**Mathematical Complex Analysis**
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Module - 3
Complex Integration Theory
Lecture - 4
Cauchy's Theorem for a Rectangle

Hello viewers, last time we stated the important theorem of complex analysis namely the Cauchy's theorem and in this section, we are going to prove the Cauchy's theorem. So, let me give the statement of the theorem once again.

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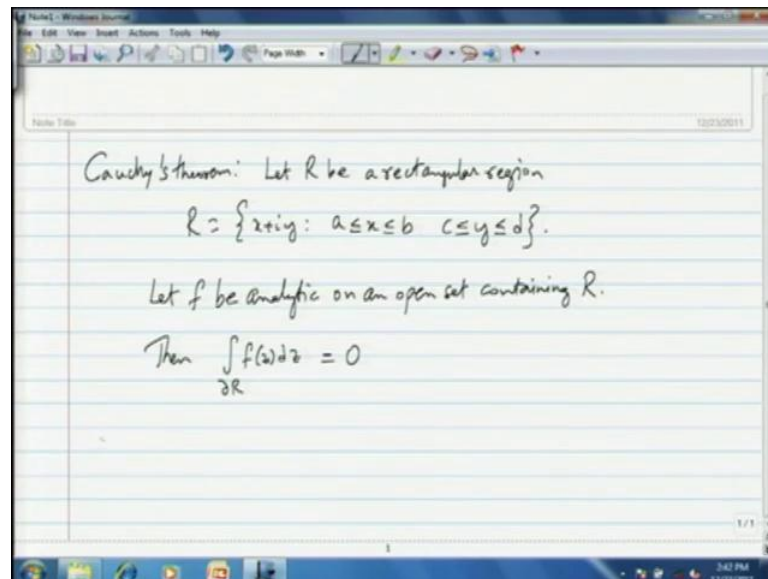
So, 1 version of Cauchy's theorem is, Cauchy's theorem says that, let r be a rectangle or rectangular region. R equals to set of all x plus i y is such that x is in between a and b and y is in between c and d. We will assume that a is not equal to b and c is not equal to d for non triviality. So, it is really a rectangular region with some area and let f be analytic on an open set containing this rectangular region R.

The conclusion is that the contour integration of f of z d z on the boundary of the rectangle R oriented in the positive sense is equal to 0. Once we prove this theorem, it does not matter how you orient boundary of the rectangle, because once its 0, then the rectangle boundary oriented in the opposite direction will also give you a contour

integral equal to 0. So, the orientation really does not matter after we prove the theorem. Hence, this notation is being casual about the orientation.

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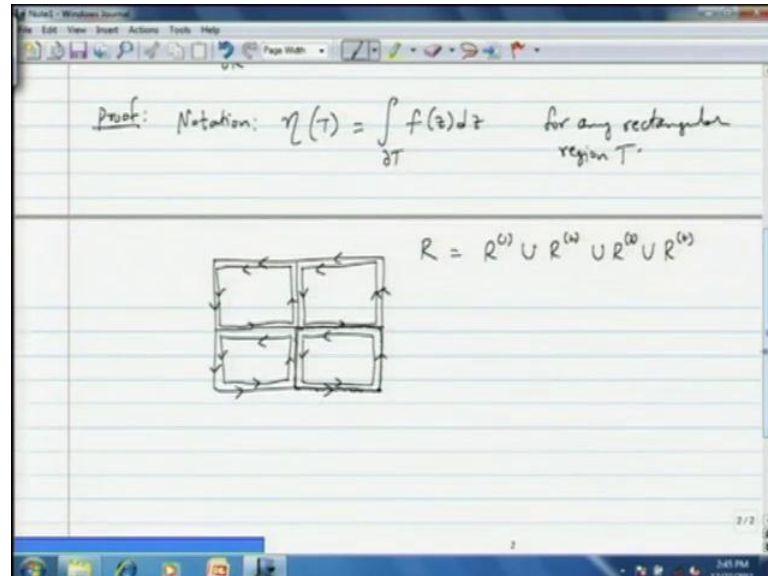
So, here is the proof of theorem. Let us start with a notation, so here is a notation just to avoid writing too much. Let me give you a notation eta of some T, where T is the rectangle. Let us say it is essentially the contour integral along the boundary of T f of z d z for any rectangular region T. So, that is the notation I will use, okay. So, let us start with a picture. So, here is the rectangular region R and let us subdivide this rectangle into 4 equal pieces in terms of area, let us say. So, here is the contour del r, which appears here in the contour integral, the statement of the theorem.

So, like I mentioned last time what we do is that we take these contours, that is the rectangle R or the rectangular region R, which is divided into 4 rectangular regions and then we take these contours, okay. So, this is just a picture. What are these 4 contours actually? They traverse the sides of the rectangle and those inside lines. So, just for lack of space, I am drawing a slightly smaller rectangle inside to indicate that the contour is going on the rectangle and on the lines which are a part of those subdivisions. So, for example, here is a contour like this, which actually should have been this contour which is wriggly right now, okay?

So, instead of drawing such wriggly things, I just drew them inside, so R is now union of four rectangular regions, R 1 union R 2 union R 3 union R 4. So, I am naming them R

super script parenthesis 1, 2, 3, 4 etcetera. Since, when you consider the boundaries of this R 1, R 2, R 3, R 4, you see that the inside lines have contour integration of f in 1 direction and then in the opposite direction. So, for example, this line right here has a contour integration along this and along this. So, they cancel each other.

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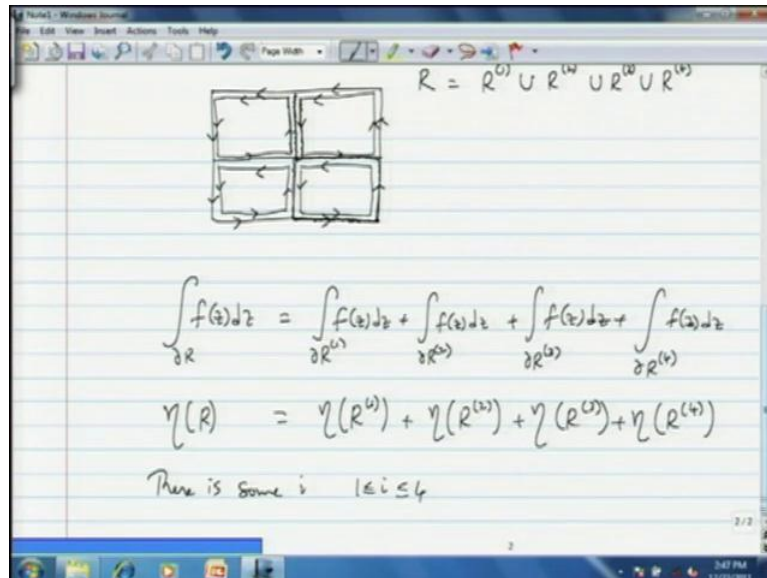
So, what I mean is that when you consider the contour integral of f of z d z on del R, this is equal to the contour integral on del R 1 of f of z d z plus the contour integral on del R 2 f of z d z. I apologize that I am taking R 1 on top, R 2 like that, del R 3 like that, and f of z d z plus integral dou R 4 f of z d z. Since, like I mentioned the contour integral on the inside cancel due to opposite directions, all we are left with contour integral along the boundary of R. So, it is easy to see why this equality is true from the above picture and from our notation we had up there, eta of R is eta of R 1 plus eta of R 2 plus eta of R 3 plus eta of R 4.

So, we can conclude that there is some i, 1 less than or equal to i less than or equal to 4. So, essentially 1 of this is at least one forth of eta of R in terms of absolute value.

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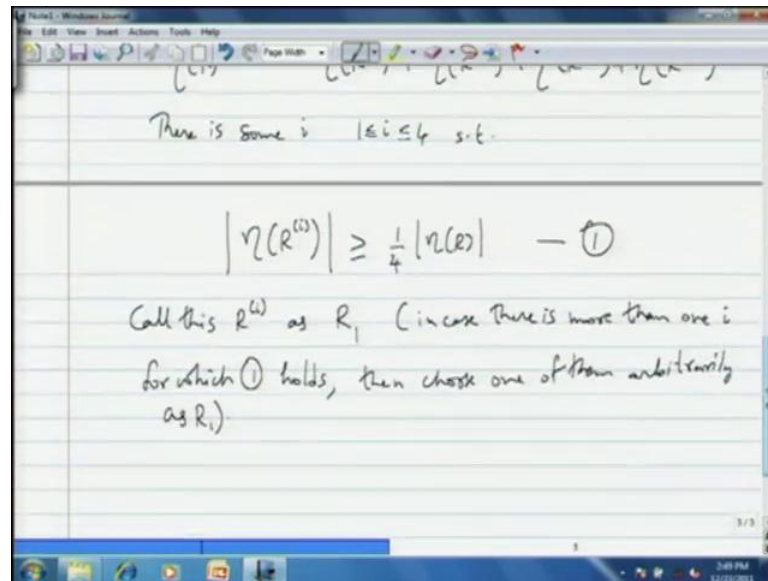
So, there is some i, such that the modulus of eta of R i is greater than or equal to one-forth of modulus of eta R. That is clear because if the modulli of any of this is strictly less than one forth of modulus of eta of R. Then you get a contradiction from triangle inequality. So, there should be 1 i, such that this should hold.

So, we will preserve this inequality. So, let us call this R i as R 1. Now, I am saying that that particular R i for which the above holds, will call that rectangle R 1. Now, there could be more than 1 rectangle for which this holds among R 1, R 2, R 3, R 4. There could be more than 1 for which this holds, and in that case you just choose 1 of them. You can randomly pick 1 of them and call that R 1. So, in case there is more than 1 i. Let me call this inequality 1, for which 1 holds, then choose 1 of them arbitrarily. It does not matter. So, we will call that R 1 and then let us pretend that the top left rectangle was R 1. So, let me go back to this picture. Let us pretend that there is R 1.

So, in that event you take R 1 and do the same. Divide it into 4 equal pieces and continue the process. So divide R 1 into 4 equal rectangles. By equal, I mean equal in terms of area. Let us say so, and call this R 1 1 R 1 2 R 1 3 and R 1 4. Once again, there exists for the same triangle inequality, there exists an i 1 less than or equal to i less than or equal to 4, such that the modulus of eta of R 1 i is at least one fourth of the modulus of eta of R 1 this time. R 1 was chosen, so that it is modulus was greater than or equal to one fourth of modulus of eta of R, rather. So, this is greater than or equal to one forth times one forth of modulus of eta R, okay?

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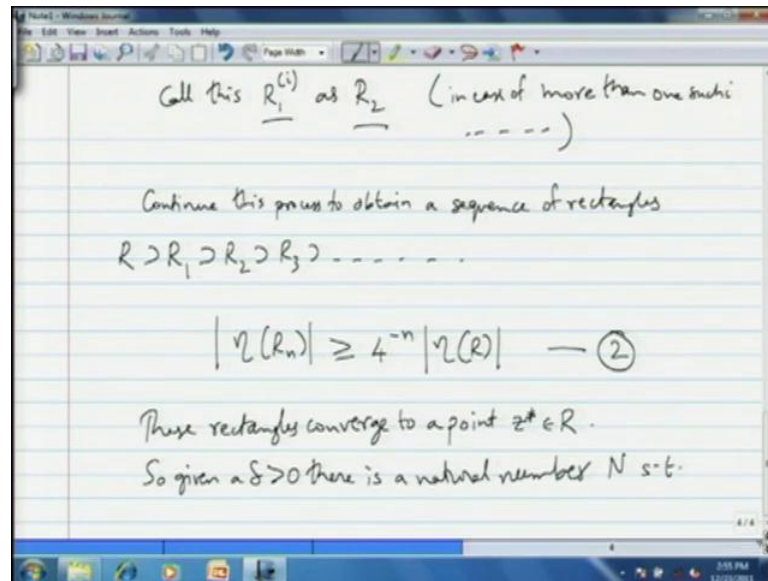
So, call this R 1 i as R 2, so once again in case of more than one such i etcetera etcetera, do the same. You just pick 1 arbitrary and then you have R 1 R 2, which is produced like that. So, continue this process to obtain this sequence of rectangles R 1 R 2 etcetera. So, R 1 of course, it contains R 2. Initially R 1 was contained in R, and then R 2 contains R 3 etcetera. So, continue this process now. So, we get this nested sequence of rectangles. What is nice about this is that, we have an estimate that eta of R n of the next rectangle is at least 4 power minus n times the modules of eta of R.

So, for example here you observe that eta of R 2, you are going to call this R 2. So, eta of R 2 in modulus is greater than or equal to 4 power minus 2 modulus of eta of R. So, continuing this you get eta of R n is at least 4 powers minus n in modules of eta of R. Then, we will preserve this inequality. We call these, let us say 2, okay? Now, then these are nested sequence of rectangular regions R 1 R 2 etcetera. So, from theorem in topology, we know that the intersection of this is non empty and these rectangles are shrinking in area each 1 is one-fourth of the previous 1.

So, they converge to a certain point because their intersection is non empty. So, we will say that these rectangles converge to a point z star. The star of course, is the inside the rectangular region R. It is in the following. So, given a delta positive, there is a natural number capital N such that R capital N is contained in set of all z, such that the modulus of z minus z star is strictly less than delta.

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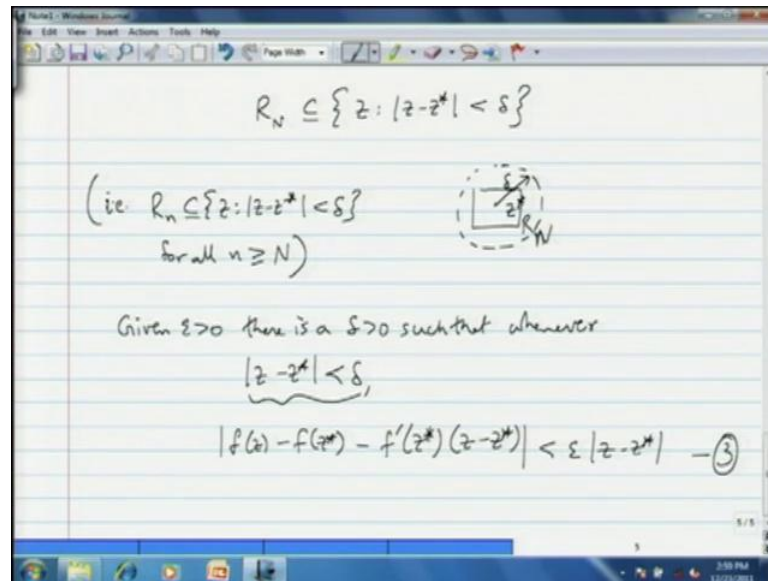
So, if you consider z star which is the point of the convergence of this rectangles and you consider a delta ball of radius delta around it, then there is a natural number n such that the nth rectangles contained in this ball, R capital. Once R n is contained in there, you know that R n plus 1 is which is contained in R n, is also contained in them. So, I notice R n is contained in z such that minus z star is strictly less than delta for all n greater than or equal to capital N, because these are nested sequence of rectangles. So, if capital R n is in there then, all rectangles are in there. So, the convergence of these rectangles to the point z star is in this particular sense.

So, now we are free to choose a delta and given any delta positive, we know that there is going to be some rectangle such that all the rectangles from thereon, in that above sequence will be in this delta ball. Now, we are going to choose this delta in a certain way. So, here is the way we choose delta. So, we know that the f is analytic in an open set containing the rectangle capital R. So, given epsilon greater than 0, there is a delta positive such that whenever the modulus of z minus z star is strictly less than delta, the modulus of f of z minus f of z star minus f prime of z star times z minus z star is strictly less than epsilon times modulus of z minus z star.

This is because the function is analytic at z star, by using the definition of analyticity or differentiability at z star. So, whenever given epsilon is greater than 0, there is a delta

greater than 0 such that whenever this z are in a delta neighborhood of z star, this inequality holds. So, we will call as inequality 3.

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It is clear from definition of the differentiability that f of z minus f of z star divided by z minus z star, the limit of this as z approaches z star is equal to f prime of z star. So, from this, we get this same equality or this is a limit, means the above inequality. Now, let epsilon greater than 0 be given. Choose delta positive such that 3 holds and given that after choosing this delta, obtain a natural number n such that this here holds. So, the set R n is contained in set of all z such that mod z minus z star is strictly less than delta. We saw that given any positive delta this can be done. So, now after choosing this delta we can get such a containment, such a natural number.

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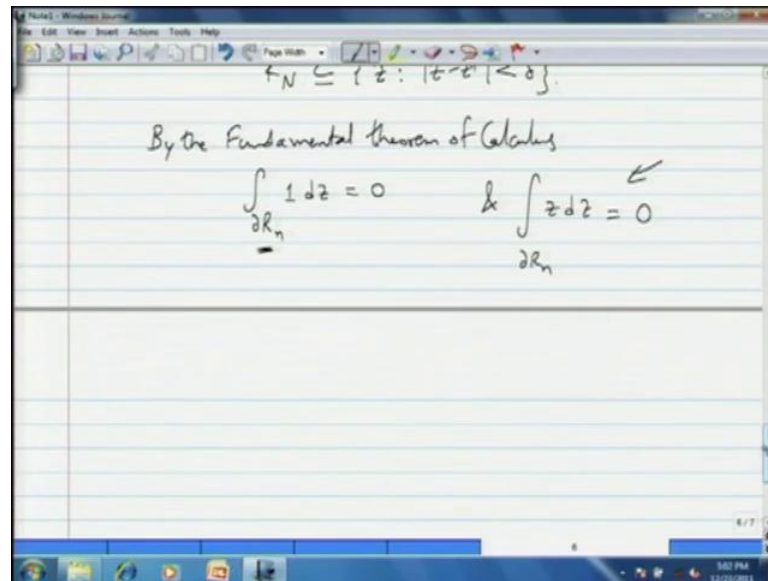
We will make some observation by the fundamental theorem of calculus, the complex version. So, by the fundamental theorem of calculus we know that the the contour integration of 1, the complex function 1 d z on the boundary of R n is 0. Because 1 is the derivative of z on all this complex plane, in particular, on an open set containing this rectangle R n. Likewise, the contour integration of the function z d z on the boundary of R n is also 0. So, we will need these 2 pieces, which are conclusions from the fundamental theorem of calculus.

Of course, the second equality or the second value of the integral follows from the fact then z is the derivative of the function z squared by 2 on all of the complex plane. So, we will use this 2 and notice the modulus of the contour integral on the boundary of R n of f of z minus f of z star minus f prime of z star times z minus z star d z is less than or equal to the modules of the contour integration on the boundary of R n of epsilon times z minus z star d z.

This inequality is right from here 3. I am using 3 and then I can conclude this thing here. So, what I get is the contour integration of f of z d z minus the contour integration on the boundary of R n f of z star d z minus the contour integration of f prime of z star times z d z plus the contour integration of f prime of z star times z star times the contour integration of 1 d z, that is in the left hand side of this inequality.

So, you notice that f prime, I mean this way because, the prime of z star is a constant and z star is a constant. So, you notice that this one and this one and this one, all this 3 are 0, because the contour integration 1 or z on the contour boundary of R n are 0. So, this is less than or equal to epsilon times the contour integration R n of the absolute or a modulus of z minus z star times modulus d z, okay?

So, these are zeroes. These yellow things are zeroes. So, what I get is that the modulus of contour integration of f z on R n is less than or equal to epsilon times. Now, I am going to use the estimation theorem, we solved in the last session. Let us say that the modulus of z minus z star in R n is at most the length of the diagonal. So, here is the picture. So, you look at z star and here is R n containing z star, z star could be anywhere in R n.
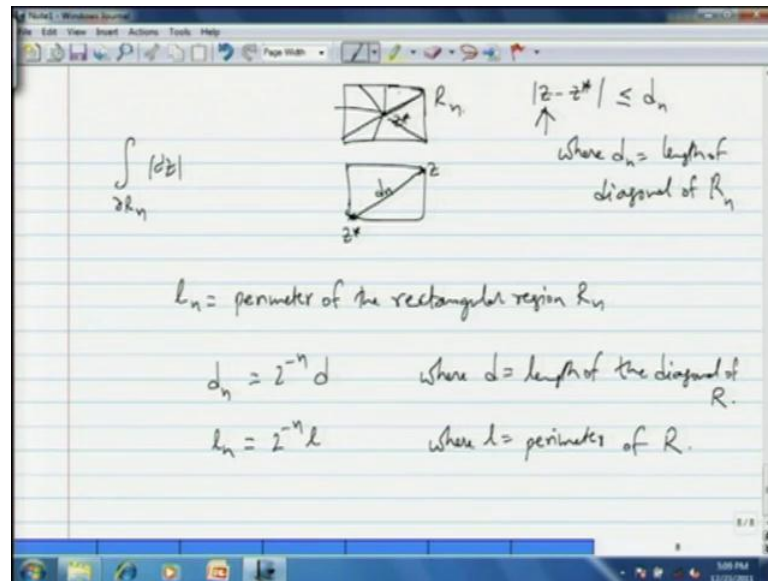
So, the maximum value of the modulus of z minus z star as z varies from the boundary of this R n is essentially the length of the diagonal because z star. For example, could be a point on this diagonal itself that is at one of the corners. If this is the point z, then z is the diagonal length away from z star. So, that is the maximum distance. So, this is less than or equal to z minus z star is at most d n, where d n equals the length of diagonal of R n.

So, this is d n and then I am left with the integration of mod d z on boundary of R n and we know that it is the length of the boundary of foreign, which is essentially the parameter. So, let me write that as l n. So, l n equals the parameter of the rectangular region R n. The way we have chosen these rectangles by subdividing them into 4 pieces of equal area, we have the following. This d n will be 2 powers minus n times d where d equals length of the diagonal of R and l n, which is the parameter of the nth rectangle is going to be 2 power minus n times l where l equals to the parameters of r.

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So, what we get is the modulus of eta of R n is less than or equal to epsilon times d n times l n, which is 2 power minus n times d two power minus n times d, which is 4 power minus n times d. So, this we will call as inequality 4. May be, I use 3, so I am going to call this 4. So, from 1 and 4, I will allow me to go back to 1. I remind you about 1 or may be 2. So, 2 tells us that the modulus of eta of R n is at least so much. 4 tells us that the modulus of eta of R n is at most so much. So, you can compare these 2 quantities.

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From 2 and 4, we conclude that the modulus of eta of R n, that is 4 to the power minus n is less than or equal to modules of eta of R n is less than or equal to epsilon times 4 to the power minus n times d l, okay? So, we will ignore the middle part namely, the modulus of eta R n. We conclude that the modules of eta of R, which we are interested is less than or equal to epsilon times d l. So, this tells us that since epsilon is arbitrary, that modulus of eta of R is actually equal to 0. Here, d and l are fixed constants. We have started at a rectangular region. So, the modulus of eta of R has to be 0, which implies eta of R itself. So, this proves the Cauchy's theorem.

So, before proceeding further, let us see the interpretation or 1 physical interpretation of Cauchy's theorem. So, I am going to give the fluid mechanics interpretation of what Cauchy's theorem is telling us. There is an equivalent version for electricity or electric field that 1 can construct.

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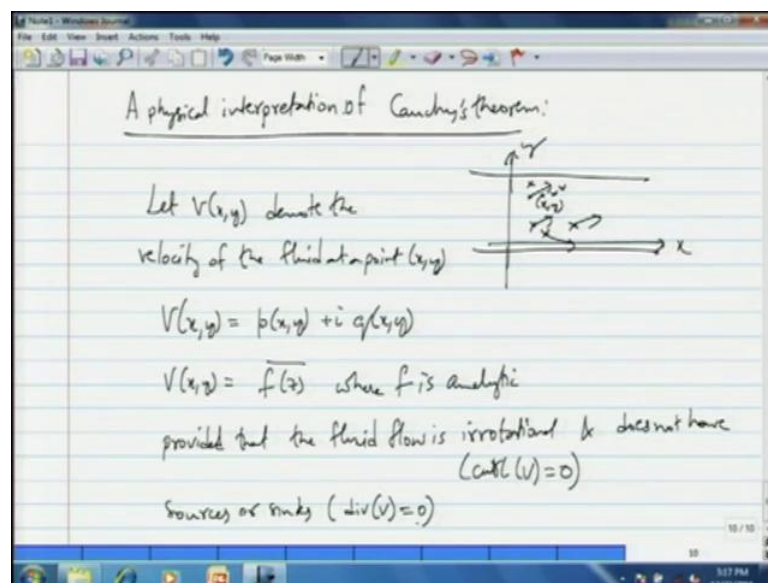
However, here there is physical interpretation of Cauchy's theorem. So, few sessions earlier we saw that when one considers that a 2 dimensional fluid flows then 1 can use complex analysis. Then, the velocity function at each point is the conjugate of the analytic function, if we assume that the fluid flow is irrotational and does not have sources or sinks.

So, irrotational just means that the fluid is not just rotating around a point roughly speaking and then it is not stagnant anywhere. Likewise the source or sink tells us that

there is no fluid produced anywhere in the flow or there is no volume of fluid which disappears during the flow. So, if you make this assumptions then we get that the velocity field of a fluid flow is the conjugate of an analytic function. So, let us first start by interpreting what the line integral means, in this context.

So, let me draw a picture first. So, here is a certain conduit of a deep channel perhaps and then we can assume that if we consider parallels strips along this conduit anywhere, then the flow is 2 dimensional. So, here is a point x y, let us modulate it as x y. Then, there is a certain velocity vector based at the point and there are points everywhere. At each point there is certain velocity to the flows. Here, v of x y denotes the velocity of the fluid at a point x comma y and we write v of x y is p of x y plus i times q of x y, where p indicates the fluid velocity parallel to the x axis.

So, we will have some imaginary or we will imagine some axis here. This is an x axis and this is the y axis etcetera in that conduit. Then we have t is the velocity along x axis and q is the velocity along y axis. Then, v of x y is a conjugate of z, where f is an analytic, provided that the fluid flow is irrotational. This is, to say that the curl of the velocity field is 0 and does not have sources or sinks. The fluid flow does not have sources of sinks sources or sinks, says that the divergence of this field is 0.

These two we saw now. These 2 conditions actually allow us to form the Cauchy Riemann equations. They allow us to conclude the Cauchy Riemann equation holds for the function v bar which is f. So, with these conditions, we have v is the conjugate of an analytic function now.
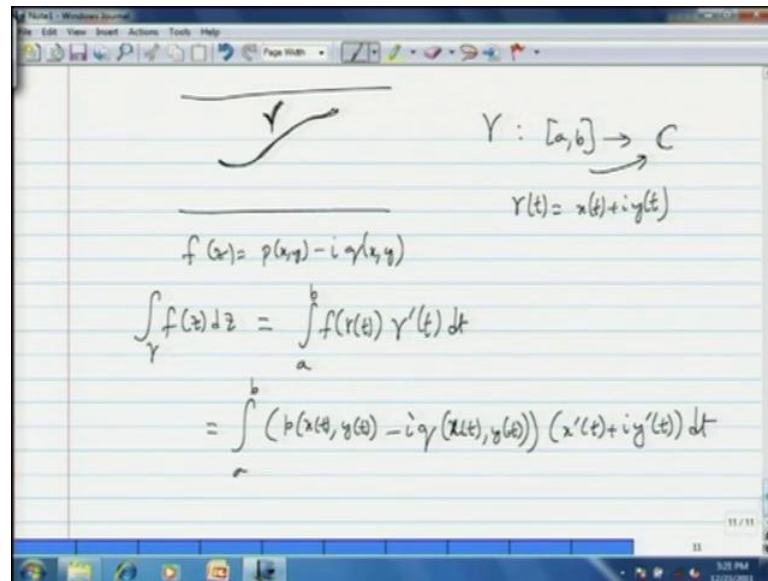
We will first start by interpreting a line integral. So, let us for simplicity, assume that the gamma is a certain arc like that, it is a curve. So, in terms of what we have been describing as gamma, what I have a drawn here is the trace of a certain gamma, which is a parameterized curve. So, remember that the interpretation of line integral is independent of how we actually parameterized the curve itself, okay.

So, I will just call this piece of string and this piece of curve here as a contour gamma. I am keeping it simply and I am not allowing it to self intersect or intersect anything. So, here is a simple piece and let us now assume f of z is p of x y minus i times q of x y from above, because v is p plus i q. So, the contour integral of f of z d z along this kind of smooth curve is the integration from a to b, where gamma is parameterized from a to b to c of f of gamma of t gamma prime of d t. So, if I call gamma of t as x of t plus i times y of t, so, gamma after all has its image in c, then I can write this line integral as a to b s is p of gamma of t gamma of t is a point in the complex plane.

So, strictly I should be writing x of t comma y f t. So, p of x of t comma y of t plus or rather minus i times q time q of x of t comma y f t. Then gamma prime so times gamma prime, where gamma prime is x prime of t plus i times y prime of t d t.

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So, I will write this for simplicity. I will ignore it. I will drop writing x of t y of t etcetera. I will simply write this as p minus i q times x prime plus i y prime. Since, what each 1 is a function is clear from the context. So, let me multiply this 2 so that a to b p x prime plus q y prime plus i times of p y prime minus q x prime times d t, all this is equal to integral r f of z times dz.

So, we can write this as or split this into integration from a to b p x prime plus q y prime plus i times integration from a to b of p y prime minus q x prime d t. Now, we will interpret each of these integrals to understand what this line integral is doing.

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So, here is curve gamma and at a point in time t here is a point gamma of t. Let us say that gamma prime of t gives you the tangent vector to this curve at a point gamma of t. So, here is a gamma prime of t which is the tangent vector to the curve and v, which is let us say, the velocity vector based at this point gamma of t, is somewhat that, this is d t here. Then, you notice that the integrand here in the first integral t x prime plus q y prime, which can be interpreted as the dot product of v with the tangent vector gamma prime of 3.

So, here is the velocity vector v. So, v dot gamma prime of t is a first integrand. So, the real part takes the dot product of the tangent vector with the velocity vector and plus i times the imaginary part. The imaginary part is the dot product and can be interpreted as the dot product of the vector p q. Let us say, with the vector y prime minus x prime, now y prime minus x prime is the vector, which is perpendicular two x prime comma y prime.

So, it is a vector perpendicular to the tangent to the curve which means, it is a normal to the curve and from the direction of y prime minus x prime, we can make out that it is a pointing inwards. So, it means that it is pointing from the left of the curve to the right of the curve. So, from the region left of the curve to the region right of curve, starting from the point gamma prime of t. This integrand in the imaginary part can be interpreted as the dot product of v with the normal in that particular direction d t. So, what is the physical interpretation of these two?
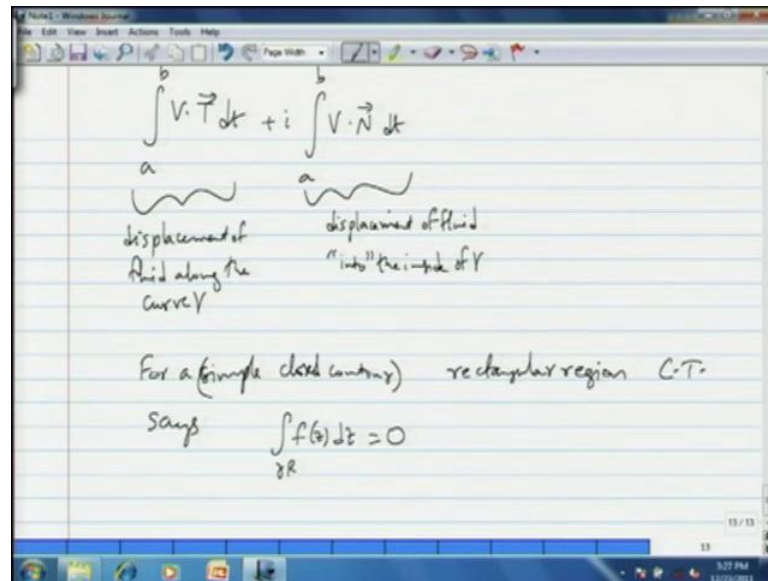
Velocity when integrated along a direction, if we assume the displacement of the fluid or the amount of fluid, then this integrand in this integration actually gives you the displacement of the fluid along the curve gamma. This imaginary part stands for the displacement of fluid into this. I mean this into does not make sense for a non closed curve, but let me just write it is into the inside of gamma. So, after writing it let me explain that, it is called the flux of this velocity vector with respect to this curve at the point gamma of t.

So, it is the component of the velocity which is perpendicular to this particular R. So, for a simple closed curve contour or somewhat, we have a proved it. So, this is more general, but for a rectangular region at least, we proved the Cauchy's theorem. So, for a rectangular region Cauchy's theorem says that the integration of f of z d z where f is analytic is equal to 0, where f is analytic on and inside the contour R.
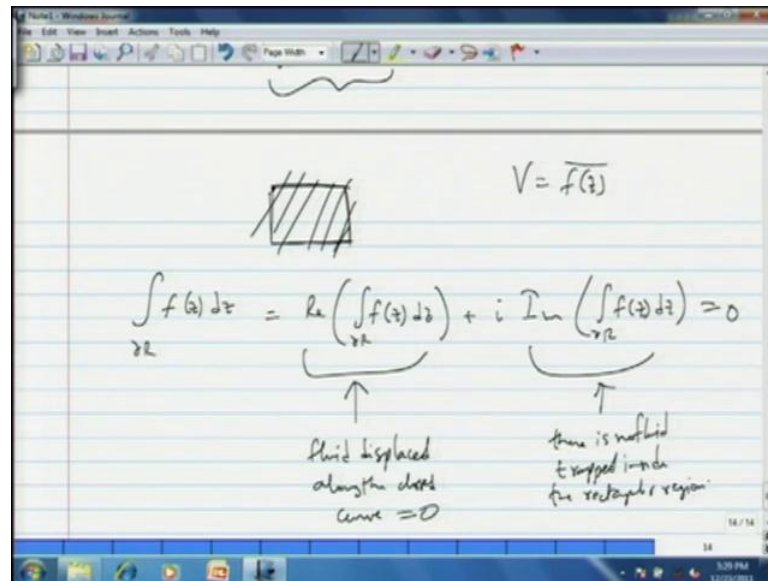
So, from the fluid flow interpretation, what we have is, if the fluid flow, I mean if the velocity field v is f of z bar for an analytic function f on and inside this rectangle region R, what we can say is that, there is no amount of fluid. This is telling you that the integration of f of z d z on dou R is equal to real part of the integration of f z d z on the boundary of R plus i times the imaginary part of this which is given to be 0, by Cauchy's theorem.

So, since the real part and the imaginary part are both 0, what this part is telling to you is that, the fluid displacement or fluid displaced along the curve along the closed curve is 0. So, there is no fluid which actual keeps flowing around a contour. So, there is no stagnation in that sense and the imaginary part being 0, tells us that once again there is no fluid trapped inside the rectangular region.

So, whatever fluid crosses the boundary of the rectangular region or into the rectangular region has to come out. So, there is no fluid which remains trapped. That is what this says because when you add up the normal components of this velocity, they give you the total fluid which remains inside, so that is 0. It tells you that there is no fluid which is trapped inside. So, this is the physical interpretation of Cauchy's theorem. Next, let us consider the following extension of Cauchy's theorem, a mild extension of Cauchy's theorem.
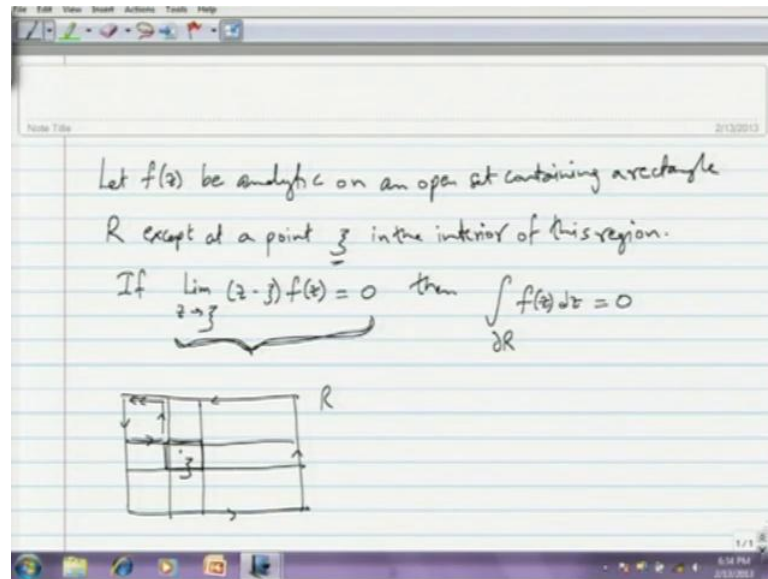
Let, f of z be analytic on and inside on an open set containing a rectangle R. By rectangular region R, I mean the rectangular region R except at a point zeta which is in the interior of this region. So, if it is true, that is, if the limit as z goes to this point, that is the special point z minus zeta times f of z. If this limit is equal to 0, then the integration on the boundary of this rectangular region R of f of z d z is still going to be 0. So, this is the modified Cauchy's theorem. So, we have a similar setting or almost the same setting except that there is a special point zeta, where f is not exactly analytic.

Here, f is not analytic perhaps it is not even defined, but what is important is that it satisfies this mysterious condition. Then, it still says that the integration on the boundary of R of f of z d z is going to give us 0. So, we will decode this later, but for now this can be considered as a slight extension of Cauchy's theorem. Let us see how or why this is true?

So, what we do here is the schematic. So, here is the rectangular region R and then there is a special point zeta, which is in the interior of R. So, what we do is that we subdivide this rectangle into further rectangles and then we consider a small enough rectangle around the special point zeta. So, we will use the contour integration on the boundary of R in the contour clock wise direction. So, this dou R is the boundary of R in the positive direction. Then using the Cauchy's theorem, we will use contour integration like that.

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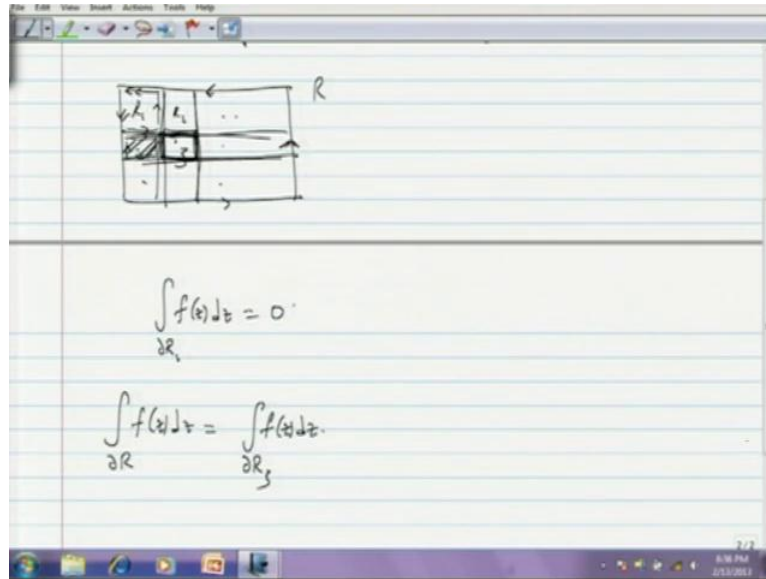
Using the Cauchy's theorem, we just proved that we will see the integration on any of this R 1 R 2 etcetera any of this rectangle, I will be a little informal about this. I will suggest the a boundary of any of R i is the integration of f of z d z, which is equal to 0. That we know, because f is analytic on an set containing this whole rectangular region, perhaps except on this little which surrounds the special point zeta.

So, for that special point zeta, I am going to make a special adjustment. So, firstly notice that the integration on the boundary of R in the contour clockwise direction is now equal to of f of z d z is equal to the integration on this smaller rectangular region, I can assume that to be a square. I can adjust these sides so that that small rectangle is actually a square so that I will say that it is R zeta, which stands for the rectangle containing zeta. So, the integration is equal to the integration of the boundary of the rectangle R zeta of f of z.

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So, we will say that this equation 1 and with the given condition, given that epsilon is greater than 0 since limit as z goes to zeta of minus zeta times of f of z is equal to 0. Given that epsilon is greater than 0, there is a delta positive such that whenever modulus of z minus zeta is strictly less than delta, the modulus of z minus zeta times the modulus of f of z is strictly less than epsilon, because the limit as z goes to zeta of this quantity 0, the modules of this is arbitrary less as z goes to zeta. So, I can do this and what this is telling me is that the modulus of f of z is strictly less than epsilon divided by modulus of z minus zeta, whenever this occurs. So, given epsilon, I can choose this delta so that two occurs.

The Pre estimation theorem that we have in the last session, the modulus of the integration over dou R zeta of f of z d z is less than or equal to the integration over dou R zeta of the modules of f of z modulus d z. This is now by 2, strictly less than epsilon or I can say this less than or equal to epsilon times integration over dou R zeta of modules of d z by modules of z minus zeta. By choosing R zeta to be small enough, I notice that I can adjust these vertical lines to make this R zeta as small as possible, that is here R zeta is rectangle R zeta. I can make it is as a square centered at zeta and I can make it as small as I like.

So, then I have this estimate. So, when R zeta is a square centered at zeta and your z varies on the boundary of R zeta, what we have is the modulus of z minus zeta for z belonging to dou R zeta is such that the modules of z minus zeta is greater than or equal to s by 2, where s is the side of length of the rectangular region R zeta, okay?

So, when this occurs, 1 by modules of z minus zeta is less than or equal to 2 by s. So, the modulus of d zeta d z on the boundary of R zeta divided by modulus of z minus zeta is less than or equal to 2 by s times the integration over dou R zeta of modulus of d z. This is nothing but the length of this curve dou R zeta, which is 2 by s times 4 s, so this is 8. So, this quantity is less than or equal to 8. So, this is less than or equal to 8 epsilon.

Since epsilon is arbitrary, so, epsilon greater than zero is arbitrary. So, given any epsilon I can do this. What I conclude is that the modulus of this integration is 0. So, the only complex number with 0 modulus is 0. So, the integration over dou R f of z d z is actually R zeta, but this is equal to integration over dou R of f of z d z by the equation 1. So, this is equal to 0. So, that proves the condition. So, with this I will stop this lecture here.