



This was introduced by the mathematician Argand and this is also called the Argand diagram or Argand plane. We often denote the set of all complex numbers by this kind of C. C is set of all complex numbers it is the set of all a plus b I, such that a comma b are real numbers. So, and since and there is no repetition writing a plus b i a comma b belongs to R because a plus b i is equal to c plus b i if and only if a equals to c and b equal to d. So, that is the set of complex numbers, the set of all complex numbers. So, please note that we write a complex as a plus b i. I also write it as a plus i b, so one at the same. I interchangeably use a plus b i and a plus i b.

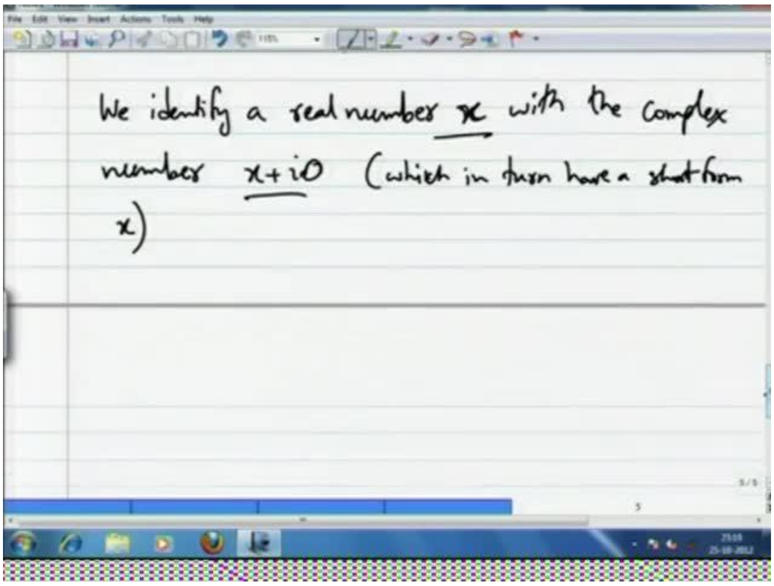
(Refer Slide Time: 11:48)



So, to this and what we will do is we will make the following table complex number and we will use many short forms from time to time. So, here is small table, where I will write its complex number and its short form. So, number of the form a plus i times 0 is often written just a. So, if I just write a real number a, and call it as a complex number I am referring to the complex number a plus i times 0. This is the short form and a number of the form 0 plus i times b is often written as i b or b i like said I am going to write i b or b i in that form.

Then the 0 plus i times, one is often written as i simply, this is the specific complex number. So, this is not number of form, but a specific complex number, I write it as i. 0 plus i minus 1 as minus i and a number of the form a plus i times minus b is often written as a minus i b. By that I mean a plus i times minus b with i mean b is that is for the time being assumed it is positive real number. Then I write this way and then so this short forms should be noted and we will use them from time to time. So, one should know what the short form represents, okay?

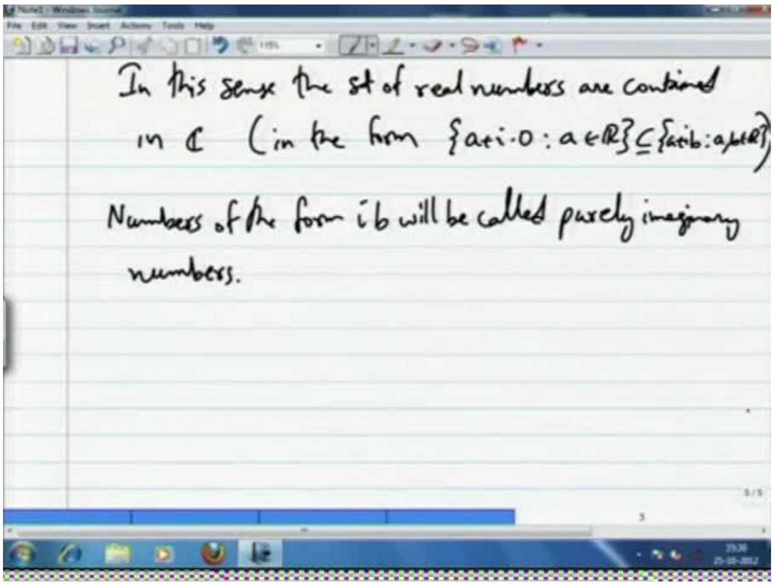
(Refer Slide Time: 13:36)



So we identify real number x with the complex number x plus i times 0. So, for the time being this plus and dot are only the symbols. So, far I have not made any sense of it. I only told that at a complex number is represented as a plus i b. So, they are just symbols for the time being. Later we will introduce operations on, will review the operations on the complex numbers, namely complex multiplication and complex addition and then we will see that, this is indeed are the complex addition and the multiplication. So, for now it is just symbol and then x plus i times 0 will represent a real number x and we identify the real numbers with this kind of complex numbers, which intern have a short form x.

So, note the short form and number of the form x plus i tan 0 is written as x. So, notice that the real numbers are are not technically the complex numbers x plus i times 0, but we identify the real numbers as been contains in the complex numbers. This identification is faithful in the sense that, later when we introduced the operations of complex numbers addition and complex number multiplication, we see that the real number multiplication and the real number addition, tally with the complex number multiplication complex number addition, when we consider numbers of the form x plus i times 0. So, in that senses the representations of real number x as the complex numbers x plus i times 0 is faithful. But for now we identify this numbers that way.

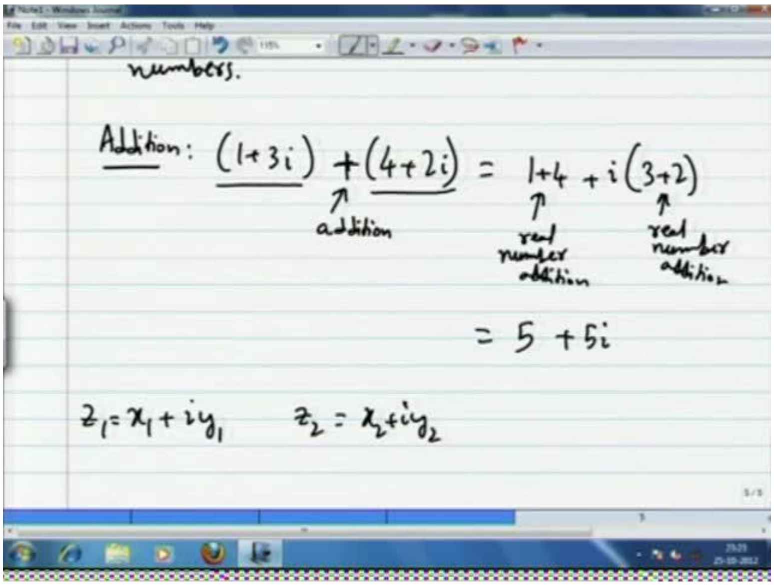
(Refer Slide Time: 16:03)



Also in this sense the the set of real numbers are contained in C, in the form in the form of the set e plus i times 0 like I pointed out, such that a belong R. Contained in set of all a plus i b, such that a comma b belong to this is the set of complex numbers. Then numbers of the form i b will be called purely imaginary numbers. The numbers of the following form 0 plus i b are denoted by i b or b i and we will call them purely imaginary numbers, because there is no real part to this numbers. So, I want to once again pause here to mention that, there is nothing imaginary about complex numbers.

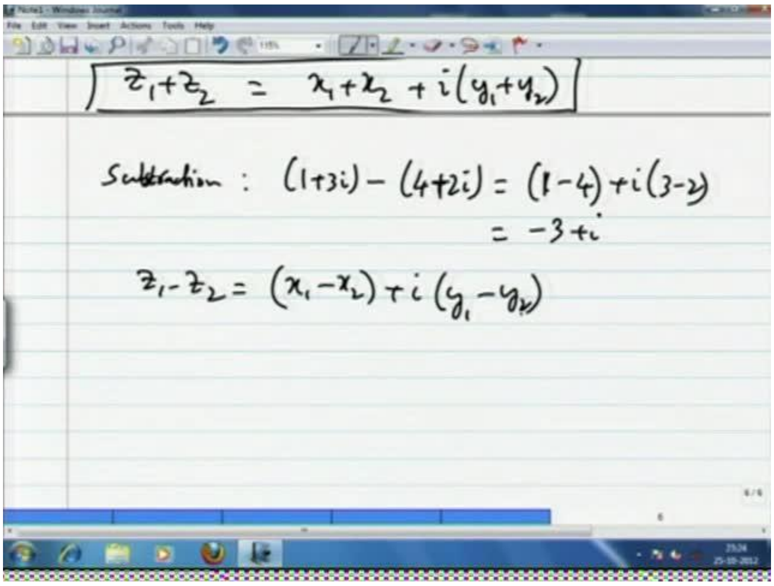
So, it is only that history as given the name imaginary to them. Then that name somehow got stuck, but there is nothing imaginary, they are as concrete as real numbers. There is a definite arithmetic to it, there is defiant rules of arithmetic. Then they have further properties as we will see. So, that you should remember is only a historical thing, so that that imaginary numbers is just a historical things and numbers of the form i b will be called a purely imaginary.

(Refer Slide Time: 18:13)



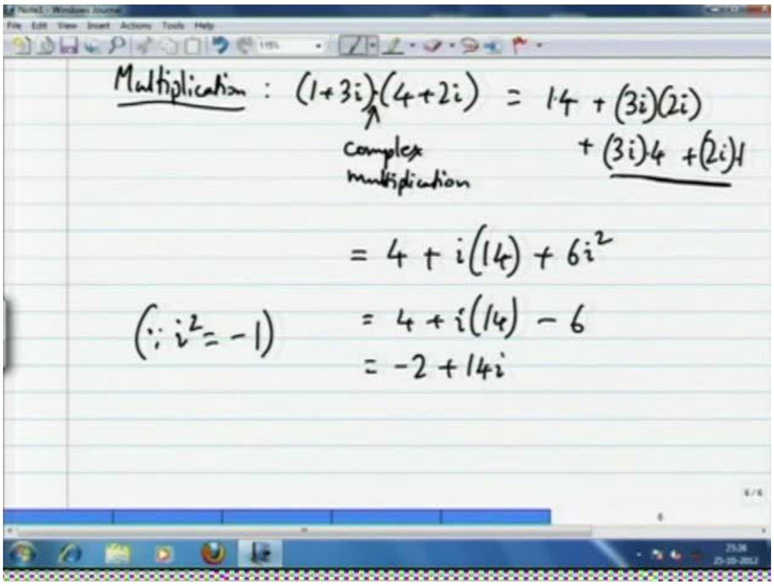
Then, let us review the rules of addition complex number, addition, multiplication etcetera. So, addition or subtraction is as follows. A number 1 plus 3 i for example, can be added to number 4 plus 2 i as follows. So, this is the complex numbers addition, addition we are adding a complex number 1 plus 3 i to the complex number 4 plus 2 i. In the in the intuitive way we will add 1 plus 4, this is real number addition because the real part is real number. Then plus i times, so once again this, time is is an artificial thing, it is just a notation for now. Plus i times 3 plus 2, so you add the imaginary parts, which are once again real numbers. So, this addition is real number addition, to get 5 plus 5 i or i i times 5.

(Refer Slide Time: 19:51)



So, that is the addition you add x 1 and number complex number z 1 equal x 1 plus i y 1 two complex number x 2 plus i y 2 to get x 1 plus x 2 plus i times y 1 plus y 2, that is the rule for the addition. Then subtraction is similar subtraction is similar 1 plus 3 i minus 4 plus 2 i is a 1 minus 4. You subtract in the proper order, one real number from the other plus i times 3 minus 2, which gives you minus 3 plus i. So, z 1 minus z 2 is x 1 minus x 2 plus i times y 1 minus y 2 and there is complex number multiplication.

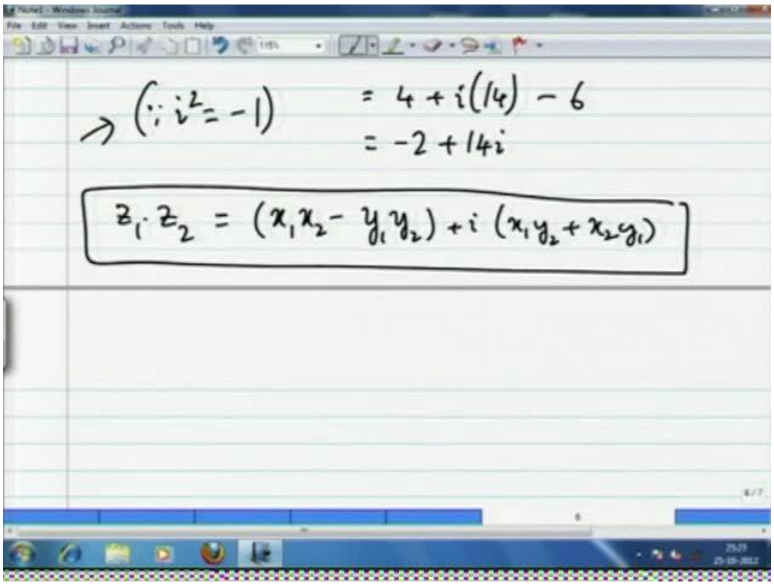
(Refer Slide Time: 21:03)



You can multiply two complex numbers z 1 and z 2 in the following manner. So, normally if you treat this 2 as binomials to complex numbers, 1 3 plus 2 i times 4 plus 2 i. So, that is the complex multiplication, that is 1 times 4 plus 3 i times 2 i plus 3 i times 4 plus 2 i times 1. So, this is like binomial multiplications and then this interns, if we treat this is as real numbers and multiply 1 times 4 gives us 4. Then plus, let us collect this this two terms here, we get i times intuitively.

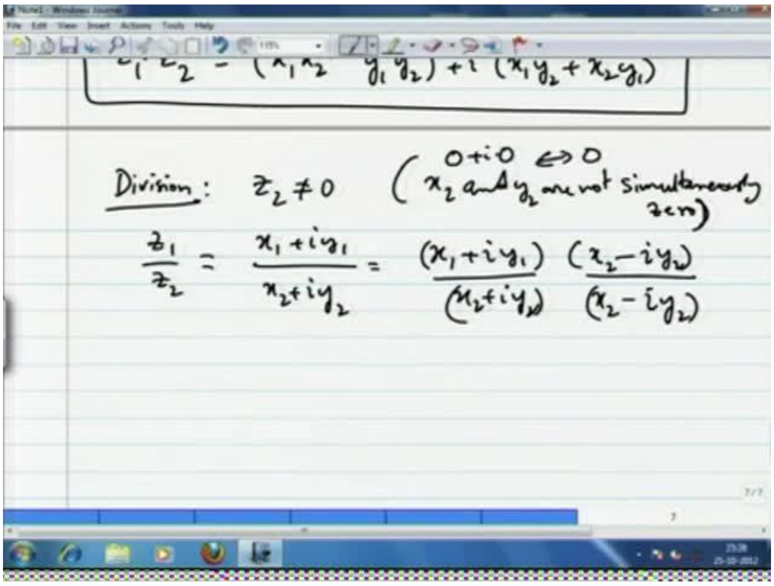
What we should do is, well at least by binomial multiplication, what we should do is this, 12 plus 2 namely 14 plus 3 i times 2 i should give us 6 i square, but i was the entity, which was the square root of minus 1. So, i square should be minus 1, so using i square equal to minus 1, what we get is… Since, i square is minus 1 we get 4 plus i times 14 and then substituting i square, it is minus 1 we get minus 6 here. So, this should be minus 2 plus 14 i, so yes.

(Refer Slide Time: 23:20)



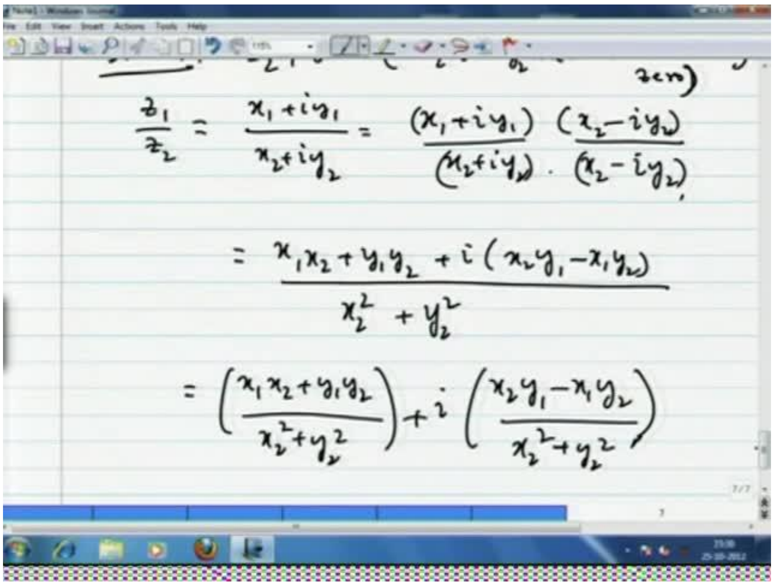
So, using this what we will do is, we will multiply two complex numbers z 1 and z 2 as x 1 x 2 minus y 1 y 2, where z 1 is like this, z 1 is x 1 plus y 1 z 2 is x 2 plus y by 2. So, we have the following rule x 1 x 2 minus y 1 y 2 plus i times plus x 1 y 2 plus x 2 y 1, so that is your complex number multiplication.

(Refer Slide Time: 24:01)



So, then we can also divide complex numbers division, so what we can do is, if we have z 2 is not 0, so we can divide only. When we have z 2 not 0 z 1 by z 2 is x 1 plus i y 1 divided by x 2 plus i y 2 is equal to well z 2 is not 0. Recall this I mean, what this means is neither x 2 and y 2 are not simultaneously 0, x 2 and y 0 are not simultaneously 0. One of them can be, but not both because the complex number 0 is 0 plus i times 0, which we are abbreviating as 0. Then x 1 plus i y 1 divided by x 2 plus i y 2 can be re-written as x 1 plus i y 1 times. Well you will see in a moment why I am doing the following, I will multiply by x 2 minus y 2 in the numerator and the denominator, okay.

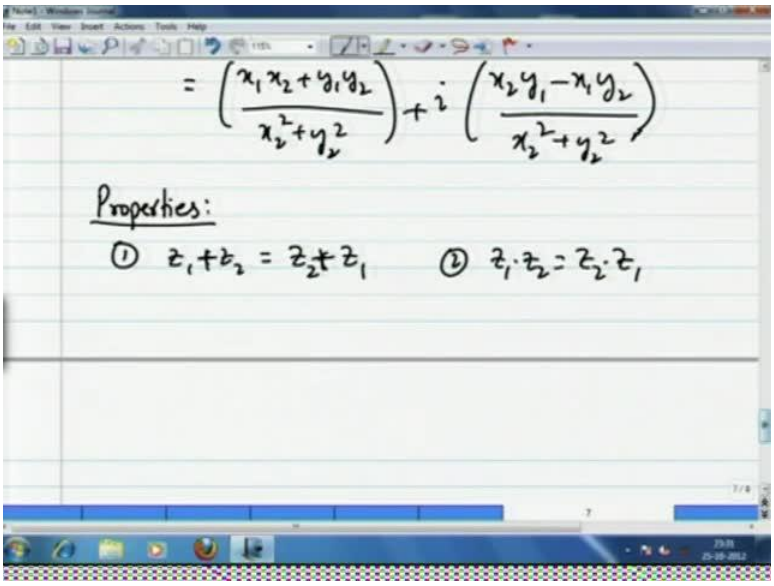
(Refer Slide Time: 25:31)



So, this is this gives me x 1 x 2 plus y 1 y 2 plus i times x 1 sorry, plus i times x 2 y 1 minus x 1 y 2 and in the denominator I get x 2 square. Then minus x 2 y 2 i n plus x 2 y 2 cancel, so I get plus and i times minus i gives me plus 1, so I get plus y 2 square. So, using complex number multiplication on this is 2 and when I multiply them I get x 2 square plus y 2 square. So, then we can divide this into real and imaginary parts x 1 x 2 plus y 1 y 2 divided by x 2 square plus y 2 square plus i times x 2 y 1 minus x 1 y 2 divided by x 2 square plus y 2 square.

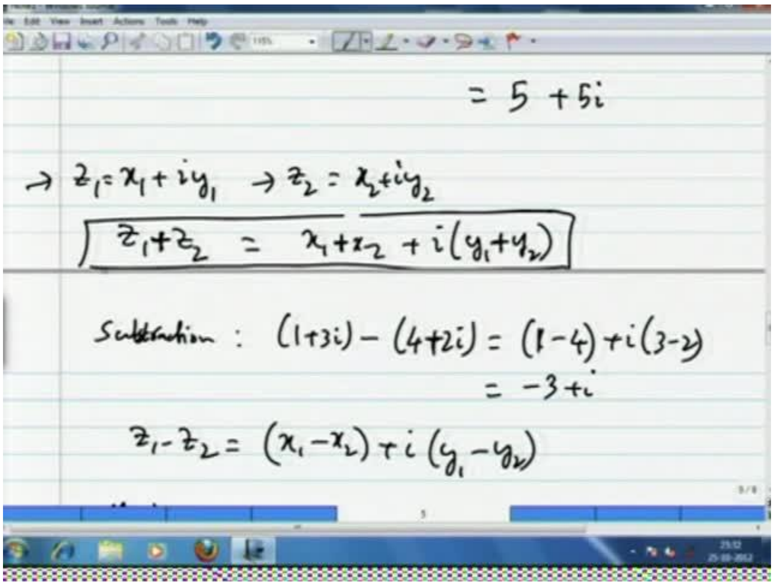
So, we we are now able to write z 1 by z 2 in the form a plus i b by using this rule, by using this process. So, notice that z 1 minus z 2 and z 2 is not 0 is already a complex number. There is a nothing more 1 is to do. All we are trying to do is put it in the form a plus b i which is how we said that, we are going to represent the complex number. So, this is your rule for division of 2 of a complex number z 1 by non zero complex number z 2. So, this are the rules for addition, subtraction, multiplication and division and they have the following properties.

(Refer Slide Time: 27:37)



They follow the following properties, which might be familiar to the viewer from an earlier course on complex numbers. So, firstly z 1 plus z 2 is equal to z 2 plus z 1 and likewise second z 1 times z 2 is equal to z 2 times z 1. So, yes so this properties this two properties are called commutatively, so the commutative property of addition and multiplication so let say that.

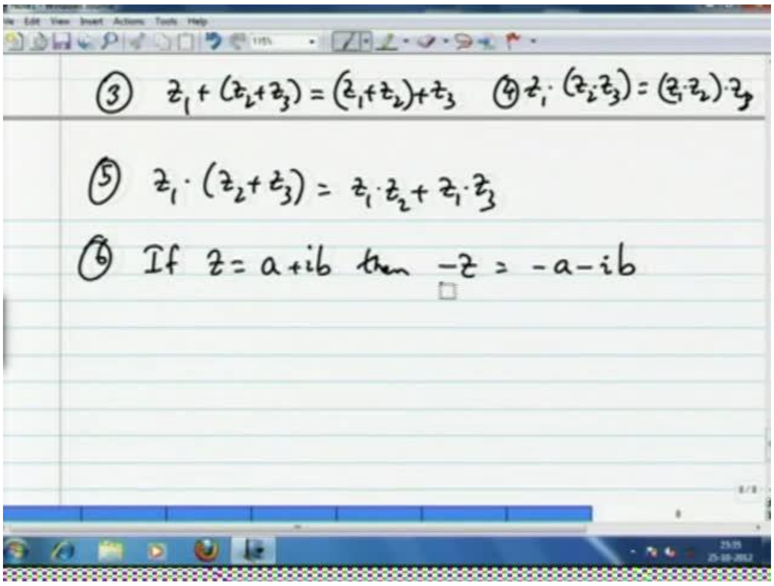
(Refer Slide Time: 28:29)



So, once again let me let me go back to this rules and point out to the viewer that, this addition etcetera I will not point out, but put point at there, but this addition which appears here or this addition, which appears here which which takes the rule are real number addition, addition which the viewer is already familiar with. This addition here is telling you how to add the complex number z 1 and z 2. And likewise this multiplication here is the new complex numbers multiplication.

Then this multiplication here or this multiplication here or this subtraction etcetera, this multiplication here, this addition this multiplication there, are all a real number multiplication, additions and subtractions, which the viewer is already a familiar with real numbers. So, we are trying to define a complex number multiplication in terms of real numbers addition, multiplication, subtraction, etcetera. Likewise for divisions, the operations which appear here in this here, within the real part or the imaginary part of z 1 by z 2 are real number operations, which one is already familiar with, so having said that, these properties pertain to complex number addition and multiplication. So, so there are there are both, the complex number addition, multiplication are commutative.

(Refer Slide Time: 30:02)

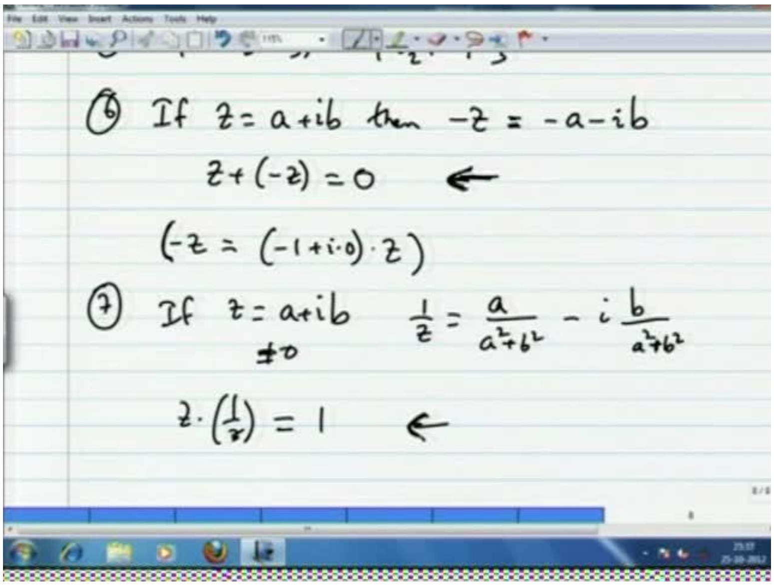


Secondly, 3 z 1 plus z 2 plus z 3, so if you add z 2 to z 3 and then add z 1 to it, it is the same as adding z 1 and z 2 first and then adding z 3 to it. So, is the case for multiplication, if you multiply z 2 and z 3 first and then multiply z 1 to it, it gives the same result as multiplying z 1 and z 2 first and then multiplying z 3 and so this kind so 3 and 4 are called associative properties for for addition and multiplication respectively 5 z

1 times z 2 plus z 3. So, multiplication distributes through addition z 1 times z 2 plus z 3 is the same as z 1 time z 2 plus z 1 times z 3.

Notice that addition does not distribute over multiplication. So, so it is not true that z 1 plus z 2 times z 3 is z 1 plus z 2 times z 1 plus z 3. So, this is multiplication distributing over a addition and then if z plus z is a plus i b or x plus i y. Let us say a plus i b, then minus z represent a complex numbers. So, this is a notation minus z stands for the complex number minus a minus i b.

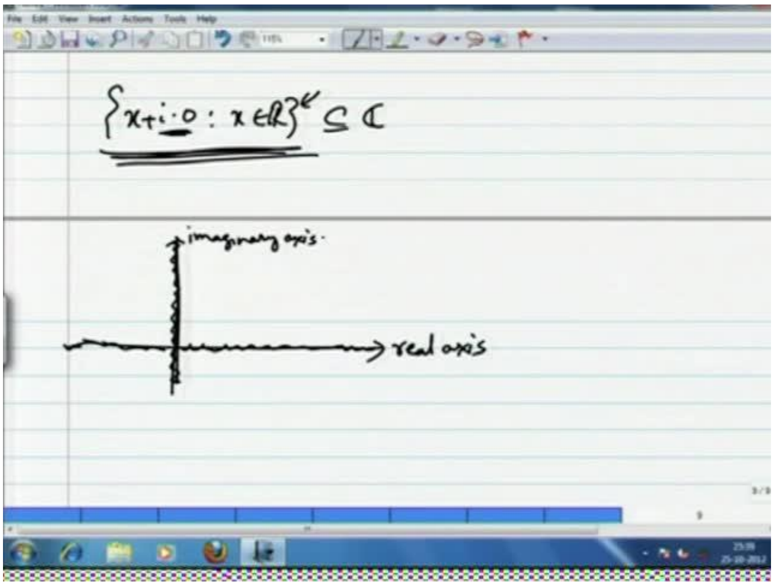
(Refer Slide Time: 31:59)



So, then z plus minus z, notice is a 0 and it is true that z minus z is actually minus 1 a times in the complex number minus 1. So, it is minus 1 plus i times 0 times the complex number z. So, that minus z although I said is denoted by e plus minus a minus i b, its stands for the complex number minus a minus i b. It is true that minus z is minus 1 times z. Then seventh one, if if z is a plus i b, then 1 by z is your complex number. a by a square plus b square well you want z non zero, then 1 by z is the complex number a by a square plus b square minus i times b by b square plus a square plus b square.

So, it is the division of the complex number 1 by the complex number z and then z times 1 by z is equal to 1. So, this so there are, this are inverse properties, there is an additive inverse and then there is a multiplicative inverse to complex a non non zero complex numbers. So, additive inverse exists for any complex number and then multiplicative inverse exists for non zero complex numbers and you see that with this properties and with this rules for addition, multiplication and subtraction, division that when you when you restrict this operations to numbers of the form x plus i times 0, such that x belongs to R like I remarked earlier.

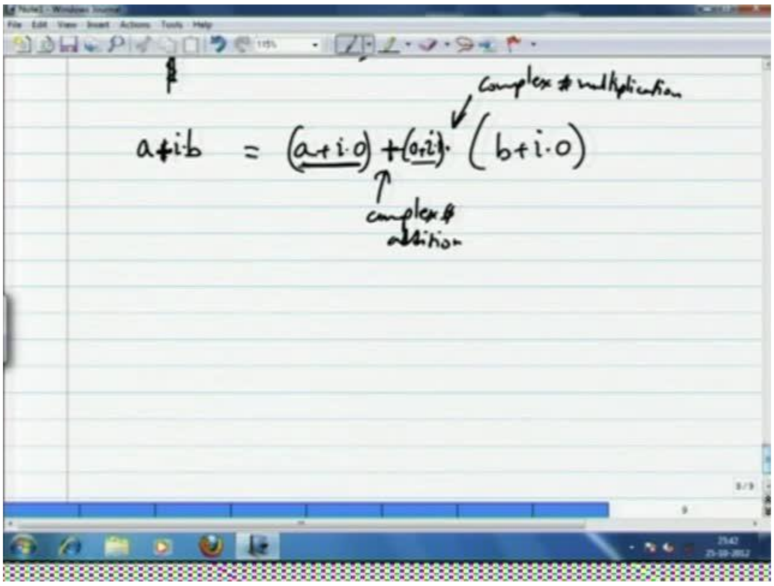
(Refer Slide Time: 33:56)



When you restrict this operations to this set contained in the complex number, you get all you retrieve, all your real number operations. So, this set indeed faithfully represents your real numbers contained in the complex number set. So, so going to that fact you call this numbers notice are represented on the x axis, right? This all are numbers whose imaginary part is 0, so this represented on the x axis in the complex plane.

So, this is called as the real axis sometimes because the numbers on it the complex numbers on the x axis, represent the real numbers contained in the complex numbers. The purely imaginary numbers are on the y axis in the x y plane. So, this is this axis, is sometimes called the imaginary axis because the numbers on it are purely imaginary and now notice another thing.

(Refer Slide Time: 35:36)

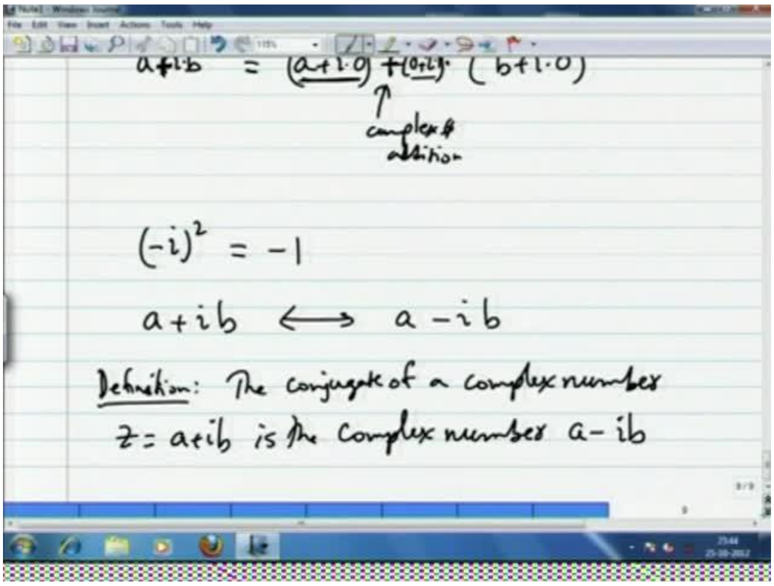


Another thing about this multiplication that this rules about addition and multiplication that a number a plus i b, we have started by representing a complex number in this fashion. This is indeed the complex number a, which is the i mean plus i timers 0 plus a complex number i times the complex number b plus i times 0. So, now this is a symbolism, it is the representation of the complex number in the form a plus i b and this is the genuine plus.

Now, this is the complex number addition and this is the complex number multiplication, when you multiply the complex number I, which is 0 plus i times 1 that is again a notation and you multiply that to the complex number b plus i times 0 complex number multiplication. Then add it with the complex number rules to a plus i 0 you get back your complex number i a plus i times b.

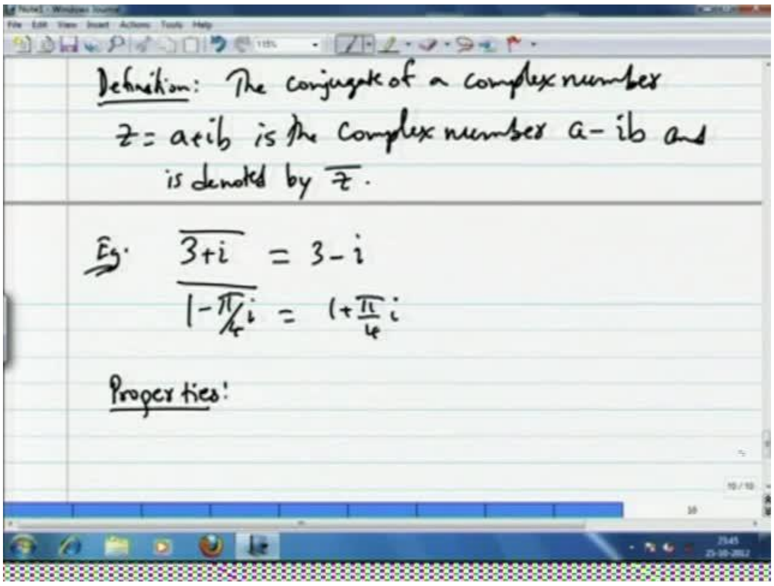
So, in that sense this plus and the a dot represent your complex addition and multiplication. And from now on, we will suppress, you know this being a symbol. We will consider them as genuine complex number, addition and multiplication in this. So, now we are going to look at the conjugation of a complex number. So, notice that a plus i b well notice first that minus i square is also equal to minus 1.

(Refer Slide Time: 37:49)



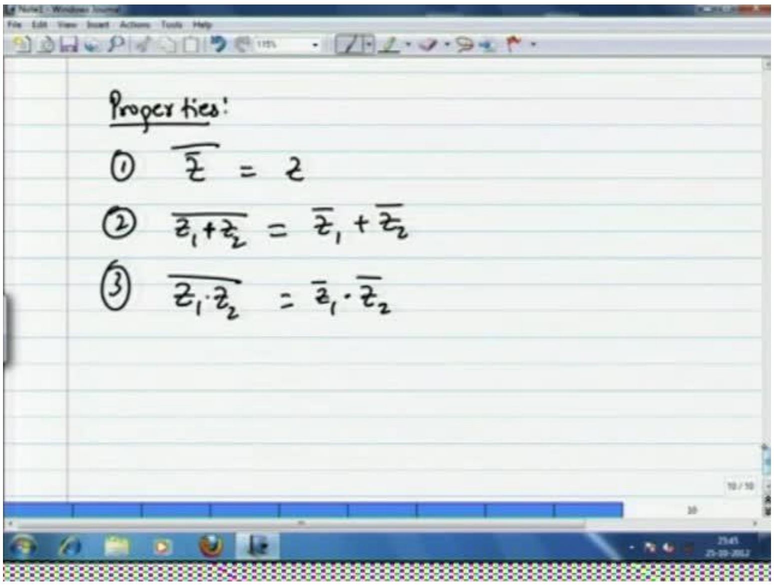
So, minus i could have equally done the job as hided. So, with that as a motivation we will look at a plus i b transforming itself as a minus i b. So, we could have started of by a parallel definition or specification of a complex number as a minus i b instead of a plus i b. So, going to that motivation what we will do is, we define.

(Refer Slide Time: 38:56)



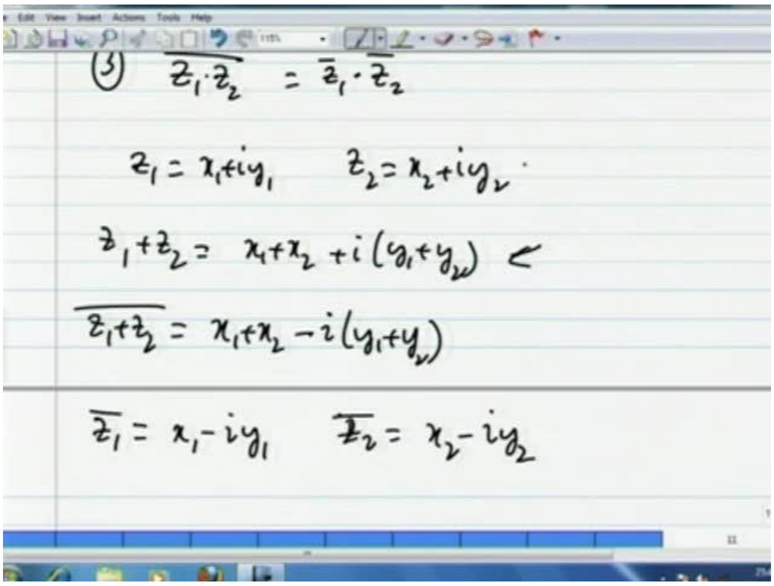
So, definition the conjugate of a complex number z equals a plus i b is the complex numbers a minus i b and and is denoted by z bar. So, z given a complex number z, z bar stands for the conjugate of the complex number. For example, given a complex number, all you have to do to find its conjugate, is just reverse the sign on the imaginary part. So, 3 minus i and then 1 plus or 1 minus pi by 4 i conjugate is just 1 plus pi by 4 i and we will see some properties of this conjugation.

(Refer Slide Time: 39:45)



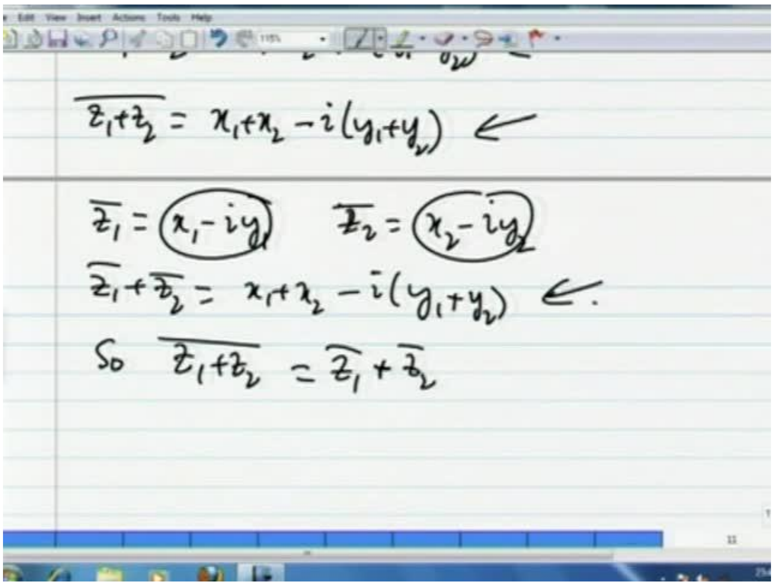
So, firstly the conjugate of a conjugate is the complex number itself. So, that is because if you reverse the sign on the imaginary part twice, you get back the complex number and the second property is that conjugation distribute over addition. So, z 1 plus z 2 bar is z 1 bar plus sorry z 2 bar. Third, z 1 z 2 bar the conjugate, if you multiply to complex numbers and then you take the conjugate, it is the same as taking the conjugate of the complex numbers z 1 and z 2 first and then multiplying them. So, the viewer is encouraged to verify these properties by by computing the left hand side and right hand side in terms of x 1 y 1 and x 2 and y 2.

(Refer Slide Time: 40:51)



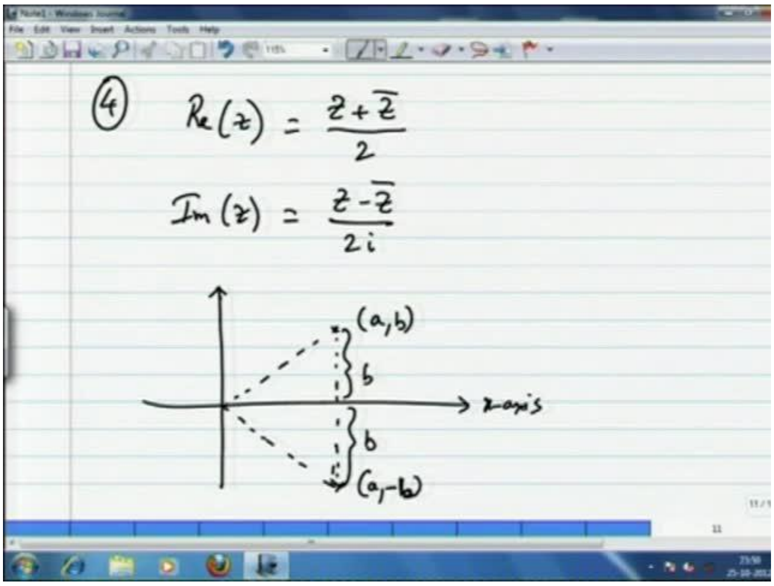
So, let me take z 1 is x 1 plus i y 1 can direct to equals x 2 plus i y 2. So, z 1 plus z 2 is x 1 plus x 2 plus i times y 1 plus y 2. So, that is the complex number addition rule. So, z 1 plus z 2 bar is going to be conjugate of this complex number, which is the x 1 plus x 2 minus i times y 1 plus y 2. Then z 1 bar is z 1 is x 1 plus i y 1 so z 1 bar x 1 minus i y 1 and z 2 bar is x 2 minus i y 2.

(Refer Slide Time: 41:42)



So, z 1 bar plus z 2 bar is the addition of this two complex numbers, which gives you x 1 plus x 2 minus i times y 1 plus y 2. It is plus i times minus y 1 plus minus y 2, which is the same as minus i times y 1 plus y 2. So, now you see that this tally is with this. So, z 1 plus z 2 bar is equal to z 1 bar plus z 2 bar. So, its likewise routine to verify 3 and 1. So, you can just take that 1 as z 1 as x 1 plus i y 1 z two is x 2 plus i y 2 and performed operations in the proper order and verify 1 and 3.

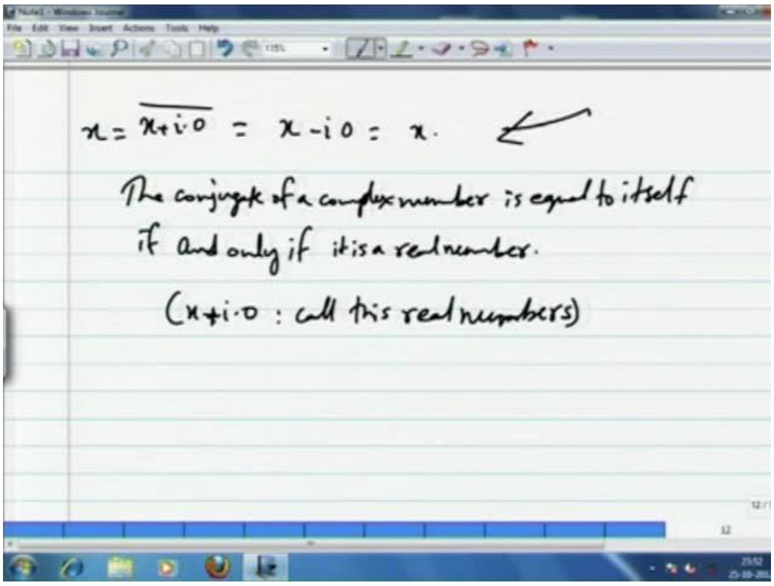
(Refer Slide Time: 42:32)



So, the real part of complex number z is equal to the sum of the complex number with its conjugate divided by 2. So, that gives you the real part and likewise the imaginary part of a complex number z is z minus z bar by 2 i. So, you can retrieve the real and imaginary parts of a complex number from a from a complex number and its complex number and its conjugate form and and on the complex plane, this conjugate is just the reversal of the sign on the imaginary part. So, if you take a complex number a comma b, then the complex number a comma minus b lies, right here.

So, it lies or it is the reflection of the complex number a comma b via on the x axis. So, when you reflect at the point a b through the x axis, so this distance off course b and this depth. Now, is b this is minus, so this distances are equal and you have a comma minus b. So, that is the representation of conjugation in the complex plane. So, that is telling you that well it is also clear from the definition that if you reflect a point on the x axis through the x axis, you get backup the very same point.

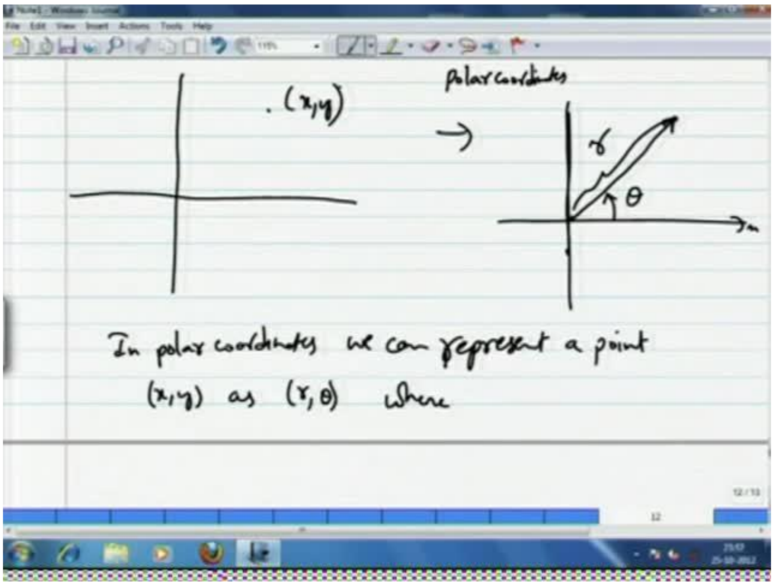
(Refer Slide Time: 44:35)



So, the conjugation of a real number in the complex plane is the is the real number itself. So, the conjugate of x plus i times 0 is x itself is x minus i times 0, which is x. So, conjugate of a complex number is equal to itself, if and only if it is a real number. By that I mean, it is a real number contained in the complex number system. So, we will often call the numbers of the form x plus i times 0 as real numbers, because we will call this real numbers, that is because we saw that the set of all x plus i times 0 is a faithful representation of real numbers in the complex numbers.

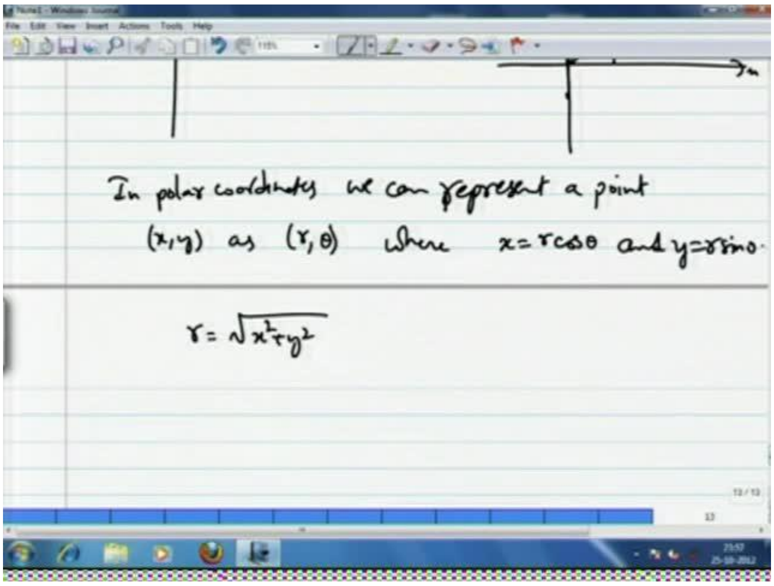
So, from now we are, when we say real number in the context of complex numbers, this is what we need. Then in the conjugate is equal to itself if and only if it is real number. I have shown here, that if you take a real number, it is equal to its conjugate and then here from here, you can infer that if a complex number is equal to its conjugate. So, that imaginary part is 0. What that means is, it is a real number. So, it is an if and only if statement.

(Refer Slide Time: 46:23)

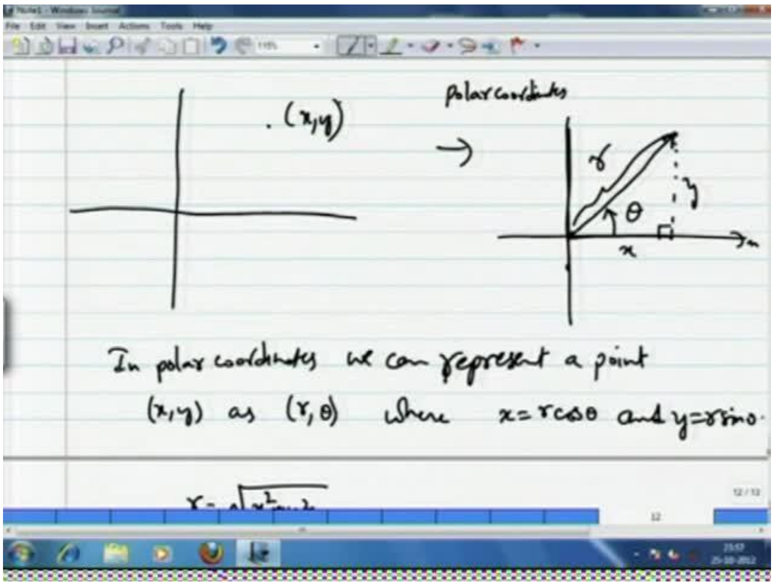


So, now let us look at the complex plane again and let us recall that in co-ordinate geometry, that we can represent points in the plane also by using polar coordinates. So, let say x and y instead of a and b does not matter. So, using polar coordinates we had a representation of this point as r comma theta, where r refer to the length of this vector this is r, which starts at the origin and whether hence at this point x comma y the length of that line segment and theta was the angle of the opening the positive x axis. So, a in polar coordinates, recall we could represent we can represent a point x comma y as r comma theta where, where x and y satisfy the equations x equal r cosine theta and y equal r sine theta or if you want to go from x y to r theta r is given by square root of x square plus y square.

(Refer Slide Time: 47:55)

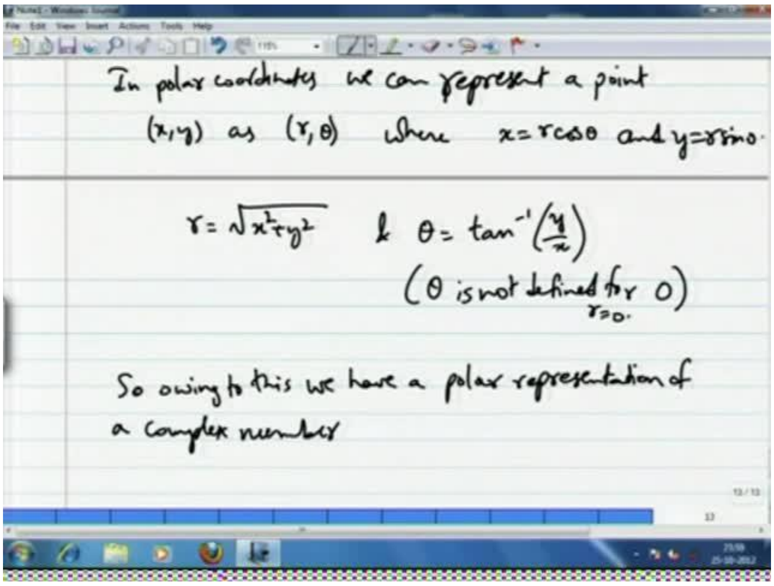


(Refer Slide Time: 48:19)



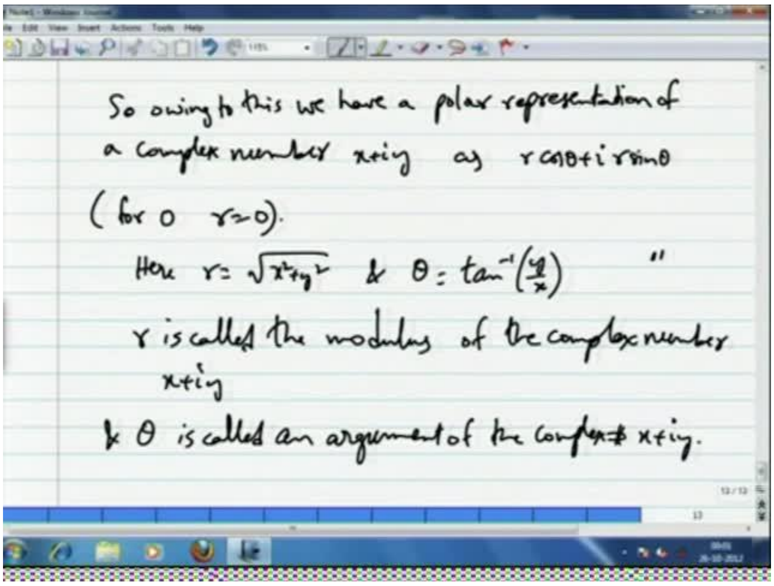
As you can see from here r is the hypotenuse of this right angle triangle, this is height is y and this is x. So, r square is x square plus y square or r is square root of x square plus y square.

(Refer Slide Time: 48:34)



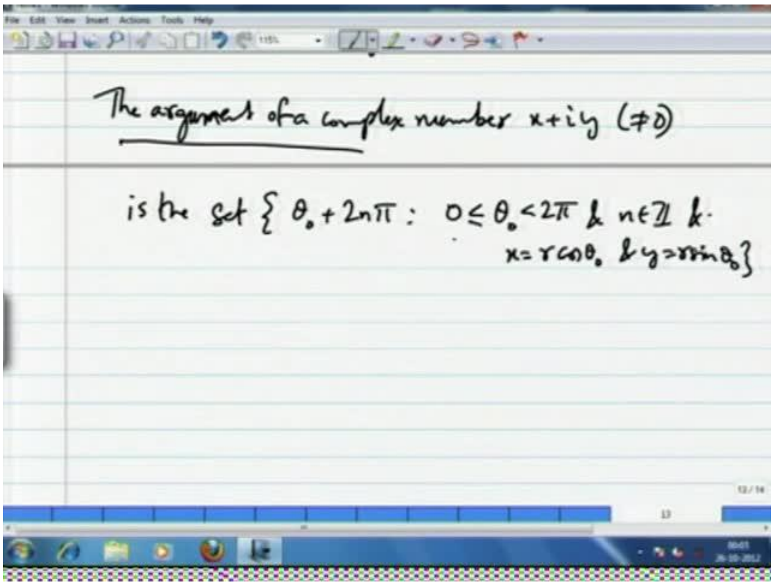
Theta the angle of opening is given by from this right angle triangle, again by tan inverse of y by x and this works more generally even if the point is in in the other quadrants. Even if the point is on is 1, the imaginary axis where x is 0. So, there you have to make the appropriate adjustments and theta is not defined for for a 0 for the complex number 0, there is a region rate triangle. So, this formula would not work there r is a simply 0, that represent and the complex number 0 for non zero complex numbers. We have r is this and theta is tan inverse of y by x except for then x is 0, when x is 0 when x is 0 depending upon the sine of y, you decide whether y is pi by 2 or 3 pi by 2 and that was polar coordinates. Likewise we have a polar representation, so going into this we have a polar representation of a complex number.

(Refer Slide Time: 50:19)



As x plus i y as r cosine theta plus i times r sine theta. For 0, we make an exception we just say r is 0 for 0 r is just 0. Then here r is equal to square root of x square by y square like above of a the complex number x plus i y and theta is called an argument of a the complex number x plus i y. So, notice that there is a certain lake of uniqueness, when you go from the rectangular coordinates to the polar coordinates. So, by that I mean theta can be more, then just one angle because cosine and sine are 2 pi periodic. So, any given any one value of theta between 0 and 2 pi any addition of integral of multiple of 2 pi to the triangle still satisfies x equal r cosine theta and y equals r r sine theta.

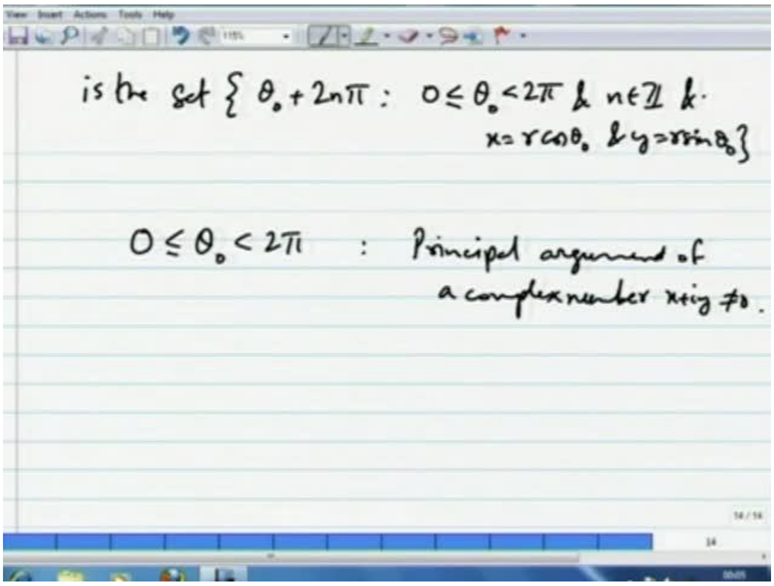
(Refer Slide Time: 52:09)



So, there is a lake of uniqueness, so the argument, so going to this the argument of a complex number x plus i y, we define this only for non zero, not equal to 0, right it in a one way is the set of all theta naught plus 2 n pi, such that theta naught 0 or less than equal to theta naught less than 2 pi. So, I will take the interval 0 comma 2 pi closed at 0 open at 2 pi. And n belongs to z and x equals r cosine theta naught and y equal to r sine theta for some r here. r is an important for now, that we know is already the modules of the complex number, but set of all such theta naught plus 2 n pi will be called the argument of a complex number.

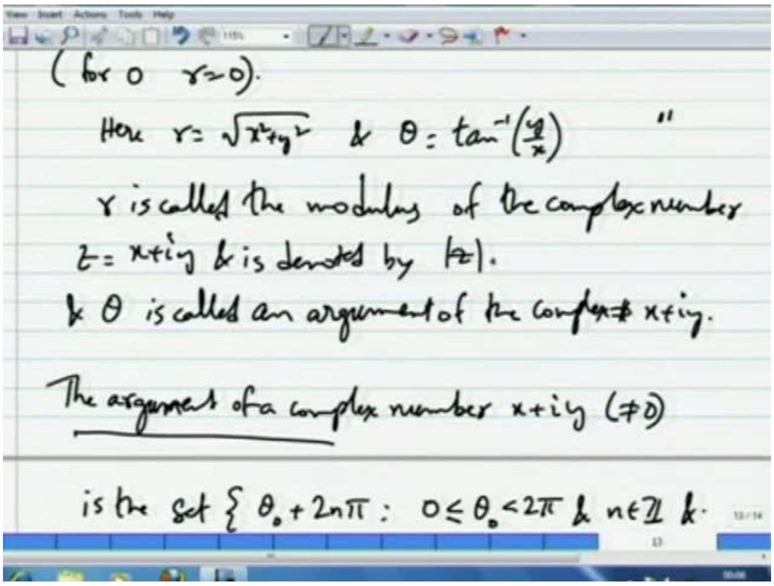
So, generally when we say the argument of a complex number we mean this particular set, which is 2 n pi the addition to the angle between 0 and 2 pi, which can be which can retrieved from this expression tan inverse of an y by x, where the complex number is x plus i y. So, but when we say n argument of complex number we peak one value from the set and that we generate into say is argument of a complex number. Also this many valued ness of this argument, we will see has a deeper consequences. We will see that during the this course.

(Refer Slide Time: 54:22)



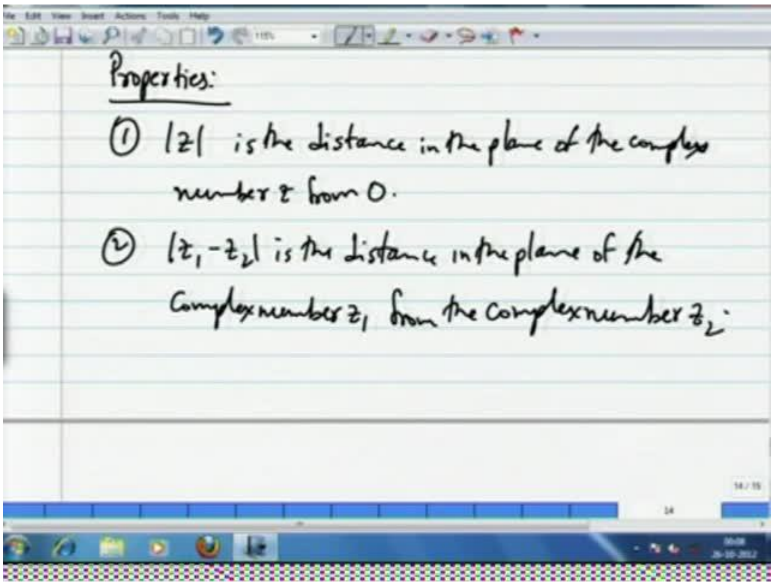
Theta note when restricted to the interval 0 2 pi is commonly referred to as the principle argument of a complex number. x plus i y once again not equal to 0, so for 0 we do not define the argument. Then we will see some properties of the modules, so we will continue the discussion about argument further on this course. There are properties of the modules that we will see. Modules of z has a geometric meaning, it has it is distance of the point a comma b or x comma y from the origin. So, modules of z, I did not probably, I just say r is for the modules of the complex number x plus i y z equal to x plus i y and is denoted by a mode set.

(Refer Slide Time: 55:26)



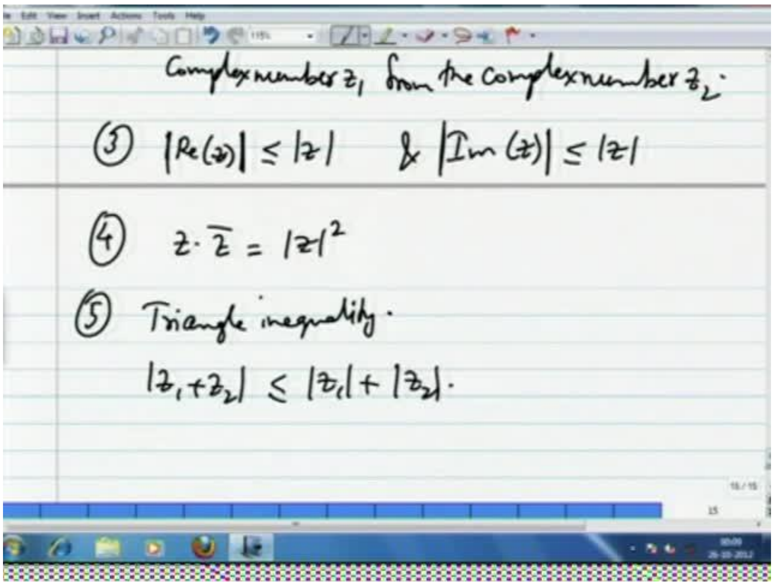
So, it reminds the viewer of the absolute value of a real number and it tallies with the notion of the absolute value of a real number. So, by abuse of notation, by abuse of a terminology, I might sometimes say absolute value of z instead of modules of z. By the time mean the modules looks of z. Then the modules of z satisfies the following properties.

(Refer Slide Time: 56:05)



Modules of z is the distance, of in the plane of the complex number z from the point 0. This is the usual (( )) distance and then the modules of z 1 minus z 2. Then is the distance in the plane of the complex number z 1 from the complex number z 2 or a vice versa.

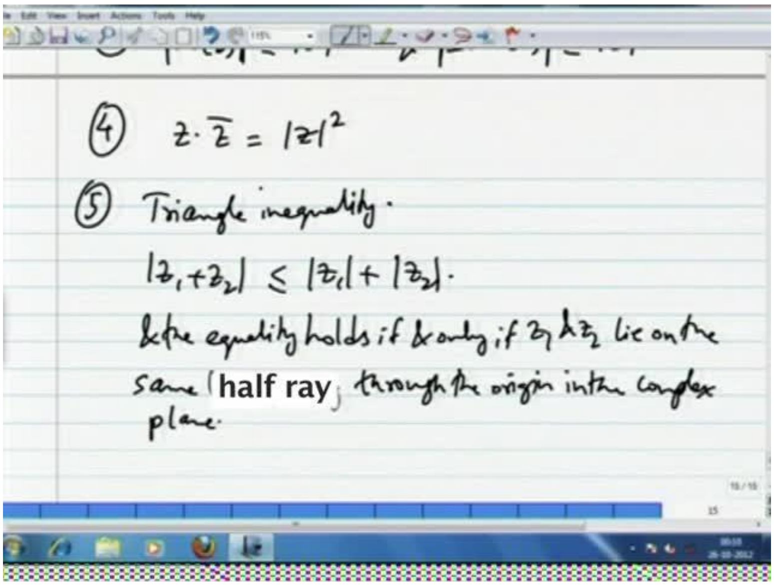
(Refer Slide Time: 57:08)



The modules of real part of z third property is that, this is less than or equal to the modules of z that is clear because the square root of x square plus y square anyway dominates the value x it is a. That is the modules and that is greater than or equal to x itself. So, that is, that and likewise it is also greater than or equal to y. So, the modules the absolute value of imaginary z is less than or equal to modules of z. I apologize this is the absolute value real and imaginary parts of a complex number are real numbers.

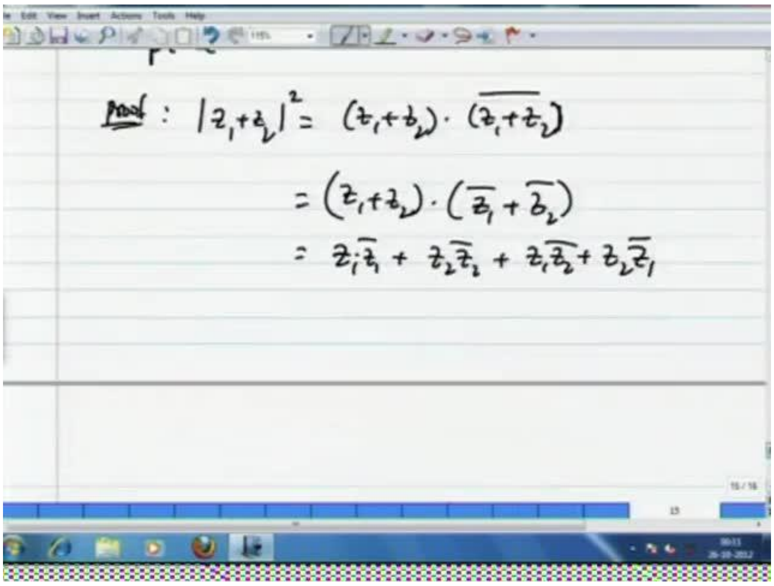
So, the absolute value of real or the imaginary part of z are less than or equal to the modules. Fourth of the properties is that the multiplication of a complex number by its conjugate gives the square of its modules. So, this is an exercise please verify this multiply number with it its conjugate and then try to show that, it is equal to the square of the modules. In the fifth and one important property of modules is the triangle inequality. This says that the modules of z 1 plus z 2 is less than or equal to the modules of z 1 plus the modules of z 2.

(Refer Slide Time: 58:37)



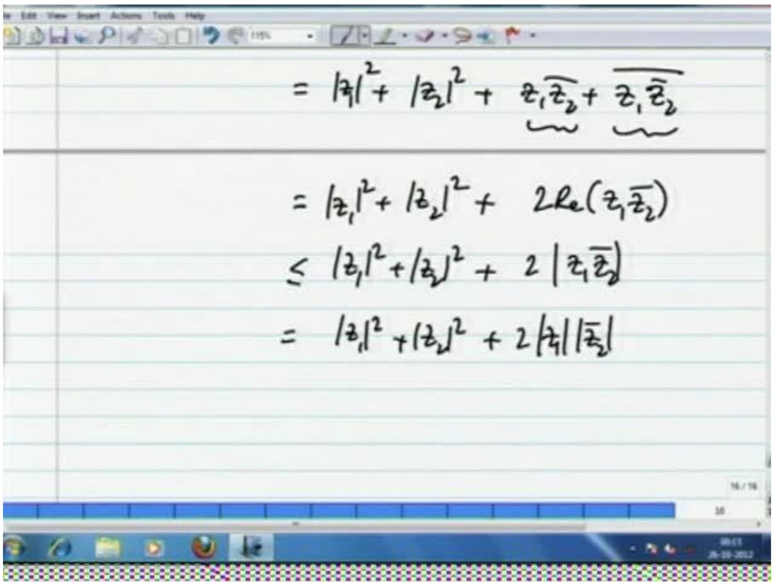
Inequality holds, if and only if z 1 and z 2 lie on the same line passing through the origin in the complex plane in the complex plane in terms of in terms of the argument. This is the z 1 and z 2, the arguments of z 1 and z 2 differ by integral multiple of 2 pi. So, only then the equality holds. So, if and only if statement, so this equality holds here in this equality, if and only if the argument of z 1 and z 2 differ by 2 n pi.

(Refer Slide Time: 59:46)



So, a proof is easy, so proof of this fact is easy. A lonely proof inequality and equality part, I will let it as a exercise to the viewer. The modules of z 1 plus z 2 square is equal to the by the previous property, its equal to the product of the complex number z 1 and z 2 with its conjugate and that gives you the product of z 1 and z 2 times by the properties of the conjugation z 1 bar plus z 2 bar. So, when you multiply this out you get z 1 z 1 bar plus z 2 z 2 bar plus z 1 z 2 bar plus z 2 z 1 bar.

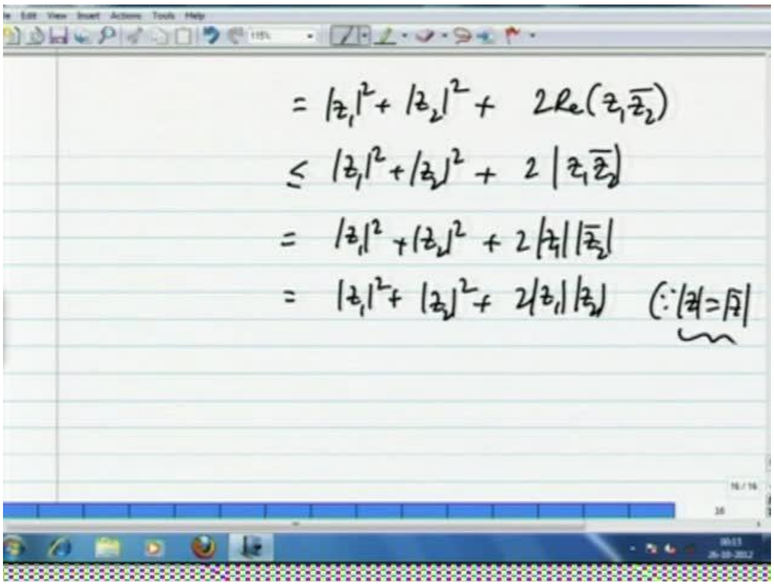
(Refer Slide Time: 01:00:32)



This we know is the modules of z 1 square and this is the modules of z 2 square by the previous properties and then z 1 z 2 bar can be z 2 z 1 bar can be seen as z 1 z 2 bar conjugate. So, I will write that down plus this second term can be seen as z 1 z 2 bar it is the conjugate of this term. So, number plus its conjugate is equal to, the twice the real part that is by the previous properties. So, this is plus twice the real part of the complex number z 1 z 2 bar.

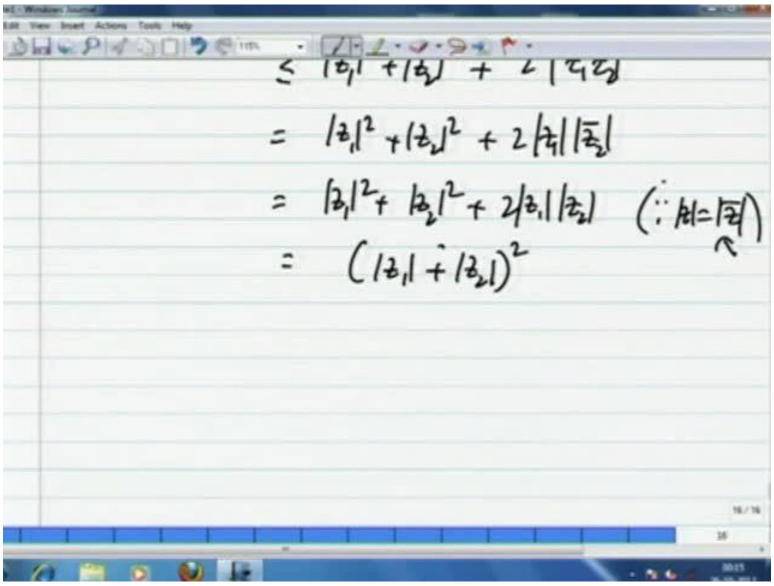
Then this by the by property, we saw previously is less than or equal to modules of z 1 square plus of z 2 square plus, then the real part is less than or equal to the absolute value of the real part, which intern is less than or equal to the modules of the complex number z 1 z 2 bar. And then once again the modules divides over product, so distributes, so the products this is less than or equal to, well this is equal to 2 times modules of z 1 times modules of z 2 bar.

(Refer Slide Time: 01:01:53)



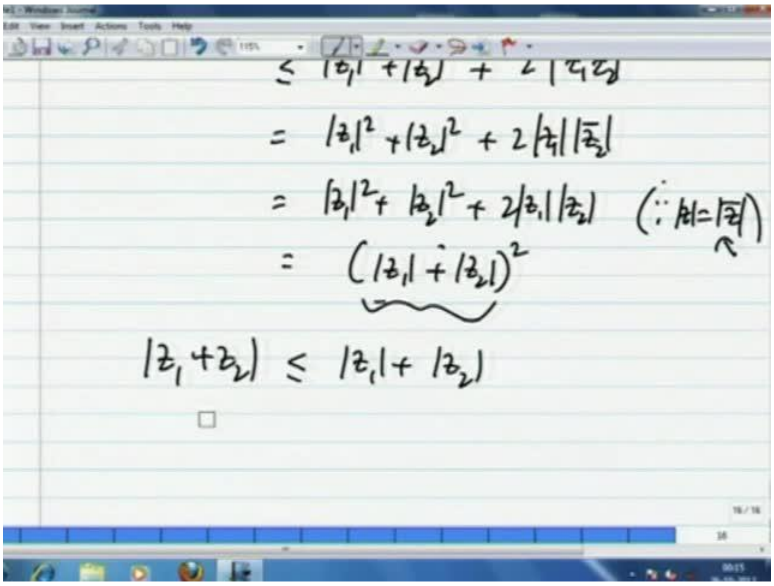
Then the modules of complex number is equal to the modules of its conjugate. So, this gives us z 1 square mod z 1 square plus mod z 2 square plus 2 times mod z 1 mod z 2. That is because mod z is equal to mod z z bar, please verify this again.

(Refer Slide Time: 01:02:16)



This intern is equal to the modules of z 1 square plus the modules of z 2 square plus 2 times the modules of z 1 times the modules of z 2. Well the the modules of complex number is equal to the modules of its conjugate, please verify this fact. So, this directly follows from the definition of modules and conjugate. Then this is the square of the modules of z 1, so modules of z 1 plus modules of z 2. So, this is sorry, is the square of modules of z 1 plus modules of z 2. It is the square of this two real numbers, the the sum of this two real numbers. So, since this modules of z 1 plus z 2 is a real number, the square of this is less than or equal to the square of this.

(Refer Slide Time: 01:03:18)



So, this we conclude that the modules of z 1 plus z 2, since both this are positive real numbers, are the negative of real numbers, the this and this. So, we conclude this is less than or equal to modules of z 1 plus modules of z 2, which is the triangle in equality. The other part of the statement is an exercise to the viewers. So, this note I will stop here.