Complex Analysis
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Module - 3
Complex Integration Theory
Lecture - 5
Cauchy's theorem Part-II

The anti derivative theorem: So, let f be a continuous function on a region G. Then the following statements are equivalent, and the statements are - one: f has an anti derivative capital F throughout the region G. What that means is that i.e. there is a function capital F defined on all of G such that capital F is analytic at every point in G, a analytic on G and the derivative of F is little f of z for all z for any z in G, that is what having an anti derivative means.

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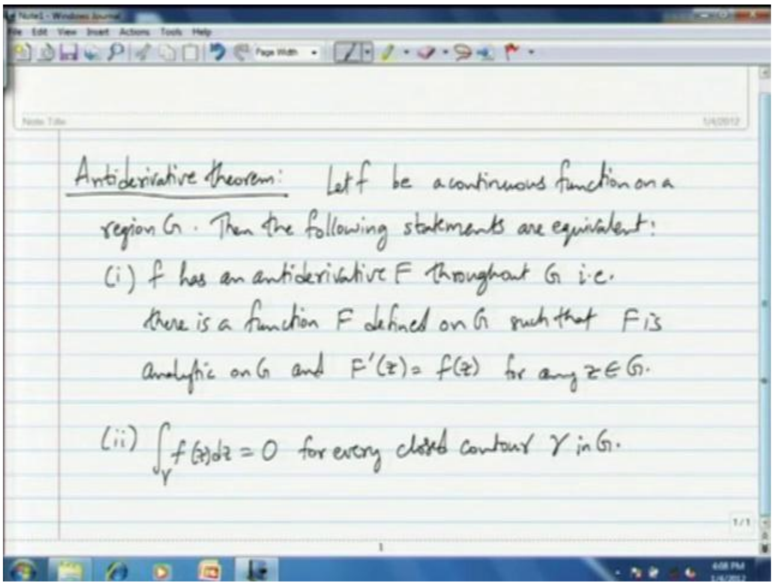

So, this statement is equivalent to the following second statement that the contour integral of f of z d z is equal to 0 on any contour gamma, which is closed. So, for every closed contour gamma, which is completely contained in G; so gamma should be G. So, the first statement is equivalent to the second statement, which in turn is equivalent to the following.

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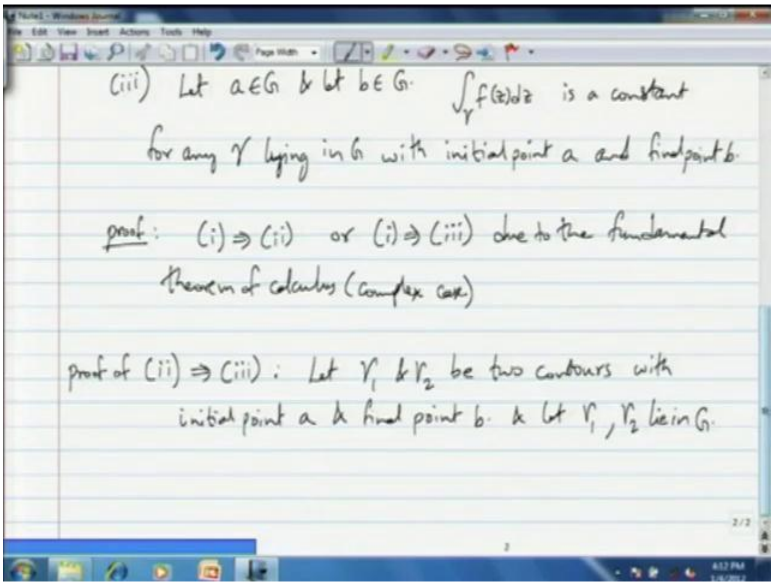

Third statement the let a be a point in G and let b be another point in G. So, then the contour integral of f of z d z over gamma, is a constant for any gamma lying in G with initial point a and final point. What does it means is that if a is a initial point and b is the initial point of the contour gamma then, then it does not matter how gamma travels in G. So, in other words, the contour integral is called path independent. So, the contour integral f only depends on the end points of the n p. So, this statement is equivalent to one and two. So, that is the assumption of this theorem.

So, let us see a proof of this. So, you will immediately recall the fundamental theorem of calculus a complex version which we have proved earlier. It immediately gives you that the first statement that there exists an anti derivative immediately gives you two and three. So, a one implies two or one implies three due to the fundamental theorem of calculus. So, let me write the fundamental theorem of calculus the complex version of course, the relevant version complex version.

So, right because if if little f where the derivative of an analytic function then the integration on close contours be proved will be 0 and we also proved that the integration line integral will depend only on the end points. So, the value in part three of the line integral will be the value of the function capital F at the point b minus the value of the function capital F at the point a. So, this we did earlier.

So now, we need to well there are various ways we can prove this equivalence of these three statements. What we will do is we will prove that two implies three. So, proof of this once again is very easy the proof that two implies three also follows very immediately. What you can do is let gamma 1 and gamma 2 be two parts or two contour's with initial point a and final point b.

So, we will take any two contours with initial point a and final point b and let gamma 1 gamma 2 lie in G. So, we allow them, we will make sure that lie in G and their initial point is a and the final point is b, then what we want to show is that the line integral of f on gamma 1 is equal to the line integral on gamma 2 by assuming that two is true. We are trying to prove that two implies three. So, we will assume two is true.

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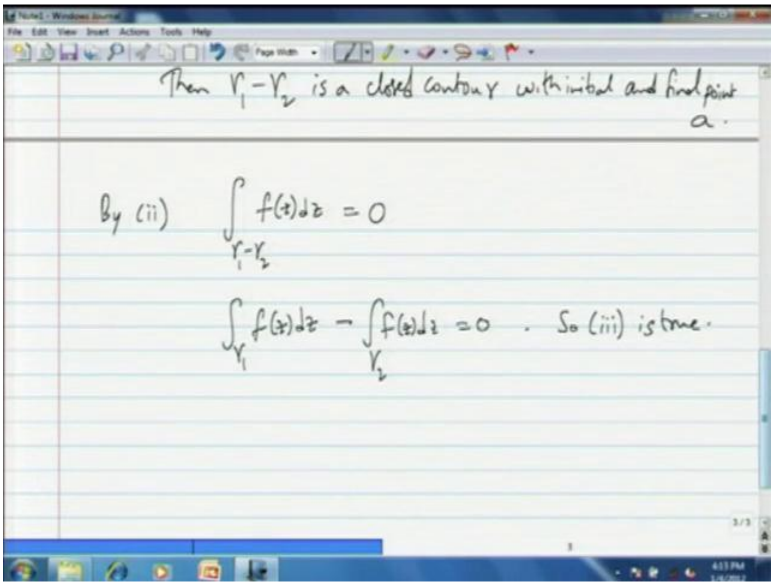

So, then gamma 1 minus gamma 2 is going to be a closed contour is a closed contour recall what minus gamma 2 means you trace gamma 2 in the opposite direction. So, gamma 2 is a close contour with initial and final points a, initial and final point will be a itself.

So, that two tells us by two, we are assuming two, so by two the integral the line integral f of z d z on the contour gamma 1 minus gamma 2 is 0. So, by properties of line integral the left hand side is f of z d z on gamma 1 minus the line integral on gamma 2 of f of z d z is equal to 0. So, three is true three is true. So, that tells that the contour integral does not depend on the path from a to b. So, that is your proof of two implies three that is very easy.

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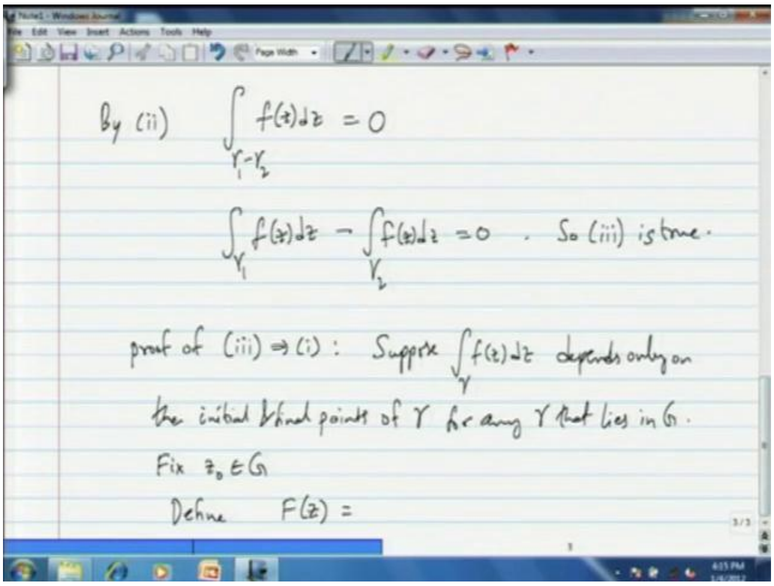

So the, the, the more difficult part of this theorem is to prove that, we will we will prove proof of three implies one. So, we will try to show that statement three that the contour integral is a path independent will imply that there is an anti derivative for the function for the continuous function little f. So, in order to prove this well we will construct a function and then we will show that is analytic. So, suppose, suppose f of z d z on the line integral of that on gamma depends only on the initial point and initial and final points of gamma. That is we are assuming three for any gamma that lies in G lies in the region G.

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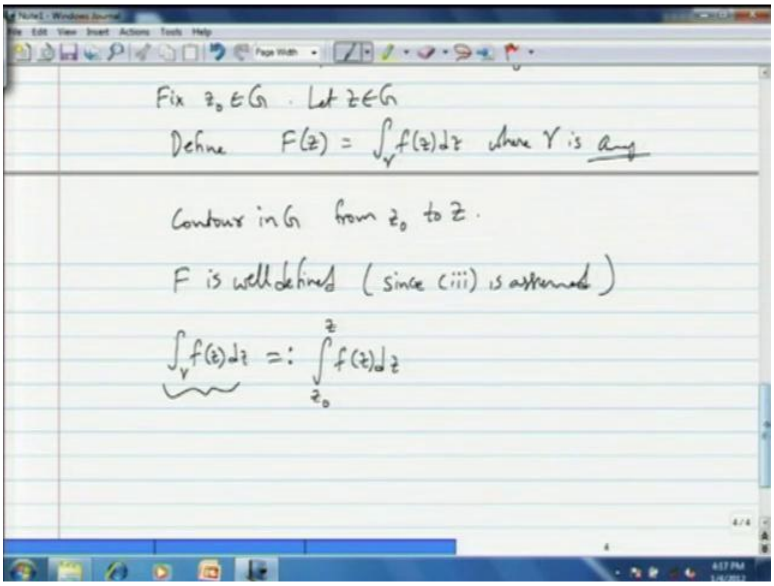

Now, fix a point z naught belongs to G you fix a point and define capital F of z to be to be the integration over gamma of f of z d z. Where, gamma is any contour in G, which lies completely in G from z naught to z. Where this z should belong to G of course, so fix z naught belongs to G let z belong to G z is a point in G. So, define capital F of z to be that.

Firstly some notes here f is well defined because of our assumption, does not matter which contour you you pick gamma is any contour G, but capital F is well defined because we are assuming three here. So, F is well defined in a since three is assumed statement three is assumed that is this is path independent the contour integral is path independent there is the first note.

And then the next is we will introduce a notation we will say this we will are we will write integral over gamma of f of z d z, as as we will write this as integration from z naught to z f of z d z. What that means is we do not care about which path we take from z naught to z as long as that path lies in G. So, you you take the contour integral from z naught to z and that this notation on the right hand side means that you take this.

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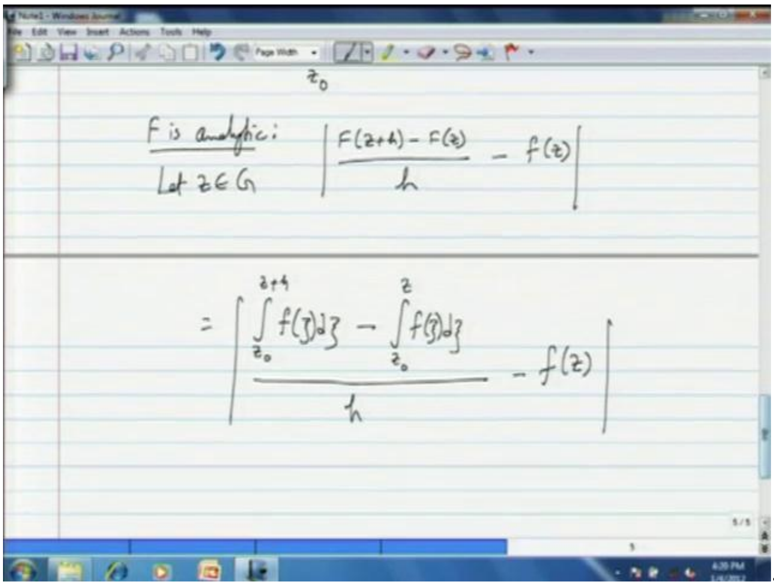

So with this we will rewrite the definition of f of z f of z is the contour integral form z naught to z of f of z d z. So, the only catch here is that, you should make sure that the path from z naught to z or the contour from z naught to z lies within G. So, with that agreement we will introduce this notation. So, we want to show that f is analytic this function. So, we have constructed a function now, we want to show that is analytic first let us estimate or let us see what the modulus of f of z plus h minus f of z by h minus f of z. We are doing this estimation because we suspect that F prime capital F prime of z is going to be little f of z.

So, we first taken z belongs to G and consider modulus of f of z plus h minus f of z by h minus littlie f of z. So, this is going to be by definition of capital f this is the integration this is the line integral from z naught to z plus h of f of I need a different variable here zeta d zeta minus the line integral from z naught to z of f of zeta d zeta that is your definition of capital f of z plus h in capital f of z divided by h minus f of z. So, once again we will assume that the contours from z naught to z plus h and z naught to z lie in G.

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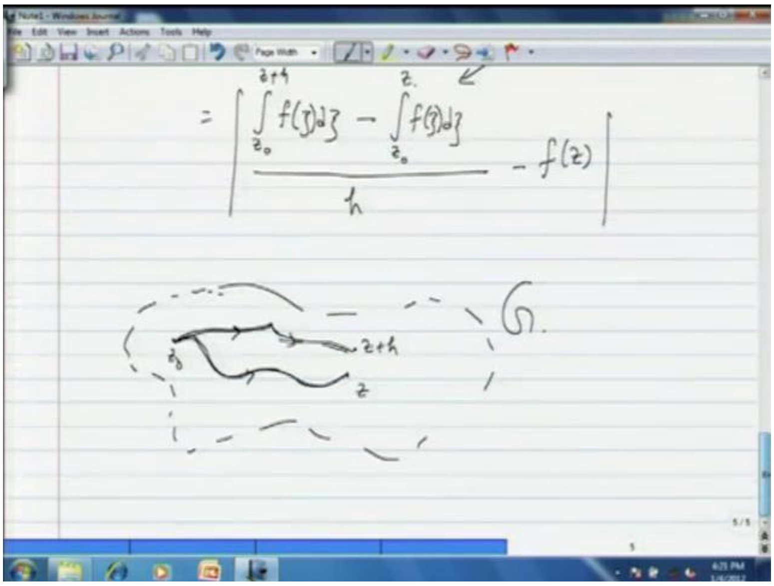

So, here is a schematic picture. Here is your region G is the region G and z naught is a fixed point and your z is here and we want z plus h eventually we are going to let z plus h be very close to z. So, then you take any contour from z naught to z and some other contour z naught to z plus h and then and then you are looking at the difference of these line integrals in the estimate. So, it is clear from this expression here that this is the same as going back on this path and then coming along with this path will give you the path from z to z plus h or a path from z to z plus h.

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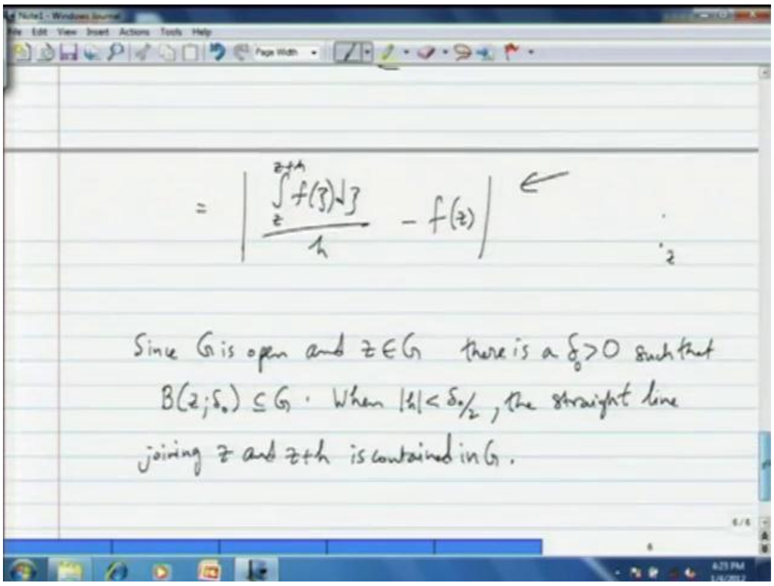

So, the above estimate the above estimate is equal to the integration from z to z plus h of f zeta d zeta by h minus f of z. We are able to do this because the line integrals are path independent in G. The line integrals of little f are path independent otherwise we not be able to go from that step to this step which I have just written.

So, using this a what we can say is that, l since since G is open and z is a point in G, what we do is we can find a small ball around z completely contained in G. So, we will say that a delta ball lies in G. So, there is there is a delta positive let me call that delta naught positive such that b z delta naught a ball of radius delta naught an open ball of radius delta naught centered at z is contained in G. So, when when the modulus of h is less than delta naught by 2 the straight line joining z and z plus h is contained in G. So, here in in this estimate we do not care what path we take from z to z plus h here is z and here is z plus h.

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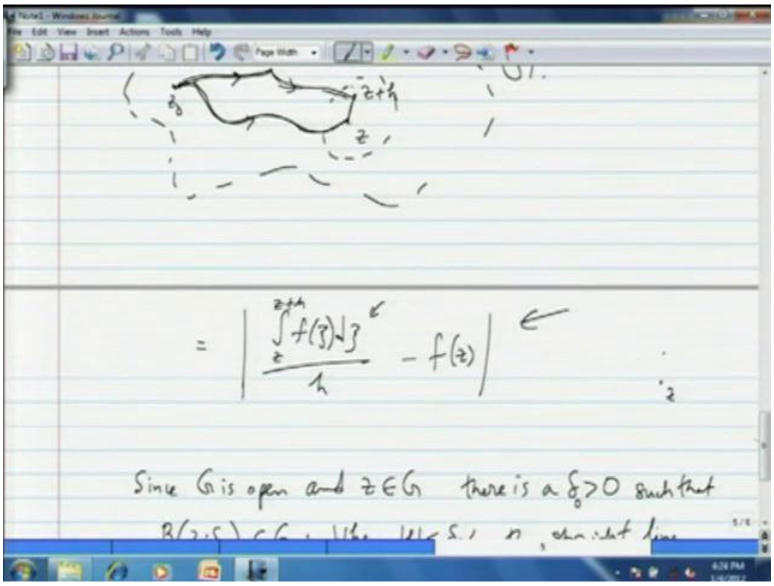

So, let us let me go back to the above picture we do not care what path we take from z to z plus h we are interested in the cont orient integral of f along any path from z to z plus h. So, what we are since G is open when when you consider h to be small enough like this the straight line from z to z plus h is of course, contain completely within G.

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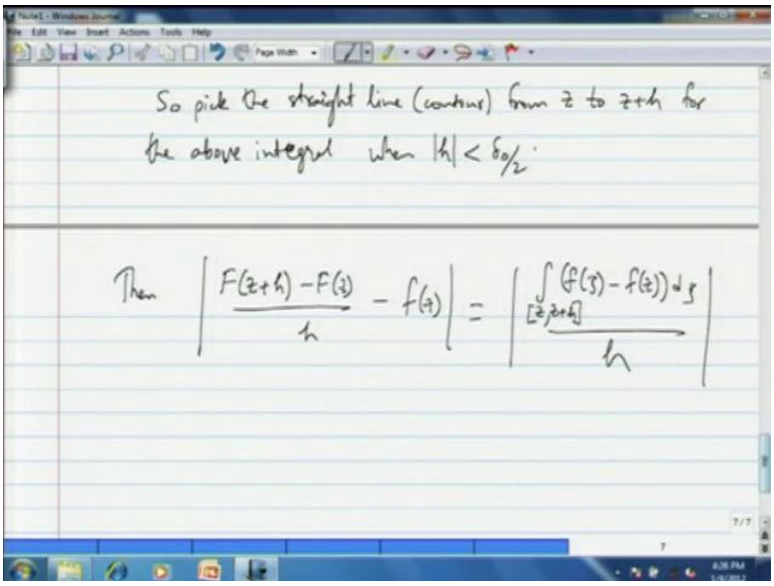

So, that we can pick so pick the straight line path or straight line contour from z to z plus h for the above integral when modulus of h is less than delta naught by 2. So, then this estimate we were making F of z plus h minus F of z by h minus little f of z. This estimate is going to be your modulus from z to z plus h. Now, I need a notation for straight line path let me say z comma z plus h in this kind of interval notation will indicate a straight line path. So, f of zeta minus f of z now, what I can do is say this is f of z times h when I pull it to the numerator. Now, because Ii am considering a straight line path I can include this f of z into the integration. So, this is d zeta divided by h.

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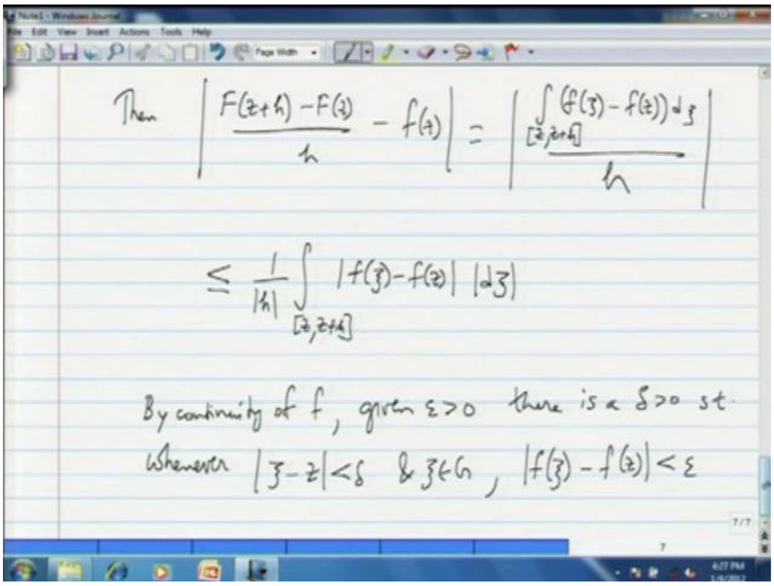

And now, this is less than are equal to by the estimation theorem we had earlier this is less than are equal to 1 by the modulus of h times the integration on the straight line from z to z plus h of the modulus of f of zeta minus f of z times the modulus d zeta. Now, little f is assumed to be continuous that is hypothesis of this theorem so by continuity the integral will be shrunk, that is the idea.

So, by continuity of f given any epsilon positive there is a delta positive such that whenever modulus of zeta minus z is strictly less than delta and zeta belongs to G, modulus of f of zeta minus f of z is strictly less than epsilon by continuity we can find such a delta.

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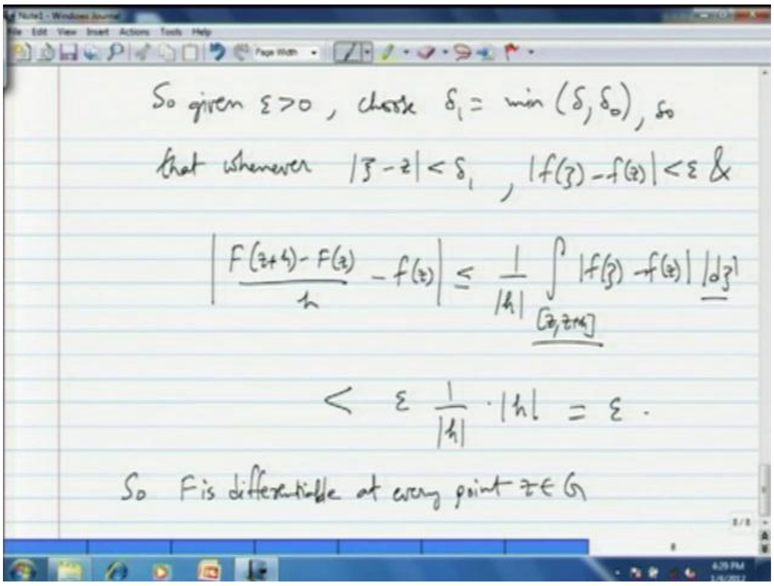

So, given… So if we are given any epsilon positive choose delta 1 to be the minimum of this delta and the delta naught, such that the delta naught ball around z was in G, recall delta naught is from here so, delta naught is from here. So, pick a delta one to be the minimum of these two so that two things are satisfied. So, that whenever modulus of zeta minus z is strictly less than delta 1 two things are true modulus of f of zeta minus f of z is strictly less than epsilon and the modulus of f of z plus h minus capital f of z by h minus f of z which we where estimating above is less than are equal to 1 by modulus of h now, the integral is strictly less than epsilon.

So, I will first write this z comma z plus h integrant modulus f of zeta minus f of z times mod d zeta, the integrant is strictly less than epsilon. So, I have 1 by mod h modulus of h and then the integration of on the straight line of modulus of d zeta will give me the length of the straight line which is mod h. So, this is equal to epsilon. Since, epsilon is arbitrary so f is differentiable at every point every point z belongs to G, z was arbitrary remember and f prime of z we have also proved is equal to little f of z

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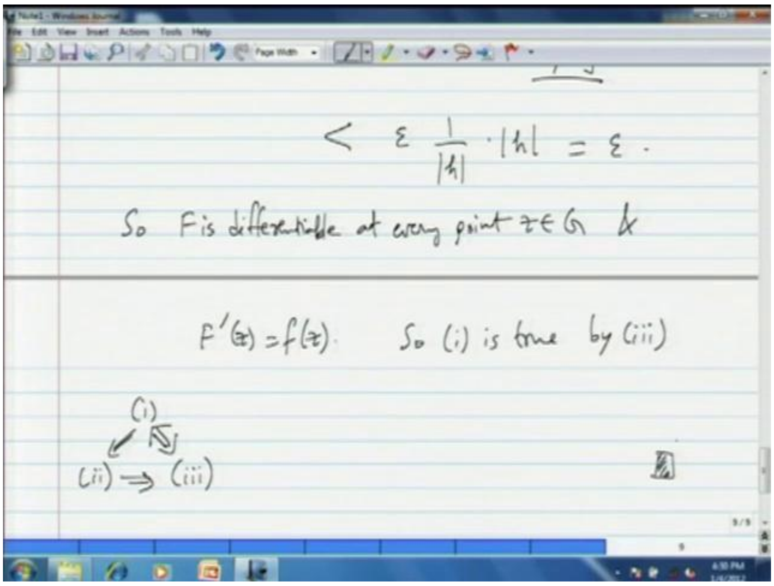

So, f is analytic and then the the differentiation of capital f... So, that shows that f has an anti derivative and that proves one, so one is true by three. So, one implies two and one implies three by your fundamental theorem of calculus and we showed that the two implies three and we also showed that three implies one.

So now, we can go from one to two or two to three and all three to one or anywhere we want. So, all these three statements are equivalent. So, that proves this theorem. So, this tool is useful as we will see further. So, this anti derivative theorem will, will be used constantly.

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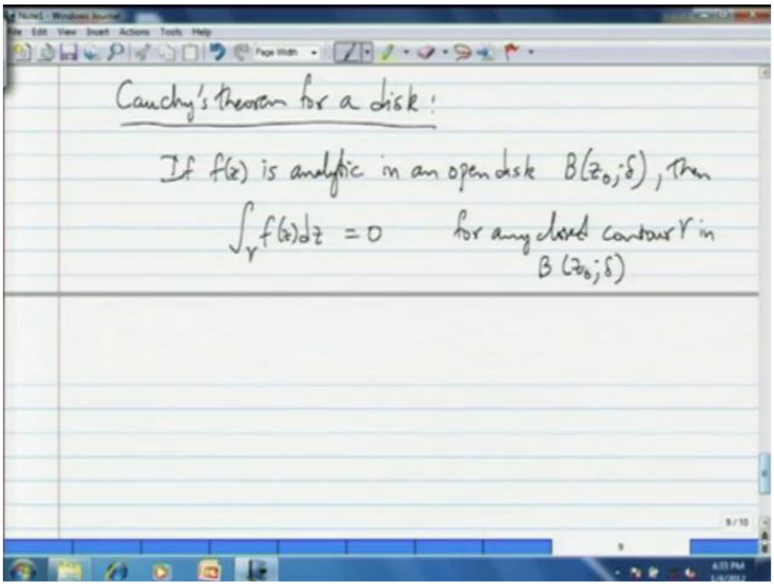

And now, we are we will see another version of Cauchy's theorem. So, we showed that Cauchy's theorem is true when you consider a rectangular region. Now, we are going to show that if you can somehow fit in a rectangle with diagonally opposite ends being you know a fixed point z naught and some variable point z then your Cauchy's theorem will through, the same proof or a similar proof will go through. You you essentially use the fact that Cauchy's theorem works on a rectangle and then apply it here.

So, here is Cauchy's theorem another version. So, Cauchy's theorem for a disk let us say. So, this should be considered as a slight generalization of the earlier version of Cauchy's theorem namely a Cauchy's theorem for a rectangle. So, if f of z is analytic in an open disk B z naught delta, So, this is an open disk center at z naught and of radius delta. Then the (( )) of f of z d z on gamma is 0 for any closed curve for any closed contour gamma in B z naught delta. So, that is the Cauchy's theorem for a disk.

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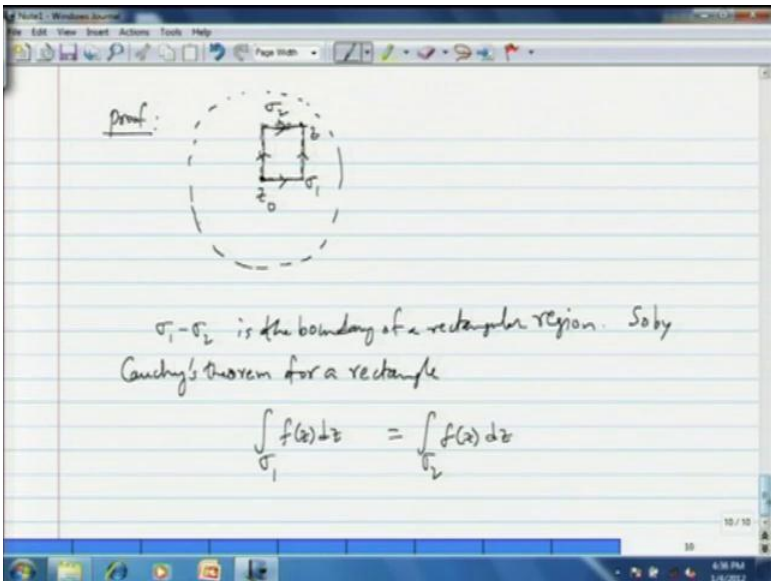

And the proof of this goes as follows and so, here is a disk centered at z naught and of radius delta. So, you consider any point z in the in the disk. So, it could be right above z naught and horizontally on the line on a horizontal line from z naught does not matter, but you can put a rectangle like that. So, in the two cases that I mentioned you will get a degenerate rectangle, but in all other cases you will get a genuine a rectangle like that does not matter and then you consider two contours one is this call that sigma one so you start from z naught travel horizontally and then go up. So, it is a trace of a contour you can parameterize that trace to get sigma one and then you parameterize these two and then you get sigma two.

So, sigma 1 minus sigma 2 is a is the boundary of a rectangular region of course. So, by Cauchy's theorem of course, we are assuming that f is analytic right f is analytic on this on this rectangular region. So, by Cauchy's theorem earlier version so on for a rectangle the integration on sigma 1 f of z d z is equal to the integration on sigma 2 f of z d z.

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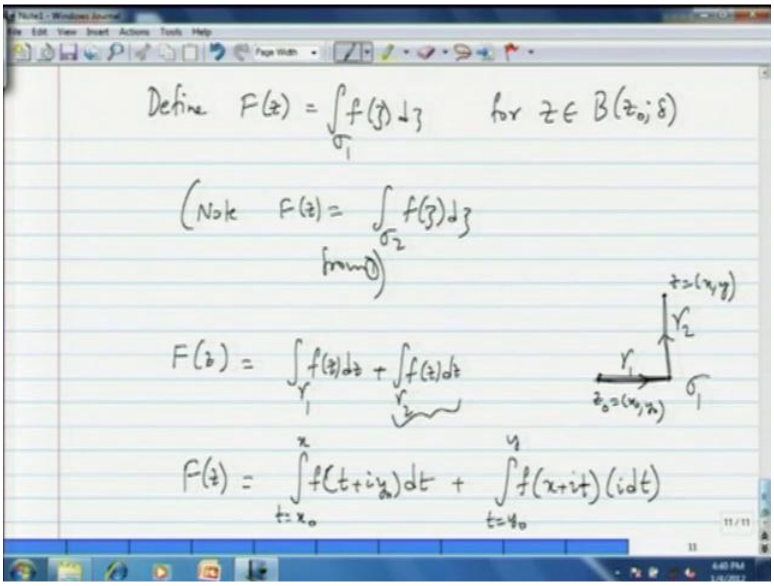

So we will store this fact we will need this fact now, what we do is we will define capital F of z is equal to the integration on sigma 1 of f of zeta d zeta for z belongs to b z naught delta. So, if z is any point in the disk we define capital f of z to be the integration along the sigma 1, sigma 1 is a fixed path. And also in centrally this is note that this is the same as the integration on sigma 2 of f zeta d zeta by the above note by one from one itself from one. So now, what we can observe is that f of z we will decode f of z on sigma 1.

So, on sigma 1, which is first horizontal and then vertical, if you consider your z naught as x naught comma y naught, then your y value stays constant on the horizontal line. So, the integration along sigma 1 goes in two portions; firstly where you have gamma 1 f of z d z plus integration on gamma 2 f of z d z. Where, gamma 1 is this portion and then gamma 2 is the other portion, so sigma one is actually this gamma 1 plus gamma 2.

So, strictly speaking I am talking about the traces of these parts. But we can parameterize them in such a way and the way we parameterize does not matter for the contour integration that we saw earlier. So now, we have split this integral into two portions; on the first portion we note that y is constant and on the second portion y varies.

So, with that observation i can say that if i partially differentiate capital F with respect to y, y is wearing only on the second portion. So, let me first back track let me write f of z equals integration on integration from t equals x 0 to let us say x. So, I am calling this point z equals x comma y in this picture. So when this is f of t plus i y 0 and then d z will be on the horizontal part d z is d x plus i d y, but since, y is not varying you only have d x and I am using the parameter t for x. So I can get a d t really here. And then plus for the second portion Ii have y is varying from y 0 to y and f starts at x plus i y 0 and goes central x plus i y. So, for the y pat I will use a variable t and then I have d x plus i d y is now i d t d x 0, so i get i d t.

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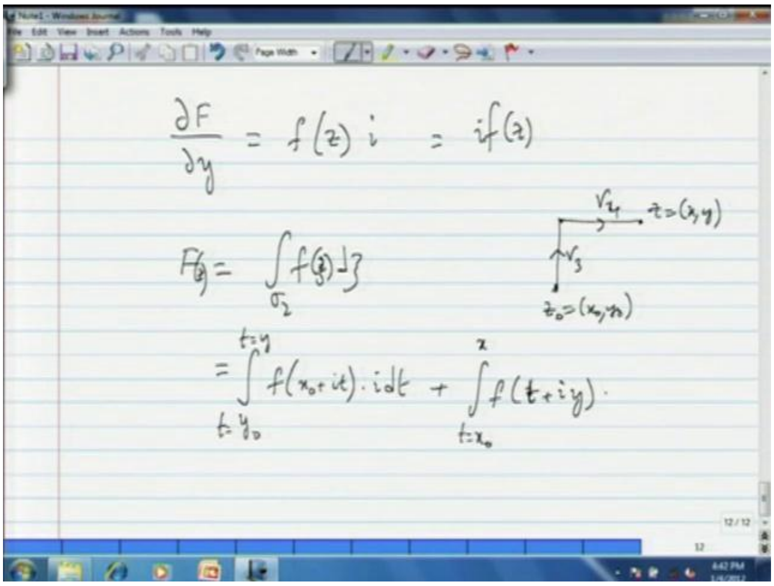

So since, y is not varying on the first portion, when I calculate the partial derivative of capital f with respect to y only the second portion of the centerigal survives and then I get this is f of your y goes in by the fundamental theorem of calculus. So, you get f of z times i, that i there is a multiplier i, so we will get I which is i f of z. Likewise since, F is capital F is also equal to the integration on sigma 2 of f of z of of zeta d zeta. So, the f of z is integration on sigma 2 here is z equals x y and here is z 0 equals x 0 comma y 0.

So, by similar argument you can say that this is equal to this is gamma 3, let us split this into two portions, gamma 3 and gamma 4. So, the integration on sigma 2 is equal to the sum of the integrations on gamma 3 and gamma 4. So, on gamma 3 you get a y is varying, so, y goes from t equals y 0 to t equals y of f of x 0 plus i t times i d t plus on gamma 4 you can use a parameter t and x goes from x 0 to x of f of t plus i y, y value has already reached y from y 0 and then you have d x plus i d y becomes d t now.

So, from this it is clear that the partial derivative of f with respect to x, this representation of capital f is convenient to calculate partial of f with respect to x and partial of f with respect to x from the second portion is clearly f of x plus i y namely f of z. Because, the first portion in the first portion you know your integral is a constant with respect x.

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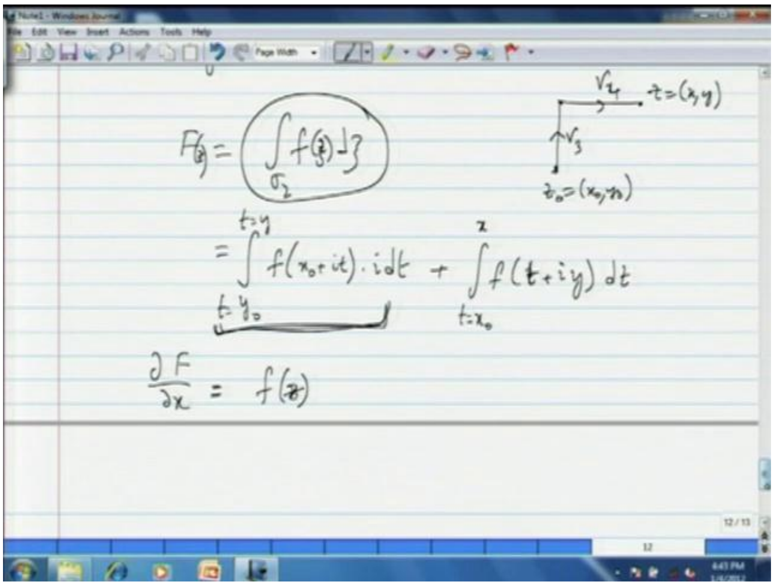

So, so then you see that dou f by duo y is i times f of z, which is dou f by dou x. So, f satisfies the C R the Cauchy Riemann equations and the differentiation of f it can be it can be found out either using dou f by dou x or do f by dou y. So, sticking to do f by dou x we see that the differentiation of f is nothing, but little f. So, and and the partial derivatives partials of f are continuous.

So, by the theorem we had earlier so, f is analytic and we conclude that what do we conclude we conclude that f has little f as f is analytic and of course, f prime of z equals little f of z. One way to calculate f prime remember is just calculating dou f by dou x. We had four different formulae for computing dou f by i mean f prime of z. So, by one form we have do f by dou x is f prime of z equals little f of z.

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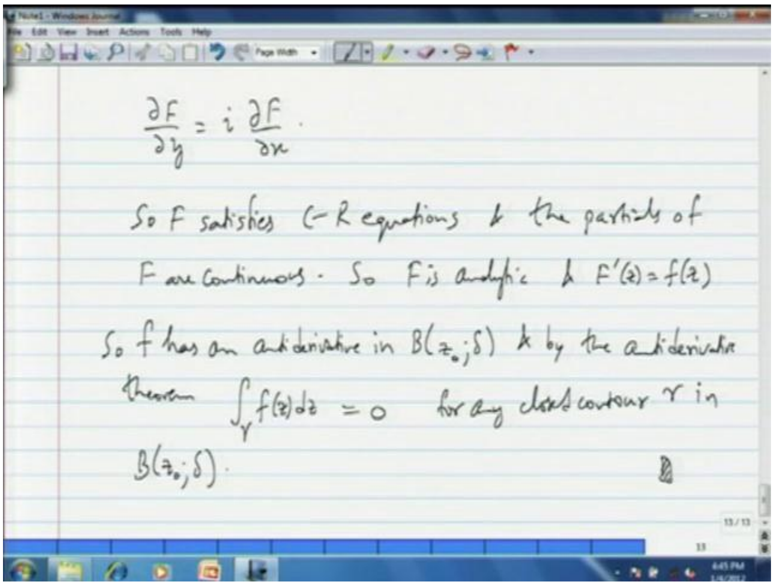

So, f has an anti derivative in B z 0 delta and by the anti derivative theorem therefore, which we just proved the integration of f of z d z on any closed contour is equal to 0, for any closed contour gamma, which lies in B z 0 delta and that proves the Cauchy's theorem for a disk. So, next in order to come up with integral formulae Cauchy's integral formula which, which actually helps us to compute the values of an analytic function by taking it is values on on a circle surrounding that point or value of a analytic function at a point by using the values on a circle surrounding that point.

What we will do is we will want to define first a quantity called an index which measures, how many times roughly speaking how many times a curve goes around a point, closed curve goes around a point. So, the intuition is that we are trying to measure given a point and a closed curve we are trying to measure how many times this closed contour goes around that given point.

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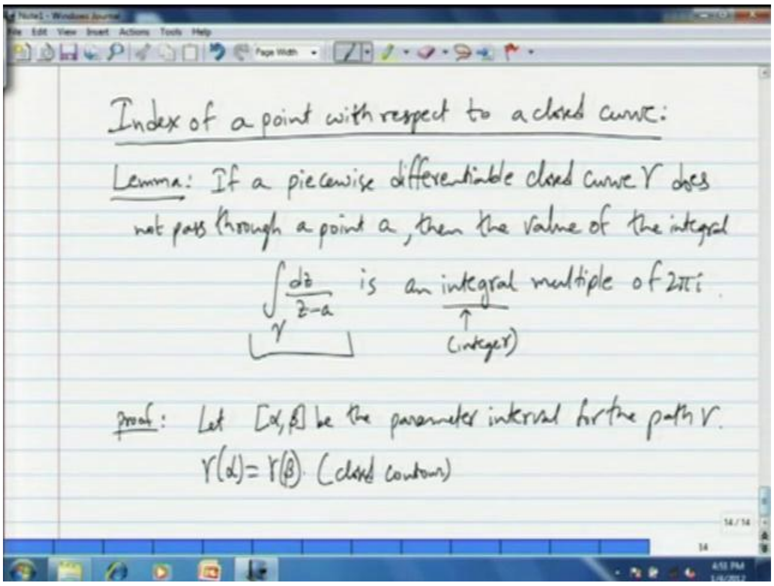

So, here we will start discussing index of a point with respect to a closed curve, So, in order to define or make the correlation between how many times the curve surrounds the point and the following line integral, which I am going to state we need the following lemma.

So, if a piecewise differentiable closed curve gamma does not pass through a point a, then the value of the integral d z by z minus a over gamma is an integral multiple of 2 pi i. So, in short I could have said if a piece if a closed contour gamma does not pass through a point a. So, let me word it this way let me store this and we want to see that this special integral actually is an integral multiple this integral is a integral as an integer multiple of two pi i so, this is to say the this is an integer the integer multiple of two pi i and that integer actually roughly speaking is the measure of the number of times the curve goes around a point and integers have signs of course, so, indirectly we are associating a sign. So, a curve could go around minus 2 times for example, around a point.

So, we are keeping track of the orientation of the curve as well, that is what this integer tells us. So, let us start by proving this lemma proof of it is as follows; now, let alpha comma beta be the parameter interval for the path or the contour gamma it is a closed contour and gamma of alpha is equal to gamma of beta because it is a closed contour.

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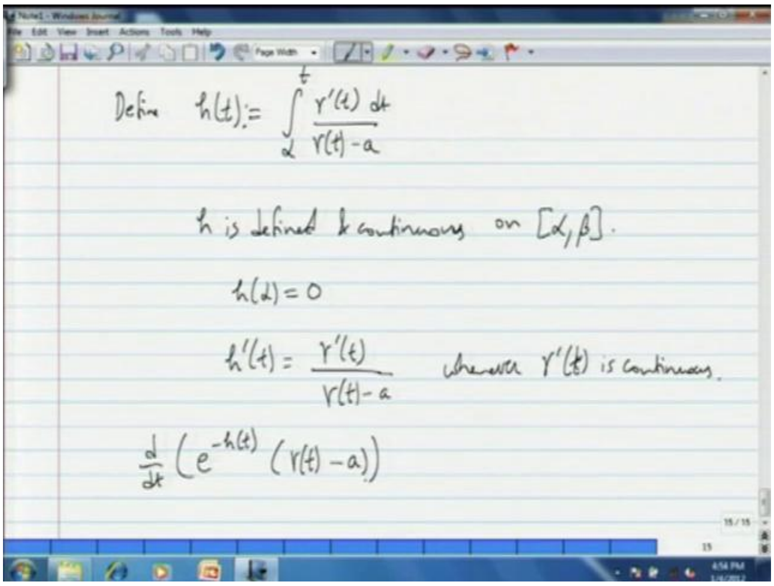

Now, what we will do is we will try to define h of t. So, define h of t to be the integration from alpha to t of gamma prime of t d t by gamma of t minus a. Firstly h is defined, before I say h prime h is defined and continuous, continuous on the closed interval alpha comma beta that is clear from the definition. Also h of alpha is 0 you are integrating from alpha to alpha h of alpha is 0.

So, then h prime of t is simply gamma prime of t by gamma of t minus a by the fundamental theorem of calculus. So, this is true of course, whenever gamma prime of t is continuous. We have we have a piecewise differentiable piecewise smooth curve. So, we have to be careful this is true only when gamma prime of t is continuous. So now, notice that if we want to find the derivative of e raise to minus h of t gamma of t minus a then we get zero.

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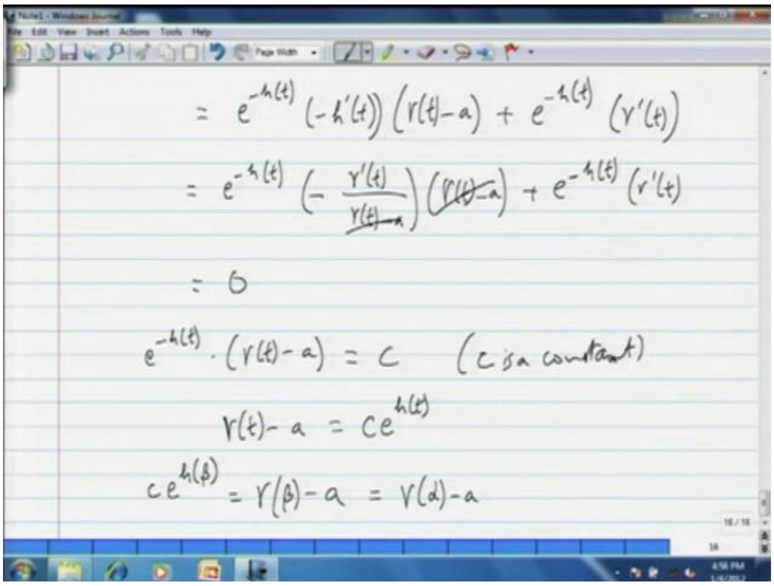

So, how is that, so, let us use differentiation are the product rule for differentiation you get e power minus h of t times minus h prime of t the differentiation of e power minus h of t times gamma of t minus a plus e raise to minus h of t times the differentiation of gamma of t minus a is gamma prime of t. So, this gives you e raise to minus h of t times minus h prime we just saw what h prime is and h prime is gamma prime of p by gamma of t minus a times gamma of t minus a plus e power minus h of t gamma prime of t.

So, these cancel notice that a is never equal to gamma of t. So, these cancel and then you get this is equal to 0. So, e raise to minus h of t times gamma of t minus a is going to give you a constant, because it is derivative is 0 some constant c is a constant, since it is the derivative of this function is 0, you get a constant and so, e power of gamma of t minus a is actually a function c e raise to h of t.

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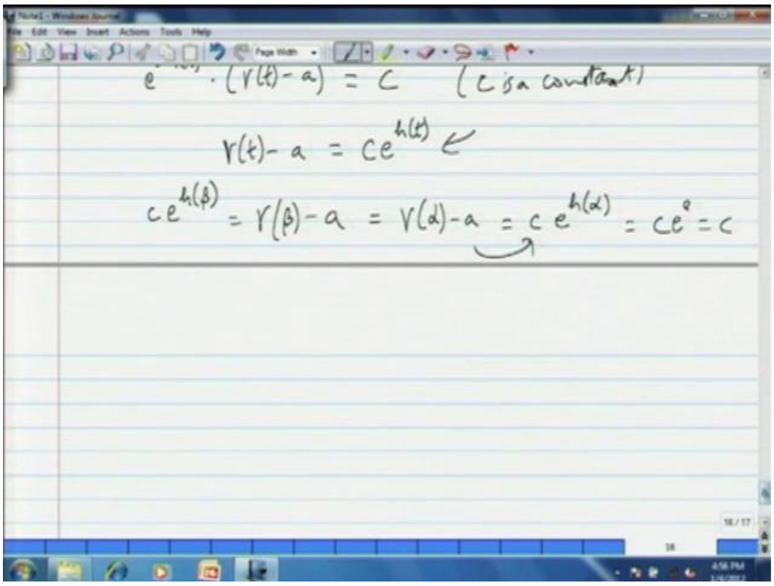

And so, c e raise to h of beta is gamma of beta minus a, which is gamma of alpha minus a because gamma of beta is equal to gamma of alpha, you are dealing with a simple closed or rather a closed contour and then that gives you c times e raise to h of alpha by definition or by this equation here this is your c e raise to h of alpha and h of alpha we know is 0. So, this is c e raise to 0 or this is equal to c.

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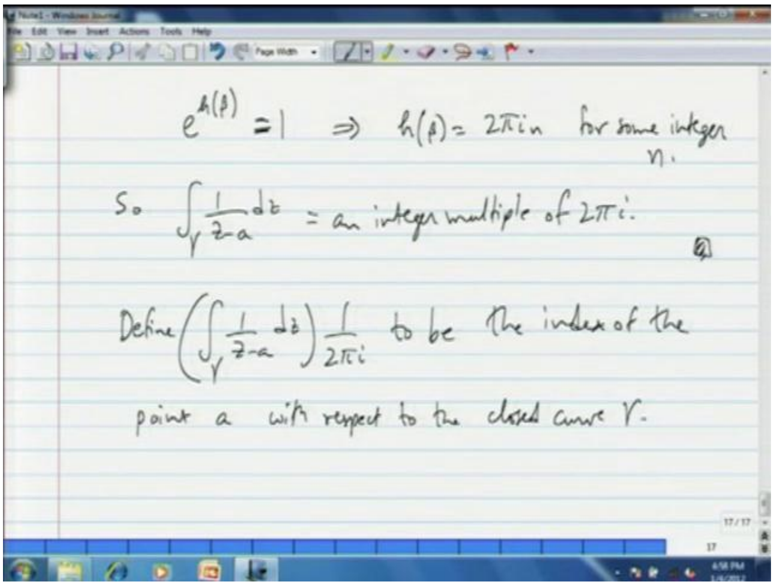

That tells us that e raise to h of beta is equal to 1 and we know precisely when e raise to something is 1. So, this implies that h of beta is 2 pi i n for some integer n, and just what we want. So, so this integration 1 by z minus a, then we parameterize it we get this h of h of t, h of t is nothing but the parameterization of this integral and or h of beta is the parameterization of this integral, and then that is an integral multiple, an integer multiple of 2 pi i.

So, that shows the lemma. So now, we define this integral, the value of this integral divided by 1 by 2 pi i to be the index of the point a with respect to the curve, the closed curve gamma. So, that integer we are going to call that as gamma, and the intuition you will have is that you are keeping track of how many times gamma goes around a with with direction in mine. So, if, if you have that the index is minus 2, then your, your curve is going around a point a in the clockwise direction etcetera.